



Certain Expansion Formulae for Incomplete \mathbb{W} -Function and Other Related Special Functions

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ABSTRACT: This research work establishes expansion formulae for the generalised incomplete \mathbb{W} -functions and several of their important reductions, including the incomplete \mathbb{Y} -functions and incomplete I -functions. Using the generalised Taylor expansion together with fractional derivatives, series representations that express each function in terms of higher-order \mathbb{W} -type functions with shifted parameter sets are derived. The identities presented here extend these results and provide a unified expansion framework for the broader \mathbb{W} -family. The results obtained through this study can find their applications in fractional calculus, integral equations, and other areas involving Mellin–Barnes kernels.

Keywords: Incomplete \mathbb{W} -Function, Taylor’s series, Mellin–Barnes contour integral, fractional derivatives.

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1. Introduction

Special functions defined through Mellin–Barnes type contour integrals play an essential role in many branches of mathematical analysis, particularly in integral transforms, fractional calculus, and the theory of differential and integral equations [1,7,4,10]. The systematic development of such functions has led to a wide range of generalised classes, among which the Meijer \mathbb{G} -function, the Fox \mathbb{H} -function, and their various extensions occupy a central position [12,7]. These functions unify several classical special functions under a single analytic framework and provide a powerful tool for deriving functional identities and transformation formulas.

In recent years, increasing attention has been given to the study of incomplete forms of special functions. These incomplete versions arise naturally when classical Mellin–Barnes representations are modified by replacing certain complete gamma factors with incomplete gamma functions [11]. Such extensions lead to new analytic structures and allow the derivation of refined identities that do not follow directly from their complete counterparts. Inspired by this line of investigation, several authors have introduced incomplete versions of the hypergeometric, Fox–Wright, and Fox \mathbb{H} -functions, together with their associated properties and applications [3,8].

The incomplete gamma functions are defined [11], respectively, by

$$\gamma(a, x) = \int_0^x t^{a-1} e^{-t} dt, \quad \Gamma(a, x) = \int_x^\infty t^{a-1} e^{-t} dt,$$

where $a \in \mathbb{C}$ with $\Re(a) > 0$ and $x \geq 0$. These functions satisfy the familiar decomposition formula

$$\gamma(a, x) + \Gamma(a, x) = \Gamma(a),$$

which forms the basis for defining incomplete versions of many Mellin–Barnes type special functions.

Within this line of development, the generalised \mathbb{W} -function was introduced as a broad extension of the known Mellin–Barnes families [14]. This function incorporates multiple groups of parameters in

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the gamma factors and contains, as special cases, the Meijer \mathbb{G} -function, the Fox \mathbb{H} -function, the Yang \mathbb{Y} -function, and the I -function [5,6,13,14]. The \mathbb{W} -function is defined by

$$\mathbb{W}_{r,s}^{p,q}[z; y; x] = \mathbb{W}_{r,s}^{p,q} \left[\begin{matrix} (A_m, k_m, K_m)_r \\ (B_n, l_n, L_n)_s \end{matrix} \middle| z; y; x \right] = \frac{1}{2\pi i} \int_{\omega} \Theta_{r,s}^{p,q}(\sigma) z^{-\sigma} e^{-y\sigma} \sigma^x d\sigma, \quad (1)$$

where

$$\Theta_{r,s}^{p,q}(\sigma) = \frac{[\prod_{n=1}^p \Gamma(l_n - L_n \sigma)]^{B_n} [\prod_{m=1}^q \Gamma(1 - k_m + K_m \sigma)]^{A_m}}{[\prod_{n=p+1}^r \Gamma(1 - l_n + L_n \sigma)]^{B_n} [\prod_{m=q+1}^s \Gamma(k_m - K_m \sigma)]^{A_m}}. \quad (2)$$

Here x, y , and z are complex variables and ω denotes a suitable contour in the complex σ -plane. The parameters p, q, r , and s are nonnegative integers satisfying $0 \leq p \leq r$ and $0 \leq q \leq s$. The scale parameters $K_m > 0$ and $L_n > 0$, while the remaining parameters $k_m, l_n \in \mathbb{C}$ and $A_m, B_n \in \mathbb{N}$ for $m \in \{1, 2, \dots, r\}$ and $n \in \{1, 2, \dots, s\}$. These parameters are assumed to satisfy the following noncoincidence conditions:

$$K_m(l_n + t) \neq L_n(k_m - t' - 1),$$

where $t, t' \in \mathbb{N}_0 = \{0, 1, 2, \dots\}$ and $m \in \{1, 2, \dots, r\}$, $n \in \{1, 2, \dots, s\}$.

