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A Study of Hybrid Fixed Points in Transformation Semigroups: Existence and Uniqueness

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ABSTRACT: The goal of this study is to establish the existence and uniqueness of hybrid FPs in semigroups of transformations defined on metric spaces. By including both contractive and nonexpansive mappings, we derive important hybrid FP results under appropriate contractive conditions. Hypothetical results are supported by illustrations that determine the existence of hybrid FPs in transformation semigroups.

Keywords: Fixed point, contractive, metric space.

Contents

1. Introduction

A FP (FP) in mathematics is a value that remains unchanged when a function or mapping is functional to it. Basically, a FP a point that is mapped X to X . FP theory explores the existence, uniqueness, and properties of these FPs, and their applications across various fields like mathematics, economics, and computer science. A pioneering works about FPs is Henri Poincaré [10] , which was proposed as the first effort almost FPs in 1886. Even though the vital conception of metric FP theory was identified to others earlier, the Polish mathematician Stefan Banach is acknowledged with construction it usable and well-known. The Banach [2] FP Theorem (also known as the contraction mapping theorem or contraction mapping principle) is a suitable tool in the knowledge of metric spaces. Erection on earlier results, comprehensive Banach's Contraction Principle (BCP) to put on to multivalued mappings in metric spaces by Nadler in 1969, expressively broadening the possibility of FP theory. Takahashi [11] (1970) donated further by examining nonexpansive mappings in Hilbert spaces, establishing fundamental results on the existence of FPs in more all-purpose topological settings. Later, Kirk [5] (2008) discovered F theorems in metric spaces, highlighting the inference of asymptotic regularity in iterative methods. These enlargements reputable a vigorous connection between FP attitude and functional analysis, enabling wide-ranging requests in operator theory and computational mathematics. Cutting-edge enhancements in FP theory have gradually focused on contractive and nonexpansive mappings inside transformation semigroups, acquiescent important results about the existence and convergence of FPs in numerous mathematical contexts. Especially, contractive mappings in semigroups have been shown to guarantee the convergence of iterative sequences, an initial component in the proof of hybrid FP presence. Hybrid contractions, which confidante adapting the usual contractive condition finished auxiliary mappings or limitations, have been scientifically reconnoitred in both metric and Banach spaces. This oversimplification enables the forming of more malleable operator behaviours, essentially relevant to iterative procedures and computational structures. [1] Augmentation hybrid FP moments to semigroups of transformations has expanded novel intervals for evacuated the behaviour of multipart mappings and their enduring dynamics. The iterative

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procedures arising in such semigroups every so often exhibit asymptotic regularity and strong convergence belongings under apposite contractive conditions. The literature sustains that semigroups composed of contractive transformations admit hybrid FPs, offering a collective basis that links FP theory with semigroup dynamics, metric space theory, and operator analysis. For more results on fixed point theory we referred [3,6,7,8,9]

2. Preliminaries

This segment initiate by proposing the necessary definitions and significant results mandatory for main result.

Definition 2.1 *[1] Let $S_T = A|A : T \to T$ denote the set of all such transformation.S_T is said to be semigroup transformation under binary operation $*$ satisfies following conditions For$r, s \in S_T$ then $r * s \in S_T$(Closure property) For $r, s, t \in S_T$ then $(r * s) * t = r * (s * t)$ (Associative property) then S_T is called as semigroup of transformation on S .*

Definition 2.2 *[1] Consider two mappings $\sigma : TXT \to T$ and $\lambda : T \to T$ and $T \neq \phi$ A point $h \in T$ is said to be a hybrid FP satisfies the following condition $\sigma(\lambda(h)) = h$.*

Definition 2.3 *Let $T : H \to H$ be a function. If there exists a constant $\mu \in [0, 1)$ then T is said to be contractive mapping satisfies the following condition $d(Tg, Th) \leq \mu d(g, h) \forall g, h \in H$.*

Lemma 2.1 *Let $\Psi : C \to C$ complete metric space(CMS) with d then Ψ has a unique FP.*
Proof:
Let $\sigma_0$be a primary point.
Consider sequence σ_p by $\sigma_(p + 1) = \sigma_p$.
By contractive condition
$d(\sigma_(p + 1), \sigma_p) = d(\psi\sigma_p, \psi\sigma(p - 1)) \leq \mu d(\sigma_p, \sigma(p - 1))$ *Since*$\mu \in [0, 1)$
σ_n*Cauchy sequence and convergence to point a.*
Taking limit on both sides we get $\psi\sigma = \sigma$
Hence proved uniqueness of the FP.

