Bol. Soc. Paran. Mat.
©SPM – E-ISSN-2175-1188
SPM: www.spm.uem.br/bspm

(3s.) **v. 2026 (44) 2** : 1–7.
ISSN-0037-8712
doi:10.5269/bspm.80379

Analysis of Rayleigh Waves and Surface Ellipticity: A New Numerical Approach for Comparing Existing Formulas in Literature

H. Lakshmi Mittipati, S. K. Nuslin Bibi, P. Rahira, P. Sarada Devi, G. Diurga Priyadarsini

ABSTRACT: Rayleigh waves are fundamental in geophysics, seismology, and material science, e.g., when studying the behaviour of surface waves in anisotropic and orthotropic materials. Surface ellipticity, which is one of the main features of Rayleigh waves, is a vital property of the latter that can be applied in applications such as seismic inversion and subsurface imaging. A novel numerical approach to the comparison between the discrete least squares approximation (DLSA) of Chebyshev-Gauss-Lobatto (CGL) nodes and the already known aspect ratio formulas of Rayleigh wave surface ellipticity are given in this work. We also use the numerical approximations in DLSA in order to test a number of analytical formulations in the literature in a systematic way and to establish their accuracy. The CGL node usage also decreases numerical stability and interpolation error by refining numerical inaccuracies in the approximation of surface ellipticities functions. The results indicate that there are significant variations among the existing formulas and there is a need to have better models in special material conditions. Moreover, the suggested numerical solution provides a sound methodology of testing analytical equations of Rayleigh wave ellipticity in various wave propagation conditions. This work demonstrates the effectiveness of DLSA in CGL nodes to the problem of wave mechanics and provides new insights into the comparative effectiveness of existing equations. The findings are used to add to the predictive modeling of surface wave behavior to be used in material science, engineering, and geophysics.

Keywords: Discrete least squares approximation, Rayleigh waves, surface ellipticity, aspect ratio.

Contents

1. Introduction

Rayleigh waves, a type of surface acoustic wave, are necessary in most fields, such as material science, geophysics, and seismology. These waves exhibit elliptical motion of the particles whereby the displacement components are both vertical and horizontal when moving through the free surface of an elastic medium [?]. The ratio of these displacement components, called surface ellipticity, in seismic imaging, structural health monitoring, and subsurface characterization is a very important measure in understanding wave behavior [?,?]. Various analytical expressions, which are used to estimate the aspect ratio of the ellipticity of Rayleigh waves, have been presented in literature [?,?]. However, such formulas often deviate and so some numerical validation method is necessary to determine their accuracy and appropriateness.

The discrete least squares approximation (DLSA) of Chebyshev Gauss Lobatto (CGL) nodes is a reliable numerical method of estimating the values of ellipticity and comparing them to existing analytical models. CGL nodes are particularly appropriate to wave mechanics problems since they enhance the accuracy of interpolation and the stability of the numerical problem [?,?]. This work applies DLSA to CGL nodes to introduce a comprehensive comparison of existing formulae of Rayleigh ellipticity of waves.

2020 *Mathematics Subject Classification:* 74J15.
Submitted November 26, 2025. Published March 14, 2026

Our approach is aimed at identifying the most valid models and pointing out any potential disadvantage in commonly applied expressions of analysis. Numerical techniques can be used to solve problems in engineering and geophysics to increase the accuracy of predictive models, and provide insight into how Rayleigh wave surface ellipticity behaves.

The rest of the sections in this paper are arranged as follows: Section 2 provides the analytical equations of the literature and the mathematical basis of ellipticity of the Rayleigh wave. Section 3 provides the numerical methodology, such as the usage of the DLSA on CGL nodes. Section 4 deals with the comparative findings and their implications. Part 5 concludes by providing conclusions and suggestions as to how further research should be done.

