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On the Arithmetic-Geometric-Harmonic-Mean Inequalities

M. Al-Hawari*, Ayat Almomani and Mohammad A. Bani Abdelrahman

ABSTRACT: In this article, we collect some inequalities concerning Arithmetic, Geometric, Young and Heinz inequalities, the goal of the article is to investigate Young and Heinz inequalities for scalars and matrices. Improvements to the inequality studied of arithmetic and geometric means for scalars and several other auxiliary results, we provided some improvements to the inequality of arithmetic and geometric means by using the convex functions. A fundamental relationship between positive real numbers and the v-weighted arithmetic-geometric mean inequality is presented for any two scalars. Also, we introduce the largest and the smallest eigenvalues inequalities for matrices by using our subjects.

Keywords: Young's inequality, Heron mean, geometric mean, arithmetic mean.

Contents

1. Introduction

The classical Young inequality for any two scalars is the v-weighted arithmetic–geometric mean inequality, which is a fundamental relation between two positive real numbers, and this inequality as follows [2,3] if $c, d \geq 0$ and $0 \leq v \leq 1$, then

$$c^v d^{1-v} \leq vc + (1-v)d \tag{1}$$

With equality if and only if $c = d$. If $v = 2$, we get the arithmetic–geometric mean inequality

$$\sqrt{cd} \leq \frac{c+d}{2} \tag{2}$$

If $e, f > 1$ such that $\frac{1}{e} + \frac{1}{f} = 1$, then the inequality (1) can be written as

$$cd \leq \frac{c^e}{e} + \frac{d^f}{f} \tag{3}$$

For two positive real numbers c, d and $v \in [0, 1]$, the quantity [3,4]

$$F_v(c, d) = (1-v)\sqrt{cd} + v\frac{c+d}{2} \tag{4}$$

Also,
If $c, d \geq 0$, and $0 \leq v \leq 1$, then [6,7]

$$\frac{1}{2}v(1-v)(\sqrt{c} - \sqrt{d})^2 + \sqrt{cd} \leq F_v(c, d) \tag{5}$$

$$F_v(c, d) \leq \frac{c+d}{2} - \frac{1}{2}v(1-v)(\sqrt{c} - \sqrt{d})^2 \tag{6}$$

Young-type inequalities have also been extensively developed for matrices and positive operators;see, for example, [1,5,6,7,8]. In particular, reverse Young/Heinz inequalities for matrices were established in 6,7],and related matrix mean-difference inequalities can be found in [8].

* Corresponding author.
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2. Main Results

Our goal in this section is to improve the inequality of the arithmetic-Geometric mean for scalar results and other complementary results.

Theorem 2.1 *If $c, d \geq 0$ and $0 \leq v \leq 1$, then*

$$\sqrt{cd} \leq \frac{c+d}{2} - v(1-v)(\sqrt{c}-\sqrt{d})^2 \leq \frac{c+d}{2} \tag{7}$$

Proof: From the inequalities (5) and (6), we get

$$\frac{1}{2}v(1-v)(\sqrt{c}-\sqrt{d})^2 + \sqrt{cd} \leq \frac{c+d}{2} - \frac{1}{2}v(1-v)(\sqrt{c}-\sqrt{d})^2$$

$$\sqrt{cd} \leq \frac{c+d}{2} - v(1-v)(\sqrt{c}-\sqrt{d})^2 \leq \frac{c+d}{2}$$

□

Theorem 2.2 *If $c, d \geq 0$ and $0 \leq v \leq 1$, then*

$$\sqrt{cd} \leq \frac{c+d}{2} - \frac{1}{2}(1-v)(\sqrt{c}-\sqrt{d})^2 \leq \frac{c+d}{2} \tag{8}$$

Proof: From the inequalities (4) and (5), we obtain

$$\frac{1}{2}v(1-v)(\sqrt{c}-\sqrt{d})^2 + \sqrt{cd} \leq (1-v)\sqrt{cd} + v\left(\frac{c+d}{2}\right)$$

$$\frac{1}{2}v(1-v)(\sqrt{c}-\sqrt{d})^2 + v\sqrt{cd} \leq v\left(\frac{c+d}{2}\right)$$

$$\frac{1}{2}(1-v)(\sqrt{c}-\sqrt{d})^2 + \sqrt{cd} \leq \left(\frac{c+d}{2}\right)$$

$$\sqrt{cd} \leq \left(\frac{c+d}{2}\right) - \frac{1}{2}(1-v)(\sqrt{c}-\sqrt{d})^2$$

□

Theorem 2.3 *If $c, d \geq 0$ and$0 \leq v \leq 1$, then*

$$\sqrt{cd} \leq \frac{c+d}{2} - \frac{1}{2}v(\sqrt{c}-\sqrt{d})^2 \leq \frac{c+d}{2} \tag{9}$$

Proof: By using the inequalities (4) and (6), we have

$$(1-v)\sqrt{cd} + v\left(\frac{c+d}{2}\right) \leq \frac{c+d}{2} - \frac{1}{2}(1-v)(\sqrt{c}-\sqrt{d})^2$$

$$(1-v)\sqrt{cd} \leq (1-v)\frac{c+d}{2} - \frac{1}{2}(1-v)(\sqrt{c}-\sqrt{d})^2$$

$$\sqrt{cd} \leq \left(\frac{c+d}{2}\right) - \frac{1}{2}v(\sqrt{c}-\sqrt{d})^2$$

□

Example 2.1 Take $v = \frac{1}{4}$, $c = 4$ and $d = 9$, then

$$\sqrt{cd} = \sqrt{36} = 6, \qquad \frac{c+d}{2} = 6.5, \qquad \frac{c+d}{2} - v(1-v)(\sqrt{c} - \sqrt{d})^2 = 6.3125$$

$$\frac{c+d}{2} - \frac{1}{2}(1-v)(\sqrt{c} - \sqrt{d})^2 = 6.125$$

Also,

$$\frac{c+d}{2} - \frac{1}{2}v(\sqrt{c} - \sqrt{d})^2 = 6.375$$

In the previous example we saw that the inequality (8) is better than the others.

