Bol. Soc. Paran. Mat.
©SPM – E-ISSN-2175-1188
SPM: www.spm.uem.br/bspm

(3s.) **v. 2026 (44) 5** : 1–10.
ISSN-0037-8712
doi:10.5269/bspm.80421

On Lucas-Balancing-Like Polynomials

E. Özkan* and B. Tekeoğlu Akkaya

ABSTRACT: We present a systematic study of a polynomial generalization of the classical Lucas–balancing numbers, which we call Lucas–balancing–like polynomials and obtain their Binet-type closed form, analyze several structural and algebraic identities, and establish a connection with Chebyshev polynomials of the first kind. Moreover, we demonstrate that fundamental relations that are satisfied by balancing and Lucas–balancing numbers extend naturally to the polynomial setting. Several new identities, including generalized Catalan-type identities, are derived. This work provides a unified framework for recurrence-based polynomial sequences and suggests further research directions within orthogonal polynomials and Diophantine structures.

Keywords: Balancing polynomials, Fibonacci sequence, Chebyshev polynomials.

Contents

1. Introduction

Second-order linear recurrence sequences play a central role in number theory, algebra, combinatorics, and approximation theory. The most classical examples are the Fibonacci and Lucas sequences, defined respectively by:

$$F_n = F_{n-1} + F_{n-2}$$

with $F_0 = 0$ and $F_1 = 1$.

$$L_n = L_{n-1} + L_{n-2}$$

with $L_0 = 2$ and $L_1 = 1$.
Both admit Binet-type formulas and satisfy numerous algebraic identities. The limit of the ratio of consecutive Fibonacci numbers yields the golden ratio. Accordingly, this limit is given by [4]:

$$\lim_{n\to\infty} \frac{F_{n+1}}{F_n} = \frac{1+\sqrt{5}}{2}$$

Metallic ratios arise in numerous branches of science [6]. Table 1 presents several of the most intriguing metallic ratios.

Similarly, as stated in [4],

$$\lim_{n\to\infty} \frac{L_{n+1}}{L_n} = \frac{1+\sqrt{5}}{2}.$$

The Fibonacci and Lucas sequences are closely related integer sequences. The algebraic identities that describe these relationships are presented below.

$$F_{n-1} + F_{n+1} = L_n,$$
$$F_n + L_n = 2F_{n+1}.$$

* Corresponding author.
2020 *Mathematics Subject Classification:* 11B39 , 11C08, 33C45.
Submitted November 27, 2025. Published February 14, 2026

Table 1: Metallic Ratios

Ratio	Value
Platinum	$\frac{0+\sqrt{4}}{2}$
Golden	$\frac{1+\sqrt{5}}{2}$
Silver	$\frac{2+\sqrt{8}}{2}$
Bronze	$\frac{3+\sqrt{13}}{2}$
Copper	$\frac{4+\sqrt{20}}{2}$
Nickel	$\frac{5+\sqrt{29}}{2}$
Aluminum	$\frac{6+\sqrt{40}}{2}$
Iron	$\frac{7+\sqrt{53}}{2}$
Tin	$\frac{8+\sqrt{68}}{2}$
Lead	$\frac{9+\sqrt{85}}{2}$

Notable examples include the balancing numbers, introduced by Behera and Panda, and the corresponding Lucas–balancing numbers defined through a similar recurrence. Balancing sequence is defined by the following recurrence relation [16]:

$$B_n = 6B_{n-1} - B_{n-2}$$

with $B_0 = 0$ and $B_1 = 1$. Likewise, as stated in [14],

$$\lim_{n\to\infty} \frac{B_{n+1}}{B_n} = 3 + \sqrt{8}.$$

For the nth balancing number B_n, the expression $8B_n^2 + 1$ is a perfect square and the square root of this expression, $C_n = \sqrt{8B_n^2 + 1}$ is referred to as the nth Lucas–balancing number [14]. The Lucas-balancing numbers satisfy the recurrence relation presented below.

$$C_n = 6C_{n-1} - C_{n-2}$$

with $C_0 = 1$ and $C_1 = 3$.