3. Main Theorems

We present and show main results on hybrid FPs in semigroups of transformations

Theorem 3.1 *Let G_s be a semigroup transformation on Y with metric space d such that $A \in G_s$ for satisfies a contractive condition $d(AJ, AX) \leq \sigma(d(J, AJ) + d(X, AX))$ for every $J, X \in Y$ and $\sigma \in [0, \frac{1}{2})$ then A has a hybrid FP (HFP).*
Proof:Take λ_0 is an arbitrary initial point in Y
$\{\lambda_0\}$ is a sequence defined by $\lambda_{(0+1)} = A\lambda_0$ for all positive integers .
Take $d(\lambda_(\varphi + 1), \lambda_(\varphi + 2)) = d(A\lambda_(\varphi), A\lambda_(\varphi + 1))$
by the contraction mapping
$d(\lambda_(\varphi + 1), \lambda_(\varphi + 2)) = d(A\lambda_(\varphi), A\lambda_(\varphi + 1)) \leq \sigma(d(\lambda_(\varphi), A\lambda_(\varphi))) + d(A\lambda_(\varphi), A\lambda_(\varphi + 1))$
$d(\lambda_(\varphi + 1), \lambda_(\varphi + 2)) = d(A\lambda_(\varphi), A\lambda_(\varphi + 1)) \leq \sigma(d(\lambda_(\varphi), \lambda_(\varphi + 1))) + d(\lambda_(\varphi + 1), \lambda_(\varphi + 2))$
$d(\lambda_(\varphi + 1), \lambda_(\varphi + 2)) \leq \sigma(d(\lambda_(\varphi), \lambda_(\varphi + 1))) + \sigma d(\lambda_(\varphi + 1), \lambda_(\varphi + 2))$
$d(\lambda_(\varphi + 1), \lambda_(\varphi + 2)) - \sigma d(\lambda_(\varphi + 1), \lambda_(\varphi + 2)) \leq \sigma(d(\lambda_(\varphi), \lambda_(\varphi + 1)))$
$(1 - \sigma)(d(\lambda_(\varphi + 1), \lambda_(\varphi + 2))) \leq \sigma(d(\lambda_(\varphi), \lambda_(\varphi + 1)))$
$(d(\lambda_(\varphi + 1), \lambda_(\varphi + 2))) \leq \frac{\sigma}{(1-\sigma)}(d(\lambda_(\varphi), \lambda_(\varphi + 1)))$
put $\frac{\sigma}{(1-\sigma)} = \gimel$ *and* $\gimel < 1$ *since* $\sigma \in [0, \frac{1}{2})$
$(d(\lambda_(\varphi + 1), \lambda_(\varphi + 2))) \leq \gimel(d(\lambda_(\varphi), \lambda_(\varphi + 1)))$
by the mathemetical induction
$(d(\lambda_(\varphi + 1), \lambda_(\varphi + 2))) \leq \gimel^m(d(\lambda_(1), \lambda_(0))$
applying the triagular inequality, for $\psi > 0)$
$(d(\lambda_(\psi), \lambda_(0)) \leq (d(\lambda_(\psi), \lambda_(\psi - 1)) + (d(\lambda_(\psi - 1), \lambda_(\psi - 2)) + ... + (d(\lambda_(\varphi + 1), \lambda_((\varphi)))$
$(d(\lambda_(\psi), \lambda_(0)) \leq \Sigma_{f=v}^{u-1} d(\lambda_(f + 1), \lambda_(f))$

$(d(\lambda_(\psi), \lambda_(0)) \leq \Sigma_{f=v}^{u-1} \gimel^m d(\lambda_(1), \lambda_(0))$
$(d(\lambda_(\psi), \lambda_(0)) \leq \frac{\gimel^m}{1-\gimel} d(\lambda_(1), \lambda_(0))$
taking $m \to \infty, \gimel^v \to 0$
$(d(\lambda_(\psi), \lambda_(0)) \to 0,$
Therefore , the sequence $\{lambda_(\varphi)\}$ *is a Cauchy's Sequence.*
Hence $\{\lambda_(0))\}$ *converges to a point* p^*.
$A(p^*) = A(lim_(m \to \infty)\lambda_(0)) = lim_(m \to \infty A\lambda_(0))$
$A(p^*) = lim_(m \to \infty\lambda_(0)$
$A(p^*) = p^*.$
hence conclude that A hybrid FP with to the identify function $L(h) = h$.