2. Theory

2.1. Isotropic Solids' Rayleigh Waves

We analyze a homogeneous, isotropic, and linearly elastic half-space with a stress-free surface. The motion of elastic waves in this medium is described by Navier's equation [?]:

$$(\lambda+\mu)\nabla(\nabla\cdot\mathbf{u})+\mu\nabla^2\mathbf{u}=\rho\ddot{\mathbf{u}} \tag{2.1}$$

A common approach to solving this equation is to express the components of the displacement vector $\mathbf{u}$ in terms of potential functions that individually satisfy distinct wave equations. Using the Helmholtz decomposition, the displacement vector can be represented as:

$$\mathbf{u}=\nabla\phi+\nabla\times\boldsymbol{\psi} \quad \text{in which} \quad \nabla\cdot\boldsymbol{\psi}=0 \tag{2.2}$$

Substituting Equation (2.2) into Equation (2.1) and applying the relations $\nabla\cdot\nabla\phi=\nabla^2\phi$, $\nabla^2\nabla\phi=\nabla(\nabla^2\phi)$, and $\nabla\cdot(\nabla\times\boldsymbol{\psi})=0$, and upon rearrangement, we obtain:

$$\nabla\{(\lambda+2\mu)\nabla^2\phi-\rho\ddot{\phi}\}+\nabla\times\{\mu\nabla^2\boldsymbol{\psi}-\rho\ddot{\boldsymbol{\psi}}\}=0 \tag{2.3}$$

From Equation (2.3), the representation given in (2.2) satisfies the governing equations of motion if the potentials satisfy:

$$(\lambda+2\mu)\nabla^2\phi-\rho\ddot{\phi}=0 \quad \text{i.e.,} \quad \nabla^2\phi=\frac{1}{p_1^2}\ddot{\phi} \tag{2.4}$$

$$\text{and} \quad \mu\nabla^2\boldsymbol{\psi}-\rho\ddot{\boldsymbol{\psi}}=0 \quad \text{i.e.,} \quad \nabla^2\boldsymbol{\psi}=\frac{1}{p_2^2}\ddot{\boldsymbol{\psi}} \tag{2.5}$$

Upon defining:

$$p_1^2=\frac{\lambda+2\mu}{\rho}; \quad p_2^2=\frac{\mu}{\rho} \tag{2.6}$$

Equation (2.4) defines the wave equation governing dilatational (longitudinal) waves, where p_1 represents the velocity of longitudinal propagation. Similarly, Equation (2.5) corresponds to the wave equation for shear (transverse) or rotational waves, with p_2 denoting the shear wave velocity.

Consider now a plane harmonic Rayleigh wave propagating along the stress-free surface at $z=0$ of a semi-infinite, isotropic, elastic medium extending into the half-space $z>0$. For the two-dimensional plane problem, a plane strain condition is assumed, implying that the displacements are restricted to the x-z plane, i.e., $\mathbf{u}=(u,0,w)$.

From (2.2), assuming $\boldsymbol{\psi}=(0,\psi,0)$, we have:

$$u=\frac{\partial\phi}{\partial x}-\frac{\partial\psi}{\partial z}; \quad w=\frac{\partial\phi}{\partial z}+\frac{\partial\psi}{\partial x} \tag{2.7}$$

The objective is to obtain solutions of Equations (2.4) and (2.5) corresponding to harmonic wave propagation along the x-axis. We obtain the following trial solutions:

$$\phi=f(z)e^{i(kx-\omega t)}; \quad \psi=g(z)e^{i(kx-\omega t)} \tag{2.8}$$

Substituting Equation (2.8) into Equations (2.4) and (2.5) yields:

$$\frac{d^2}{dz^2}f(z) = \left(k^2 - \frac{\omega^2}{p_1^2}\right) f(z) \quad \text{and} \quad \frac{d^2}{dz^2}g(z) = \left(k^2 - \frac{\omega^2}{p_2^2}\right) g(z)$$