Similarly,

$$\lim_{n\to\infty} \frac{C_{n+1}}{C_n} = 3 + \sqrt{8}.$$

Balancing sequence and Lucas-balancing sequence are closely related sequences. For instance [13]:

$$C_{4n+1} - 3 = 16B_{2n}B_{2n+1}$$
$$C_{4n+3} - 3 = 32B_{n+1}C_{n+1}B_{2n+1}.$$

Balancing numbers have been the subject of extensive investigation in the literature. A considerable amount of research has been carried out on this topic in the literature. For instance, Uysal, Özkan, and Shannon introduced the concept of dual bicomplex balancing numbers [17]. Taşçı introduced the Gaussian balancing and Gaussian Lucas balancing numbers [15]. In 2012, Panda and Rout introduced balancing-like sequences and demonstrated that they possess properties analogous to those of the classical balancing sequences.

Balancing-like sequences are defined by the following recurrence relation [10]:

$$x_n = Ax_{n-1} - x_{n-2}$$

with $x_0 = 0$ and $x_1 = 1$.

Lucas-balancing-like sequences are defined by the following recurrence relation:

$$y_n = Ay_{n-1} - y_{n-2}$$

with $y_0 = 1$ and $y_1 = \frac{A}{2}$.

Subsequently, polynomial forms corresponding to these integer sequences were constructed. Balancing polynomials are defined by the following recurrence [3]:

$$B_n(x) = 6xB_{n-1}(x) - B_{n-2}(x)$$

with $B_0(x) = 0$ and $B_1(x) = 1$. Similarly, Lucas-balancing polynomials are defined by the following recurrence [3]:

$$C_n(x) = 6xC_{n-1}(x) - C_{n-2}(x)$$

with $C_0(x) = 1$ and $C_1(x) = 3x$.

Balancing-like polynomials are defined by the following recurrence [2]:

$$x_m(t) = Atx_{m-1}(t) - x_{m-2}(t)$$

with $x_0(t) = 0$ and $x_1(t) = 1$. The Binet formula of balancing-like polynomials are:

$$x_m(t) = \frac{\beta_1^m(t) - \beta_2^m(t)}{\beta_1(t) - \beta_2(t)},$$

$\beta_1(t) = \frac{At+\sqrt{A^2t^2-4}}{2}$ and $\beta_2(t) = \frac{At-\sqrt{A^2t^2-4}}{2}$.

The polynomial analogues, such as balancing polynomials and balancing-like polynomials, have been studied in recent years. The purpose of the present paper is to introduce and develop the parallel theory for the Lucas–balancing–like polynomials and to examine their properties, explicit formulas, and structural connections. We also show that many fundamental identities extend naturally from the numerical setting to the polynomial domain.

2. Lucas-balancing-like Polynomials

In this section, we will define Lucas-balancing-like polynomials. In addition, we obtain many properties of these polynomials.

Definition 2.1 *Let A be a fixed real parameter. Define the sequence of polynomials $\{y_m(t)\}$ by:*

$$y_m(t) = Aty_{m-1}(t) - y_{m-2}(t) \tag{2.1}$$

with initial conditions $y_0(t) = 1$ and $y_1(t) = \frac{A}{2}t$.

We refer to this sequence as the Lucas–balancing–like polynomials. The first few polynomials are

$$y_0(t) = 1, y_1(t) = \frac{A}{2}t, y_2(t) = \frac{A^2t^2}{2} - 1, y_3(t) = \frac{A^3t^3}{2} - \frac{3At}{2}, y_4(t) = \frac{A^4t^4}{2} - 2A^2t^2 + 1, \cdots$$

Setting $A = 6$ yields the sequence of Lucas-balancing polynomials. Obviously, $y_m(1) = y_m$.

The characteristic equation is:

$$\beta^2(t) - At\beta(t) + 1 = 0.$$

Its roots are:

$$\beta_1(t) = \frac{At + \sqrt{A^2t^2 - 4}}{2}$$

$$\beta_2(t) = \frac{At - \sqrt{A^2t^2 - 4}}{2}$$

The relationship between these roots is given below:

$$\beta_1(t)+\beta_2(t)=At, \quad \beta_1(t)-\beta_2(t)=\sqrt{A^2t^2-4}, \quad \beta_1(t)\beta_2(t)=1.$$

In the following theorem, the Binet formula for Lucas-balancing-like polynomials is obtained.

