Bol. Soc. Paran. Mat.
©SPM – E-ISSN-2175-1188
SPM: www.spm.uem.br/bspm

(3s.) **v. 2026 (44) 5** : 1–8.
ISSN-0037-8712
doi:10.5269/bspm.80523

On Ideal Lacunary Statistical Convergence of Order α in Seminormed Space

Ekrem SAVAŞ

ABSTRACT: In this paper, the concept of ideal lacunary statistical convergence of order α, where $0 < \alpha < 1$ with respect to seminorm q has been introduced by following very important results of [4,5] and several results associated with this set has also been proven.

Keywords: Ideal, ideal statistical convergence, lacunary statistical convergence of order α, seminorm.

Contents

1. Introduction

Fast [5] and Schoenberg [29] introduced the concept of statistical convergence and related it with summability methods. it was further investigated and studies by Friday [7], Salat [13] and many others in sequence spaces, where the idea was to generalize the usual notion of convergence

Definition 1.1 *We say that a sequence $\xi = (\xi_k)$ statistically converges to γ, when, for every positive number $\epsilon > 0$,*

$$\lim_m \tfrac{1}{m} |\{k \le m : |\xi_k - \gamma| \ge \epsilon\}| = 0.$$

On a related note, $\mathcal{I}$-convergence, which serves as a broader framework for statistical convergence, was initially proposed by Kostyrko et al. in their work [9]. This concept is rooted in the notion of an ideal $\mathcal{I}$ associated with a subset of the natural numbers. Kostyrko et al. delved into the concept of $\mathcal{I}$ -convergence. After this work, several researchers have helped to enhance the work on ideal convergence in different directions(see, [4,11,12,16,17,18,19,23,24,26,30]) and others. It should be note that the statistical convergence of order α was defined in [1,2].

A lacunary sequence is an increasing integer sequence $\theta = \{v_r\}_{r\in\mathbb{N}\cup\{0\}}$ such that $v_0 = 0$ and $\phi_r = v_r - v_{r-1} \to \infty$, as $r \to \infty$. We define the intervals $I_r = (v_{r-1}, v_r]$,(see, [6]).
In [8], lacunary statistically convergent was defined as: If for any $\varepsilon > 0$,

$$\lim_{r\to\infty} \frac{1}{\phi_r} |\{k \in I_r : |\xi_k - \gamma| \ge \epsilon\}| = 0,$$

we say that the sequence $\{\xi_k\}$ is lacunary statistically convergent to γ (or, S_θ-convergent to γ). For the detail on this convergence, the reader may consult the papers [10,14,20,21,22,25,27,28].

In what follows, the sequence $\xi = \{\xi_k\}$ will denote a sequence of real numbers.

2. Main Results

Consider a nonempty set X. A family of subsets $\mathcal{I} \subset P(X)$ is termed an ideal on X provided that it meets the following conditions:

2020 *Mathematics Subject Classification:* 35B40, 35L70.
Submitted November 30, 2025. Published April 09, 2026

Definition 2.1 *([9]). Consider a nonempty set X. A family of subsets $\mathcal{I} \subset \wp(X)$ is termed an ideal on X if it satisfies the following conditions:*
(a) $X_1, X_2 \in \mathcal{I}$ imply $X_1 \cup X_2 \in \mathcal{I}$,
(b) $X_1 \in \mathcal{I}, X_2 \subset X_1$ imply $X_2 \in \mathcal{I}$.

Further $\mathcal{I}$ is called to be admissible if for each $x \in X, \{x\} \in \mathcal{I}$ and it is said to be non-trivial if $\mathcal{I} \neq \phi$ and $X \notin \mathcal{I}$.

Definition 2.2 *([9]). A family $\mathcal{F} \subset 2^X$ of subsets of a nonempty set X is deemed a filter in X if it satisfies the following conditions:*
(a) The empty set ϕ does not belong to $\mathcal{F}$,
(b) $X_1, X_2 \in \mathcal{F}$ imply $X_1 \cap X_2 \in \mathcal{F}$,
(c) $X_1 \in \mathcal{F}, X_1 \subset X_2$ imply $X_2 \in \mathcal{F}$.

