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Enumeration of Subsets with Closedness in Finite Fields of Characteristic 2

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ABSTRACT: The additive closedness in the subset of an additive group is termed as r-value. The nature of closedness in different subsets of fixed size is observed as a spectrum of r-values. We enumerate r-values of subsets in finite fields of characteristic 2 and represent them as the spectrum of values. Based on these values the subsets can be further studied as partial Steiner triple systems, sum-free sets, Sidon sets, and Schure triples.

Keywords: r-values, sum-free sets, Sidon sets, spectrum, characteristic 2.

Contents

1. Introduction

In an additive group G, a subset is associated with a non-negative integer called r-value, representing the amount of closedness in it. This concept was initially defined and studied by Sophie et al. in 2009 [11]. Formally, for a subset A of an additive group G, r-value of A denoted by $r(A)$ is defined as the number of ordered pairs in A, such that for each pair sum of its elements is again an element of A. For prime p, Sophie et al. studied r-values of subsets of $\mathbb{Z}_p$ and presented as a spectrum of r-values of fixed subset size in $\mathbb{Z}_p$ [11]. Continuing, r-closed sets of natural number particularly r-values of intervals $[1, N]$ in $\mathbb{N}$ were studied by Sophie in 2014 [10]. In 2024, Nithish et al. defined $r(A, B, C)$, where A, B, C are subsets of an additive group G. They obtained r-values of subsets of $\mathbb{Z}_n$ by representing each subset as a union of intervals in $\mathbb{Z}_n$ and deriving formula for r-values of intervals in $\mathbb{Z}_n$.

The zero valued subsets, commonly known as Sum-free sets can be regarded as the initial approach to study r-values of subsets. In a sequence of integers, properties like Sum-free sets and Sidon sets connected with r-values were studied even before 1965, as mentioned by P. Erdös in [8]. Terence and Vu recently addressed Erdös' problems in their study on Sum-free sets in groups [16]. In between, Sum-free sets were studied in view of the maximum cardinality of zero valued sets [See [6], [1]], geometric study of Sum-free sets in integer lattice grid [See [12], [7]]. Zero valued sets were used in the process of discovering new APN(Almost Perfect Nonlinear) exponents [2] and also results on F-saturated graphs for large complete graphs [17].

Cardinality of Sidon sets and the generalization of Sidon's problem on Sidon set is related to r-closed set [See [13], [4]]. In [14], the authors introduced the concept of a relative anti-closure property

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for subsets to study the mathematical model of the biological concept of mutation. This anti-closure property is connected to r-closed when r is zero. Datskovsky in [5] discussed r-values over complete residue modulo n as the number of Schur triples. The minimum number of additive tuples in groups of prime order is studied in 2019 by Ostap et al. [3].

For a subset A of $\mathbb{F}_{2^n}\setminus\{0\}$, let $\mathcal{R}$ be the set of all 3-subsets of A such that in each 3-subset sum of two elements is equal to the remaining one element. If $a, b, c \in A$ with $a + b = c$ then the triplet $\{a, b, c\}$ contribute 6 to $r(A)$. If another triplet $\{d, e, f\}$ contributing to $r(A)$, then the intersection of these two triplets can be atmost a singleton set. Thus $(A, \mathcal{R})$ is a partial Steiner triple system of order $|A|$ [see [9]]. This interlink motivated to explore r-values of subsets of $\mathbb{F}_{2^n}$.

Section 2 covers the basic results and observations of r-values over subsets of a finite additive group. Section 3 illustrates the results which generate the spectrum of r-values of $\mathbb{F}_{2^3}$. Section 4 presents the results for $n \geq 3$, r-values in $\mathbb{F}_{2^{n+1}}$ is generated using r-values in $\mathbb{F}_{2^n}$.

2. Preliminary

In this section, we will provide some definitions and fundamental results. In addition, we present a few basic results that we established.

Definition 2.1 [15] For any subsets A, B, C of a finite additive group G, we define $r(A, B, C)$ as the cardinality of the set $\{(a, b) \mid a \in A,\ b \in B,\ a + b \in C\}$.

If $A = B$, then we denote $r(A, B, C) = r(A, C)$ and if $A = B = C$ then $r(A, B, C) = r(A)$, which reduce to r-value of A.

If A, B and C are three disjoint subsets of an additive abelian group G, using the property $r(A, B, C) = r(B, A, C)$ we have $r(A \cup B) = r(A) + r(A, B) + 2r(A, B, A) + 2r(A, B, B) + r(B, A) + r(B)$.

In the above definition, we call the set associated with $r(A, B, C)$ as R-set of A, B, C.

Definition 2.2 For any subsets A, B, C of an additive group G, we define R-set of A, B, C denoted by $R(A, B, C)$ as the set $\{(a, b, c) \in A \times B \times C \mid a + b = c\}$.

Note that $r(A, B, C) = |R(A, B, C)|$. The following result gives a bound of $r(A, B, C)$.

Proposition 2.1 *Let A, B and C are three subsets of a finite additive abelian group G with $|A| \leq |B|$. Then $r(A, B, C) \leq |A| \cdot min\{|B|, |C|\}$.*

Proof: For a fixed a in A, there are $|B|$ number of pairs (a, b) with $b \in B$ giving $|B|$ distinct elements $a+b$ of G. Suppose $|C| \leq |B|$, then among $|B|$ distinct elements atmost $|C|$ elements lies in C. Therefore, $r(\{a\}, B, C) \leq |C|$ and hence $r(A, B, C) \leq |A||C|$. Similarly suppose $|B| < |C|$, we have $r(A, B, C) \leq |A||B|$. □

The following result connect the r-value of a set and its compliment in an additive abelian group G.

Theorem 2.1 *[11] Let G be a finite abelian group of order g. Let k be a positive integer with $0 \leq k \leq g$, and let A be a subset of G of size k. Let $\overline{A}$ be the complement of A in G. Then $r(A)+r(\overline{A}) = g^2-3gk+3k^2$.*

The proof of this result follows from the proof of 4 in Corollary 2.1. The following Theorem and Corollary connect the relation of r-values of sets and its compliment with respect to the Definition 2.1. The relation 1 of Theorem 2.2 gives the partition of $A \times B$ with the sum of each pair either lies in C or not with respect to the set $R(A, B, C)$ and its r-values. In relation 2, $G \times B$ is partitioned into two sets, in which for each pair with sum in C either first element in A or not.

Theorem 2.2 *Let A, B, and C are three subsets of a finite additive abelian group G. Let $\overline{A}$ and $\overline{C}$ are compliments of A and C respectively. Then*

1. $r(A, B, C) + r(A, B, \overline{C}) = |A||B|$.
2. $r(A, B, C) + r(\overline{A}, B, C) = |B||C|$.

Proof: Given C and $\overline{C}$ are compliments, we have $R(A,B,C) \cap R(A,B,\overline{C}) = \emptyset$ and $R(A,B,C) \cup R(A,B,\overline{C}) = R(A,B,G)$. Therefore, $r(A,B,C)+r(A,B,\overline{C}) = |R(A,B,C)|+|R(A,B,\overline{C})| = |R(A,B,G)| = r(A,B,G) = |A||B|$. Similarly for A and its compliment $\overline{A}$ we get $r(A,B,C)+r(\overline{A},B,C) = |B||C|$. □

Corollary 2.1 *Let A, B, and C are three subsets of a finite additive abelian group G. Let $\overline{A}$, $\overline{B}$ and $\overline{C}$ are compliments of A, B and C in G respectively. Then*

1. $r(A,B,C) + r(A,\overline{B},C) = |A||C|$.
2. $r(A,B,C) - r(\overline{A},B,\overline{C}) = (|A| - |\overline{C}|)|B|$.
3. $r(A,B,C) - r(\overline{A},\overline{B},C) = (|B| - |\overline{A}|)|C|$.
4. $r(A,B,C) + r(\overline{A},\overline{B},\overline{C}) = |B||C| - |\overline{A}||C| + |\overline{A}||\overline{B}|$.

Proof:

1. We have $r(A,B,C) + r(A,\overline{B},C) = r(B,A,C) + r(\overline{B},A,C)$. Using 2 of Theorem 2.2, we have $r(B,A,C) + r(\overline{B},A,C) = |A||C|$.
2. Using 2 of Theorem 2.2 in $r(A,B,\overline{C})$, we have $r(A,B,\overline{C}) + r(\overline{A},B,\overline{C}) = |B||\overline{C}|$. Now combining $r(A,B,\overline{C}) + r(\overline{A},B,\overline{C}) = |B||\overline{C}|$ and 1 of Theorem 2.2, we get $r(A,B,C) - r(\overline{A},B,\overline{C}) = (|A| - |\overline{C}|)|B|$.
3. Similarly follows as above.
4. Using 1 of Theorem 2.2, we have $r(\overline{A},\overline{B},C)+r(\overline{A},\overline{B},\overline{C}) = |\overline{A}||\overline{B}|$. Substituting for $r(\overline{A},\overline{B},C)$ from 3, we get $r(A,B,C) + r(\overline{A},\overline{B},\overline{C}) = |B||C| - |\overline{A}||C| + |\overline{A}||\overline{B}|$.