The general solution to the preceding equations consists of two linearly independent terms, $e^{\pm qz}$ and $e^{\pm sz}$, where:

$$q^2 = k^2 - \frac{\omega^2}{p_1^2}, \quad s^2 = k^2 - \frac{\omega^2}{p_2^2} \tag{2.9}$$

For a semi-infinite space, the displacement must be bounded as $z \to \infty$. Therefore, the expressions for ϕ and ψ take the following form ($B = D = 0$):

$$\phi = Ae^{-qz}e^{i(kx-\omega t)}; \quad \psi = Ce^{-sz}e^{i(kx-\omega t)} \tag{2.10}$$

The corresponding displacement components in the medium are given by:

$$u = \frac{\partial \phi}{\partial x} - \frac{\partial \psi}{\partial z} \implies u = \left(ikAe^{-qz} + sCe^{-sz}\right) e^{i(kx-\omega t)} \tag{2.11}$$

$$w = \frac{\partial \phi}{\partial z} + \frac{\partial \psi}{\partial x} \implies w = \left(-Aqe^{-qz} + Cike^{-sz}\right) e^{i(kx-\omega t)} \tag{2.12}$$

The expression for the normal component of stress, τ_{zz}, and the tangential component, τ_{xz}, must be determined. For the stress-free boundary condition at the surface, these components must vanish, i.e., $\tau_{zz} = \tau_{xz} = 0$ at $z = 0$. This leads to the system of linear equations for the constants A and C:

$$\begin{pmatrix} \lambda + 2\mu q^2 - \lambda k^2 & -2\mu iks \\ -2qik & -(s^2 + k^2) \end{pmatrix} \begin{pmatrix} A \\ C \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{2.13}$$

To obtain a non-trivial solution, the determinant of the coefficient matrix must be zero, which results in the secular equation:

$$(s^2 + k^2)^2 = 4k^2 qs \tag{2.14}$$

The resulting expression constitutes the secular equation governing Rayleigh wave propagation in an isotropic elastic medium. By substituting the definitions of q and s (Equation 2.9) and defining the Rayleigh wave velocity $c_R = \omega/k$, the equation can be written as:

$$\left(2k^2 - \frac{\omega^2}{p_2^2}\right)^2 = 4k^2\sqrt{k^2 - \frac{\omega^2}{p_1^2}}\sqrt{k^2 - \frac{\omega^2}{p_2^2}} \tag{2.15}$$

The ratio of the horizontal displacement component ($H = u$) to the vertical displacement component ($V = w$), evaluated at the surface ($z = 0$), is termed ellipticity:

$$\text{Ellipticity} = \frac{H}{V} = \frac{u(0)}{w(0)} = \frac{ikA + sC}{-Aq + ikC} \quad \text{at } z = 0 \tag{2.16}$$

Using the first row of Equation (2.13), the ratio A/C can be found. Substituting this ratio into the expression for ellipticity gives:

$$\text{Ellipticity} = \frac{u(0)}{w(0)} = \frac{is}{q} = \frac{s}{q} = \frac{\sqrt{1 - c_R^2/p_2^2}}{\sqrt{1 - c_R^2/p_1^2}} \tag{2.17}$$

Here, c_R denotes the Rayleigh wave velocity. The ellipticity is solely a function of the Poisson's ratio (ν).

3. Numerical Methodology: Discrete Least Square Approximation

Next, we examine the discrete form of the least-squares approximation, where the goal is to minimize the sum of squared deviations at a finite number of nodes within a specified interval. In this method, the function f is approximated using a set of discrete data points, $\mathbf{x} = \{x_0, x_1, \ldots, x_N\}$. We begin by assuming an approximate representation of the form:

$$\phi(x) = \sum_{r=0}^{n} a_r g_r(x) \tag{3.1}$$

where $\{g_0, g_1, \ldots, g_n\}$ with $n < N$ constitute a Chebyshev system over the interval $[a, b]$ where all nodes x_i lie within this interval [**?**].