If $\mathcal{I}$ is a proper non-trivial ideal in Y, then the family of sets $\mathcal{F}(\mathcal{I}) = \{A \subset Y : \exists\, B \in \mathcal{I} : A = Y \setminus B\}$ constitutes a filter in Y. This filter is commonly referred to as the filter associated with the ideal $\mathcal{I}$.

Throughout the paper we present $\mathcal{I}$ as a proper admissible ideal.

We now present our main definitions.

Definition 2.3 *We call that a sequence $\xi = (\xi_k)$ is $\mathcal{I}_q$-statistically convergent of order α to γ, $(0 < \alpha \leq 1)$, if for each $\varepsilon > 0$ and $\tau > 0$,*

$$\{m \in \mathbb{N} : \frac{1}{m^\alpha} |\{k \leq m : q(\xi_k - \gamma) \geq \epsilon\}| \geq \tau\} \in \mathcal{I},$$

By $S_q^\alpha(\mathcal{I})$, we shall present the set consisting of all $\mathcal{I}_q$-statistically convergent sequences of order α.

Remark 2.1 Suppose $\mathcal{I} = \mathcal{I}_{fin}$, then $\mathcal{I}_q$- statistical convergence of order α reduces to statistical convergence of order α with respect to seminorm q. For $\alpha = 1$ and an arbitrary ideal $\mathcal{I}$, it reduces to $\mathcal{I}_q$- statistical convergence. For $\theta = 1$ and $\mathcal{I} = \mathcal{I}_{fin}$, it is mainly statistical convergence with respect to seminorm q.

Example 2.1 Let us consider the sequence $\{\sigma_m\}_{m \in \mathbb{N}}$ where $\sigma_m = 1$ for $m = 1$ to 10 and $\sigma_m = m - 10$ for all $m \geq 10$, and consider $\mathcal{I} = \mathcal{I}_d$ (the ideal of density zero sets of N) and let $D = \{1^2, 2^2, 3^2, 4^2, 5^2, ...\}$.

Define $\xi = \{\xi_k\}_{k \in \mathbb{N}}$ by

$$\xi_k = \begin{cases} k & \text{for } m - [\sqrt{\sigma_m^\alpha}] + 1 \leq k \leq m, m \notin D \\ k & \text{for } m - \sigma_m + 1 \leq k \leq m, m \in D \\ 0 & \text{otherwise.} \end{cases}$$

Then for every $\epsilon > 0$ $\ (0 < \epsilon < 1)$ since

$$\frac{1}{\sigma_m^\alpha} |\{k \in J_m : q(\xi_k - 0) \geq \epsilon\}| = \frac{[\sqrt{\sigma_m^\alpha}]}{\sigma_m^\alpha} \to 0$$

as $m \to \infty$ and $m \notin D$, where $J_m = [m - \sigma_m + 1, m]$, so for every $\delta > 0$,

$$\left\{ m \in \mathbb{N} : \frac{1}{\sigma_m^\alpha} |\{k \in J_m : q(\xi_k - 0) \geq \epsilon\}| \geq \tau \right\} \subset D \cup \{1, 2, ..., n_1\} \ldots\ldots (1)$$

for some $n_1 \in N$. Suppose $\tau > 0$ is considered. Take that $\lim_m \frac{m - \sigma_m}{m^\alpha} = 0$, and so we can pick $n_2 \in N$ such that $\frac{m - \sigma_m}{m^\alpha} < \frac{\tau}{2}$ for all $m \geq n_2$. Note for the above $\varepsilon > 0$,

$$\begin{aligned} \frac{1}{m^\alpha} |\{k \leq m : q(\xi_k - 0) \geq \epsilon\}| &= \frac{1}{m^\alpha} |\{k \leq m - \sigma_m : q(\xi_k - 0) \geq \epsilon\}| + \frac{1}{m^\alpha} |\{k \in J_m : q(\xi_k - 0) \geq \epsilon\}| \\ &\leq \frac{m - \sigma_m}{m^\alpha} + \frac{1}{m^\alpha} |\{k \in J_m : q(\xi_k - 0) \geq \epsilon\}| \\ &\leq \frac{\tau}{2} + \frac{1}{{\sigma_m}^\alpha} |\{k \in J_m : q(\xi_k - 0) \geq \epsilon\}| \end{aligned}$$