□

Note that the relation in 4 of Corollary 2.1 is further reduced to $r(A,B,C) + r(\overline{A},\overline{B},\overline{C}) = |G|^2 - (|A| + |B| + |C|)|G| + |A||B| + |A||C| + |B||C|$.

As mentioned by Sophie et al. in [11], empty sets are considered zero valued sets. Singleton sets and subgroups have r-values as follows.

Proposition 2.2 *[11] Let A be a subset of a finite abelian group with $|A| = k$.*

1. *If $k = 1$, then* $r(A) = \begin{cases} 1 & \text{if } A = \{0\}, \\ 0 & \text{otherwise.} \end{cases}$
2. *A is a subgroup of G if and only if $r(A) = k^2$.*

3. r-values of subsets of $\mathbb{F}_{2^n}$

Consider a finite field $\mathbb{F}_{2^{n-1}}$, where n is a positive integer. Let $[\alpha_1, \alpha_2, \ldots, \alpha_{n-1}]$ and $[\beta_1, \beta_2, \ldots, \beta_{n-1}]$ be two elements of $\mathbb{F}_{2^{n-1}}$, where each $\alpha_i, \beta_i \in \{0,1\}$. If $[\alpha_1, \alpha_2, \ldots, \alpha_{n-1}] + [\beta_1, \beta_2, \ldots, \beta_{n-1}] = [\gamma_1, \gamma_2, \ldots, \gamma_{n-1}]$ in $\mathbb{F}_{2^{n-1}}$, then $[0, \alpha_1, \alpha_2, \ldots, \alpha_{n-1}] + [0, \beta_1, \beta_2, \ldots, \beta_{n-1}] = [0, \gamma_1, \gamma_2, \ldots, \gamma_{n-1}]$ in $\mathbb{F}_{2^n}$. Using this, for a subset A of $\mathbb{F}_{2^{n-1}}$, if every element of A is prefixed with 0 and calculate its r-value in $\mathbb{F}_{2^n}$ is same as r-value of A in $\mathbb{F}_{2^{n-1}}$. Thus throughout the discussion we treat $\mathbb{F}_{2^{n-1}}$ as subset of $\mathbb{F}_{2^n}$.

Let A be subset of $\mathbb{F}_{2^n}$, where n is a positive integer. The following result gives the lower bound of r-values for a subset of $\mathbb{F}_{2^n}$ which contains zero element.

Proposition 3.1 *If $A \subseteq \mathbb{F}_{2^n}$ and $0 \in A$, then $r(A) \geq 3|A| - 2$.*

Proof: For all $a \in A$, we have $a + 0 = a$, $0 + a = a$ and $a + a = 0$. Hence contributing $3|A| - 2$ pairs to $r(A)$. □

The proposition 3.2 and 3.3 gives the spectrum of r-values of subsets of size 2 and 3 in $\mathbb{F}_{2^n}$.

Proposition 3.2 *If $A \subseteq \mathbb{F}_{2^n}$ with $|A| = 2$, say $A = \{a, b\}$ then* $r(A) = \begin{cases} 0 & \text{if } a \neq 0 \text{ and } b \neq 0 \\ 4 & \text{otherwise.} \end{cases}$

Proof: Suppose $a \neq 0$ and $b \neq 0$. We have $a + a = 0$ and $b + b = 0$, but then $0 \notin A$. Also $a + b = a$ implies $b = 0$, which is not possible. similarly $a + b = b$ is not possible and hence $a + b \notin A$.
Suppose $a = 0$, then by Proposition 3.1, we have $3 \cdot 2 - 2 \leq r(A) \leq 4$ giving $r(A) = 4$. □

Proposition 3.3 *If $A \subseteq \mathbb{F}_{2^n}$ with $|A| = 3$, then $r(A)$ belongs to $\{0, 6, 7\}$.*

Proof: Suppose $0 \in A$, say $A = \{0, b, c\}$. Note that if $b + c = 0$, then $b = c$, which is not possible. Similarly if $b + c = b$ or $b + c = c$ then $c = 0$ or $b = 0$. Thus $b + c \notin A$, hence in this case $r(A) = 7$.

Suppose $0 \notin A$. Say $A = \{a, b, c\}$. Using the Cayley table below,

	a	b	c
a	0	a+b	a+c
b	b+a	0	b+c
c	c+a	c+b	0

if $a + b$, $a + c$ and $b + c$ are not in A, then $r(A) = 0$. Suppose $a + b \in A$, then by above arguments $a + b \neq a$ and $a + b \neq b$. But then $a + b = c$, which also implies $a + c = b$ and $b + c = a$. Hence $r(A) = 6$. □

Theorem 3.1 *For any subsets A of $\mathbb{F}_{2^n} \backslash \{0\}$, $r(A) \equiv 0 \pmod 6$.*

Proof: Suppose $r(A) > 0$. Then there exist $\{a, b, c\}$ in A such that $a + b = c$. But then $a + c = b$ and $b + c = a$. Hence by abalian property $\{a, b, c\}$ contribute 6 pairs to $r(A)$. Suppose $\{d, e, f\}$ is another pair which is disjoint from $\{a, b, c\}$, then $\{d, e, f\}$ is also contribute 6 to $r(A)$. If $\{a, b, c\} \cap \{d, e, f\} \neq \emptyset$, then either both are equal or one element is common. If two elements are common, that is $a = d$ and $b = e$, then $c = a + b = d + e = f$. Suppose $a = d$, then again $\{a, e, f\}$ contribute 6 to $r(A)$ without overlapping with the combinations of $\{a, b, c\}$. Hence, the proof. □

The proof of above result ensures that, if $\mathcal{R} = \{set(B) | B \in R(A, A, A)\}$, that is elements of R-set of A is considered as set, then $\mathcal{R}$ is a blocks of A to forms partial Steiner triple system with $|\mathcal{R}| = r(A)/6$.

Proposition 3.4 *Let $B = \{a \in \mathbb{F}_{2^n} | Tr(a) = 1\}$. Then $r(B) = 0$.*

Proof: Let $a, b \in B$, then $Tr(a + b) = Tr(a) + Tr(b) = 1 + 1 = 0$, which implies $a + b \notin B$. Thus $r(B) = 0$. □

In the above proposition, note that $0 \notin B$ and every subset of B is sum-free set. In other words, for each subset of size m with $0 \leq m \leq |B| = 2^{n-1}$ there is a subset of $\mathbb{F}_{2^n}$ of zero r-value.

Theorem 3.2 *For any subsets A of $\mathbb{F}_{2^n} \backslash \{0\}$ with $|A| = l$, we have $r(A \cup \{0\}) = r(A) + 3l + 1$.*

Proof: We have $r(A \cup \{0\}) = r(A) + r(A, \{0\}) + 2r(A, \{0\}, A) + 2r(A, \{0\}, \{0\}) + r(\{0\}, A) + r(\{0\})$. Since for any $a \in A$, its additive inverse is a itself, we have $r(A, \{0\}) = |A| = l$. Now 0 is the identity element and $0 \notin A$, we have $r(A, \{0\}, A) = |A| = l$, $r(A, \{0\}, \{0\}) = 0$ and $r(\{0\}, A) = 0$. Hence $r(A \cup \{0\}) = r(A) + 3l + 1$. □

The above theorem help us to calculate the r-value of a subset A of $\mathbb{F}_{2^n}$ with $0 \in A$ by knowing the r-value of $A \backslash \{0\}$. The following result gives the r-values of size 4 subsets in $\mathbb{F}_{2^n} \backslash \{0\}$.

Proposition 3.5 *If $A \subseteq \mathbb{F}_{2^n}$ with $|A| = 4$ and $0 \notin A$, then $r(A)$ is either 0 or 6.*

Proof: Suppose A is a subset of B, where B is as defined in Proposiion 3.4. Then $r(A) = 0$. Suppose $r(A) > 0$, then using Proposition 3.3, there exits a three elements $\{a, b, c\}$ of A contribute 6 pairs to $r(A)$. If there exists another set $\{d, e, f\}$ then as seen from proof of Theorem 3.1, there can be atmost one element common in $\{a, b, c\}$ and $\{d, e, f\}$. Which contradicts $|A| = 4$. Hence, the proof. □

The following Theorem 3.3 gives the upper bound for the r-values of subsets of $\mathbb{F}_{2^n}\backslash\{0\}$.