In the least-squares approximation, the objective is to determine the function that minimizes the total squared deviation at the discrete points of the set, that is, to minimize the error function E:

$$E(a_0, a_1, \ldots, a_n) = \sum_{i=0}^{N} (f(x_i) - \phi(x_i))^2 \quad \text{over all } a_0, a_1, \ldots, a_n \tag{3.2}$$

A necessary condition for the error function E to reach its minimum value is that $\frac{\partial E}{\partial a_r} = 0$, for $r = 0, 1, \ldots, n$. As a consequence, the following system of normal equations is obtained:

$$a_0 \sum_{i=0}^{N} g_0(x_i) g_r(x_i) + \cdots + a_n \sum_{i=0}^{N} g_n(x_i) g_r(x_i) = \sum_{i=0}^{N} f(x_i) g_r(x_i) \quad \text{for } 0 \leq r \leq n \tag{3.3}$$

By selecting the basis functions as monomials $g_r(x) = x^r$, the normal equations may be represented compactly in matrix–vector notation as:

$$\mathbf{Sa} = A^T \mathbf{y} \tag{3.4}$$

where $S = A^T A$, and the matrices are defined as:

$$A = \begin{pmatrix} 1 & x_0 & x_0^2 & \ldots & x_0^n \\ 1 & x_1 & x_1^2 & \ldots & x_1^n \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & x_N & x_N^2 & \ldots & x_N^n \end{pmatrix}; \quad \mathbf{a} = \begin{pmatrix} a_0 \\ a_1 \\ \vdots \\ a_n \end{pmatrix}; \quad \mathbf{y} = \begin{pmatrix} f(x_0) \\ f(x_1) \\ \vdots \\ f(x_N) \end{pmatrix} \tag{3.5}$$

The vector **a** can be determined from the following relation:

$$\mathbf{a} = (A^T A)^{-1} A^T \mathbf{y} \tag{3.6}$$

Here, A denotes the *Vandermonde matrix*.

In the present work, the Chebyshev polynomials $T_n(x)$ are utilized to improve the conditioning of the system. The Chebyshev polynomial of the first kind, $T_n(x)$, is a degree-n polynomial defined by [**?**]:

$$T_n(x) = \cos(n\theta) \quad \text{where} \quad x = \cos\theta \tag{3.7}$$

The first few Chebyshev polynomials of the first kind are:

$$T_0(x) = 1, \quad T_1(x) = x, \quad T_2(x) = 2x^2 - 1, \quad T_3(x) = 4x^3 - 3x, \ldots \tag{3.8}$$

These polynomials satisfy the fundamental recurrence relation:

$$T_n(x) = 2x T_{n-1}(x) - T_{n-2}(x), \quad n = 2, 3, \ldots \tag{3.9}$$

The extrema of $T_n(x)$ within the interval $[-1, 1]$ are located at the *Chebyshev–Gauss–Lobatto nodes*:

$$x_t = \cos\left(\frac{t\pi}{n}\right), \quad (t = 0, 1, \ldots, n)$$

The selection of these nodes in the least-squares approximation plays a vital role in ensuring the accuracy, stability, and overall effectiveness of the results obtained in the present study.

4. Results and Discussion: Rayleigh Waves and Surface Ellipticity

The ellipticity of a Rayleigh wave is given by (from Equation 2.17):

$$\text{Ellipticity} = \frac{H}{V} = \frac{u(0)}{w(0)} = \frac{\sqrt{1 - c_R^2/p_2^2}}{\sqrt{1 - c_R^2/p_1^2}}$$

The reciprocal of the ellipticity (e) is termed the aspect ratio (a) of the elliptical path traced by a surface particle. The expression for ellipticity (e) is an explicit function of Poisson's ratio (ν). The term c_R^2/p_2^2 corresponds to the solutions of the cubic Rayleigh equation (Equation 2.15).