Theorem 3.2 *Let* P *be a semigroup transformation on* X*such that* $B \in P$ *satisfies a contractive condition* $d(\eta v, \eta\mu) \leq \sigma(d(v, \eta v) + d(\mu, \eta\mu)) + d(v, \mu))$ *for every* $\eta, \mu \in X$ *and* $\sigma \in [0, \frac{1}{2})$ *then* η *has a HFP.*

Proof :For an arbitrary initial point λ_0 in X and
Express λ_n be a sequence by $\lambda_(n + 1) = B(\lambda_n)$ for all positive integers
Take $d(\lambda_{n+1}, \lambda_{n+2}) = d(B\lambda_n, B\lambda_{n+1})$
By the contraction mapping
$d(\lambda_{n+1}, \lambda_{n+2}) = d(B\lambda_n, B\lambda_{n+1}) \leq \sigma(d(\lambda_n, B\lambda_n)) + d(\lambda_{n+1}, B\lambda_{n+1}) + d(\lambda_n, \lambda_{n+1})$ $d(\lambda_{n+1}, \lambda_{n+2}) = d(B\lambda_n, B\lambda_{n+1}) \leq \sigma(d(\lambda_n, \lambda_{n+1})) + \sigma d(\lambda_{n+1}, \lambda_{n+2}) + \sigma d(\lambda_n, \lambda_{n+1})$
$d(\lambda_{n+1}, \lambda_{n+2}) \leq \sigma(d(\lambda_n, \lambda_{n+1})) + \sigma d(\lambda_{n+1}, \lambda_{n+2}) + \sigma d(\lambda_n, \lambda_{n+1})$ $d(\lambda_{n+1}, \lambda_{n+2}) - \sigma d(\lambda_{n+1}, \lambda_{n+2}) \leq 2\sigma(d(\lambda_n, \lambda_{n+1}))$
$(1 - \sigma)d(\lambda_{n+1}, \lambda_{n+2}) \leq 2\sigma(d(\lambda_n, \lambda_{n+1}))$
$d(\lambda_{n+1}, \lambda_{n+2}) \leq \frac{2\sigma}{(1-\sigma)}(d(\lambda_n, \lambda_{n+1}))$ put $\frac{2\sigma}{(1-\sigma)} = K$ and $K < 1$ since $\sigma \in [0\frac{1}{2})$
$d(\lambda_{n+1}, \lambda_{n+2}) \leq K(d(\lambda_n, \lambda_{n+1}))$
By Mathematical Induction , we get
$d(\lambda_{n+1}, \lambda_{n+2}) \leq K^l d((\lambda_n, \lambda_{n+1}))$
applying triangular inequality, for $\alpha > \beta$, we get
$d(\lambda_\alpha, \lambda_\beta) \leq d(\lambda_\alpha, \lambda_{\alpha-1}) + d(\lambda_{\alpha-1}, \lambda_{\alpha-2}) + ... + d(\lambda_{\beta+1}, \lambda_\beta)$
$d(\lambda_\alpha, \lambda_\beta) \leq \Sigma_{m=l}^{k-1} d(\lambda_{n+1}, \lambda_n))$
$d(\lambda_\alpha, \lambda_\beta) \leq \Sigma_{m=l}^{k-1} K^m d(\lambda_1, \lambda_0))$
$d(\lambda_\alpha, \lambda_\beta) \leq (\frac{K^m}{1-K})d(\lambda_1, \lambda_0)$
taking $m \to \infty$, $K^m \to 0$
$d(\lambda_\alpha, \lambda_\beta) \to 0$
Therefore $\{\lambda_l\}$ is a Cauchy sequence.
by the given statement Y is CMS.
Hence $\{\lambda_l\}$ convergent to a point $q*$
$B(q^*) = B(lim_{m\to\infty}\lambda_(l)) = lim \to \infty B\lambda_(l))$
$B(\lambda q^*) = q^*$
which conclude that B hybrid F P with to the infinity function.