By replacing a suitable gamma factor in the kernel $\Theta_{r,s}^{p,q}(\sigma)$ defined in (2) with the incomplete gamma functions $\gamma(\cdot, \cdot)$ or $\Gamma(\cdot, \cdot)$, the corresponding incomplete \mathbb{W} -functions are defined, respectively, by

$$\gamma \mathbb{W}_{r,s}^{p,q}[z; y; x] = \frac{1}{2\pi i} \int_{\omega} \gamma \Theta_{r,s}^{p,q}(\sigma; \tau) z^{-\sigma} e^{-y\sigma} \sigma^x d\sigma,$$

and

$$\Gamma \mathbb{W}_{r,s}^{p,q}[z; y; x] = \frac{1}{2\pi i} \int_{\omega} \Gamma \Theta_{r,s}^{p,q}(\sigma; \tau) z^{-\sigma} e^{-y\sigma} \sigma^x d\sigma,$$

where $\gamma \Theta_{r,s}^{p,q}(\sigma; \tau)$ and $\Gamma \Theta_{r,s}^{p,q}(\sigma; \tau)$ denote the kernels obtained from $\Theta_{r,s}^{p,q}(\sigma)$ by replacing the selected gamma factor with $\gamma(\cdot, \cdot)$ and $\Gamma(\cdot, \cdot)$, respectively.

A fundamental tool in the derivation of expansion formulas for incomplete special functions is the generalised Taylor series of fractional order introduced by Osler [9]. This expansion expresses a function in terms of fractional derivatives of arbitrary order evaluated at a fixed point. When this technique is applied to Mellin–Barnes type functions, each fractional derivative produces a function of the same family with systematically modified parameter sets. This approach has been successfully used in earlier studies for incomplete \mathbb{H} -functions and related classes [3].

The present investigation is motivated by the recent expansion results obtained for the incomplete \mathbb{H} - and \mathbb{H} -functions by Jangid et al. in 2021 [3]. The objective of the present paper is to investigate, by means of the generalised Taylor series, a unified class of expansion formulas for the incomplete \mathbb{W} -functions. The resulting identities represent the original functions as infinite series involving higher-order \mathbb{W} -functions with shifted parameters. As direct consequences, corresponding expansions are obtained for the incomplete \mathbb{Y} - and incomplete I -functions [2] through appropriate reductions.

2. Main Results

The following theorem establishes an expansion formula for the incomplete special function $\Gamma \mathbb{W}_{r,s}^{p,q}[z; y; x]$.

Theorem 2.1 *Let $\xi > 0$, $p - 1 \leq \Re(\rho n + \eta) \leq p$, $\eta \in \mathbb{C}$, $0 < \rho \leq 1$, where c is the arbitrary constant and n is the integer over the summation, then*

$$\begin{aligned} & \Gamma \mathbb{W}_{r,s}^{p,q} \left[cz^\xi; y; x \middle| \begin{matrix} (A_1, k_1, K_1, \tau), (A_m, k_m, K_m)_2^r \\ (B_n, l_n, L_n)_1^s \end{matrix} \right] \\ &= \sum_{n=-\infty}^{\infty} \frac{\rho(z-w)^{\rho n + \eta}}{\Gamma(\rho n + \eta + 1)} z^{-\rho n - \eta} \Gamma \mathbb{W}_{r+1, s+1}^{p, q+1} \left[cz^\xi; y; x \middle| \begin{matrix} (A_1, k_1, K_1, \tau), (1, 0, -\xi), (A_m, k_m, K_m)_2^r \\ (B_n, l_n, L_n)_1^s, (1, \rho n + \eta, -\xi) \end{matrix} \right]. \end{aligned}$$