for all $m \geq n_2$. Further

$$\left\{m \in N : \frac{1}{m^{\alpha}} | \{k \leq m : q(\xi_k - 0) \geq \epsilon\} | \geq \delta\right\}$$
$$\subset \left\{m \in N : \frac{1}{{\sigma_m}^{\alpha}} |\{k \in J_m : q(\xi_k - 0) \geq \epsilon\}| \geq \frac{\tau}{2}\right\} \cup \{1, 2, 3, ..., m_2\} \subset D \cup \{1, 2, \ldots, n\}$$

from (1) where $n = max\{n_1, n_2\}$. It is easy to see that $\xi = \{\xi_k\}_{k\in\mathbb{N}}$ is $\mathcal{I}$- statistically convergent of order α to 0 in seminormed space but ξ is not statistically convergent of order α to 0 in seminormed space.

Definition 2.4 *Suppose (X, q) is a seminormed space with the seminorm q. Let θ be a lacunary sequence. We call that the sequence $\xi = \{\xi k\}$ is $\mathcal{I}_q$-lacunary statistically convergent of order α to γ or $S_q^{\theta}(\mathcal{I}^{\alpha})$-convergent to γ, if for each $\epsilon > 0$ and $\tau > 0$,*

$$\{r \in \mathbb{N} : \frac{1}{\phi_r^{\alpha}} |\{k \in I_r : q(\xi_k - \gamma) \geq \epsilon\}| \geq \tau\} \in \mathcal{I}.$$

By $S_q^{\theta}(\mathcal{I}^{\alpha})$, we present the set consisting of all $\mathcal{I}_q$- lacunary statistically convergent sequences of order α.

Remark 2.2 For $\alpha = 1$, the definition coincides with $\mathcal{I}_q$-lacunary statistical convergence. Further it must be noted in this context that lacunary statistical convergence of order α with respect to seminorm q has not been studied till now. Obviously lacunary statistical convergence of order α with respect to seminorm q is a special case of $\mathcal{I}_q$-lacunary statistical convergence of order α with respect to seminorm q, when we take $I = I_{fin}$. So properties of lacunary statistical convergence of order α with respect to seminorm q can be easily obtained from our results with obvious modifications.

Theorem 2.1 *Suppose (X, q) is a seminormed space with the seminorm q. Let $0 < \alpha \leq \beta \leq 1$. Then $S_q(\mathcal{I}^{\alpha}) \subset S_q(\mathcal{I}^{\beta})$ and the inclusion is strict for at least those α, β for which there is a $k \in \mathbb{N}$ such that $\alpha < \frac{1}{k} < \beta$ and when $I = I_{fin}$.*

Proof: Let $0 < \alpha \leq \beta \leq 1$. Then

$$\frac{|k \leq m : q(\xi_k - \xi) \geq \epsilon\}|}{m^{\beta}} \leq \frac{|\{k \leq m : q(\xi_k - \gamma) \geq \epsilon\}|}{m^{\alpha}}$$

and so for any $\delta > 0$,

$$\{m \in \mathbb{N} : \frac{|\{k \leq m : q(\xi_k - \gamma) \geq \epsilon\}|}{m^{\beta}} \geq \tau\} \subset \{m \in \mathbb{N} : \frac{|\{\{k \leq m : q(\xi_k - \gamma) \geq \epsilon\}|}{m^{\alpha}} \geq \tau\}.$$

It is obvious that $S_q(\mathcal{I}^{\alpha}) \subset S_q(\mathcal{I}^{\beta})$. To show that the inclusion is strict for α, β which is mentioned above, let us define the sequence $\xi = \{\xi_k\}$ as

$$x_k = 1, \text{if } k = j^n$$

$$x_k = 0, \text{if } k \neq j^n, j \in \mathbb{N}.$$

So $S_q(\mathcal{I})^{\beta} - \lim \xi_k = 0$, that is $\xi \in S_q(\mathcal{I}^{\beta})$ but $\xi \notin S_q(\mathcal{I}^{\alpha})$ where $\mathcal{I} = \mathcal{I}_{fin}$.