Theorem 3.3 *For any subsets A of $\mathbb{F}_{2^n}\backslash\{0\}$ with $|A| = k$, then $r(A) \leq \lfloor\frac{k(k-1)}{6}\rfloor 6$. Moreover $r(A) = k(k-1)$ holds if and only if $A \cup \{0\}$ is a subgroup of $\mathbb{F}_{2^n}$.*

Proof: Clearly we have $r(A) \leq k^2 - k$. But by Theorem 3.1, $r(A)$ is a multiple of 6. Thus $r(A) \leq \lfloor\frac{k(k-1)}{6}\rfloor 6$. Moreover, $A \cup \{0\}$ is a subgroup of $\mathbb{F}_{2^n}$ if and only in $r(A \cup \{0\}) = (k+1)^2$. By Theorem 3.2 $r(A \cup \{0\}) = (k+1)^2$ if and only if $r(A) = (k+1)^2 - (3k+1) = k(k-1)$. □

3.1. Spectrum of r-values in $\mathbb{F}_{2^3}$

The above listed results are enough to generate a spectrum of r-values in $\mathbb{F}_{2^3}$. The Figure 1 gives the

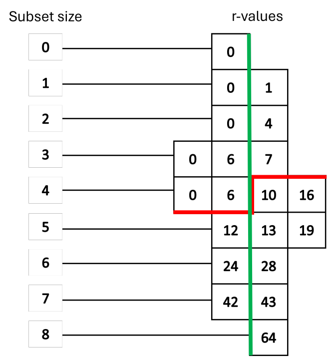

Figure 1: Spectrum of r-values in $\mathbb{F}_{2^3}$

all possible r-values of each subset size.

The r-values of subsets of $\mathbb{F}_{2^3}\backslash\{0\}$ is tabulated in the left side of vertical green coloured line. The values in the right side of vertical green coloured line gives the r-values of subsets containing 0. Moreover the right side values are calculated using Theorem 3.2. For example to calculate the r-values of subsets of size 4 containing 0 is obtained by adding $3 \times 4 - 2 = 10$ to each r-values of the subsets of size 3 in $\mathbb{F}_{2^3}\backslash\{0\}$. The values listed below the red coloured line can be computed using Theorem 2.1. Thus by knowing r-values of subsets of $\mathbb{F}_{2^3}\backslash\{0\}$ of size upto 4, it is possible to generate spectrum of r-values for each size in $\mathbb{F}_{2^3}$.

4. Spectrum of r-values in $\mathbb{F}_{2^n}$

In this section we generate spectrum of r-values in $\mathbb{F}_{2^n}$, where $n \geq 4$ using spectrum of r-values in $\mathbb{F}_{2^{n-1}}$. As seen in the subsection 3.1, to generate spectrum of r-values in $\mathbb{F}_{2^n}$ it is enough to calculate the r-values of subsets of $\mathbb{F}_{2^n}\backslash\{0\}$ upto size 2^{n-1}.

The following theorem gives the condensed formula for r-value of $A \cup B$ in $\mathbb{F}_{2^n}$.

Theorem 4.1 *Suppose $A \subseteq \mathbb{F}_{2^{n-1}}\backslash\{0\}$ and $B \subseteq \mathbb{F}_{2^n}\backslash\mathbb{F}_{2^{n-1}}$, then $r(A \cup B) = r(A) + 3r(A, B, B)$.*

Proof: We have $r(A \cup B) = r(A) + r(A, B) + 2r(A, B, A) + 2r(A, B, B) + r(B, A) + r(B)$. Since $A \subseteq \mathbb{F}_{2^{n-1}}\backslash\{0\}$ and $B \subseteq \mathbb{F}_{2^n}\backslash\mathbb{F}_{2^{n-1}}$, we observed that sum of two elements in A cannot lies in B. Which gives $r(A, B) = 0$. In vector representation, the first coordinate of each element of A is 0, and the first coordinate of each element of B is 1. Thus, the sum of two elements in B cannot belongs to B, and if one element is from A and another element is from B, then the sum cannot belongs to A. Therefor $r(B) = 0$ and $r(A, B, A) = 0$.

Since each element is its additive inverse in $\mathbb{F}_{2^n}$ we have,

$$\begin{aligned}(\alpha, \beta, \gamma) \in R(A, B, B) &\Leftrightarrow (\alpha, \beta, \gamma), (\alpha, \gamma, \beta) \in R(A, B, B)\\ &\Leftrightarrow (\beta, \alpha, \gamma), (\gamma, \alpha, \beta) \in R(B, A, B)\\ &\Leftrightarrow (\beta, \gamma, \alpha), (\gamma, \beta, \alpha) \in R(B, B, A)\end{aligned}$$

Thus $|R(A, B, B)| = |R(B, A, B)| = |R(B, B, A)|$ and $r(A, B, B) = r(B, A, B) = r(B, A)$. Hence $r(A \cup B) = r(A) + 3r(A, B, B)$. □

Corollary 4.1 *Every subset of $\mathbb{F}_{2^n} \backslash \mathbb{F}_{2^{n-1}}$ has r-value zero.*

The proof of the above Corollary directly follows from the Theorem 4.1. The following result relate the r-value of a set and its compliment in $\mathbb{F}_{2^n} \backslash \{0\}$.

Proposition 4.1 *Let $S \subseteq \mathbb{F}_{2^n} \backslash \{0\}$ with size k. Then the compliment of S in $\mathbb{F}_{2^n} \backslash \{0\}$ denoted by $\overline{S}^*$ is a subset of size $2^n - 1 - k$ with r-value given by $r(\overline{S}^*) = 2^{2n} - 3(2^n - k - 1)(k + 1) - (3k + 1) - r(S)$.*

Proof: Let $S \subseteq \mathbb{F}_{2^n} \backslash \{0\}$ with size k and $\overline{S}$ is the compliment of S in $\mathbb{F}_{2^n}$. Then by Theorem 2.1, $r(\overline{S}) = 2^{2n} - 3 \cdot 2^n k + 3k^2 - r(S)$ with $|\overline{S}| = 2^n - k$. Now let $\overline{S}^* = \overline{S} \backslash \{0\}$, then $\overline{S}^*$ is compliment of S in $\mathbb{F}_{2^n} \backslash \{0\}$ with size $2^n - k - 1$ and by Proposition 3.1 $r(\overline{S}^*) = r(\overline{S}) - (3(2^n - 1 - k) + 1) = 2^{2n} - 3(2^n - k - 1)(k + 1) - (3k + 1) - r(S)$. □

Using the above result it is enough to calculate the r-values of subsets of $\mathbb{F}_{2^n} \backslash \{0\}$ upto size 2^{n-1}.

Let S be a subset of $\mathbb{F}_{2^n} \backslash \{0\}$ with size m, where $5 \leq m \leq 2^{n-1}$. Suppose $S = A \cup B$ where $A \subseteq \mathbb{F}_{2^{n-1}} \backslash \{0\}$ with $|A| = k$ and $B \subseteq \mathbb{F}_{2^n} \backslash \mathbb{F}_{2^{n-1}}$ with $|B| = l$, where $0 \leq k \leq 2^{n-1} - 1$ and $0 \leq l \leq 2^{n-1}$.

We continue by considering different cases on k and l. The case $k = 0$ is follows from the Proposition 3.4 and Corollary 4.1 by giving zero as an r-value in each subset size from 1 to 2^{n-1} in $\mathbb{F}_{2^n}$. The following result gives the r-value when $l = 0$ and $l = 1$.

Proposition 4.2 *Let $A \subseteq \mathbb{F}_{2^{n-1}} \backslash \{0\}$ with size k. Then $r(A)$ is a r-value of two subsets of size k and $k + 1$ in $\mathbb{F}_{2^n} \backslash \{0\}$.*

Proof: Let $A = \{a_1, a_2, \ldots, a_k\}$ be a subset of $\mathbb{F}_{2^{n-1}} \backslash \{0\}$. Then $r(A)$ in $\mathbb{F}_{2^n}$ is equal to $r(A)$ in $\mathbb{F}_{2^{n-1}}$. In other words, each r-values occurring in $\mathbb{F}_{2^{n-1}} \backslash \{0\}$ is a r-value of subsets in $\mathbb{F}_{2^n} \backslash \{0\}$ with same subset size.