Theorem 2.1 ***(Binet Formula)*** *For all $m \geq 0$,*

$$y_m(t)=\frac{\beta_1^m(t)+\beta_2^m(t)}{2}. \tag{2.2}$$

Theorem 2.2 *Let $m \in \mathbb{Z}^+$,*

$$y_{m+1}(t)=\sum_{i=0}^{\lfloor \frac{m+1}{2} \rfloor} \binom{m+1}{2i} \left(\frac{At}{2}\right)^{m+1-2i} \left(\frac{A^2t^2-4}{4}\right)^i \tag{2.3}$$

Through this theorem, a closed-form summation expression for the $(m+1)$th Lucas-balancing-like polynomials are obtained, simplifying the computation by eliminating recursive dependence.

Corollary 2.1 ***(Symmetry Properties)***
Let $m \in \mathbb{Z}^+$,

$$i) \quad y_{-m}(t)=y_m(t)$$
$$ii) \quad y_m(-t)=(-1)^m y_m(t).$$

This corollary follows directly from the Binet formula.

There is a close relationship between Lucas-balancing-like polynomials and Chebyshev polynomials; the subsequent theorem articulates this connection.

Theorem 2.3 ***(Connection with Chebyshev Polynomials)***
For each $m \in \mathbb{Z}^+$, we have the following identity:

$$y_m(t)=T_m\left(\frac{At}{2}\right). \tag{2.4}$$

where $T_m(t)$ is the Chebyshev polynomial of the first kind.

Proof. The Chebyshev polynomials of the first kind may also be defined through the following recurrence relation: For $m \geq 2$,
$T_m(x)=2xT_{m-1}(x)-T_{m-2}(x)$ with the initials $T_0(x)=1$, $T_1(x)=x$. Comparing the definition proves the theorem.

Theorem 2.4 *For $t > \frac{2}{A}$,*

$$\lim_{m\to\infty}\frac{y_{m+1}(t)}{y_m(t)}=\beta_1(t). \tag{2.5}$$

Proof: By using the Binet formula,

$$\lim_{m\to\infty}\frac{y_{m+1}(t)}{y_m(t)}=\lim_{m\to\infty}\frac{\beta_1^{m+1}(t)+\beta_2^{m+1}(t)}{\beta_1^m(t)+\beta_2^m(t)}$$
$$=\lim_{m\to\infty}\frac{\beta_1(t)+\left(\frac{\beta_2(t)}{\beta_1(t)}\right)^m\beta_2(t)}{1+\left(\frac{\beta_2(t)}{\beta_1(t)}\right)^m}.$$

Since $\beta_2(t) < \beta_1(t)$ for every $t > \frac{2}{A}$, we obtain

$$\lim_{m\to\infty}\left(\frac{\beta_2(t)}{\beta_1(t)}\right)^m = 0$$

and this yields the desired result. □

The limit of successive terms in an integer sequence has long been regarded as a significant computation. This limit becomes particularly significant when it yields metallic ratio. Here, by substituting the values of A and t from Table 2, we observe that the corresponding metallic ratios are obtained

Table 2: Metallic Ratios for Lucas-Balancing-Like Polynomials

A	t	Ratio	Value
1	$\sqrt{5}$	Golden	$\frac{1+\sqrt{5}}{2}$
1	$\sqrt{8}$	Silver	$\frac{2+\sqrt{8}}{2}$
1	$\sqrt{13}$	Bronze	$\frac{3+\sqrt{13}}{2}$
1	$\sqrt{20}$	Copper	$\frac{4+\sqrt{20}}{2}$
1	$\sqrt{29}$	Nickel	$\frac{5+\sqrt{29}}{2}$
1	$\sqrt{40}$	Aluminum	$\frac{6+\sqrt{40}}{2}$

