Bol. Soc. Paran. Mat.
©SPM – E-ISSN-2175-1188
SPM: www.spm.uem.br/bspm

(3s.) **v. 2026 (44) 2** : 1–15.
ISSN-0037-8712
doi:10.5269/bspm.80394

# Characterization of L-Fuzzy Filters in Softened Distributive and Sectionally * Semilattices with Certain Applications

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ABSTRACT: The ideas of L-fuzzy filters, ideals, and semi lattices are examined in this work along with their characteristics. By defining the softened distributive semi lattice,we expand the notion of semi lattice to each section [0, 1] and investigate the requirements for a modular semi lattice to be softened distributive.Our research provides new insights into the structure and properties of fuzzy algebraic systems, with some potential applications in fuzzy logic. The main contributions of this study consists of, the creation of sufficient and required circumstances for softened distributive semi lattices.Proved that set of dense elements are a L-fuzzy filter. The paper investigates the properties of L-fuzzy filters and ideals in distributive semi lattices, providing a foundation for their applications. In this entire manuscript we denote semi lattice by the symbol S■

Keywords: *Semi lattice, sectionally *semi lattice, modular semi lattice, L-fuzzy filter.

## Contents



## 1. Introduction

Information processing and filtering are crucial in various physical systems, particularly when dealing with uncertainty or partial information. ‘Adaptive fuzzy filters’, a type of ‘Nonlinear adaptive filter’, have been developed to address this challenge. The two primary components of these filters are an adaptive mechanism that modifies the parameters of membership functions and a set of configurable if-then rules. L-fuzzy filters of distributive semi lattices have applications in image processing, decision-making under vagueness, signal analysis, and modeling complex systems. The concept of distributive * lattices was introduced by Speed (1969) and later extended by Krishna Murthy (1980). Since then, various notions of fuzzy algebra have been developed, including fuzzy subgroups, fuzzy ideals, and fuzzy filters. However, most studies have focused on fuzzy algebraic structures with values in [0,1]. To overcome this limitation,

2020 *Mathematics Subject Classification:* 06A12, 06D72, 08A72.
Submitted November 26, 2025. Published April 29, 2026.

Goguen (1967) introduced L-sets, which can take values in a complete lattice. This study investigates L-fuzzy ideals, * semi lattices, and sectionally * semi lattices. We examine softened distributive * semi lattices and expand on the idea of * semi lattices to every segment [0,1] in a semi lattice.

Given that filtering is merely the process of extracting specifics regarding a desired amount from a collection of measured data obtained from operations in a setting without being familiar with statistics , and the term "Filter" refers to a tool that processes information, in the name of semantic principles from (human) specialists in the filter design, it was always preferable to use particular materials that were readily available, including both quantitative (numerical) and qualitative material. As a special kind of "nonlinear adaptive filters," "adaptive fuzzy filters" are made up of a pair of primary components: a collection of adjustable if-then rules and an adaptive system that uses empirical data to adjust the settings of the membership functions relating to the language parameters established on the input-output standard collections. L-fuzzy filters of distributive semi lattices find applications in various physical systems where dealing with uncertainty or partial information is crucial, including areas like image processing, decision making under vagueness, signal analysis, and modeling complex systems.

The distributive * lattice was first given by Speed (1969). The same condition was applied to a distributive lattice to make it a * lattice, which was then placed on any meet semi lattice to make it a * semi lattice. Krishna Murthy (1980) demonstrated that nearly all of Speed's findings could be extended to a more general class of distributive * semi lattice [3]. Krishna Murthy has extended the notion of *semi lattice to semi lattice [3]. Since Rosenfeld [7] first introduced the idea of fuzzy subgroups, several notions of fuzzy algebra have been developed. Additionally, Tepavcevic and Trajkovski [10] devised fuzzy lattice. However, the scope is severely constrained because the researchers in [1] created an L-fuzzy lattice by using a fuzzy set of a crisp lattice. Prime fuzzy Ideals (filters) of Latice and Almost distributive lattice were examined by Koguep et al. [2], Raj et al. [6], Sundar Raj et al. [8], and Swamy et al. [9]. However, nearly all of the aforementioned researchers worked with fuzzy substructures that had values in [0,1]. Thereby, Goguen [1] proposed L-sets, examined associated properties, and To address the limiting ability of [0,1] to offer values for generic expressions, a lattice was substituted for the preceding valuation set.

Many academics used this method on many algebraic structures, such as groups, rings, lattices, and others, after Goguen. Tepavčević and Trajkovski [10] investigated L-lattices (L-fuzzy sub lattices, L-fuzzy ideals, L-fuzzy filters) associated features, whereas Mordeson and Malik [5] presented L-groups.Congruence relations are equivalence relations in abstract algebra that are consistent with any given structure, including groups and rings. Similar to Kuroki [4], one who prospect some of its characterization on the subject of normal subgroups, Kuroki [4] established the ideas of fuzzy congruence on a groupoid and a group.

The ideas of L-fuzzy filters, ideals, and semi lattices are examined in this work along with their characteristics. We expand the notion of semi lattice to each section [0, 1] and investigate the requirements for a modular semi lattice to be softened distributive. Our research provides new insights into the structure and properties of fuzzy algebraic systems, with potential applications in fuzzy logic and artificial intelligence. The main contributions of this study consists of, the creation of sufficient and required circumstances for softened distributive semi lattices and development of new methods for constructing fuzzy filters.

## 2. Characterization of L-Fuzzy Filters

### 2.1. Fuzzy filter

**Definition 2.1** *A fuzzy-subset* $\Theta$ *is fuzzy filter for* $S_{\blacksquare}$*, if (i)* $\Theta\ (x_0) = 1$ *(ii)* $\Theta\ (\widehat{x} \wedge \widehat{y}) \geq \Theta\ (\widehat{x}) \wedge \Theta\ (\widehat{y})$*, (iii)* $\Theta\ (\widehat{x}) \leq \Theta\ (\widehat{y})$ *whenever* $\widehat{x} \leq \widehat{y}$*, where every* $\widehat{x}$*,* $\widehat{y}$ *elements of* $S_{\blacksquare}$

**Example 2.1** ***Fuzzy Filter on a Partially Ordered Set:***
*Assume* $S = \{0, \gamma, \delta, 1\}$ *a set with a "partialorderrelation"* $\leq$ *defined as:* $0 \leq 0$, $\gamma \leq \gamma$, $\delta \leq \delta$, $1 \leq 1$, $0 \leq \gamma$, $0 \leq \delta$, $\gamma \leq 1$, $\delta \leq 1$. *The partial order relation may be represented by the following Hasse diagram:*

Let $\Theta$, a fuzzy-subset for S described as: $\Theta\ (0) = 1$; $\Theta\ (\gamma) = 0.8$; $\Theta\ (\delta) = 0.7$; $\Theta\ (1) = 1$. To verify that $\Theta$ is a fuzzy-filter, we need to check the three conditions:

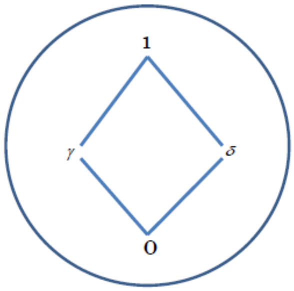


Fig.1 Hasse diagram

**Condition (i):** $\Theta\ (0_0) = 1$ : Since $0_0$ is not explicitly defined, we assume that $0_0 = 0$. Then $\Theta\ (0_0) = \Theta\ (0) = 1$. So, condition (i) is satisfied.

**Condition (ii):** $\Theta\ (x_\alpha \wedge y_\alpha) \geq \Theta\ (x_\alpha) \wedge \Theta\ (y_\alpha)$

We need to check this condition for all pairs of elements $x_\alpha$, $y_\alpha$ in S.