Let $b \in \mathbb{F}_{2^n} \backslash \mathbb{F}_{2^{n-1}}$ be fixed. Then we have $r(\{a_1, a_2, \ldots, a_k, b\}) = r(\{a_1, a_2, \ldots, a_k\}) + 3r(\{a_1, a_2, \ldots, a_k\}, \{b\}, \{b\})$. But then $a_i + b \neq b$ for all $1 \leq i \leq k$, we get $r(\{a_1, a_2, \ldots, a_k, b\}) = r(\{a_1, a_2, \ldots, a_k\})$. Here note that $\{a_1, a_2, \ldots, a_k\}$ is a subset of size k in $\mathbb{F}_{2^{n-1}} \backslash \{0\}$, giving an r-value of size $k + 1$ subset in $\mathbb{F}_{2^n} \backslash \{0\}$. □

Suppose $l = 2$, then we have the following result.

Proposition 4.3 *Suppose $S = \{a_1, a_2, \ldots, a_k, b_1, b_2\}$, where $a_1, a_2, \ldots, a_k \in \mathbb{F}_{2^{n-1}} \backslash \{0\}$, $k \geq 2$ and $b_1, b_2 \in \mathbb{F}_{2^n} \backslash \mathbb{F}_{2^{n-1}}$. Then $r(\{a_1, a_2, \ldots, a_k, b_1, b_2\}) = r(\{a_1, a_2, \ldots, a_k\})$ $+ \begin{cases} 6 & \text{if } b_1 + b_2 \in \{a_1, a_2, \ldots, a_k\}, \\ 0 & \text{otherwie.} \end{cases}$*

Proof: We have $k + 2 \leq 2^{n-1}$ then $k \leq 2^{n-1} - 2$, which implies there is an element in $\mathbb{F}_{2^{n-1}} \backslash \{0\}$ say a such that $a \notin \{a_1, a_2, \ldots, a_k\}$. But then there exists two elements $b_1, b_2 \in \mathbb{F}_{2^n} \backslash \mathbb{F}_{2^{n-1}}$ such that $b_1 + b_2 = a$. In this case we have $r(\{a_1, a_2, \ldots, a_k, b_1, b_2\}) = r(\{a_1, a_2, \ldots, a_{m-2}\})$. Suppose $a = a_i$ for some $\leq i \leq k$, then $(b_1, b_2, a_i), (b_2, b_1, a_i) \in R(\{b_1, b_2\}, \{b_1, b_2\}, \{a_1, a_2, \ldots, a_k\})$. Hence by Theorem 4.1 we have, $r(\{a_1, a_2, \ldots, a_k, b_1, b_2\}) = r(\{a_1, a_2, \ldots, a_k\}) + 6$. □

From the above result, each r-value r of subsets of $\mathbb{F}_{2^{n-1}} \backslash \{0\}$ of size k where $1 \leq k \leq 2^{n-1} - 2$, r and $r + 6$ are r-values of size $k + 2$ in $\mathbb{F}_{2^n} \backslash \{0\}$.

Theorem 4.2 *There exits two subsets A of $\mathbb{F}_{2^{n-1}}\backslash\{0\}$ and B of $\mathbb{F}_{2^n}\backslash\mathbb{F}_{2^{n-1}}$ with same size 2^{n-2} such that $r(A,B,B)=0$. Moreover, there is no such subsets of size greater than 2^{n-2}.*

Proof: Let $A=\{[\alpha_1,\alpha_2,\dots,\alpha_n]\in\mathbb{F}_{2^n}\mid \alpha_1=0,\alpha_n=1\}$ and $B=\{[\alpha_1,\alpha_2,\dots,\alpha_n]\in\mathbb{F}_{2^n}\mid \alpha_1=1,\alpha_n=0\}$. Clearly $|A|=|B|=2^{n-2}$. Let $C=\{[\alpha_1,\alpha_2,\dots,\alpha_n]\in\mathbb{F}_{2^n}\mid \alpha_1=\alpha_n=1\}$. Note that for any $a\in A$ and $b\in B$ we have $a+b\in C$. Therefore $r(A,B,B)=0$.

Suppose $A\subseteq\mathbb{F}_{2^{n-1}}\backslash\{0\}$ with $|A|=2^{n-2}+1$. Then there is an element a in A of the form $a=[0,\alpha_2,\dots,\alpha_{n-1},0]$. But then for any $[1,\beta_2,\dots,\beta_{n-1},0]\in B$, we have $[0,\alpha_2,\dots,\alpha_{n-1},0]+[1,\beta_2,\dots,\beta_{n-1},0]=[1,\alpha_2+\beta_2,\dots,\alpha_{n-1}+\beta_{n-1},0]\in B$. Which gives $r(A,B,B)>0$.

Similarly suppose $B\subseteq\mathbb{F}_{2^n}\backslash\mathbb{F}_{2^{n-1}}$ with $|B|=2^{n-2}+1$. Then there is an element b in B of the form $b=[1,\beta_2,\dots,\beta_{n-1},1]$. But then for any $[0,\alpha_2,\dots,\alpha_{n-1},1]\in A$, we have $[0,\alpha_2,\dots,\alpha_{n-1},1]+[1,\beta_2,\dots,\beta_{n-1},1]=[1,\alpha_2+\beta_2,\dots,\alpha_{n-1}+\beta_{n-1},0]\in B$ giving $r(A,B,B)>0$. □

Suppose $k=1$, then we have the following result.

Proposition 4.4 *Suppose $S=\{a_1,b_1,\dots,b_{m-1}\}$, where $a_1\in\mathbb{F}_{2^{n-1}}\backslash\{0\}$ and $b_1,\dots,b_{m-1}\in\mathbb{F}_{2^n}\backslash\mathbb{F}_{2^{n-1}}$. Then $r(\{a_1,b_1,\dots,b_{m-1}\})$ attain each values in the set $\left\{6i\ |0\le i\le\left\lfloor\frac{m-1}{2}\right\rfloor\right\}$ whenever $m-1\le 2^{n-2}$ and whenever $m-1>2^{n-2}$, $r(\{a_1,b_1,\dots,b_{m-1}\})$ attain each values in the set $\left\{6i\ |(m-1)-2^{n-2}\le i\le\left\lfloor\frac{m-1}{2}\right\rfloor\right\}$.*

Proof: Suppose $m-1\le 2^{n-2}$. If $a_1\in A$ and $\{b_1,\dots,b_{m-1}\}\subseteq B$, where A and B as defined in Theorem 4.2, then $r(\{a_1\},\{b_1,\dots,b_{m-1}\},\{b_1,\dots,b_{m-1}\})=0$ giving $r(\{a_1,b_1,\dots,b_{m-1}\})=0$. We have if $(a_1,b_i,b_j)\in R(\{a_1\},\{b_1,\dots,b_{m-1}\},\{b_1,\dots,b_{m-1}\})$ for some $1\le i,j\le m-1$ then $(a_1,b_j,b_i)\in R(\{a_1\},\{b_1,\dots,b_{m-1}\},\{b_1,\dots,b_{m-1}\})$. Also for each $a_i\in\mathbb{F}_{2^{n-1}}\backslash\{0\}$ there are 2^{n-2} pairs $\{b_t,b_s\}$, where $b_t,b_s\in\mathbb{F}_{2^n}\backslash\mathbb{F}_{2^{n-1}}$ such that $a_i=b_t+b_s$. Using this, among the $m-1$ elements $\{b_1,\dots,b_{m-1}\}$, atmost $\lfloor\frac{m-1}{2}\rfloor$ possible to give sum as a_1. Hence by Theorem 4.1 $r(\{a_1,b_1,\dots,b_{m-1}\})$ attain each values in the set $\left\{6i\ |0\le i\le\left\lfloor\frac{m-1}{2}\right\rfloor\right\}$.

Suppose $m-1>2^{n-2}$. For distinct i and j with $1\le i,j\le m-1$, we know that a_1+b_i and a_1+b_j distinct and they all lies in $\mathbb{F}_{2^n}\backslash\mathbb{F}_{2^{n-1}}$. Since $m-1>2^{n-2}$ and $|\mathbb{F}_{2^n}\backslash\mathbb{F}_{2^{n-1}}|=2^{n-1}$ there are atleast $2((m-1)-2^{n-2})$ number of i's between 1 and $m-1$ such that $a_1+b_i\in\{b_1,\dots,b_{m-1}\}$. Now as seen in above case, among the $m-1$ elements $\{b_1,\dots,b_{m-1}\}$, atmost $\lfloor\frac{m-1}{2}\rfloor$ possible to give sum as a_1. Hence in this case $r(\{a_1,b_1,\dots,b_{m-1}\})$ attain each values in the set $\left\{6i\ |(m-1)-2^{n-2}\le i\le\left\lfloor\frac{m-1}{2}\right\rfloor\right\}$. □

The Proposition 4.5,4.6,4.7 and 4.8 gives r value in the case $l=3$.