Proof: Using the generalised Taylor series formula given by **Osler** [9], we get,

$$\begin{aligned}
 & \Gamma_{\mathbb{W}_{r,s}^{p,q}} \left[cz^\xi; y; x \left| \begin{array}{c} (A_1, k_1, K_1, \tau), (A_m, k_m, K_m)_2^r \\ (B_n, l_n, L_n)_1^s \end{array} \right. \right] \\
 &= \sum_{n=-\infty}^{\infty} \frac{\rho(z-w)^{\rho n + \eta}}{\Gamma(\rho n + \eta + 1)} \left\{ D_z^{\rho n + \eta} \left(\Gamma_{\mathbb{W}_{r,s}^{p,q}} \left[cz^\xi; y; x \left| \begin{array}{c} (A_1, k_1, K_1, \tau), (A_m, k_m, K_m)_2^r \\ (B_n, l_n, L_n)_1^s \end{array} \right. \right] \right) \right\} \\
 &= \sum_{n=-\infty}^{\infty} \frac{\rho(z-w)^{\rho n + \eta}}{\Gamma(\rho n + \eta + 1)} \left\{ \frac{1}{2\pi i} \int_{\omega} \Gamma_{\Theta_{r,s}^{p,q}}(\sigma, \tau) c^{-\sigma} D_z^{\rho n + \eta} (z^{-\xi \sigma}) e^{\sigma y} \sigma^x d\sigma \right\} \\
 &= \sum_{n=-\infty}^{\infty} \frac{\rho(z-w)^{\rho n + \eta}}{\Gamma(\rho n + \eta + 1)} \left\{ \frac{1}{2\pi i} \int_{\omega} \Gamma_{\Theta_{r,s}^{p,q}}(\sigma, \tau) c^{-\sigma} \frac{\Gamma(1 - \xi \sigma)}{\Gamma(1 - \rho n - \eta - \xi \sigma)} (z^{-\rho n - \eta - \xi \sigma}) e^{\sigma y} \sigma^x d\sigma \right\} \\
 &= \sum_{n=-\infty}^{\infty} \frac{\rho(z-w)^{\rho n + \eta}}{\Gamma(\rho n + \eta + 1)} z^{-\rho n - \eta} \Gamma_{\mathbb{W}_{r+1,s+1}^{p,q+1}} \left[cz^\xi; y; x \left| \begin{array}{c} (A_1, k_1, K_1, \tau), (1, 0, -\xi), (A_m, k_m, K_m)_2^r \\ (B_n, l_n, L_n)_1^s, (1, \rho n + \eta, -\xi) \end{array} \right. \right].
 \end{aligned}$$

□

It is well established in prior studies [5,8] that incomplete \mathbb{Y} -functions and incomplete I -functions are special cases of incomplete \mathbb{W} -functions. The reductions are given as follows:

$$\Gamma_{\mathbb{W}_{r,s}^{p,q}} \left[z; y; x \left| \begin{array}{c} (1, k_1, K_1, \tau), (1, k_m, K_m)_2^r \\ (1, l_n, L_n)_1^s \end{array} \right. \right] = \Gamma_{\mathbb{Y}_{r,s}^{p,q}} \left[z; y; x \left| \begin{array}{c} (k_1, K_1, \tau), (k_m, K_m)_2^r \\ (l_n, L_n)_1^s \end{array} \right. \right],$$

and

$$\Gamma_{\mathbb{W}_{r,s}^{p,q}} \left[\frac{1}{z}; 0; 0 \left| \begin{array}{c} (A_1, k_1, K_1, \tau), (A_m, k_m, K_m)_2^r \\ (B_n, l_n, L_n)_1^s \end{array} \right. \right] = \Gamma_{I_{r,s}^{p,q}} \left[z \left| \begin{array}{c} (A_1, k_1, K_1, \tau), (A_m, k_m, K_m)_2^r \\ (B_n, l_n, L_n)_1^s \end{array} \right. \right].$$

Hence, the following corollaries can be derived as particular cases of the result established above:

Corollary 2.1 Let $\xi > 0$, $p - 1 \leq \Re(\rho n + \eta) \leq p$, $\eta \in \mathbb{C}$, $0 < \rho \leq 1$, where c is the arbitrary constant and n is an integer, then

$$\begin{aligned}
 & \Gamma_{\mathbb{Y}_{r,s}^{p,q}} \left[cz^\xi; y; x \left| \begin{array}{c} (k_1, K_1, \tau), (k_m, K_m)_2^r \\ (l_n, L_n)_1^s \end{array} \right. \right] \\
 &= \sum_{n=-\infty}^{\infty} \frac{\rho(z-w)^{\rho n + \eta}}{\Gamma(\rho n + \eta + 1)} z^{-\rho n - \eta} \Gamma_{\mathbb{Y}_{r+1,s+1}^{p,q+1}} \left[cz^\xi; y; x \left| \begin{array}{c} (k_1, K_1, \tau), (0, -\xi), (k_m, K_m)_2^r \\ (l_n, L_n)_1^s, (\rho n + \eta, -\xi) \end{array} \right. \right].
 \end{aligned}$$

Corollary 2.2 Let $\xi > 0$, $p - 1 \leq \Re(\rho n + \eta) \leq p$, $\eta \in \mathbb{C}$, $0 < \rho \leq 1$, where c is the arbitrary constant and n is an integer, then

$$\begin{aligned}
 & \Gamma_{I_{r,s}^{p,q}} \left[cz^\xi \left| \begin{array}{c} (A_1, k_1, K_1, \tau), (A_m, k_m, K_m)_2^r \\ (B_n, l_n, L_n)_1^s \end{array} \right. \right] \\
 &= \sum_{n=-\infty}^{\infty} \frac{\rho(z-w)^{\rho n + \eta}}{\Gamma(\rho n + \eta + 1)} z^{-\rho n - \eta} \Gamma_{I_{r+1,s+1}^{p,q+1}} \left[cz^\xi \left| \begin{array}{c} (A_1, k_1, K_1, \tau), (1, 0, \xi), (A_m, k_m, K_m)_2^r \\ (B_n, l_n, L_n)_1^s, (1, \rho n + \eta, \xi) \end{array} \right. \right].
 \end{aligned}$$

Following the lines of proof, as done in Theorem 2.1, we have the following result for the incomplete special function ${}^\gamma \mathbb{W}_{r,s}^{p,q} [z; y; x]$

Theorem 2.2 *Let $\xi > 0$, $p - 1 \leq \Re(\rho n + \eta) \leq p$, $\eta \in \mathbb{C}$, $0 < \rho \leq 1$, where c is the arbitrary constant and n is an integer, then*

$$\begin{aligned} & \gamma \mathbb{W}_{r,s}^{p,q} \left[cz^\xi; y; x \left| \begin{array}{c} (A_1, k_1, K_1, \tau), (A_m, k_m, K_m)_2^r \\ (B_n, l_n, L_n)_1^s \end{array} \right. \right] \\ &= \sum_{n=-\infty}^{\infty} \frac{\rho(z-w)^{\rho n + \eta}}{\Gamma(\rho n + \eta + 1)} z^{-\rho n - \eta} \gamma \mathbb{W}_{r+1, s+1}^{p, q+1} \left[cz^\xi; y; x \left| \begin{array}{c} (A_1, k_1, K_1, \tau), (1, 0, -\xi), (A_m, k_m, K_m)_2^r \\ (B_n, l_n, L_n)_1^s, (1, \rho n + \eta, -\xi) \end{array} \right. \right]. \end{aligned}$$

Again, it is known from [5,8] that

$$\gamma \mathbb{W}_{r,s}^{p,q} \left[z; y; x \left| \begin{array}{c} (1, k_1, K_1, \tau), (1, k_m, K_m)_2^r \\ (1, l_n, L_n)_1^s \end{array} \right. \right] = \gamma \mathbb{Y}_{r,s}^{p,q} \left[z; y; x \left| \begin{array}{c} (k_1, K_1, \tau), (k_m, K_m)_2^r \\ (l_n, L_n)_1^s \end{array} \right. \right],$$

and

$$\gamma \mathbb{W}_{r,s}^{p,q} \left[\frac{1}{z}; 0; 0 \left| \begin{array}{c} (A_1, k_1, K_1, \tau), (A_m, k_m, K_m)_2^r \\ (B_n, l_n, L_n)_1^s \end{array} \right. \right] = \gamma I_{r,s}^{p,q} \left[z \left| \begin{array}{c} (A_1, k_1, K_1, \tau), (A_m, k_m, K_m)_2^r \\ (B_n, l_n, L_n)_1^s \end{array} \right. \right].$$