3. Relations Between Balancing-like Polynomials and Lucas-balancing-like Polynomials

Proposition 3.1 *The balancing-like and Lucas-balancing-like polynomial families satisfy the evaluations given below:*

$$x_m(0) = \begin{cases} 0, & m \text{ even} \\ (-1)^{\frac{m-1}{2}}, & m \text{ odd} \end{cases}$$

$$y_m(0) = \begin{cases} (-1)^{\frac{m}{2}}, & m \text{ even} \\ 0, & m \text{ odd} \end{cases}$$

$$x_m\left(\frac{2}{A}\right) = \text{is undefined.}$$

$$y_m\left(\frac{2}{A}\right) = 1,$$

$$x_m(\sqrt{5}) = \sqrt{5}F_m,$$

$$y_m(\sqrt{5}) = \frac{L_m}{2}.$$

Theorem 3.1 *For integer m the following relations are valid:*

$$i)\quad y_m^2(t) - \left(\frac{A^2t^2-4}{2}\right)x_m^2(t) = 1$$

$$ii)\quad x_{m+1}(t) - \frac{A}{2}tx_m(t) = y_m(t)$$

Proof. i) By using the Binet formula of balancing-like polynomials and Lucas-balancing-like polynomials we get,

$$\frac{A^2t^2-4}{2}x_m^2(t) + 1 = y_m^2(t).$$

ii)By using the Binet formula of balancing-like polynomials and Lucas-balancing-like polynomials we get,

$$\begin{aligned} x_{m+1}(t)-\frac{A}{2}tx_m(t) &= \frac{\beta_1^{m+1}(t)-\beta_2^{m+1}(t)}{\beta_1(t)-\beta_2(t)}-\frac{A}{2}t\frac{\beta_1^m(t)-\beta_2^m(t)}{\beta_1(t)-\beta_2(t)} \\ &= \frac{\beta_1^m(t)(2\beta_1(t)-At)+\beta_2^m(t)(At-2\beta_2(t))}{2(\beta_1(t)-\beta_2(t))} \\ &= y_m(t). \end{aligned}$$

The first identity in the preceding theorem may be regarded as the Pell-type equation for balancing-like polynomials.

Theorem 3.2 *For $m \in \mathbb{Z}^+$,*

$$\frac{x_{m+1}(t)-x_{m-1}(t)}{2} = y_m(t). \tag{3.1}$$

Proof: By using the Binet formula and since $\beta_1(t)\beta_2(t) = 1$ we obtain,

$$\begin{aligned} x_{m+1}(t)-x_{m-1}(t) &= \frac{\beta_1^{m+1}(t)-\beta_2^{m+1}(t)}{\beta_1(t)-\beta_2(t)}-\frac{\beta_1^{m-1}(t)-\beta_2^{m-1}(t)}{\beta_1(t)-\beta_2(t)} \\ &= \frac{\beta_1^m(t)(\beta_1(t)-\beta_2(t))+\beta_2^m(t)(\beta_1(t)-\beta_2(t))}{\beta_1(t)-\beta_2(t)} \\ &= \beta_1^m(t)+\beta_2^m(t). \end{aligned}$$

□

Theorem 3.3

$$\frac{A}{2}tx_m(t)-x_{m-1}(t) = y_m(t) \tag{3.2}$$

Proof: Using the Binet representations of both the balancing-like and Lucas–balancing-like polynomials, we derive

$$\begin{aligned} \frac{A}{2}tx_m(t)-x_{m-1}(t) &= \frac{A}{2}t\frac{\beta_1^m(t)-\beta_2^m(t)}{\beta_1(t)-\beta_2(t)}-\frac{\beta_1^{m-1}(t)-\beta_2^{m-1}(t)}{\beta_1(t)-\beta_2(t)} \\ &= \frac{\beta_1^m(t)(At-2\beta_2(t))+\beta_2^m(t)(2\beta_1(t)-At)}{2(\beta_1(t)-\beta_2(t))} \\ &= \frac{\beta_1^m(t)+\beta_2^m(t)}{2} \\ &= y_m(t). \end{aligned}$$

□

Theorem 3.4 *For $m, n \in \mathbb{Z}^+$,*

$$i) \quad y_{m+n}(t) = y_m(t)y_n(t)+\frac{A^2t^2-4}{4}x_m(t)x_n(t) \tag{3.3}$$

$$ii) \quad x_{m+n}(t) = y_m(t)x_n(t)+x_m(t)y_n(t). \tag{3.4}$$

Proof: From the Binet formulas of the balancing-like and Lucas–balancing-like polynomials,

$$\begin{aligned} \sqrt{\frac{A^2t^2-4}{4}}x_m(t) &= \frac{\sqrt{A^2t^2-4}}{2}\frac{\beta_1^m(t)-\beta_2^m(t)}{\beta_1(t)-\beta_2(t)} \\ &= \frac{\beta_1^m(t)-\beta_2^m(t)}{2}. \end{aligned}$$