Proposition 4.5 *Let $A\subseteq\mathbb{F}_{2^{n-1}}\backslash\{0\}$ with size $k\ge 3$ and positive r-value. Then $r(A)+18$ is a r-value of size $k+3$ subset in $\mathbb{F}_{2^n}\backslash\{0\}$.*

Proof: Let $A=\{a_1,a_2,\dots,a_k\}\subseteq\mathbb{F}_{2^{n-1}}\backslash\{0\}$ with size $k\ge 3$ and positive r-value. Since $r(A)>0$, there exists three elements say a_1,a_2,a_3 such that $a_1+a_2=a_3$ in $\mathbb{F}_{2^n}$.

Let $b_1=a_1+[1,0,0,\dots,0]$, $b_2=a_2+[1,0,0,\dots,0]$ and $b_3=a_3+[1,0,0,\dots,0]$. Note that $b_1,b_2,b_3\in\mathbb{F}_{2^n}\backslash\mathbb{F}_{2^{n-1}}$ with $b_1+b_2=a_3$, $b_1+b_3=a_2$ and $b_2+b_3=a_1$ in $\mathbb{F}_{2^n}$. Thus if $B=\{b_1,b_2,b_3\}$, then $r(B,B,A)=6$. Hence by Theorem 4.1 $r(A\cup B)=r(A)+18$. □

Proposition 4.6 *Let $A\subseteq\mathbb{F}_{2^{n-1}}\backslash\{0\}$ with size $k\ge 2$ and $r(A)\ne k(k-1)$. Then $r(A)+12$ is a r-value of size $k+3$ subset in $\mathbb{F}_{2^n}\backslash\{0\}$.*

Proof:

Given $k\ge 2$ and $r(A)\ne k(k-1)$, there exist two elements say a_1,a_2 such that $a_1+a_2\notin A$. Let $b_1=a_1+[1,0,0,\dots,0]$, $b_2=a_2+[1,0,0,\dots,0]$ and $b_3=a_1+a_2+[1,0,0,\dots,0]$. Note that $b_1+b_3=a_2\in A$ and $b_2+b_3=a_1\in A$, but $b_1+b_2=a_1+a_2\notin A$. Thus $r(B,B,A)=4$ and by Theorem 4.1 $r(A\cup\{b_1,b_2,b_3\})=r(A)+12$. □

Proposition 4.7 *Let $A \subseteq \mathbb{F}_{2^{n-1}} \backslash \{0\}$ with size $0 < k < 2^{n-1} - 3$ and $\overline{A}^*$ is the compliment of A in $\mathbb{F}_{2^{n-1}} \backslash \{0\}$ with $r(\overline{A}^*) \neq |\overline{A}^*|(|\overline{A}^*| - 1)$. Then $r(A) + 6$ is a r-value of size $k + 3$ subset in $\mathbb{F}_{2^n} \backslash \{0\}$.*

Proof:
Given $r(\overline{A}^*) \neq |\overline{A}^*|(|\overline{A}^*| - 1)$, there exist three elements say $\overline{a}_1, \overline{a}_2$ in $\overline{A}^*$ and $a_3 \in A$ such that $\overline{a}_1 + \overline{a}_2 = a_3$. Let $b_1 = \overline{a}_1 + [1, 0, 0, \dots, 0]$, $b_2 = \overline{a}_2 + [1, 0, 0, \dots, 0]$ and $b_3 = a_3 + [1, 0, 0, \dots, 0]$. Note that $b_1 + b_2 = a_3 \in A$, $b_1 + b_3 = \overline{a}_2 \notin A$ and $b_2 + b_3 = \overline{a}_1 \notin A$. Thus $r(B, B, A) = 2$ and by Theorem 4.1 $r(A \cup \{b_1, b_2, b_3\}) = r(A) + 6$. □

Proposition 4.8 *Let $A \subseteq \mathbb{F}_{2^{n-1}} \backslash \{0\}$ with size $k < 2^{n-1} - 4$ and $\overline{A}^*$ is the compliment of A in $\mathbb{F}_{2^{n-1}} \backslash \{0\}$ with $r(\overline{A}^*) > 0$. Then $r(A)$ is a r-value of size $k + 3$ subset in $\mathbb{F}_{2^n} \backslash \{0\}$.*

Proof: Given $r(\overline{A}^*) > 0$, there exist three elements say $\overline{a}_1, \overline{a}_2, \overline{a}_3$ in $\overline{A}^*$ such that $\overline{a}_1 + \overline{a}_2 = \overline{a}_3$. Let $b_1 = \overline{a}_1 + [1, 0, 0, \dots, 0]$, $b_2 = \overline{a}_2 + [1, 0, 0, \dots, 0]$ and $b_3 = \overline{a}_3 + [1, 0, 0, \dots, 0]$. Note that $b_1 + b_2 = \overline{a}_3$, $b_1 + b_3 = \overline{a}_2$ and $b_2 + b_3 = \overline{a}_1$. Thus $r(B, B, A) = 0$ and by Theorem 4.1 $r(A \cup \{b_1, b_2, b_3\}) = r(A)$. □

When $l = 4$, r-values of subset size $k + 4$ is obtained using the results 4.9,4.10,4.11 and 4.12.

Proposition 4.9 *Let $A \subseteq \mathbb{F}_{2^{n-1}} \backslash \{0\}$ with size k and positive r-value. Then $r(A) + 36$ is a r-value of size $k + 4$ subset in $\mathbb{F}_{2^n} \backslash \{0\}$.*

Proof: Given $r(A) > 0$, there exist three elements say $a_1, a_2, a_3 \in A$ such that $a_1 + a_2 = a_3$. Let $b \in \mathbb{F}_{2^n} \backslash \mathbb{F}_{2^{n-1}}$ be fixed and $b_1 = b + a_1$, $b_2 = b + a_2$ and $b_3 = b + a_3$. Note that all three b_1, b_2, b_3 are distinct. If $B = \{b, b_1, b_2, b_3\}$, then $r(B, B, A) = 12$. Hence by Theorem 4.1 $r(A \cup B) = r(A) + 36$. □

Proposition 4.10 *Let $A \subseteq \mathbb{F}_{2^{n-1}} \backslash \{0\}$ with size $k \geq 2$ and $r(A) \neq k(k - 1)$. Then $r(A) + 24$ is a r-value of size $k + 4$ subset in $\mathbb{F}_{2^n} \backslash \{0\}$.*

Proof: Given $k \geq 2$ and $r(A) \neq k(k - 1)$, there exist two elements say $a_1, a_2 \in A$ such that $a_1 + a_2 \notin A$. Let $b \in \mathbb{F}_{2^n} \backslash \mathbb{F}_{2^{n-1}}$ be fixed and $b_1 = b + a_1$, $b_2 = b + a_2$ and $b_3 = b + a_1 + a_2$. Note that all three b_1, b_2, b_3 are distinct. If $B = \{b, b_1, b_2, b_3\}$, then $r(B, B, A) = 8$. Hence by Theorem 4.1 $r(A \cup B) = r(A) + 24$. □

Proposition 4.11 *Let $A \subseteq \mathbb{F}_{2^{n-1}} \backslash \{0\}$ with size $0 < k < 2^{n-1} - 3$ and $\overline{A}^*$ is the compliment of A in $\mathbb{F}_{2^{n-1}} \backslash \{0\}$ with $r(\overline{A}^*) \neq |\overline{A}^*|(|\overline{A}^*| - 1)$. Then $r(A) + 12$ is a r-value of size $k + 4$ subset in $\mathbb{F}_{2^n} \backslash \{0\}$.*

Proof:
Given $r(\overline{A}^*) \neq |\overline{A}^*|(|\overline{A}^*| - 1)$, there exist three elements say $\overline{a}_1, \overline{a}_2 \in \overline{A}^*$ and $a_3 \in A$ such that $\overline{a}_1 + \overline{a}_2 = a_3$. Let $b \in \mathbb{F}_{2^n} \backslash \mathbb{F}_{2^{n-1}}$ be fixed and $b_1 = b + \overline{a}_1$, $b_2 = b + \overline{a}_2$ and $b_3 = b + a_3$. Note that all three b_1, b_2, b_3 are distinct. If $B = \{b, b_1, b_2, b_3\}$, then $r(B, B, A) = 4$. Hence by Theorem 4.1 $r(A \cup B) = r(A) + 12$. □

Proposition 4.12 *Let $A \subseteq \mathbb{F}_{2^{n-1}} \backslash \{0\}$ with size $k < 2^{n-1} - 4$ and $\overline{A}^*$ is the compliment of A in $\mathbb{F}_{2^{n-1}} \backslash \{0\}$ with $r(\overline{A}^*) > 0$. Then $r(A)$ is a r-value of size $k + 4$ subset in $\mathbb{F}_{2^n} \backslash \{0\}$.*

Proof: Given $r(\overline{A}^*) > 0$, there exist three elements say $\overline{a}_1, \overline{a}_2, \overline{a}_3 \in \overline{A}^*$ such that $\overline{a}_1 + \overline{a}_2 = \overline{a}_3$. Let $b \in \mathbb{F}_{2^n} \backslash \mathbb{F}_{2^{n-1}}$ be fixed and $b_1 = b + \overline{a}_1$, $b_2 = b + \overline{a}_2$ and $b_3 = b + \overline{a}_3$. Note that all three b_1, b_2, b_3 are distinct. If $B = \{b, b_1, b_2, b_3\}$, then $r(B, B, A) = 0$. Hence by Theorem 4.1 $r(A \cup B) = r(A)$. □

The following result cover the case $k = 2$ and $l = 5$ to give r-values of subsets of size 7 in $\mathbb{F}_{2^n} \backslash \{0\}$.