Hence, the following corollaries can be seen as particular cases of the result established above:

Corollary 2.3 *Let $\xi > 0$, $p - 1 \leq \Re(\rho n + \eta) \leq p$, $\eta \in \mathbb{C}$, $0 < \rho \leq 1$, where c is the arbitrary constant and n is an integer, then*

$$\begin{aligned} & \gamma \mathbb{Y}_{r,s}^{p,q} \left[cz^\xi; y; x \left| \begin{array}{c} (k_1, K_1, \tau), (k_m, K_m)_2^r \\ (l_n, L_n)_1^s \end{array} \right. \right] \\ &= \sum_{n=-\infty}^{\infty} \frac{\rho(z-w)^{\rho n + \eta}}{\Gamma(\rho n + \eta + 1)} z^{-\rho n - \eta} \gamma \mathbb{Y}_{r+1, s+1}^{p, q+1} \left[cz^\xi; y; x \left| \begin{array}{c} (k_1, K_1, \tau), (0, -\xi), (k_m, K_m)_2^r \\ (l_n, L_n)_1^s, (\rho n + \eta, -\xi) \end{array} \right. \right]. \end{aligned}$$

Corollary 2.4 *Let $\xi > 0$, $p - 1 \leq \Re(\rho n + \eta) \leq p$, $\eta \in \mathbb{C}$, $0 < \rho \leq 1$, where c is the arbitrary constant and n is an integer, then*

$$\begin{aligned} & \gamma I_{r,s}^{p,q} \left[cz^\xi \left| \begin{array}{c} (A_1, k_1, K_1, \tau), (A_m, k_m, K_m)_2^r \\ (B_n, l_n, L_n)_1^s \end{array} \right. \right] \\ &= \sum_{n=-\infty}^{\infty} \frac{\rho(z-w)^{\rho n + \eta}}{\Gamma(\rho n + \eta + 1)} z^{-\rho n - \eta} \gamma I_{r+1, s+1}^{p, q+1} \left[cz^\xi \left| \begin{array}{c} (A_1, k_1, K_1, \tau), (1, 0, \xi), (A_m, k_m, K_m)_2^r \\ (B_n, l_n, L_n)_1^s, (1, \rho n + \eta, \xi) \end{array} \right. \right]. \end{aligned}$$

3. Conclusion

This work establishes a unified expansion scheme for the generalised incomplete \mathbb{W} -functions by employing Taylor series together with fractional derivatives. The main results of this paper provide expansion formulae for the incomplete \mathbb{W} -functions and their reductions. The same mechanism yields corresponding expansions for the incomplete \mathbb{Y} - and I -function reductions through standard parameter specialisations. It is well known that the \mathbb{W} -function is a generalisation of both \mathbb{H} and I -functions and since $\overline{\mathbb{H}}$ and \overline{I} [3,8] are special cases of \mathbb{H} and I -functions respectively, the results discussed in this paper would hold for them as well.

Furthermore, we may obtain the following function by putting $B_n = 1$ in (1) and (2), for $n \in \{1, 2, \dots, p\}$:

$$\overline{\mathbb{W}}_{r,s}^{p,q} [z; y; x] = \mathbb{W}_{r,s}^{p,q} \left[z; y; x \left| \begin{array}{l} (A_1, k_1, K_1, \tau), (A_m, k_m, K_m)_2^r \\ (1, l_n, L_n)_1^p, (B_n, l_n, L_n)_{p+1}^s \end{array} \right. \right].$$

The $\overline{\mathbb{W}}$ -function and its incomplete counterparts inherit the same Mellin–Barnes structure as the \mathbb{W} -functions. Consequently, all expansion results proved in this paper extend directly to ${}^\gamma\overline{\mathbb{W}}$, and ${}^\Gamma\overline{\mathbb{W}}$ without modification.

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