Example 2.2 Take $v = \frac{3}{4}$, $c = 4$ and $d = 9$, then we get,

$$\sqrt{cd} = \sqrt{36} = 6, \qquad \frac{c+d}{2} = 6.5,$$

$$\frac{c+d}{2} - v(1-v)(\sqrt{c} - \sqrt{d})^2 = 6.3125$$

$$\frac{c+d}{2} - \frac{1}{2}(1-v)(\sqrt{c} - \sqrt{d})^2 = 6.375$$

Also,

$$\frac{c+d}{2} - \frac{1}{2}v(\sqrt{c} - \sqrt{d})^2 = 6.125$$

In the previous Example, we say that the inequality (9) is better than the others.
Also, the inequality (7) is better than the inequality (8)

Theorem 2.4 *Suppose* $t^n + c_1 t^{n-1} + c_2 t^{n-2} + \ldots + c_{n-1} t + c_n = 0$ *is the characteristic Equation for* $C_{n\times n}$, *then*

- $trace(C) = t_1 + t_2 + \ldots + t_n = -c_1$.
- $det(C) = t_1 t_2 \ldots t_n = (-1)^n c_n$.

Now we rewrite the previous results by using the trace and the determinates as

Theorem 2.5 *Suppose* $tr(C), tr(D) \geq 0$ *and* $0 \leq v \leq 1$, *then*

$$\sqrt{tr(C)\, tr(D)} \leq \frac{tr(C) + tr(D)}{2} - \frac{1}{2}(1-v)\left(\sqrt{tr(C)} - \sqrt{tr(D)}\right)^2 \leq \frac{tr(C) + tr(D)}{2} \tag{11}$$

Theorem 2.6 *Suppose* $tr(C), tr(D) \geq 0$ *and* $0 \leq v \leq 1$, *then*

$$\sqrt{tr(C)\, tr(D)} \leq \frac{tr(C) + tr(D)}{2} - \frac{1}{2}v\left(\sqrt{tr(C)} - \sqrt{tr(D)}\right)^2 \leq \frac{tr(C) + tr(D)}{2} \tag{12}$$

Theorem 2.7 *For* $C_{n\times n}$, *be a positive semi-definite Hermitian matrix* C, *with the largest and the smallest eigenvalues* t_1 *and* t_n *respectively, and* $n \times 1$ *vector* z *such that* $z^* z = 1$, *we have*

$$(z^* C z)(z^* C^{-1} z) \leq \frac{(t_1 + t_n)^2}{4 t_1 t_n}.$$

With equality, if and only if,

$$z^* C z = \frac{(t_1 + t_n)}{2}, \qquad \text{and} \qquad z^* C^{-1} z = \frac{(t_1 + t_n)}{2 t_1 t_n}.$$

Theorem 2.8 *For $C_{n\times n}$, be a positive semi definite Hermitian matrix C, with the largest and the smallest eigenvalues t_1 and t_n respectively, and $n\times 1$ vector z such that $z^*z=1$, we have*

$$1 \leq \frac{(t_1+t_n)^2}{4t_1t_n}. \tag{13}$$

With equality if and only if,

$$(z^*Cz) = \frac{t_1+t_n}{2t_1t_n} \qquad And \qquad (z^*C^{-1}z) = \frac{t_1+t_n}{2}. \tag{14}$$

Proof:
By using Theorem2.7, we have

$$(z^*Cz)(z^*C^{-1}z) \leq \frac{(t_1+t_n)^2}{4t_1t_n}$$

$$z^*Cz\; z^*C^{-1}z \leq \frac{(t_1+t_n)^2}{4t_1t_n}$$

$$z^*CC^{-1}z \leq \frac{(t_1+t_n)^2}{4t_1t_n}$$

$$z^*z \leq \frac{(t_1+t_n)^2}{4t_1t_n}$$

$$1 \leq \frac{(t_1+t_n)^2}{4t_1t_n}.$$

This complete the proof. □

Corollary 2.1 *Let the conditions of Theorem2.8 be fulfilled. Then*

$$1 \leq \frac{t_1^2+t_n^2}{2t_1t_n}.$$

With equality, if and only if, $t_1 = t_n$.

Proof: By Theorem2.8, we have

$$4t_1t_n \leq t_1^2 + 2t_1t_n + t_n^2$$

$$2t_1t_n \leq t_1^2 + t_n^2$$

$$1 \leq \frac{t_1^2+t_n^2}{2t_1t_n}$$

□

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Mohammad Alhawari,
Department of Mathematics, Faculty of Science,
Ajloun National University, P.O. Box 43, Ajloun 26810, JORDAN
E-mail address: mh.hawari@anu.edu.jo , analysis2003@yahoo.com

and

Ayat Almomani,
Department of Mathematics, Faculty of Science,
Ajloun National University, P.O. Box 43, Ajloun 26810, JORDAN
E-mail address: ayatalmomani877@gmail.com

and

Mohammad A. Bani Abdelrahman,
Department of Mathematics, Faculty of Science,
Ajloun National University, P.O. Box 43, Ajloun 26810, JORDAN
E-mail address: maabdelrahman991@gmail.com