Proposition 4.13 *Suppose $S = \{a_1, a_2, b_1, b_2, b_3, b_4, b_5\}$, where $a_1, a_2 \in \mathbb{F}_{2^{n-1}} \backslash \{0\}$ and $b_1, b_2, b_3, b_4, b_5 \in \mathbb{F}_{2^n} \backslash \mathbb{F}_{2^{n-1}}$. Then for $n \geq 5$, $r(S)$ will attain the values 0,6,12,18 and 24. For $n = 4$, $r(S)$ will attain the values 0,12,18 and 24.*

Table 1: Type 1

	b_1	b_2	b_3	b_4	b_5
b_1	0	x_1	x_2	x_3	x_4
b_2	x_1	0	x_1+x_2	x_1+x_3	x_1+x_4
b_3	x_2	x_1+x_2	0	x_2+x_3	x_2+x_4
b_4	x_3	x_1+x_3	x_2+x_3	0	x_3+x_4
b_5	x_4	x_1+x_4	x_2+x_4	x_3+x_4	0

Proof: Let $A=\{a_1,a_2\}$ and $B=\{b_1,b_2,b_3,b_4,b_5\}$. Using the following Cayley Table 1, $r(B,B,A)$ depends on $r(\{x_1,x_2,x_3,x_4\},\{x_1,x_2,x_3,x_4\},A)$.

Since $x_1,x_2,x_3,x_4\in\mathbb{F}_{2^{n-1}}\backslash\{0\}$, we have $r(\{x_1,x_2,x_3,x_4\})$ is 0 or 6. If $r(\{x_1,x_2,x_3,x_4\})=0$, then $x_1+x_2, x_1+x_3, x_1+x_4\notin\{x_1,x_2,x_3,x_4\}$. Say $x_1+x_2=x_5, x_1+x_3=x_6$ and $x_1+x_4=x_7$.

Suppose r-value of $\{x_5,x_6,x_7\}$ is 0, then $x_5+x_6, x_5+x_7, x_6+x_7$ not in $\{x_5,x_6,x_7\}$ and $\{x_1,x_2,x_3,x_4\}$. Say $x_5+x_6=x_2+x_3=x_8$, $x_5+x_7=x_2+x_4=x_9$ and $x_6+x_7=x_3+x_4=x_{10}$. This is possible only if $n\geq 5$ (See Cayley Table 2). If $\{a_1,a_2\}\not\subset\{x_1,x_2,\ldots,x_{10}\}$, then $r(S)=0$. If one element of $\{a_1,a_2\}$ in $\{x_1,x_2,\ldots,x_{10}\}$ and other one not in $\{x_1,x_2,\ldots,x_{10}\}$, then $r(S)=6$. If $\{a_1,a_2\}\subset\{x_1,x_2,\ldots,x_{10}\}$, then $r(S)=12$.

Table 2: Type 2

	b_1	b_2	b_3	b_4	b_5
b_1	0	x_1	x_2	x_3	x_4
b_2	x_1	0	x_5	x_6	x_7
b_3	x_2	x_5	0	x_8	x_9
b_4	x_3	x_6	x_8	0	x_{10}
b_5	x_4	x_7	x_9	x_{10}	0

Suppose r-value of $\{x_5,x_6,x_7\}$ is 6, then Cayley Table 2 becomes Table 3, If $\{a_1,a_2\}\not\subset\{x_1,x_2,x_3,x_4,x_5,x_6,x_7\}$, then $r(S)=0$. If one element of $\{a_1,a_2\}$ in $\{x_1,x_2,x_3,x_4\}$ and other one not in $\{x_1,x_2,x_3,x_4,x_5,x_6,x_7\}$, then $r(S)=6$. If one element of $\{a_1,a_2\}$ in $\{x_5,x_6,x_7\}$ and other one not in $\{x_1,x_2,x_3,x_4,x_5,x_6,x_7\}$, then $r(S)=12$. If $\{a_1,a_2\}\subset\{x_1,x_2,x_3,x_4\}$, then $r(S)=12$. If $\{a_1,a_2\}\subset\{x_5,x_6,x_7\}$, then $r(S)=24$. If one element of $\{a_1,a_2\}$ in $\{x_1,x_2,x_3,x_4\}$ and other one in $\{x_5,x_6,x_7\}$, then $r(S)=18$.

Table 3: Type 3

	b_1	b_2	b_3	b_4	b_5
b_1	0	x_1	x_2	x_3	x_4
b_2	x_1	0	x_5	x_6	x_7
b_3	x_2	x_5	0	x_7	x_6
b_4	x_3	x_6	x_7	0	x_5
b_5	x_4	x_7	x_6	x_5	0

If $r(\{x_1,x_2,x_3,x_4\})=6$, then Cayley table will be similar to following Table 4.
This is similar to the above case; hence $r(S)$ will be one of the values in the set $\{0,6,12,18,24\}$.

□

Table 4: Type 4

	b_1	b_2	b_3	b_4	b_5
b_1	0	x_1	x_2	x_3	x_4
b_2	x_1	0	x_3	x_2	x_5
b_3	x_2	x_3	0	x_1	x_6
b_4	x_3	x_2	x_1	0	x_7
b_5	x_4	x_5	x_6	x_7	0

The following two results holds the case $l = k$ and $l = k + 1$.

Proposition 4.14 *Let $A \subseteq \mathbb{F}_{2^{n-1}} \backslash \{0\}$ with size $k \geq 3$. Then $4r(A)$ is a r-value of size $2k$ subset in $\mathbb{F}_{2^n} \backslash \{0\}$.*

Proof: Let $A = \{a_1, a_2, \ldots, a_k\} \subseteq \mathbb{F}_{2^{n-1}} \backslash \{0\}$ with size $k \geq 3$. Let $B = \{b_i = a_i + [1, 0, 0, \ldots, 0] | \ 1 \leq i \leq k\}$. Now note that $(a_i, a_j, a_t) \in R(A, A, A)$ if and only if $(b_i, b_j, a_t) \in R(B, B, A)$. Thus $|R(A, A, A)| = |R(B, B, A)|$ and by Theorem 4.1 $r(A \cup B) = 4r(A)$. □

Proposition 4.15 *Let $A \subseteq \mathbb{F}_{2^{n-1}} \backslash \{0\}$ with size k. Then $4r(A) + 6k$ is a r-value of a subset of size $2k + 1$ in $\mathbb{F}_{2^n} \backslash \{0\}$.*

Proof: Let $A = \{a_1, a_2, \ldots, a_k\}$ be a subset of $\mathbb{F}_{2^{n-1}} \backslash \{0\}$ with size k. Let $b \in \mathbb{F}_{2^n} \backslash \mathbb{F}_{2^{n-1}}$ be fixed. For each $a_i \in A$, there exist $b_i \in \mathbb{F}_{2^n} \backslash \mathbb{F}_{2^{n-1}}$ such that $b + b_i = a_i$. Say $B = \{b, b_1, b_2, \ldots, b_k\}$. Now for all $1 \leq i \leq k$, $b + b_i = b_i + b = a_i \in A$, giving $2k$ to $r(B, B, A)$. For $1 \leq i, j \leq k$ with $i \neq j$, we have $b_i + b_j = a_i + a_j$. Thus, $b_i + b_j \in A$ if and only if $a_i + a_j \in A$. Hence $r(B, B, A) = r(A) + 2k$ and by Theorem 4.1 $r(A \cup B) = 4r(A) + 6k$. □

4.1. An algorithm to generate r-values of $\mathbb{F}_{2^n}$ from r-values of $\mathbb{F}_{2^{n-1}}$

The Algorithm 1 generate r-values of $\mathbb{F}_{2^n} \backslash \{0\}$ from given r-values of $\mathbb{F}_{2^{n-1}} \backslash \{0\}$. The remaining values are obtained by Theorem 3.2 and Theorem 2.1. The pair $(k, r(A))$ in Input and Output represents the size k subset A of r-value $r(A)$. For each subsets size, all possible r-values are termed as a spectrum of r-values. The following subsections 4.2 and 4.3 gives the spectrum in $\mathbb{F}_{2^4}$ and $\mathbb{F}_{2^5}$ respectively.