- For $x_\alpha = 0$, $y_\alpha = \gamma$: $0 \wedge\gamma = 0$, so $\Theta\ (0 \wedge\gamma) = \Theta\ (0) = 1 \geq 1 \wedge 0.8 = \Theta\ (0) \wedge \Theta\ (\gamma)$

- For $x_\alpha = 0$, $y_\alpha = \delta$: $0 \wedge\delta = 0$, so $\Theta\ (0 \wedge\delta) = \Theta\ (0) = 1 \geq 1 \wedge 0.7 = \Theta\ (0) \wedge \Theta\ (\delta)$

- For $x_\alpha = a$, $y_\alpha = \delta$: $\gamma\wedge\delta = 0$, so $\Theta\ (\gamma\wedge\delta) = \Theta\ (0) = 1 \geq 0.8 \wedge 0.7 = \Theta\ (\gamma) \wedge \Theta\ (\delta)$

- For $x_\alpha = a$, $y_\alpha = 1$: $\gamma\wedge 1 = \gamma$, so $\Theta\ (\gamma\wedge 1) = \Theta\ (\gamma) = 0.8 \geq 0.8 \wedge 1 = \Theta\ (\gamma) \wedge \Theta(1)$

- For $x_\alpha = b$, $y_\alpha = 1$: $\delta\wedge 1 = \delta$, so $\Theta\ (\delta\wedge 1) = \Theta\ (\delta) = 0.7 \geq 0.7 \wedge 1 = \Theta\ (\delta) \wedge \Theta(1)$

So, condition (ii) is satisfied.

**Condition (iii):** $\Theta\ (x_\alpha) \leq \Theta\ (y_\alpha)$ whenever $x_\alpha \leq y_\alpha$

We need to check this condition for all pairs of elements $x_\alpha$, $y_\alpha$ in S such that $x_\alpha \leq y_\alpha$.

- For $x_\alpha = 0$, $y_\alpha = \gamma$: $0 \leq \gamma$, so $\Theta\ (0) = 1 \geq 0.8 = \Theta\ (\gamma)$

- For $x_\alpha = 0$, $y_\alpha = \delta$: $0 \leq \delta$, so $\Theta\ (0) = 1 \geq 0.7 = \Theta\ (b)$

- For $x_\alpha = \gamma$, $y_\alpha = 1$: $\gamma \leq 1$, so $\Theta\ (\gamma) = 0.8 \leq 1 = \Theta(1)$

- For $x_\alpha = \delta$, $y_\alpha = 1$: $\delta \leq 1$, so $\Theta\ (\delta) = 0.7 \leq 1 = \Theta(1)$

So, condition (iii) is satisfied. Therefore, $\Theta$ is applied to the partially ordered set S as a fuzzy filter.

### 2.2. L-fuzzy ideal:

**Definition 2.2** *Bounded lattice $\mathcal{B}$ and L-fuzzy subset $\Theta$ is L-fuzzy ideal for $\mathcal{B}$,wherein $\Theta_\alpha$ is ideal of $\mathcal{B}$ for each $\alpha$ in L, where $\Theta_\alpha = \{\alpha \in\mathcal{B}\ /\ \alpha\leq\ \Theta\ (x)\}$ is the $\alpha$ cut (parametric cut) of $\Theta$.*

**Example 2.2** *In other words, $\Theta$, a L-fuzzy ideal of $\mathcal{B}$ if every $\alpha$-cut of $\Theta$ is an ideal in the classical sense.). Let $\mathcal{B} = \{0, \gamma, \delta, 1\}$ be a bounded lattice with the following order relation:*

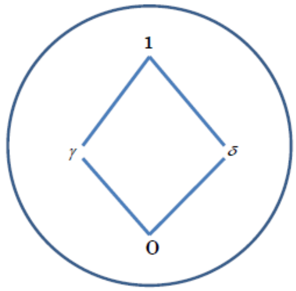


Fig.2 order relation

Defining an L-fuzzy subset Θ of $\mathcal{B}$ as: Θ (0) = 0.4; Θ (γ) = 0.8; Θ (δ) = 0.6; Θ (1) = 1. Let the lattice of numbers that are real between 0 and 1 be represented by L = [0, 1].

Now, consider the α-cuts of Θ: $\Theta_0 = \{x \in \mathcal{B} \mid \Theta(x) \geq 0\} = \{0, \gamma, \delta, 1\}$

$\Theta_{0.4} = \{x \in \mathcal{B} \mid \Theta(x) \geq 0.4\} = \{0, \gamma, \delta, 1\}$

$\Theta_{0.6} = \{x \in \mathcal{B} \mid \Theta(x) \geq 0.6\} = \{\delta, \gamma, 1\}$

$\Theta_{0.8} = \{x \in \mathcal{B} \mid \Theta(x) \geq 0.8\} = \{\gamma, 1\}$

Each of these α-cuts is an ideal in the lattice $\mathcal{B}$:

$\Theta_0$ is the entire lattice, which is an ideal.

$\Theta_{0.4}$ is the ideal {0, γ, δ, 1}.

$\Theta_{0.6}$ is the ideal { δ, 1, γ }.

$\Theta_{0.8}$ is the ideal { γ, 1}.

Therefore, Θ become L-fuzzy ideal for the lattice $\mathcal{B}$.

### 2.3. L-fuzzy ideal of semi lattice

**Definition 2.3** *If $\Theta_\alpha$ is an ideal of $S_\blacksquare$ for all $\alpha \in L$, then (L-fuzzy subset) $\Theta \subset S_\blacksquare$ is "L-fuzzy ideal" for semi lattice $S_\blacksquare$. "L-fuzzy ideal" Θ of a semi lattice$S_\blacksquare$ must satisfy the following conditions:*

*(i) Θ ($x_0$) = 1 for some $x_0$ in $S_\blacksquare$ (ii) Θ is an antitone (iii) Θ is directed above.*

### 2.4. L-fuzzy filter of semi lattice

**Definition 2.4** *L-fuzzy subset T of semi lattice ($S_\blacksquare$) is defined as L-fuzzy filter for $S_\blacksquare$. For $\dot{\mu}$, $\dot{\rho}$ in $S_\blacksquare$, $T(\dot{\mu} \wedge \dot{\rho}) \leq T(\dot{\mu}) \wedge T(\dot{\rho})$ and $T(\dot{\mu}_0) = 1$, for some $\dot{\mu}_0 \in (S_\blacksquare, \leq)$*

**Theorem 2.1** *A fuzzy-subset T of $S_\blacksquare$ is L-fuzzy filter for $S_\blacksquare$, whenever T is fuzzy filter.*

**Proof:**
Let T, a L- fuzzy filter in $S_\blacksquare$, when $\dot{\mu}$, $\dot{\rho}$ are in $S_\blacksquare$,

We have T ($\dot{\mu}_0$) =1 and T ($\dot{\mu}\wedge\dot{\rho}$) = T ($\dot{\mu}$) $\wedge$T ($\dot{\rho}$)

Thus, we have T ($\dot{\mu}\wedge\rho$) $\geq$ T ($\dot{\mu}$) $\wedge$T ($\dot{\rho}$).let $\dot{\mu}$, $\dot{\rho}\in$S$_\blacksquare$

such as $\dot{\mu}\leq\dot{\rho}$, implies $\dot{\mu}$ $\wedge\dot{\rho}=\dot{\mu}$, implies T ($\dot{\mu}\wedge\dot{\rho}$) = T ($\dot{\mu}$).

Therefore, T ($\dot{\mu}$) = T ($\dot{\mu}\wedge\dot{\rho}$) = T ($\dot{\mu}$) $\wedge$T ($\dot{\rho}$).

Hence T ($\dot{\mu}$) $\leq$ T ($\dot{\rho}$).