Theorem 3.3 *If* U_s *is a Banach space and* Δ *transformation maps from* m *to* m*. If* Δ *is weak contractive condition* $||U_s g - |U_s h|| \leq ||g - U_s g|| + ||h - U_s h|| - \psi(||g - U_s g|| + ||h - U_s h)||$ *for all* $g, h \in \Delta$ *Here* ψ *is an increasing and continuous from* $[0, \infty) \to [0, \infty)$ *and* $\psi(t) = t$ *if* $t = 0$ *then* B *has a HFP.*

Proof: Let us take Δ_0 be an initial point and also form sequence $U_s\Delta_a = \Delta_{a+1}$ for $a = 0, 1, 2, 3, ...$
applying the contractive inequality
$||\Delta_{a+1} - \Delta_a|| = ||U_s\Delta_a - U_s\Delta_{a-1}||$
$||U_s\Delta_a - U\Delta_{a-1} \leq ||\Delta_a - U_s\Delta_a|| + ||\Delta_{a-1} - U_s\Delta_{a-1}|| + \psi(||\Delta_a - U_s\Delta_a)|| + ||\Delta_{a-1} - U_s\Delta_{a-1}||$
$||\Delta_{a+1} - \Delta_a|| \leq ||\Delta_a - \Delta_{a+1}|| + ||\Delta_{a-1} - \Delta_a|| - \psi(||\Delta_a - \Delta_{a+1} + ||\Delta_{a-1} - \Delta_a||)$
put $||\Delta_a - \Delta_{a+1}|| = \nabla_a$ and $||\Delta_{a-1} - \Delta_a|| = \nabla_{a-1}$

So that above inequality become
$\nabla_a \leq \nabla_a + \nabla_{a-1} - \psi(\nabla_a + \nabla_{a-1})$
Subtracting from G_k from above equation $0\ le\nabla_{a-1} - \psi(\nabla_a + \nabla_{a-1})$
$\psi(\nabla_a + \nabla_{a-1}) \leq \nabla_{a-1}$
By the definition ψ is non-decreasing and $\psi(t) > 0$ ans if $\nabla_a + \nabla_{a-1} \to 0$
From above inequality both $\nabla_a \to 0$
and $\nabla_{a-1} \to 0$
Taking limit as $a \to \infty$ Finally $\lim_{a\to\infty} \nabla_a = 0$
Now, we have to prove that ∇_a is a Cauchy sequence .
$||\nabla_a - \nabla_b|| = \Sigma_{n=a}^{b-1} ||\nabla_{a+1} + \nabla_a|| = \Sigma_{n=a}^{b-1} \nabla_a$
We know that $\nabla_a \to 0$ That implies $||\nabla_a - \nabla_b|| \to 0$
Finally, ∇_a is a Cauchy sequence and ∇ is a CMS hence $\{\nabla_a\}$convergence to E^*.
$U_s(\lim_{a\to\infty} \nabla_a) = U_s\nabla^*$
$(\lim_{a\to\infty} = U_s\nabla_a) = U_s\nabla^*$
$\lim_{a\to\infty} \nabla_{a+1} = U_s\nabla^*$
$\nabla^* = U_s\nabla^*$
We conclude that, by the definition of E^* hybrid FP with respect to identity function $E = h(E)$.

4. Examples

Let ${}^0C = [0,1]$with the standard metric space and use transformation$U_s({}^0C) = \frac{{}^0C}{2}$ Based on above theorem
$||U_s({}^0C) - U_s\delta \leq ||({}^0C) - U_s({}^0C)|| + ||\delta - U_s(\delta) - \psi(||({}^0C) - U_s({}^0C)|| + ||\delta - U_s(\delta)||)$

$$||U_s({}^0C) - U_s\delta|| \leq ||({}^0C) - \frac{{}^0C}{2}|| + ||\delta - \frac{\delta}{2} - \psi(||{}^0C - \frac{{}^0C}{2}|| + ||\delta - \frac{\delta}{2}||)$$

$$||U_s({}^0C) - U_s\delta|| \leq ||\frac{{}^0C}{2}|| + ||\frac{\delta}{2} - \psi(||{}^0C - \frac{{}^0C}{2}|| + ||\frac{\delta}{2}||)$$

$$||\frac{{}^0C}{2} - \frac{\delta}{2}|| \leq ||\frac{{}^0C}{2}|| + ||\frac{\delta}{2} - \psi(||{}^0C - \frac{{}^0C}{2}|| + ||\frac{\delta}{2}||)$$

$||\frac{{}^0C}{2} - \frac{\delta}{2}|| \leq ||\frac{{}^0C}{2} - \frac{\delta}{2}||$, therefore the inequality holds.

5. Conclusion

This study gives to FPT by establishing the existence and uniqueness of FPs for transformations on Banach spaces that gratify a weak contractive condition. By using a generalized inequality concerning a strictly positive function verified that the iterative sequence made by the transformation converges intensely to a unique FP. The result extends classical Banach-type contractions by including a more flexible condition that admits a broader class of non-linear operators.

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