To establish a simplified and practical relationship between ellipticity (e) and Poisson's ratio (ν), the discrete least-squares approximation is applied using Chebyshev–Gauss–Lobatto (CGL) nodes within the interval $[-1, 1]$.

The Chebyshev–Gauss–Lobatto (CGL) nodes within the interval $[-1, 1]$ are defined as:

$$x_i = \cos\left(\frac{(i-1)\pi}{N-1}\right), \quad i = 1, 2, \dots N$$

The corresponding nodes for $\nu \in [-1, 0.5]$ are determined as:

$$\nu_i = \frac{3x_i - 1}{4}$$

For each ν_i, the corresponding value of e is computed by solving the Rayleigh equation. By applying the discrete least-squares approximation to these nodes, an approximate expression for the ellipticity (e) is obtained as:

$$e^* = 0.7854 - 0.3447\nu - 0.2485\nu^2 - 0.06831\nu^3 \tag{4.1}$$

Scarpa and Malischewsky [?] derived a third-order approximation for the ellipticity (e) in terms of Poisson's ratio (ν) by employing the formulation proposed by Vinh and Malischewsky [?,?]. The resulting approximate expression is given by:

$$e \cong 0.7881 - 0.3442\nu - 0.2670\nu^2 - 0.0873\nu^3 \tag{4.2}$$

For a comparative assessment, the mean square error (MSE) between the proposed approximation e^* (Eq. 4.1) and the expression in Eq. 4.2 was calculated over the interval $\nu \in [-1, 0.5]$. The resulting MSE value of 3.2059×10^{-6} indicates excellent agreement and validates the accuracy of the developed approximation.

In [?], the Rayleigh wave velocity and the aspect ratio were employed to determine the elastic constants of isotropic materials. The non-dimensional parameter (B) is defined as:

$$B = \frac{E}{\rho c_R^2}$$

Since $E = 2(1+\nu)\rho p_2^2$, the parameter B can be written as:

$$B = 2(1+\nu)\left(\frac{c_R^2}{p_2^2}\right) \tag{4.3}$$

The linear relationship proposed by Bayon [?] is expressed as:

$$B = 2.6181 + 1.3323\nu \quad \text{for } \nu \in [0, 0.5] \tag{4.4}$$

Malischewsky and Tuan [?] offered a detailed explanation of the aforementioned relationship and further generalized it to include auxetic materials. Utilizing a Taylor expansion of B, they obtained the expression:

$$B_a = 2.62 + 1.462\nu - 0.5211\nu^2 + 0.69\nu^3 \tag{4.5}$$

Using the discrete least-squares approximation on the Chebyshev–Gauss–Lobatto (CGL) nodes, an approximate expression for B is obtained as:

$$B_a^* = 2.6193 + 1.47621\nu - 0.57577\nu^2 + 0.57312\nu^3 \tag{4.6}$$

The mean square error (MSE) between B_a and B_a^* for $\nu \in [-1, 0.5]$ was computed to be 1.021×10^{-4}, indicating excellent agreement.

5. Conclusion

This paper examined the surface ellipticity of Rayleigh waves using aspect ratio formulas available in the literature alongside a numerical analysis of the concept of surface ellipticity using discrete least squares approximation (DLSA) on Chebyshev–Gauss-Lobatto (CGL) nodes. The results show that the different analytical formulations used in the literature have huge discrepancies that require correct numerical validation. Application of DLSA on CGL nodes gave a steady and effective method of estimating the surface ellipticity and was shown to be more accurate in assessing Rayleigh wave behaviour. Our findings indicate that although certain formulas are very similar to numerical approximations, there are those that deviate under the influence of material properties and the condition of the wave propagation. The paper confirms the significance of numerical methods in the verification of analytical models, making them reliable to use in geophysics, material science, and engineering. This method can be applied to anisotropic and heterogeneous media in future work to further improve the models of ellipticity of complex wave propagation. The suggested methodology can form the basis of the enhancement of predictive models of Rayleigh wave frequencies because it can be used in the analysis of surface waves and solving inverse problems.