Therefore T is fuzzy filter. On the other hand, let T be S$_\blacksquare$'s fuzzy filter, $\dot{\mu}$, $\dot{\rho}\in$S, since $\dot{\mu}\wedge\dot{\rho}\leq\dot{\mu}$ and $\dot{\mu}\wedge\dot{\rho}\leq\dot{\rho}$, also T ($\dot{\mu}_0$) =1 for some $\dot{\mu}_0\in$S$_\blacksquare$,

Then T ($\dot{\mu}\wedge\dot{\rho}$) $\leq$ T ($\dot{\mu}$) and T ($\dot{\mu}\wedge\dot{\rho}$) $\leq$ T ($\dot{\rho}$), thus T ($\dot{\mu}$ $\wedge\dot{\rho}$) $\leq$ T ($\dot{\mu}$) $\wedge$T ($\dot{\rho}$), which concludesT is a L-fuzzy filter.

□

**Theorem 2.2** *A fuzzy-subset* T *for S$_\blacksquare$ is L fuzzy-filter iff parametric cut($\alpha$ cut)*T$_t$ *a Filter.*

**Proof:** Suppose T is L- fuzzy filter of S$_\blacksquare$, thus T ($\varepsilon_0$) =1, clearly 1 $\in$T$_t$,
where T$_t$ is parametric cut ($\alpha$ cut)of T, for every t $\in$ [0,1].

First proving that T$_t$ is a fuzzy filter of S$_\blacksquare$. Let $\ddot{\varepsilon}$, $\ddot{\omega}\in$T$_t$, then T($\ddot{\varepsilon}$)$\wedge$T($\ddot{\omega}$) $\geq$ t. As T is L-fuzzy filter of S$_\blacksquare$, for t $\in$T ($\ddot{\varepsilon}$ $\wedge\ddot{\omega}$) $\subseteq$T ($\ddot{\varepsilon}$) $\wedge$T ($\ddot{\omega}$).

Thus $\ddot{\varepsilon}\wedge\ddot{\omega}$ $\in$T$_t$ . Let $\ddot{\varepsilon}\in$T$_t$and $\ddot{z}$ $\in$S$_\blacksquare$ such that $\ddot{\varepsilon}\leq\ddot{z}$, then$\ddot{\varepsilon}\wedge$ $\ddot{z}=\ddot{\varepsilon}$, implies T$_t$($\ddot{\varepsilon}$) = T$_t$($\ddot{\varepsilon}\wedge$ $\ddot{z}$) = T$_t$($\ddot{\varepsilon}$) $\wedge$T$_t$($\ddot{z}$).

Thus T$_t$($\ddot{z}$) $\geq$ T$_t$($\ddot{\varepsilon}$) $\geq$ t, implies T$_t$($\ddot{z}$) $\geq$ t.

Therefore, $\ddot{z}\in$T$_t$. Hence,T$_t$ is a filter.

Conversely suppose, that T$_t$ be a fuzzy filter for t$\in$ [0,1], then T ($\varepsilon_0$) = 1 for some $\varepsilon_0$ in S$_\blacksquare$.

Let $\ddot{\varepsilon}$, $\ddot{\omega}\in$S$_\blacksquare$, such that T ($\ddot{\varepsilon}$) $\geq$ $\ddot{\alpha}$ andT ($\ddot{\omega}$) $\geq$ $\dddot{\beta}$, then $\ddot{\varepsilon}\in$T$_\alpha$ and$\ddot{\omega}\in$T$_\beta$. Since $\ddot{\alpha}$, $\dddot{\beta}\in$ [0,1] , then either $\ddot{\alpha}$ $\leq$ $\dddot{\beta}$ or $\dddot{\beta}$ $\leq$ $\ddot{\alpha}$

by taking $\ddot{\alpha}$ $\leq$ $\dddot{\beta}$, such as T$_\alpha$ $\subseteq$ T$_\beta$, which implies $\ddot{\varepsilon}$, $\ddot{\omega}\in$T$_\alpha$ and also T$_\alpha$ is a filter, we have $\ddot{\varepsilon}\wedge\ddot{\omega}\in$T$_\alpha$ as T$_\alpha$ $\subseteq$T. Therefore T ($\ddot{\varepsilon}\wedge\ddot{\omega}$) =T ($\ddot{\varepsilon}$) $\wedge$T ($\ddot{\omega}$). Hence T a fuzzy filter of S$_\blacksquare$.

□

### 2.5. Antitone

**Definition 2.5** *For every $\alpha$, $\omega$ in S$_\blacksquare$ with $\alpha\leq\omega$, implies* T *($\omega$)* $\leq$ T *($\alpha$), then the L-fuzzy subset T of S$_\blacksquare$ is assumed as antitione.*

### 2.6. Directed-above

**Definition 2.6** *If there is a $z_\varepsilon$ in S$_\blacksquare$ such that $z_\varepsilon\geq x_\varepsilon$, $y_\varepsilon$and $(x_\varepsilon]\wedge(y_\varepsilon]<(z_\varepsilon]$ for any $x_\varepsilon$, $y_\varepsilon$ in S$_\blacksquare$.*

### 2.7. Softened Distributive semi lattice

**Definition 2.7** *If* $(\overrightarrow{a})^*$ *an L-fuzzy ideal in* $S_\blacksquare$*, assumed as* $(\overrightarrow{a})^* = \{ x^\diamond \in S / x^\diamond \wedge a = 0; a \in S\}$*, then semilattice*$S_\blacksquare$ *is a softened distributive semi lattice when the following conditions met by* $(\overrightarrow{a})^*$*:*

*(i) for some* $\vartheta_0 \in S_\blacksquare$*,* $x^\diamond \wedge \vartheta_0 = 1$*;*

*(ii) for* $x^\diamond \leq y^\diamond$*,* $y^\diamond \wedge a \leq x^\diamond \wedge a$*; and*

*(iii) for* $z^\diamond \geq x^\diamond, y^\diamond$*, and* $(x^\diamond \wedge a) \wedge (y^\diamond \wedge a) \leq (z^\diamond \wedge a)$*.*

**Example 2.3** *Let* $S_\varphi = \{0, a^\diamond, b^\diamond, 1\}$ *be a semi lattice with operations*$(\wedge)$

| $\wedge$ | 0 | $a^\diamond$ | $b^\diamond$ | 1 |
|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 0 |
| $a^\diamond$ | 0 | $a^\diamond$ | 0 | $a^\diamond$ |
| $b^\diamond$ | 0 | 0 | $b^\diamond$ | $b^\diamond$ |
| 1 | 0 | $a^\diamond$ | $b^\diamond$ | 1 |

*Table-1*

*For any* $a^\diamond \in S_\varphi$*, define* $(a^\diamond)^*$ *as:* $(a^\diamond)^* = \{ \tilde{x} \in S_\varphi \mid \tilde{x} \wedge a^\diamond = 0\}$ *( L-fuzzy ideal) of* $S_\varphi$

*For example:* $(0)^* = \{0, a^\diamond, b^\diamond, 1\}$*;* $(a^\diamond)^* = \{0, b^\diamond\}$*;* $(b^\diamond)^* = \{0, a^\diamond\}$*;* $(1)^* = \{0\}$

*Now, let's check if* $(a^\diamond)^*$ *satisfies the conditions: There exists* $x_0 \in S_\varphi$ *like* $\tilde{x} \wedge x_0 = 0 \; \forall \; \tilde{x} \in S_\varphi$*. (where* $x_0 = 0$*)*

*If* $\tilde{x} \leq \tilde{y}$*, it results* $\tilde{y} \wedge a^\diamond \leq \tilde{x} \wedge a^\diamond$*. (This condition holds to all* $\tilde{x}, \tilde{y}, a^\diamond \in S_\varphi$*)*

*If* $\tilde{z} \geq \tilde{x}, \tilde{y}$*, it results* $(\tilde{x} \wedge a^\diamond) \wedge (\tilde{y} \wedge a^\diamond) \leq \tilde{z} \wedge a^\diamond$*. (This condition holds to all* $\tilde{x}, \tilde{y}, \tilde{z}, a^\diamond \in S_\varphi$*)*

*Since* $(a^\diamond)^*$ *satisfies all the conditions,* $S_\varphi$ *is a softened distributive semi lattice.*

### 2.8. *semi lattice:

A semi lattice $S_\blacksquare$ is said to be a * semi lattice if and only if for any a in $S_\blacksquare$, we can find $a^\nabla$ in $S_\blacksquare$ such as $(a)^* = (a^\nabla)^{**}$ where $a \wedge a^\nabla = 0$.