Then,

$$y_m(t) + \sqrt{\frac{A^2t^2-4}{4}}x_m(t) = \beta_1^m(t).$$

Upon replacing m with $m+n$ in this equation,

$$y_{m+n}(t) + \frac{\sqrt{A^2t^2-4}}{2}x_{m+n}(t) = \beta_1^{m+n}(t).$$

On the other hand,

$$\left(y_m(t) + \frac{\sqrt{A^2t^2-4}}{2}x_m(t)\right)\left(y_n(t + \frac{\sqrt{A^2t^2-4}}{2}x_n(t))\right) = \beta_1^m(t)\beta_2^n(t) = \beta_1^{m+n}(t).$$

From the combination of these last two relations we obtain,

$$\begin{aligned} &i) \quad y_{m+n}(t) = y_m(t)y_n(t) + \frac{A^2t^2-4}{4}x_m(t)x_n(t) \\ &ii) \quad x_{m+n}(t) = y_m(t)x_n(t) + x_m(t)y_n(t). \end{aligned}$$

□

Corollary 3.1 *For $m \in \mathbb{Z}^+$,*

$$i) \quad x_{2m}(t) = 2x_m(t)y_m(t) \tag{3.5}$$

$$ii) \quad y_{2m}(t) = y_m^2(t) + \frac{A^2t^2-4}{4}x_m^2(t). \tag{3.6}$$

Theorem 3.5 *For $m, n \in \mathbb{Z}^+$,*

$$i) \quad y_{m-n}(t) = y_m(t)y_n(t) - \frac{A^2t^2-4}{4}x_m(t)x_n(t) \tag{3.7}$$

$$ii) \quad x_{m-n}(t) = x_m(t)y_n(t) - y_m(t)x_n(t). \tag{3.8}$$

Proof: From the Binet formulas of the balancing-like and Lucas–balancing-like polynomials,

$$\begin{aligned} y_m(t) - \sqrt{\frac{A^2t^2-4}{4}}x_m(t) &= \frac{\beta_1^m(t)+\beta_2^m(t)}{2} - \frac{\beta_1^m(t)-\beta_2^m(t)}{2} \\ &= \beta_2^m(t). \end{aligned}$$

Upon replacing m with $m-n$ in this equation,

$$y_{m-n}(t) - \frac{\sqrt{A^2t^2-4}}{2}x_{m-n}(t) = \beta_2^{m-n}(t).$$

On the other hand,

$$\beta_2^{m-n}(t) = \beta_2^m(t)\beta_1^n(t) = \left(y_m(t) - \frac{\sqrt{A^2t^2-4}}{2}x_m(t)\right)\left(y_n(t) + \frac{\sqrt{A^2t^2-4}}{2}x_n(t)\right).$$

From the combination of these last two relations we obtain,

$$\begin{aligned} &i) \quad y_{m-n}(t) = y_m(t)y_n(t) - \frac{A^2t^2-4}{4}x_m(t)x_n(t) \\ &ii) \quad x_{m-n}(t) = x_m(t)y_n(t) - y_m(t)x_n(t). \end{aligned}$$

□

The fundamental relationship between balancing and Lucas-balancing numbers discussed in the introduction similarly extends to the balancing-like and Lucas–balancing-like polynomials. This connection is made explicit in the theorem that follows.

Theorem 3.6 *For* $m \in \mathbb{Z}^+$,

$$y_m^2(t) = \frac{A^2t^2-4}{4}x_m^2(t)+1. \tag{3.9}$$

The following result demonstrates that the mth Lucas–balancing-like polynomial can be computed directly from the balancing-like polynomials, without resorting to the recurrence relation.