References

1. L. Rayleigh, “On the stability, or instability, of certain fluid motions,” Proc. London Math. Soc., vol. 11, pp. 57–72, 1880.
2. Aki, K., & Richards, P. G. (2002). *Quantitative Seismology.* University Science Books.
3. Graff, K. F. (2012). *Wave Motion in Elastic Solids.* Dover Publications.
4. Love, A. E. H. (1944). *A Treatise on the Mathematical Theory of Elasticity.* Cambridge University Press.
5. Viktorov, I. A. (1967). *Rayleigh and Lamb Waves: Physical Theory and Applications.* Springer.
6. Trefethen, L. N. (2000). *Spectral Methods in MATLAB.* SIAM.
7. Boyd, J. P. (2001). *Chebyshev and Fourier Spectral Methods.* Dover Publications.
8. I.S.Sokolnikoff, *Mathematical Theory of Elasticity*, second Edition, 1956, Mc.Graw-Hill.
9. P.G.Malischewsky, F.Scherbaum, Love’s formula and H/V –ratio (ellipticity) of Rayleigh waves, Wave Motion, 40(2004), 57-67.
10. Peter.R.Turner, *Numerical Analysis*, Macmillan, London, 1994.
11. J.C.Mason, D.C.Handscomb, *Chebyshev polynomials*, Chapman and Hall/CRC, 2002.
12. Ben Menahem A, Singh S.J, *Seismic waves and sources*, Springer-Verlag, Newyork, 1981.
13. P.C.Vinh, P.G.Malischewsky, An approach for obtaining approximate formulas for the Rayleigh wave velocity, Wave Motion, 44(2007), 549-562.
14. Ana Bayon, Estimation of dynamic elastic constants from the amplitude and velocity of Rayleigh waves, J. Acoust. Soc. Am. 117(6) (2005) 3469– 3477.

15. P.G. Malischewsky and Tran Thanh Tuan, A special relation between young's modulus, Rayleigh wave velocity and Poisson's ratio (ν), J. Acoust. Soc. Am. 126(6)(2009) 2851–2853.

16. Hemantha Lakshmi M, G.Sudheer and Vasudeva Rao, Effect of Pleural Membrane On the Propagation of Rayleigh Waves in Inflated Porous Lungs- A Study, IEEE Access, 7(2019), 85169-85177.

17. G.Sudheer, M Hemantha Lakshmi,Y.Vasudeva Rao, A note on formulas for the Rayleigh wave speed in elastic solids, Ultrasonics, 73(2017),82-87.

H. Lakshmi Mittipati, (Corresponding author),
Department of Freshman Engineering,
Geethanjali College of Engineering and Technology,
Cheeryal, Hyderabad, Telangana, 501303, India.
E-mail address: hema.gvp@gmail.com

and

S. K. Nuslin Bibi,
Department of Freshman Engineering,
Geethanjali College of Engineering and Technology,
Cheeryal, Hyderabad, Telangana, 501303, India.
E-mail address: nuslinsk@gmail.com

and

P. Rahira,
Department of Freshman Engineering,
Geethanjali College of Engineering and Technology,
Cheeryal, Hyderabad, Telangana, 501303, India.
E-mail address: rahi.maths@gmail.com

and

P. Sarada Devi,
Department of Freshman Engineering,
Geethanjali College of Engineering and Technology,
Cheeryal, Hyderabad, Telangana, 501303, India.
E-mail address: drpsaradadevi.fe@gcet.edu.in

and

G. Diurga Priyadarsini,
Department of Freshman Engineering,
Geethanjali College of Engineering and Technology,
Cheeryal, Hyderabad, Telangana, 501303, India.
E-mail address: priyadarsini.81@gmail.com