□

Corollary 2.1 *Being $\mathcal{I}_q$- statistically convergent of order α to γ (for $0 < \alpha \leq 1$) implies $\mathcal{I}_q$- l statistical convergence to γ that is $S_q(\mathcal{I}^{\alpha}) \subset S_q(\mathcal{I})$.*

The proof of the next theorem is analogous.

Theorem 2.2 *Let $0 < \alpha \leq \beta \leq 1$. Then*
(i) $S_{(\theta,q)}(\mathcal{I}^{\alpha}) \subset S^{\theta}_{(\theta,q)}(\mathcal{I}^{\beta})$.
(ii) Specifically $S^{\theta}_{q}(\mathcal{I}^{\alpha}) \subset S^{\theta}_{q}(\mathcal{I})$
the inclusion becomes strict if there exists a natural number $k \in \mathbb{N}$ such that $\alpha < \frac{1}{k} < \beta$ and when $\mathcal{I} = \mathcal{I}_{fin}$.

The proofs of the following theorems follow from standard techniques and are thus omitted.

Theorem 2.3 *Suppose (X, q) is a seminormed space with the seminorm q. Let $\alpha \in (0,1]$ and $\theta = (v_r)$ be a lacunary sequence. Suppose that $S^{\theta}_{q}(\mathcal{I}^{\alpha}) - \lim \xi_k = \gamma_1$, $S^{\theta}_{q}(\mathcal{I}^{\alpha}) - \lim y_k = \gamma_2$ and $c \in \mathbb{R}$, then*

i) $S^{\theta}_{q}(\mathcal{I}^{\alpha}) - \lim c\xi_k = c\gamma_1$,
ii) $S^{\theta}_{q}(\mathcal{I}^{\alpha}) - \lim (\xi_k + y_k) = \gamma_1 + \gamma_2$,

Theorem 2.4 *Suppose (X, q) is a seminormed space with the seminorm q. Let $\alpha \in (0,1]$ and $\theta = (v_r)$ be a lacunary sequence. Then, the limit of any sequence that is $S^{\theta}_{q}(\mathcal{I}^{\alpha})-$convergent is uniquely determined.*

Theorem 2.5 *Suppose (X, q) is a seminormed space with the seminorm q. Let $\alpha \in (0,1]$ and $\theta = (v_r)$ be a lacunary sequence. Let $\xi = (\xi_k)$, $y = (y_k)$ and $z = (z_k)$ be real sequences such that $\xi_k \leq y_k \leq z_k$. If $S^{\theta}_{q}(\mathcal{I}^{\alpha}) - \lim \xi_k = \gamma = S^{\theta}_{q}(\mathcal{I}^{\alpha}) - \lim z_k$ then $S^{\theta}_{q}(\mathcal{I}^{\alpha}) - \lim y_k = \gamma$.*

Definition 2.5 *Suppose (X, q) is a seminormed space with the seminorm q. Let $\theta = (v_r)$ be a lacunary sequence and let p be a positive real number. If for every $\epsilon > 0$*

$$\{r \in \mathbb{N} : \frac{1}{\phi_r^{\alpha}} \sum_{k \in I_r} q(\xi_k - \gamma)^p \geq \epsilon\} \in \mathcal{I},$$

we say that $\xi = (\xi_n)$ is $N^{\theta}_{q}(\mathcal{I}^{\alpha})_p$-convergent to γ

It is denoted by $\xi_k \to L(N^{\theta}_{q}(\mathcal{I}^{\alpha})_p)$ and the class of such sequences will be denoted by simply $N^{\theta}_{q}(\mathcal{I}^{\alpha})_p$.

Theorem 2.6 *Let α and β be fixed real numbers such that $0 < \alpha \leq \beta \leq 1$, then $N^{\theta}_{q}(\mathcal{I}^{\beta})_p \subseteq N^{\theta}_{q}(\mathcal{I}^{\alpha})_p$ and the inclusion is strict.*

The proof follows adapting the proof of the theorem 2.2.