Corollary 3.2 *For* $m \in \mathbb{Z}^+$

$$y_m(t) = \sqrt{\frac{A^2t^2-4}{4}x_m^2(t)+1} \tag{3.10}$$

Theorem 3.7 ***(Generalized Catalan Identities)***

$$i) \quad x_{m+n}(t)x_{m-n}(t) = x_m^2(t) - x_n^2(t), \tag{3.11}$$

$$ii) \quad y_{m+n}(t)y_{m-n}(t) = y_m^2(t) + y_n^2(t) - 1. \tag{3.12}$$

Proof: i)

$$\begin{aligned}
x_{m+n}(t)x_{m-n}(t) &= \left(\frac{\beta_1^{m+n}(t)-\beta_2^{m+n}(t)}{\beta_1(t)-\beta_2(t)}\right)\left(\frac{\beta_1^{m-n}(t)-\beta_2^{m-n}(t)}{\beta_1(t)-\beta_2(t)}\right)\\
&= \frac{\beta_1^{2m}(t)-\beta_1^{2n}(t)-\beta_2^{2n}(t)+\beta_2^{2m}(t)}{(\beta_1(t)-\beta_2(t))^2}\\
&= \frac{\left(\beta_1^{2m}(t)+\beta_2^{2m}(t)-2\right)-\left(\beta_1^{2n}(t)+\beta_2^{2n}(t)-2\right)}{(\beta_1(t)-\beta_2(t))^2}\\
&= \left(\frac{\beta_1^{m}(t)-\beta_2^{m}(t)}{\beta_1(t)-\beta_2(t)}\right)^2-\left(\frac{\beta_1^{n}(t)-\beta_2^{n}(t)}{\beta_1(t)-\beta_2(t)}\right)^2\\
&= x_m^2(t)-x_n^2(t).
\end{aligned}$$

ii)

$$\begin{aligned}
y_{m+n}(t)y_{m-n}(t) &= \left(\frac{\beta_1^{m+n}(t)+\beta_2^{m+n}(t)}{2}\right)\left(\frac{\beta_1^{m-n}(t)+\beta_2^{m-n}(t)}{2}\right)\\
&= \frac{\beta_1^{2m}(t)+\beta_2^{2m}(t)+\beta_1^{2n}(t)+\beta_2^{2n}(t)}{4}\\
&= \frac{\beta_1^{2m}(t)+\beta_2^{2m}(t)}{2}+\frac{\beta_1^{2n}(t)+\beta_2^{2n}(t)}{4}-\frac{4}{4}\\
&= y_m^2(t)+y_n^2(t)-1.
\end{aligned}$$

□

Theorem 3.8 *For* $m, n \in \mathbb{Z}^+$,

$$x_{m+n}(t) = x_m(t)x_{n+1} - x_n(t)x_{m-1}(t). \tag{3.13}$$

Proof:

$$\begin{aligned}
x_m(t)x_{n+1}-x_n(t)x_{m-1}(t) &= \left(\frac{\beta_1^{m}(t-\beta_2^{m}(t))}{\beta_1(t)-\beta_2(t)}\right)\left(\frac{\beta_1^{n+1}(t)-\beta_2^{n+1}(t)}{\beta_1(t)-\beta_2(t)}\right)-\left(\frac{\beta_1^{n}(t)-\beta_2^{n}(t)}{\beta_1(t)-\beta_2(t)}\right)\left(\frac{\beta_1^{m-1}(t)-\beta_2^{m-1}(t)}{\beta_1(t)-\beta_2(t)}\right)\\
&= \frac{\left(\beta_1^{m+n}(t)-\beta_2^{m+n}(t)\right)(\beta_1(t)-\beta_2(t))}{(\beta_1(t)-\beta_2(t))^2}\\
&= \frac{\beta_1^{m+n}(t)-\beta_2^{m+n}(t)}{\beta_1(t)-\beta_2(t)}\\
&= x_{m+n}(t).
\end{aligned}$$

□

Theorem 3.9 *For $m, n \in \mathbb{Z}^+$,*

$$\left(\frac{A^2t^2-4}{4}\right)x_{m+n}(t) = y_m(t)y_{n+1}(t) - y_{m-1}(t)y_n(t). \tag{3.14}$$