***Note:*** L fuzzy filters of meet semilattice establish fuzzy connectedness by considering the relationship between pixels in the spatial domain but generally operate without using the feature information like color, intensity or texture, inherent to those pixels, but can use a specially designed L fuzzy frame work such as different types of fuzzy models in * semilattice to incorporate such features.

### 2.9. Dense element

In semi lattice $S_\blacksquare$, an element M is considered dense if $(M)^* = \{0\}$.

**Example 2.4** *Let* $S\varphi = \{0, a^\diamond, b^\diamond, 1\}$ *be a semilattice with the operation:*

| $\wedge$ | 0 | $a^\diamond$ | $b^\diamond$ | 1 |
|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 0 |
| $a^\diamond$ | 0 | $a^\diamond$ | 0 | $a^\diamond$ |
| $b^\diamond$ | 0 | 0 | $b^\diamond$ | $b^\diamond$ |
| 1 | 0 | $a^\diamond$ | $b^\diamond$ | 1 |

*Table-2*

*The element "1" is dense element because: (1)* = {0}. The only element that annihilates 1 is 0 itself.*

## 2.10. Result:

**Theorem 2.3** *If every dense elements of* $S_{\blacksquare}$*'s is defined set* $D_{\otimes}$*, then* $D_{\otimes}$ *represents an L-fuzzy filter.*

**Proof:** Assuming all dense elements of $S_{\blacksquare}$ in the set $D_{\otimes}$, defined as

$D_{\otimes} = \{x^{\odot} / (x^{\odot})^* = \{0\}\}$, for $x_0 \in D_{\otimes}$

we have, $(x_0)^* = 0$, implies that $((x_0)^*)^* = 0^*$, which implies $D_{\otimes}(x_0) = 1$.

Then $(x^{\odot})^* = \{0\}$ and $(y^{\odot})^* = \{0\}$ for $x^{\odot}, y^{\odot} \in D_{\otimes}$

Let $j^{\odot} \in S_{\blacksquare}$, such as $j^{\odot} \equiv x^{\odot}$ and $j^{\odot} \equiv y^{\odot}$

then $j^{\odot} \equiv x^{\odot} \wedge y^{\odot}$, also $(j^{\odot})^* \subseteq (x^{\odot})^*$ and $(j^{\odot})^* \subseteq (y^{\odot})^*$,

implies that $(j^{\odot})^* \subseteq (x^{\odot})^* \wedge (y^{\odot})^* = \{0\}$.

Therefore, $(j^{\odot})^* \subseteq \{0\}$ . Similarly we can verify $\{0\} \subseteq (j^{\odot})^*$.

Therefore, $(j^{\odot})^* = \{0\}$, thus $j^{\odot} \in D_{\otimes}$.

Therefore, $x^{\odot} \wedge y^{\odot} \in D_{\otimes}$, implies $D_{\otimes}(x^{\odot} \wedge y^{\odot}) = 1$, where $D_{\otimes}(x^{\odot}) = 1$ and $D_{\otimes}(y^{\odot}) = 1$.

Therefore, $D_{\otimes}(x^{\odot} \wedge y^{\odot}) = D_{\otimes}(x^{\odot}) \wedge D_{\otimes}(y^{\odot})$, gives $D_{\otimes}$ as L-fuzzy filter. □

## 2.11. Fuzzy filter

**Definition 2.8** *A fuzzy-filter* $\boldsymbol{T}$ *of* $S_{\blacksquare}$ *is known as *fuzzy filter, if* $\boldsymbol{T}(x_{\odot}) = \boldsymbol{T}(x_{\odot}^{**})$ *for all* $x_{\odot}$ *in L.*

**Example 2.5** ***Example of fuzzy filter:*** *Let* $S_{\odot} = \{0, a^{\diamond}, b^{\diamond}, 1\}$ *be a semilattice with operation* ($\wedge$) *as followed:*

| $\wedge$ | 0 | $a^{\diamond}$ | $b^{\diamond}$ | 1 |
|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 0 |
| $a^{\diamond}$ | 0 | $a^{\diamond}$ | 0 | $a^{\diamond}$ |
| $b^{\diamond}$ | 0 | 0 | $b^{\diamond}$ | $b^{\diamond}$ |
| 1 | 0 | $a^{\diamond}$ | $b^{\diamond}$ | 1 |

*Table-3*

*Define, fuzzy-filter* M *on* $S_{\odot}$ *as:* M $(0) = 0$; M $(a^{\diamond}) = 0.5$; M $(b^{\diamond}) = 0.5$; M$(1) = 1$. *Now, let's check if* M*, a fuzzy filter:*

*(i)* M $(0) = 0 \leq$ M $(m^{\diamond})$ *to all* $m^{\diamond}$ *of* $S_{\odot}$

*(ii) If* $m^{\diamond} \leq h^{\diamond}$, *it gives* M $(m^{\diamond}) \leq$ M $(h^{\diamond})$

*For example:* $a^{\diamond} \leq 1$ *implies* M $(a^{\diamond}) = 0.5 \leq$ M $(1) = 1$

*(iii)* M $(m^{\diamond} \wedge h^{\diamond}) \geq$ M $(m^{\diamond}) \wedge$M $(h^{\diamond})$

*For example:* M $(a^\diamond \wedge b^\diamond) = $ M $(0) = 0 \geq$ M $(a^\diamond) \wedge$M $(b^\diamond) = 0.5 \wedge 0.5 = 0.5$

*Since,* M *satisfies all the conditions,* M *is a "fuzzy filter".*

### 2.12. L-Fuzzy filter

**Definition 2.9** *Fuzzy-filter* M *of* $S_\blacksquare$ *known as *L- fuzzy filter if (i)* M $(x_0) = 1$, *(ii)* M $(x) = $ M $(x^{**})$ $\forall$ $x$ *in L, for some* $x_0 \in S_\blacksquare$.

**Example 2.6** *Take* $S_\odot = \{0, a^\diamond, b^\diamond, 1\}$, *semilattice having operation(∧) as:*

| ∧ | 0 | $a^\diamond$ | $b^\diamond$ | 1 |
|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 0 |
| $a^\diamond$ | 0 | $a^\diamond$ | 0 | $a^\diamond$ |
| $b^\diamond$ | 0 | 0 | $b^\diamond$ | $b^\diamond$ |
| 1 | 0 | $a^\diamond$ | $b^\diamond$ | 1 |

*Table-4*

*Define fuzzy-filter* M *on* $S_\odot$ *as:* M $(0) = 0$; M $(a^\diamond) = 0.5$; M $(b^\diamond) = 0.5$; M$(1) = 1$

*Now, let's check if* M, *a * L-fuzzy filter:*

*(i) As* M$(1) = 1$ *then for some* $x_0 \in S_\odot$ $(x_0 = 1)$, *we have* M$(x_0) = 1$.

*(ii)* M $(x^\diamond) = $ M $(x^{**})$ *for every x in* $S_\odot$:

$0^{**} = 1, \quad M(0) = 0 \neq M(1) = 1$ *(condition not met)*

*However, if we redefine* M *as:*

M$(0) = $ '1'; M$(a^\diamond) = $ '0.5'; M$(b^\diamond) = $ '0.5'; M $=$ '1' .