4.2. Spectrum of r-values in $\mathbb{F}_{2^4}$

For each subsets of $\mathbb{F}_{2^4}$, its r-value can be obtained from the results listed above. Let S be a subset of $\mathbb{F}_{2^4} \backslash \{0\}$ of cardinality m. Suppose $S = A \cup B$, where $A \subset \mathbb{F}_{2^3} \backslash \{0\}$ with cardinality k and $B \subset \mathbb{F}_{2^4} \backslash \mathbb{F}_{2^3}$ with cardinality l such that $m = k + l$. For $m = 1, 2, 3, 4$ we use Propositions 2.2,3.2,3.3,3.5 resp. For subset size 5,6,7 see Table 5, For any subset $A \subset \mathbb{F}_{2^4}$ of size 1 to 8 with $0 \in A$ is obtained by Theorem 3.2. Rest of the values for any subsets of $\mathbb{F}_{2^4}$ of size 8 to 16 is obtained by Theorem 2.1. The spectrum of each subset size in $\mathbb{F}_{2^4}$ is given as Figure 2. The red and green lines are as indicated in Section 3.1.

Algorithm 1 r-values of $\mathbb{F}_{2^n}$ from r-values of $\mathbb{F}_{2^{n-1}}$

```
Input: $n$ and $(k, r(A))$ in $\mathbb{F}_{2^{n-1}}$
Output: $(k, r(A))$, $(k+1, r(A))$, $(2k, 4r(A))$ and $(2k+1, 4r(A)+6)$ in $\mathbb{F}_{2^n}$
if $1 \leq k \leq 2^{n-1} - 2$ then
    return Output: $(k+2, r(A))$ and $(k+2, r(A)+6)$ in $\mathbb{F}_{2^n}$
end if
if $k \geq 3$ and $r(A) > 0$ then
    return Output: $(k+3, r(A)+18)$ and $(k+4, r(A)+36)$ in $\mathbb{F}_{2^n}$
end if
if $k \geq 2$ and $r(A) \neq k(k-1)$ then
    return Output: $(k+3, r(A)+12)$ and $(k+4, r(A)+24)$ in $\mathbb{F}_{2^n}$
end if
if $0 < k < 2^{n-1} - 3$ and $r(\overline{A}^*) \neq |\overline{A}^*|(|\overline{A}^*| - 1)$ then
    return Output: $(k+3, r(A)+6)$ and $(k+4, r(A)+12)$ in $\mathbb{F}_{2^n}$
end if
if $k < 2^{n-1} - 4$ and $r(\overline{A}^*) > 0$ then
    return Output: $(k+3, r(A))$ and $(k+4, r(A))$ in $\mathbb{F}_{2^n}$
end if
if n=4 then
    return Output: (7,0), (7,12), (7,18), (7,24) in $\mathbb{F}_{2^4}$
else if $n \geq 5$ then
    return Output: (7,0), (7,6), (7,12), (7,18), (7,24) in $\mathbb{F}_{2^4}$
end if
for $l = 1$ to $2^{n-1}$ do
    if $l \leq 2^{n-2}$ then
        for $i = 0$ to $\lfloor \frac{l}{2} \rfloor$ do
            return Output: $(l+1, 6i)$ in $\mathbb{F}_{2^n}$
        end for
    else if $l > 2^{n-2}$ then
        for $i = l - 2^{n-2}$ to $\lfloor \frac{l}{2} \rfloor$ do
            return Output: $(l+1, 6i)$ in $\mathbb{F}_{2^n}$
        end for
    end if
end for
```

Table 5: Spectrum of r-values in $\mathbb{F}_{2^4}$ of size $m = 5, 6, 7$.

m	k	l	$r(S)$	Spectrum
5	0	5	0	$\{0, 6, 12, 13, 19\}$
	1	4	0,6,12	
	2	3	0,6,12	
	3	2	0,6,12	
	4	1	0,6	
	5	0	12	
6	0	6	0	$\{0, 12, 18, 24, 42, 19, 25, 31, 43\}$
	1	5	6,12	
	2	4	0,12,24	
	3	3	6,12,24	
	4	2	0,6,12	
	5	1	12	
	6	0	24	
7	0	7	0	$\{0, 18, 24, 30, 42, 22, 34, 40, 46, 64\}$
	1	6	12,18	
	2	5	0,12,18,24	
	3	4	18,24,42	
	4	3	12,18,24	
	5	2	12,18	
	6	1	24	
	7	0	42	

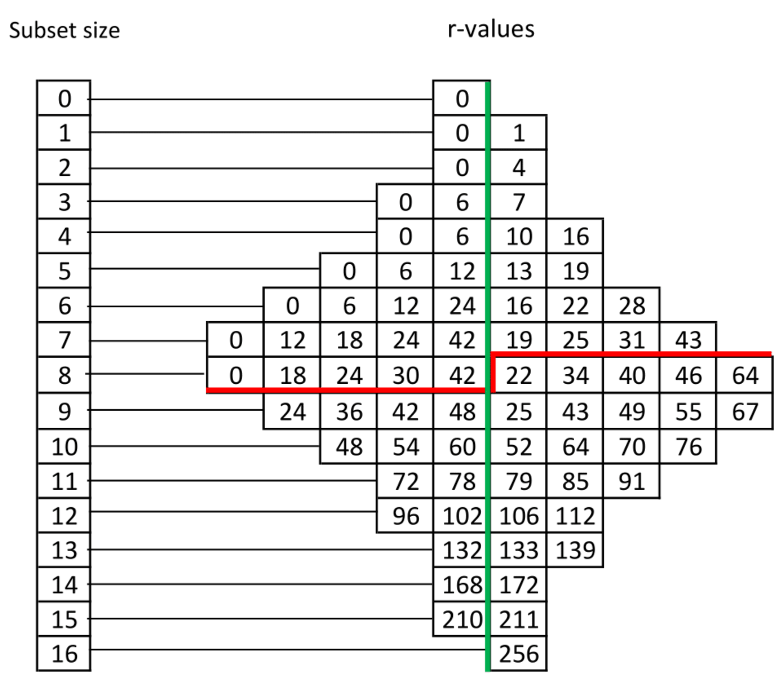

Figure 2: Spectrum of r-values in $\mathbb{F}_{2^4}$

4.3. Spectrum of r-values in $\mathbb{F}_{2^5}$

It is computationally verified that the above listed result will generate a spectrum of r-values in $\mathbb{F}_{2^5}$. In particular the following Figure 3 classifies 429,49,67,296 subsets in terms of subset size and r-value. The red and green lines are as indicated in Section 3.1.