Proof:

$$\begin{aligned} y_m(t)y_{n+1}(t) - y_{m-1}(t)y_n(t) &= \frac{\left(\beta_1^m(t)+\beta_2^m(t)\right)\left(\beta_1^{n+1}(t)+\beta_2^{n+1}(t)\right)}{4} - \frac{\left(\beta_1^{m-1}(t)+\beta_2^{m-1}(t)\right)\left(\beta_1^n(t)+\beta_2^n(t)\right)}{4} \\ &= \frac{\left(\beta_1^{m+n}(t)-\beta_2^{m+n}(t)\right)\left(\beta_1(t)-\beta_2(t)\right)}{4}. \end{aligned}$$

From here,

$$\begin{aligned} \frac{4}{\left(\beta_1(t)-\beta_2(t)\right)^2} y_m(t)y_{n+1}(t) - y_{m-1}(t)y_n(t) &= \frac{\beta_1^{m+n}(t)-\beta_2^{m+n}(t)}{\beta_1(t)-\beta_2(t)} \\ &= x_{m+n}(t) \end{aligned}$$

□

4. Conclusion

We introduced the Lucas–balancing–like polynomials, derived their Binet representation, and established their structural equivalence with Chebyshev polynomials under a suitable scaling. We proved several identities generalizing those known for balancing and Lucas–balancing numbers. These results unify various recurrence-based polynomial families and open further research directions, including orthogonal polynomial theory, recurrence-based Diophantine structures, and generalizations involving complex parameters or higher-order recurrences.

References

1. Behera, A., Panda, G.K.,*On the square roots of triangular numbers*, Fibonacci Quarterly, 37(2), 98-105, (1999).
2. Dutta, U.K., Ray, P.K., *Euler-zagier multiple l-functions involving balancing-like polynomials associated to Dirichlet characters*, Palestine Journal of Mathematics, 11, 274-289, (2022).
3. Frontczak, R., *On balancing polynomials*, *Applied Mathematical Sciences*, 13, 57-66, (2019).
4. Koshy, T., *Fibonacci and Lucas numbers with applications*, Wiley, Canada, (2001).
5. Mason, J. C., Hamdscomb, D. C., *Chebyshev polynomials*, Chapman & Hall/CRC, Florida, (2003).
6. Özkan, E., Akkuş, H., *Copper ratio obtained by generalizing the Fibonacci sequence*, AIP Advances, 14, 075207 (2024).
7. Özkan, E., Tekeoğlu Akkaya, B., *On the properties of balancing-like polynomials* (Preprint).
8. Özkan, E., Tekeoğlu Akkaya, B., *On k-balancing-like-sequences and k-Lucas-balancing-like-sequences* (Preprint).
9. Panda, G.K., *Some fascinating properties of balancing numbers*, Fibonacci Numbers and Their Applications, 10, (2006).
10. Panda, G.K., Rout, S.S., *A class of recurrent sequences exhibiting some exciting properties of balancing numbers*, International Journal of Mathematical, Computational, Physical, Electrical and Computer Engineering, 6, 4-6, (2012).
11. Panda, A.K., *Some variants of the balancing sequence*, Thesis (PhD), Department of Mathematics, National Institute of Technology, Rourkela, (2016).
12. Rout, S.S., *Some generalizations and properties of balancing numbers*, Thesis (PhD), Department of Mathematics, National Institute of Technology, Rourkela (2015).
13. Ray, P.K., *New identities for the common factors of balancing and Lucas-balancing numbers*, International Journal of Pure and Applied Mathematics, 85, 487-494, (2013).
14. Ray, P.K., *Balancing and cobalancing numbers*, Thesis(PhD), Department of Mathematics, National Institute of Technology, Rourkela (2009).
15. Taşçı, D., *Gaussian balancing numbers and Gaussian Lucas balancing numbers*, Journal of Science and Arts, 3, 661-666, (2018).

16. Tekeoğlu, B., Karadeniz Gözeri, G., *Some properties of balancing-like sequences*, Thesis (Master), Istanbul University, (2024).

17. Uysal, M., Özkan, E., Shannon, A.G., *On dual bicomplex balancing and Lucas-balancing numbers*, Journal of Science and Arts, 23, 925-938, (2023).

18. Vajda, S., *Fibonacci & Lucas numbers, and the Golden section*, Ellis Horwood Limited, West Sussex, (1989).

E. Özkan and B. Tekeoğlu Akkaya
Department of Mathematics,
Marmara University,
Turkey.
E-mail address: engin.ozkan@marmara.edu.tr