*Then: (i)* M $(0) = 1$ *for some* $0 \in S$ $(x_0 = 0)$

*(ii)* M $(x^\diamond) = $ M $(\underline{x}^{**})$ *to each x in* $S_\odot$:

$0^{**} = 1$, M$(0) = 1 = $ M$(1)$

$(a^\diamond)^{**} = b^\diamond$, M$(a^\diamond) = 0.5 = $ M $(b^\diamond)$

$(b^\diamond)^{**} = a^\diamond$, M$(b^\diamond) = 0.5 = $ M $(a^\diamond)$

$1^{**} = 0, \quad M(1) = 1 \neq M(0) = 1$ *(actually, this condition is met)*

*In this redefined example,* M *is a * L-fuzzy filter.*

**Theorem 2.4** *For any parametric cut* $M_t \subseteq$M, *a *fuzzy Filter of (Semilattice)* $S_\blacksquare$, *a fuzzy-subset M of* $S_\blacksquare$ *is a *L-fuzzy filter.*

**Proof:** Consider M, fuzzy-subset of $S_\blacksquare$, if it is a * L fuzzy filter, then

$$\mathrm{M}(x_0) = 1; \mathrm{M}(v) = \mathrm{M}(v^{**}) \tag{2.1}$$

for all v in $S_\blacksquare$.

Then we know that every position of subset $M_t$of M,for t∈[0,1] represents a filter of S.

Now to verify that $M_t$ , a * fuzzy filter in S. For t in [0,1] & x** ∈$M_t$,suchas $M_t$(x) = $M_t$(x**).

Thus, $M_t$ is a * filter of S■(from I)On the other hand, let's assume that ,each parametric cut $M_t$ is a * filter for all t ∈ [0,1], i.e. for each position subset of M , a * fuzzy filter inS■.

Proving that M, * L-fuzzy filter for S■.

Any $x_0$∈ [0,1]⊆ S■, we have M ($x_0$) = 1. For ẋ ∈$M_t$ , we have ẋ** ∈$M_t$ and as $M_t$ is fuzzy filter , then M is fuzzy filter of S■ ( by known theorem) since ẋ ≤ ẋ**, gives

$$M(\dot{x})\leq M(\dot{x}^{**}) \tag{2.2}$$

for all ẋ in S■
If M (ẋ**) = t , ∀ t ∈ [0,1] and as $M_t$(ẋ) = $M_t$(ẋ**) for ẋ ∈$M_t$, which shows that

$$M(\dot{x}^{**})\leq M(\dot{x}) \tag{2.3}$$

Therefore, from (2.2) and (2.3) we have M (ẋ) = M (ẋ**). Hence M is *L-fuzzy filter of S■. □

**Theorem 2.5** *In a softened distributive semilattice S■, the given below conditions are equivalent: (i) S■ is a * semi lattice (ii) when ξ ∈S■, exists some $\xi^1$ in S■ as ξ ∧ $\xi^1$ =0 and $(\dot{\xi}] \bigcap (\dot{\xi}^1] \subseteq D_\otimes$, here $D_\otimes$ is L-fuzzy filter.*

**Proof:**
For $\dot{\xi}$ in S■, $\dot{\xi}^1$exists in S■ such that $(\dot{\xi})^* = (\dot{\xi}^1)^{**}$ where $\dot{\xi}\wedge\dot{\xi}^1$ =0, and S■ be * semi lattice. Let $D_\otimes$be L-fuzzy filter of S■, then $D_\otimes(\dot{\xi}_0)$=1 for some $\dot{\xi}_0$ in S■ , also $D_\otimes(\dot{\xi}\wedge y) = D_\otimes(\dot{\xi}) \wedge D_\otimes(y)$ for $\dot{\xi}$, y in S■.

Now for $\dot{\xi}$ in S■, $\dot{\xi}_0$ exists in S■ such as $\dot{\xi}\wedge\dot{\xi}_0$ =0, let ẗ ∈ $(\dot{\xi}]\bigcap(\dot{\xi}^1]$, implies ẗ ∈ $(\dot{\xi}]$ and ẗ ∈ $(\dot{\xi}^1]$, this implies ẗ $\wedge\dot{\xi}\leq 0$ and ẗ $\wedge\dot{\xi}^1\leq 0$, implies ẗ ∈ $(\dot{\xi}^1)^*$ and as $D_\otimes$ is L- fuzzy filter there exists ẗ ∈ $D_\otimes$ for $\dot{\xi}$, $\dot{\xi}^1$∈ $D_\otimes$, such that $D_\otimes(\dot{\xi}\wedge\ddot{t}) = D_\otimes(\dot{\xi}) \wedge D_\otimes(\ddot{t})$ and $D_\otimes(\dot{\xi}^1\wedge\ddot{t}) = D_\otimes(\dot{\xi}^1) \wedge D_\otimes(\ddot{t})$.

Now $D_\otimes(\dot{\xi}\wedge\dot{\xi}^1\wedge\ddot{t}) = D_\otimes(\dot{\xi}\wedge t\wedge\dot{\xi}^1\wedge\ddot{t}) = D_\otimes(\dot{\xi} \wedge \ddot{t}) \wedge D_\otimes(\dot{\xi}^1\wedge\ddot{t})= D_\otimes(\dot{\xi}) \wedge D_\otimes(\ddot{t}) \wedge D_\otimes(\dot{\xi}^1 ) \wedge D_\otimes(\ddot{t})$ $= D_\otimes(\dot{\xi}) \wedge D_\otimes(\dot{\xi}^1 ) \wedge D_\otimes(\ddot{t})$, thus ẗ ∈ $D_\otimes(\dot{\xi}) \wedge D_\otimes(\dot{\xi}^1 ) \wedge D_\otimes(\ddot{t})$,

which implies $D_\otimes(\dot{\xi}) \wedge D_\otimes(\dot{\xi}^1 ) \subseteq D_\otimes(\ddot{t})$ and it leads to $(\dot{\xi}]\bigcap(\dot{\xi}^1)]\subseteq D_\otimes$.

Now (ii)→ (i): for $\dot{\xi}$ in S■, $\dot{\xi}^1$ exists in S■ such as $\dot{\xi}\wedge\dot{\xi}^1 = 0$ and $(\dot{\xi}) \bigcap(\dot{\xi}^1) \subseteq D_\otimes$, where $D_\otimes$ a L-fuzzy filter. First proving that S■is a *semi lattice. As $\dot{\xi}\wedge\dot{\xi}^1 = 0$, then $\dot{\xi}^1\in(\dot{\xi})^*$.

Let a∈ $(\dot{\xi}^1)^{**}$, implies $(a)^* \subseteq (\dot{\xi}^1)^*$ then a ∧ $(\dot{\xi}^1)^* = 0$, gives a $\wedge\dot{\xi} = 0$ for $\dot{\xi}\in (\dot{\xi}^1)^*$.

Then, a ∈ $(\dot{\xi})^*$, as well as

$$(\dot{\xi}^1)^{**}\subseteq(\dot{\xi})^* \tag{2.4}$$

Let ẗ ∈$(\dot{\xi})^*$ for ẗ in S■, then ẗ $\wedge\dot{\xi} = 0$ also a$\wedge\dot{\xi}^1 = 0$ for a in S■, then ẗ ∧ a∧ $\dot{\xi}^1 = 0\wedge\ddot{t} = 0$, implies ẗ ∈ $(\dot{\xi}^1\wedge a)^*$.