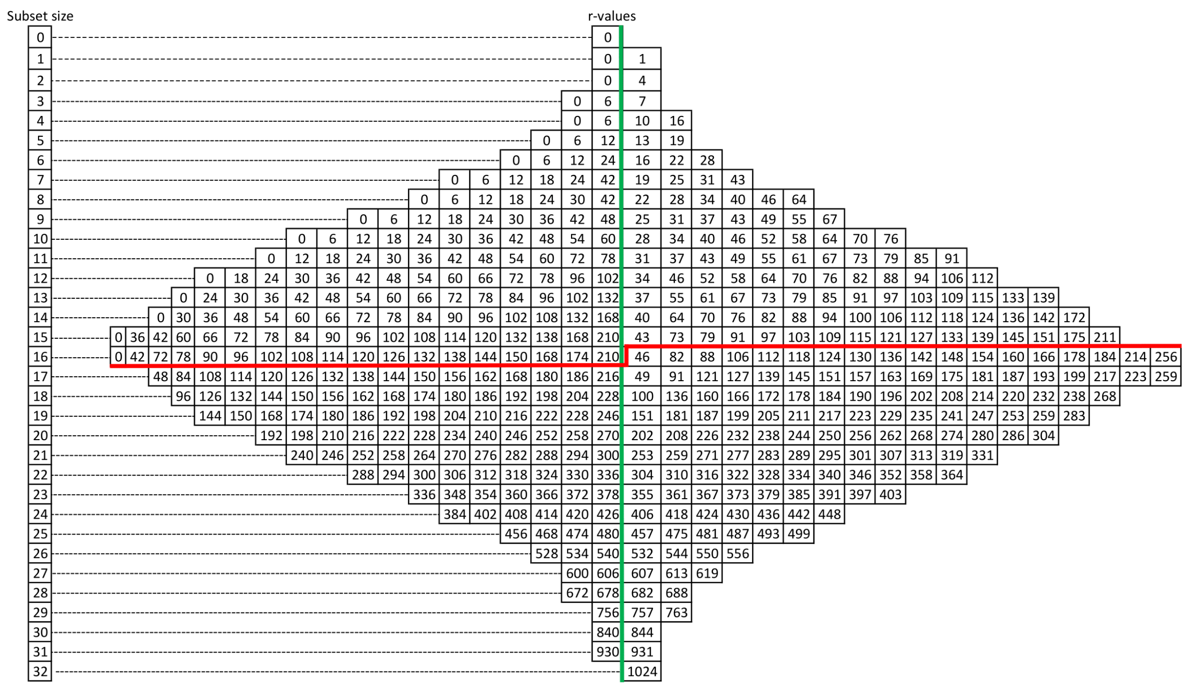

Figure 3: Spectrum of r-values in $\mathbb{F}_{2^5}$

4.4. Enumeration of r-values in $\mathbb{F}_{2^n}$, for $n \geq 6$

Using the above listed results, it is possible to observe r-values of a large number of subsets in $\mathbb{F}_{2^n}$, for $n \geq 6$. The Table 6 gives the some r-values for different subset size in $\mathbb{F}_{2^6}\backslash\{0\}$. Using these values we can further continue to observe some collection of r-values in $\mathbb{F}_{2^7}$ and so on.

Table 6: r-values of some subsets of $\mathbb{F}_{2^6} \backslash \{0\}$

Size	**r-values**
0	0
1	0
2	0
3	0, 6
4	0, 6
5	0, 6, 12
6	0, 6, 12, 24
7	0, 6, 12, 18, 24, 42
8	0, 6, 12, 18, 24, 30, 42
9	0, 6, 12, 18, 24, 30, 36, 42, 48
10	0, 6, 12, 18, 24, 30, 36, 42, 48, 54, 60
11	0, 6, 12, 18, 24, 30, 36, 42, 48, 54, 60, 72, 78
12	0, 6, 12, 18, 24, 30, 36, 42, 48, 54, 60, 66, 72, 78, 96, 102
13	0, 6, 12, 18, 24, 30, 36, 42, 48, 54, 60, 66, 72, 78, 84, 96, 102, 132
14	0, 6, 12, 18, 24, 30, 36, 42, 48, 54, 60, 66, 72, 78, 84, 90, 96, 102, 108, 132, 168
15	0, 6, 12, 18, 24, 30, 36, 42, 48, 54, 60, 66, 72, 78, 84, 90, 96, 102, 108, 114, 120, 132, 138, 168, 210
16	0, 6, 12, 18, 24, 30, 36, 42, 48, 54, 60, 66, 72, 78, 84, 90, 96, 102, 108, 114, 120, 126, 132, 138, 144, 150, 168, 174, 210
17	0, 6, 12, 18, 24, 30, 36, 42, 48, 54, 60, 66, 72, 78, 84, 90, 96, 102, 108, 114, 120, 126, 132, 138, 144, 150, 156, 162, 168, 174, 180, 186, 210, 216
18	0, 6, 12, 18, 24, 30, 36, 42, 48, 54, 60, 66, 72, 78, 84, 90, 96, 102, 108, 114, 120, 126, 132, 138, 144, 150, 156, 162, 168, 174, 180, 186, 192, 198, 204, 210, 216, 228
19	0, 12, 18, 24, 30, 36, 42, 48, 54, 60, 66, 72, 78, 84, 90, 96, 102, 108, 114, 120, 126, 132, 138, 144, 150, 156, 162, 168, 174, 180, 186, 192, 198, 204, 210, 216, 222, 228, 246
20	0, 18, 24, 30, 36, 42, 48, 54, 60, 66, 72, 78, 84, 90, 96, 102, 108, 114, 120, 126, 132, 138, 144, 150, 156, 162, 168, 174, 180, 186, 192, 198, 204, 210, 216, 222, 228, 234, 240, 246, 252, 258, 270
21	0, 24, 30, 36, 42, 48, 54, 60, 72, 84, 96, 102, 108, 114, 120, 126, 132, 138, 144, 150, 156, 162, 168, 174, 180, 186, 192, 198, 204, 210, 216, 222, 228, 234, 240, 246, 252, 258, 264, 270, 276, 282, 288, 294, 300
22	0, 30, 36, 42, 48, 54, 60, 72, 96, 108, 120, 126, 132, 138, 144, 150, 156, 162, 168, 174, 180, 186, 192, 198, 204, 210, 216, 222, 228, 234, 240, 246, 252, 258, 264, 270, 276, 282, 288, 294, 300, 306, 312, 318, 324, 330, 336
23	0, 36, 42, 48, 54, 60, 66, 114, 138, 144, 150, 156, 162, 168, 174, 180, 186, 192, 198, 204, 210, 216, 222, 228, 234, 240, 246, 252, 258, 264, 270, 276, 282, 288, 294, 300, 306, 312, 318, 324, 330, 336, 348, 354, 360, 366, 372, 378
24	0, 42, 48, 54, 60, 66, 72, 96, 120, 144, 168, 192, 198, 204, 210, 216, 222, 228, 234, 240, 246, 252, 258, 264, 270, 276, 282, 288, 294, 300, 306, 312, 318, 324, 330, 336, 342, 348, 354, 360, 366, 372, 378, 384, 402, 408, 414, 420, 426
25	0, 48, 54, 60, 66, 72, 144, 168, 192, 216, 240, 246, 252, 258, 264, 270, 276, 282, 288, 294, 300, 306, 312, 318, 324, 330, 336, 342, 348, 354, 360, 366, 372, 378, 384, 402, 408, 414, 420, 426, 456, 468, 474, 480
26	0, 54, 60, 66, 72, 96, 120, 144, 168, 192, 216, 240, 264, 288, 294, 300, 306, 312, 318, 324, 330, 336, 342, 348, 354, 360, 366, 372, 378, 384, 390, 396, 402, 408, 414, 420, 426, 432, 456, 468, 474, 480, 528, 534, 540
27	0, 60, 66, 72, 78, 174, 198, 222, 246, 270, 294, 318, 336, 342, 348, 354, 360, 366, 372, 378, 384, 390, 396, 402, 408, 414, 420, 426, 432, 438, 444, 456, 462, 468, 474, 480, 486, 528, 534, 540, 600, 606
28	0, 66, 72, 78, 120, 144, 192, 216, 240, 264, 288, 312, 336, 360, 384, 402, 408, 414, 420, 426, 432, 438, 444, 450, 456, 462, 468, 474, 480, 486, 492, 498, 528, 534, 540, 546, 600, 606, 672, 678
29	0, 72, 78, 84, 204, 228, 276, 300, 324, 348, 372, 396, 420, 444, 456, 468, 474, 480, 486, 492, 498, 504, 510, 516, 528, 534, 540, 546, 552, 558, 600, 606, 612, 672, 678, 756
30	0, 78, 84, 144, 156, 168, 240, 264, 288, 312, 336, 360, 384, 408, 432, 456, 480, 528, 534, 540, 546, 552, 558, 564, 570, 576, 600, 606, 612, 618, 624, 672, 678, 684, 756, 840
31	0, 84, 90, 156, 162, 168, 216, 222, 228, 234, 240, 258, 330, 354, 378, 402, 426, 450, 474, 498, 522, 546, 570, 600, 612, 618, 624, 630, 636, 642, 672, 684, 690, 696, 756, 762, 840, 930
32	0, 90, 168, 174, 234, 240, 246, 258, 288, 294, 300, 306, 312, 318, 330, 360, 384, 408, 432, 456, 480, 504, 528, 552, 576, 600, 672, 690, 696, 702, 708, 714, 762, 768, 774, 840, 846, 930

5. Conclusion

The study resulted in enumerating r-values of large number of subsets of $\mathbb{F}_{2^n}$ by knowing r-values in $\mathbb{F}_{2^{n-1}}$. It is observed that every subset S of $\mathbb{F}_{2^n} \backslash \{0\}$ will form a partial Steiner triple system such that the number of blocks in the partial Steiner triple system is equal to $r(A)/6$. For each size m, the maximum number of blocks possible in the partial Steiner triple system of order m is equal to the maximum r-value in the spectrum of size m in $\mathbb{F}_{2^n} \backslash \{0\}$. These results can be used to study different sets like Sum-free sets and Sidon sets in the future.

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