If a ∈ $(\dot{\xi}]\bigcap(\dot{\xi}^1]$, we have a ∈$(\dot{\xi}]$ and a ∈$(\dot{\xi}^1]$, then $\dot{\xi} \leq$ a and $\dot{\xi}^1\leq$ a, implies $\dot{\xi}$ ∧ẗ ≤ a ∧ẗ and $\dot{\xi}^1\wedge\ddot{t} \leq$ a∧ẗ, implies $\dot{\xi}$ ∧ẗ $\wedge\dot{\xi}^1\wedge\ddot{t} \leq$ a∧ẗ ∧ a ∧ẗ, implies $\dot{\xi}$ $\wedge\dot{\xi}^1\wedge\ddot{t} \leq$ a ∧ẗ, implies 0 ∧ẗ ≤ a∧ẗ, implies 0 ≤ a∧ẗ, implies ẗ ∈ (a)*.

Therefore, $\ddot{t}\in(a)^* \subseteq (\dot{\xi}^1)^{**}$. Therefore

$$(\dot{\xi})^*\subseteq(\dot{\xi}^1)^{**} \tag{2.5}$$

Hence, (2.4) and (2.5) concludes $(\dot{\xi})^* = (\dot{\xi}^1)^{**}$ which represents $S_\blacksquare$ as a *semi lattice.

L fuzzy filter in the distributive semilattice is a collection of filters and distributive semilattice provides the algebraic structure for these filters, allowing operations like meet & join that distribute over each other, which allows to create approximate filtering mechanisms that are useful for handling the data. □

**Distributive semi lattice L fuzzy filter**

**Example 2.7** *The L fuzzy filter is taken as "men that are atleast somewhat height and also quiet fat"; in this case, the words somewhat and quiet are fuzzy grades of membership; therefore, distributive structure would allow to combine these fuzzy concepts algebraically. Similarly, consider a distributive semi lattice of men with the meet operation being "And" and the join operation being "Or." The goal of the mathematical study of L fuzzy filters on meet semi lattice is to define and characterise these fuzzy filters on those structures using tools from fuzzy set theory or lattice theory. When the domain is lacking distributivity, the focus shifts from direct image processing applications to the generalised theory of L fuzzy filters (i.e., L fuzzy filters extend the concept of fuzzy filters by replacing the standard membership values from the unit interval [0,1]).*

### 2.13. Sectionally * semi lattice:

: A *semi lattice$S_\blacksquare$ is defined as sectionally *semi lattice, if interval (0,a) becomes *semi lattice to each a in $S_\blacksquare$

**Theorem 2.6** *Whenever a semi lattice is * semi lattice, then it is a sectionally * semi lattice.*

**Proof:** $S_\blacksquare$be * semi lattice, then for $x^\diamond$ of $S_\blacksquare$, $a^\diamond$ present in $S_\blacksquare$ such as $(x^\diamond)^* = (a^\diamond)^{**}$, where $x^\diamond \wedge a^\diamond = 0$.

Consider the interval $(0, w^\diamond) \subseteq S_\blacksquare$ and Let $\mathbf{X} = (0, w^\diamond)$. Let $x^\diamond\in\mathbf{X}\subseteq S_\blacksquare$, thus for $x^\diamond\in S_\blacksquare$, $a^\diamond$ exists in $S_\blacksquare$ like that $(x^\diamond)^* = (w^\diamond)^{**}$, where $w^\diamond \wedge x^\diamond = 0$, taking $x^1$ in $S_\blacksquare$ In that way, $x^1 = x^\diamond \wedge w^\diamond = 0 \in \mathbf{X}$.

Therefore, $x^\diamond\wedge x^1 = 0$, provides $x^\diamond\in (x^1)^*$ in X.

To prove that $(\mathbf{X})_{[0,w^\diamond]} = (x^1)_{[0,w^\diamond]}$. Let $y^\diamond \in (\mathbf{X})_{[0,w^\diamond]}$. Then $y^\diamond \wedge x^\diamond = 0$ for $x^\diamond \in [0, w^\diamond]$ implies $y^\diamond\in(x^\diamond)^*$, but $(x^\diamond)^* = (w^\diamond)^{**}$ in $S_\blacksquare$, implies $y^\diamond \in (w^\diamond)^{**}$, implies $y^\diamond \wedge (w^\diamond)^* = 0$, implies $y^\diamond \wedge z^\diamond = 0$ for $z^\diamond \in (w^\diamond)^*$, where $z^\diamond \wedge w^\diamond = 0$, then $y^\diamond \wedge z^\diamond \wedge w^\diamond = 0$. for $a^\diamond \in (x^1)_{[0,w^\diamond]}$, which implies $y^\diamond \in (x^1)_{[0,w^\diamond]}$.

Thus,

$$(X)_{[0,w^\diamond]} \subseteq (x^1)_{[0,w^\diamond]} \tag{2.6}$$

Similarly, we can prove $(x^1)_{[0,w^\diamond]} \subseteq (\mathbf{X})_{[0,w^\diamond]}$.

Therefore, $(0, w^\diamond)$ is *semi lattice, which shows $S_\blacksquare$ is sectionally *semi lattice. □

**Theorem 2.7** *The essential and sufficient condition of a position subset $A_t$for t∈ [0,$a^\diamond$] ⊆ [0,1] * fuzzy filter, is a fuzzy-subset A in $S_\blacksquare$, is *L fuzzy filter in sectionally *semi lattice.*

**Proof:** $S_\blacksquare$be a sectionally * semi lattice,then each interval [0,$a^\diamond$] ⊆ [0,1] represents a *semi lattice. Let fuzzy subset A of $S_\blacksquare$,be a *L fuzzy filter, results $A(x^\diamond)=A(x^{\diamond**})$ for all $x^\diamond$ in $S_\blacksquare$.

To show that each position subset $A_t$ is * fuzzy filter in $[0,a^\diamond]$, let $x^{\diamond**} \in A_t$ for $x^\diamond$ in $[0,a^\diamond]$, thus we have $x^\diamond \in A_t$ such that $A_t(x^\diamond) = A_t(x^{\diamond**})$.

Consequently, $A_t$ is a * fuzzy filter. On the other hand we assume that $A_t$ is a *fuzzy filter in $[0,a^\diamond]$, then we prove A, *L fuzzy filter for $S_\blacksquare$, by known theorem, If $A_t$ is fuzzy filter, A is fuzzy filter of $S_\blacksquare$, also as $S_\blacksquare$ is sectionally * semi lattice, thus $x^{\diamond*} = (x^{\diamond 1})^{**}$ like that $x^\diamond \wedge x^{\diamond 1} = 0$.

$x^\diamond \in [0,a^\diamond] \subseteq [0,1]$ , as $(x^{\diamond*}) \leq (x^{\diamond 1})^{**}$ , then

$$A(x^{\diamond*}) \leq A(x^{\diamond 1})^{**} \tag{2.7}$$

since, $A_t$ is a * fuzzy filter, then $A_t(x^{\diamond 1})^{**} = A_t(x^{\diamond*})$for $t^\diamond \in [0,a^\diamond]$, which shows that

$$A(x^{\diamond 1})^{**} \leq A(x^{\diamond*}) \qquad \text{for} \quad (x^{\diamond 1})^{**} \leq (x^\diamond)^* \tag{2.8}$$

Therefore (2.7) and (2.8) gives $A(x^{\diamond*}) = A(x^{\diamond 1})^{**}$ such that $x^\diamond \wedge x^{\diamond 1} = 0$, which leads to A is a *L fuzzy filter. □

### 2.14. Modular semi lattice

**Definition 2.10** *A meet semi lattice $S_\blacksquare$ ,is modular, when $c^\sim \wedge k^\sim \leq w^\diamond \leq c^\sim$, then there prevails $y^\diamond$ inn $S_\blacksquare$ like that $y^\diamond \geq k^\sim$ and $w^\diamond = c^\sim \wedge k^\sim$.*

**Example 2.8** $\widehat{\mathbf{S}}$ = { *0*, $\bar{e}$, $\bar{k}$, $\bar{d}$, *1*} *be a meet semi lattice along operation* ($\wedge$) *defined as*

| $\wedge$ | *0* | $\bar{e}$ | $\bar{k}$ | $\bar{d}$ | *1* |
|---|---|---|---|---|---|
| *0* | *0* | *0* | *0* | *0* | *0* |
| $\bar{e}$ | *0* | $\bar{e}$ | *0* | $\bar{e}$ | $\bar{e}$ |
| $\bar{k}$ | *0* | *0* | $\bar{k}$ | *0* | $\bar{k}$ |
| $\bar{d}$ | *0* | $\bar{e}$ | *0* | $\bar{d}$ | $\bar{d}$ |
| *1* | *0* | $\bar{e}$ | $\bar{k}$ | $\bar{d}$ | *1* |

*Table-5*

*Now, let's check the modular condition: Take* $\bar{e}^- = \bar{d}$, $\bar{k}$=$\bar{e}$, *and* $w_\circ = \bar{e}$: $\bar{d} \wedge \bar{e} \leq \bar{e} \leq \bar{d}$.

*There is* $y_\circ = 1 \in \widehat{\mathbf{S}}$ *such as:* $y_\circ \geq \bar{e}$ *as well as* $\bar{e} = \bar{e} \wedge 1$. *Therefore,* $\widehat{\mathbf{S}}$ *is a modular semilattice.*

**Theorem 2.8** *In a modular *semi lattice $S_\blacksquare$, we have $\theta_D{}^\circ = R$ where $R = \{ (m_\circ,l_\circ)/ \ m_\circ{}^* \equiv l_\circ{}^*\}$ and $D^\circ$ is L-fuzzy filter .*

**Proof:** For any L-fuzzy filter $D^\circ$ of semilattice $S_\blacksquare$, define $\theta_D{}^\circ = \{(m_\circ,l_\circ)/ \ D^\circ(m_\circ \wedge d_\circ) = D^\circ(l_\circ \wedge d_\circ)$, for $d_\circ \in D^\circ\}$. Since $D^\circ(m_\circ \wedge d_\circ)$=$D^\circ(m_\circ \wedge d_\circ)$, thus $m_\circ R m_\circ$ and $R = \{ (m_\circ,l_\circ)/ \ m_\circ{}^* \equiv l_\circ{}^*\}$

If $m_\circ R l_\circ$ then $D^\circ(m_\circ \wedge d_\circ) = D^\circ(l_\circ \wedge d_\circ)$, gives $D^\circ(l_\circ \wedge d_\circ) = D^\circ(m_\circ \wedge d_\circ)$, thus $l_\circ R m_\circ$ and also the relation is compatible w.r.t $\wedge$.

Therefore, for any filter $D^\circ$, $\theta_{D^\circ}$ is congruence on $S_\blacksquare$. Therefore, $\theta_{D^\circ} \subseteq$ R. suppose $(k_\circ,f_\circ) \in$R,as $k_\circ{}^* \equiv f_\circ{}^*$, implies for $d_\circ \in (k_\circ)^*$, we have $k_\circ \wedge d_\circ = 0$ also $f_\circ \wedge d_\circ = 0$ for $k_\circ$, $f_\circ$ in $S_\blacksquare$ also $D^\circ$ is L-fuzzy filter & $D^\circ(k_\circ \wedge f_\circ) = D^\circ(k_\circ) \wedge D^\circ(f_\circ)$, any dense element $d_\circ$ in $D^\circ$ defines $(d_\circ)^* = \{0\}$.

For any $k_\circ \in (d_\circ)^*$ gives $k_\circ \wedge d_\circ = 0$, then $0 \leq k_\circ \wedge d_\circ \leq k_\circ$, using modularity there is $f_\circ$ like that $f_\circ \geq d_\circ$ and $k_\circ \wedge d_\circ = k_\circ \wedge y_\circ$, implies

$$D^\circ(k_\circ \wedge f_\circ) = D^\circ(k_\circ \wedge d_\circ) \tag{2.9}$$

similarly for $f_\circ \in (d_\circ)^*$, we have $f_\circ \wedge d_\circ = 0$, then $0 \leq f_\circ \wedge d_\circ \leq f_\circ$, using modularity, $k_\circ$ exists such as $k_\circ \geq d_\circ$ as well as $f_\circ \wedge d_\circ = f_\circ \wedge x_\circ$, which results

$$D^\circ(f_\circ \wedge d_\circ) = D^\circ(f_\circ \wedge k_\circ) \tag{2.10}$$

Therefore, (2.9) and (2.10) gives $D^\circ(k_\circ \wedge d_\circ) = D^\circ(f_\circ \wedge d_\circ)$, thus $(k_\circ, f_\circ) \in \theta_{D^\circ}$ which represents $R \subseteq \theta_{D^\circ}$

Hence, $\theta_{D^\circ} = R$.

□

**Theorem 2.9** *The essential and sufficient constrain for a directed above * semi lattice $S_\blacksquare$ with respective to $\theta_{D^\circ} = R$ to be softened distributive semi lattice $S_\blacksquare$ is $(x_\triangle \cup y_\triangle] \vee D^\circ = (x_\triangle \cup D^\circ] \bigcap (y_\triangle \cup D^\circ]$.*

**Proof:** Directed above * semi lattice $S_\blacksquare$ is softened distributive semi lattice such that $\theta_{D^\circ} = R$, where $\theta_{D^\circ} = \{(x_\triangle, y_\triangle) / D^\circ(x_\triangle \wedge d_\triangle) = D^\circ(y_\triangle \wedge d_\triangle)\}$ and $R = \{ (x_\triangle, y_\triangle) / x_\triangle^* \equiv y_\triangle^*\}$ where $D^\circ$ is L-fuzzy filter.

Let $t_\triangle \in (x_\triangle \cup D^\circ] \bigcap (y_\triangle \cup D^\circ]$, then $t_\triangle \in (x_\triangle \cup D^\circ]$ and $t_\triangle \in (y_\triangle \cup D^\circ]$, implies $x_\triangle \wedge d_\triangle \leq t_\triangle \leq x_\triangle \vee d_\triangle$ and $y_\triangle \wedge d_\triangle \leq t_\triangle \leq y_\triangle \vee d_\triangle$ for $d_\triangle \in D^\circ$.

As $S_\blacksquare$ is a * semi lattice for any $t_\triangle$ in $S_\blacksquare$, $t_\triangle{}^1$ exists as $(t_\triangle)^* = (t_\triangle{}^1)^{**}$, where $t_\triangle \wedge t_\triangle{}^1 = 0$. It leads to $x_\triangle \wedge d_\triangle \wedge t_\triangle{}^1 = 0$ and $y_\triangle \wedge d_\triangle \wedge t_\triangle{}^1 = 0$.

Because, $S_\blacksquare$ is softened distributive,for some $t_\triangle$ in $S_\blacksquare$, $x_\triangle \wedge t_\triangle = 1$ and when $x_\triangle \leq y_\triangle$ provides $y_\triangle \wedge t_\triangle \leq x_\triangle \wedge t_\triangle$ and also for $z_\triangle \geq x_\triangle, y_\triangle$ we have

$$(x_\triangle \wedge t_\triangle) \wedge (y_\triangle \wedge t_\triangle) \leq (z_\triangle \wedge t_\triangle) \tag{2.11}$$

Since, $(x_\triangle)^* \equiv (y_\triangle)^*$ from R , then for $d_\triangle \in (x_\triangle)^*$ we have $x_\triangle \wedge d_\triangle = 0$, similarly for $d_\triangle \in (y_\triangle)^*$ we have $y_\triangle \wedge d_\triangle = 0$, thus $0 \leq t_\triangle \leq x_\triangle \vee d_\triangle$ and $0 \leq t_\triangle \leq y_\triangle \vee d_\triangle$ then $t_\triangle \leq (x_\triangle \vee d_\triangle) \wedge (y_\triangle \vee d_\triangle) = (x_\triangle \vee y_\triangle) \wedge d_\triangle$, since $(x_\triangle)^* = (y_\triangle)^*$ then $(x_\triangle \wedge t_\triangle)^* = (y_\triangle \wedge t_\triangle)^*$,

It provides that $x_\triangle \wedge y_\triangle \leq t_\triangle$ also $D^\circ(t_\triangle) = 1$ for some $t_\triangle$ in $S_\blacksquare$, then $D^\circ(x_\triangle \wedge y_\triangle) \subseteq D^\circ(t_\triangle)$, implies $D^\circ(x_\triangle) \wedge D^\circ(y_\triangle) \leq D^\circ(t_\triangle)$,

so,

$$D^\circ(x_\triangle \wedge t_\triangle) \wedge D^\circ(y_\triangle \wedge t_\triangle) \leq D^\circ(t_\triangle) \tag{2.12}$$

By comparing (2.11) and (2.12) provides $D^\circ(t_\triangle) = D^\circ(z_\triangle \wedge t_\triangle)$, implies $z_\triangle \wedge t_\triangle = 1$ for $z_\triangle \geq x_\triangle$ and $z_\triangle \geq y_\triangle$. For $d_\triangle \in D^\circ$, $z_\triangle \wedge t_\triangle \wedge d_\triangle = d_\triangle$. Therefore $z_\triangle \wedge t_\triangle \wedge d_\triangle = d_\triangle = z_\triangle \wedge d_\triangle$, implies $z_\triangle \wedge d_\triangle \leq t_\triangle$.

Therefore, $t_\triangle \in ((x_\triangle \wedge y_\triangle) \cup D^\circ]$ for $z_\triangle \in (x_\triangle \wedge y_\triangle)$ and $d_\triangle \in D^\circ$, obtaining $(x_\triangle \vee D^\circ) \bigcap (y_\triangle \vee D^\circ) \subseteq ((x_\triangle \bigcap y_\triangle] \cup D^\circ)$. Similarly $((x_\triangle \bigcap y_\triangle) \cup D^\circ]) \subseteq (x_\triangle \vee D^\circ] \bigcap (y_\triangle \vee D^\circ]$.

Consequently, $((x_\triangle \bigcap y_\triangle] \cup D^\circ) = (x_\triangle \cup D^\circ] \bigcap (y_\triangle \cup D^\circ]$.

Now on, assuming $((x_\triangle \bigcap y_\triangle] \cup D^\circ) = (x_\triangle \cup D^\circ] \bigcap (y_\triangle \cup D^\circ]$ holds and to prove $S_\blacksquare$ is softened distributive semi lattice. It is require to prove if $(a_\triangle)^* = \{ x_\triangle \text{ in } S_\blacksquare / x_\triangle \wedge a_\triangle = 0\}$ is a L-fuzzy ideal of $S_\blacksquare$

These results (i)let $x_\triangle, y_\triangle \in (a_\triangle)^*$ then $x_\triangle \wedge a_\triangle = 0 = y_\triangle \wedge a_\triangle$, let $x_\triangle \leq y_\triangle$ then $x_\triangle \wedge a_\triangle \leq y_\triangle \wedge a_\triangle$ , implies $0 \leq 0$ , implies $y_\triangle \wedge a_\triangle \leq x_\triangle \wedge a_\triangle$.

(ii) Now $z_\triangle \geq x_\triangle, y_\triangle$ implies $x_\triangle \wedge z_\triangle = x_\triangle$ and $y_\triangle \wedge z_\triangle = y_\triangle$ , therefore $x_\triangle \wedge z_\triangle \wedge a_\triangle = x_\triangle \wedge a_\triangle = 0$ and $y_\triangle \wedge z_\triangle \wedge a_\triangle = y_\triangle \wedge a_\triangle = 0$.

Therefore, $x_{\triangle} \wedge z_{\triangle} \wedge a_{\triangle} = 0 = y_{\triangle} \wedge z_{\triangle} \wedge a_{\triangle}$, results in $z_{\triangle} \wedge a_{\triangle} = 0$.

Hence, $(x_{\triangle} \wedge a_{\triangle}) \wedge (y_{\triangle} \wedge a_{\triangle}) \leq (z_{\triangle} \wedge a_{\triangle})$.

And (iii) also we can prove $x_{\triangle} \wedge x_{\triangle 0} = 1$ for some $x_{\triangle 0} \in D_{\circ} \subseteq S_{\blacksquare}$.

Hence, $S_{\blacksquare}$ is a softened distributive semi lattice. □

The distributive semi lattice's L fuzzy filter is a group of filters, and the semi lattice gives these filters their algebraic structure by enabling operations like meet and join that distribute over one another.This enables the creation of approximation filtering mechanisms that are helpful for managing the data.

Using L fuzzy filters on * semilattice offers more powerful approach like (i) Handling image gradients (ii) Improved handling of shading. Handling image gradients: An algorithm can calculate the image gradients (breaks in uniformity) and use the pseudo complemented structure to better model the uncertainty and ambiguity associated with edges. The fuzzy rules can define to what degree a pixel belongs to an edge based on its neighbourhood, rather than using a sharp cut-off. (iii) *Improved handling of shading:* Because the fuzzy filter can quantify the degree of membership to an edge, it can better distinguish a true edge from a subtle shading effect.

## 3. Real World Applications Towards the Characteristics of L-Fuzzy Filters

L-fuzzy filters find applications across a wide range of domains. In quantum systems, they can filter out unwanted quantum states, enabling more precise control and potentially reducing computational errors in quantum computing. They also enhance the security of quantum communication protocols by minimizing the effects of noise and interference. (i) In image processing, L-fuzzy filters assign membership degrees to pixels based on intensity or features, which aids in boundary detection and segmentation of images into regions defined by texture, colour, or other attributes. (ii) For control systems, they allow the design of controllers that can handle uncertain or imprecise inputs by associating membership degrees with possible control actions. (iii) In decision-making, fuzzy filters provide a way to represent degrees of membership across different outcomes, supporting more flexible and adaptive choices. (iv) Within signal analysis, they help identify patterns by assigning membership degrees to data points, as well as reduce noise by lowering the weight of noisy elements. (v) Finally, in complex systems, L-fuzzy filters offer a structured method to represent partial membership across different states, contributing to a clearer understanding of system dynamics.

## 4. Concluding Remarks

This article examines the conceptual results of L-fuzzy filters and ideals in distributive semi-lattices, emphasizing their role in systems that involve uncertainty and partial information. The main contributions of the study can be outlined as follows. The paper develops their theoretical basis within distributive semi-lattices, forming a foundation for applications. The idea of a * semilattice is extended to each interval [0,1] within a semilattice, enabling a finer understanding of its structure. The authors establish a necessary and sufficient condition for a modular * semilattice to qualify as a softened distributive semi-lattice, clarifying the links between these concepts. Potential uses of L-fuzzy filters in distributive semi-lattices are highlighted, particularly in image processing, decision-making under vagueness, signal analysis, and modeling of complex systems. In conclusion, the study broadens the scope of fuzzy algebraic structures and deepens the understanding of their mathematical foundations, while also pointing to practical applications across diverse domains.

## Acknowledgments

The Department of Mathematics of Siddhartha Academy of Higher Education Deemed to be University, Vijawawada, Department of Mathematics, School of Science and Technology, United States International University, Nairobi, Kenya, School of Advanced Sciences, Vellore Institute of Technology have all provided invaluable assistance. We also Thankful to reviewers for their valuable suggestions.

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