Corollary 2.2 *Let $0 < \alpha \leq 1$ be a positive real number. Then $N^{\theta}_{q}(\mathcal{I}^{\alpha})_p \subseteq N^{\theta}_{q}(\mathcal{I})_p$ for each $\alpha \in (0,1]$.*

Theorem 2.7 *Let $\theta = (v_r)$ and $\vartheta = (s_r)$ be two lacunary sequences such that $I_r \subset J_r$ for all $r \in \mathbb{N}$ and let α and β be positive real numbers such that $0 < \alpha \leq \beta \leq 1$,*

(i) If

$$\lim_{r \to \infty} \inf \frac{\phi_r^{\alpha}}{\ell_r^{\beta}} > 0 \tag{8}$$

then $N^{\vartheta}_{q}(\mathcal{I}^{\beta})_p \subseteq N^{\theta}_{q}(\mathcal{I}^{\alpha})_p$,

(ii) If

$$\lim_{n \to \infty} \frac{\ell_r}{\phi_r^{\beta}} = 1 \tag{9}$$

and $x \in l_{\infty}$, then $N^{\theta}_{q}(\mathcal{I}^{\alpha})_p \subseteq N^{\vartheta}_{q}(\mathcal{I}^{\beta})_p$.

Proof: (i) Omitted.

(ii) Let $\xi \in N_q^{\theta}(\mathcal{I}^{\alpha})_p$ and suppose that (9) holds. Since $\xi = (\xi_k) \in \ell_{\infty}$ then there exists some $M > 0$ such that $q(\xi_k - \gamma) \leq M$ for all k. Now, since $I_r \subseteq J_r$ and $\phi_r \leq \ell_r$, for all $r \in \mathbb{N}$ we may write

$$\begin{aligned}
\frac{1}{\ell_r^{\beta}} \sum_{k \in J_r} q(\xi_k - \gamma)^p &= \frac{1}{\ell_r^{\beta}} \sum_{k \in J_r - I_r} q(\xi_k - \gamma)^p + \frac{1}{\ell_r^{\beta}} \sum_{k \in I_r} q(\xi_k - \gamma)^p \\
&\leq \left(\frac{\ell_r - \phi_r}{\ell_r^{\beta}}\right) M^p + \frac{1}{\ell_r^{\beta}} \sum_{k \in I_r} q(\xi_k - \gamma)^p \\
&\leq \left(\frac{\ell_r - \phi_r^{\beta}}{\phi_r^{\beta}}\right) M^p + \frac{1}{\phi_r^{\beta}} \sum_{k \in I_r} q(\xi_k - \gamma)^p \\
&\leq \left(\frac{\ell_r}{\phi_r^{\beta}} - 1\right) M^p + \frac{1}{\phi_r^{\alpha}} \sum_{k \in I_r} q(\xi_k - \gamma)^p
\end{aligned}$$

for all $r \in \mathbb{N}$. So we have

$$\left\{ r \in \mathbb{N} : \frac{1}{\ell_r^{\alpha}} \sum_{k \in J_r} q(\xi_k - \gamma)^p \geq \tau \right\} \subseteq \left\{ r \in \mathbb{N} : \frac{1}{\phi_r^{\beta}} \sum_{k \in I_r} q(\xi_k - \gamma)^p \geq \tau \right\} \in \mathcal{I}.$$

Hence $N_q^{\vartheta}(\mathcal{I}^{\alpha})_p \subseteq N_q^{\theta}(\mathcal{I}^{\beta})_p$. Which proves the result.

□

Corollary 2.3 *Let $\theta = (v_r)$ and $\vartheta = (s_r)$ be two lacunary sequences such that $I_r \subset J_r$ for all $r \in \mathbb{N}$. If* (8) *holds then,*
(a) $N_q^{\vartheta}(\mathcal{I}^{\alpha})_p \subseteq N_q^{\theta}(\mathcal{I}^{\alpha})_p$ *for each* $0 < \alpha \leq 1$,
(b) $N_q^{\vartheta}(\mathcal{I})_p \subseteq N_q^{\theta}(\mathcal{I}^{\alpha})_p$ *for each* $0 < \alpha \leq 1$,
(c) $N_q^{\vartheta}(\mathcal{I})_p \subseteq N_q^{\theta}(\mathcal{I})_p$.
If (8) *holds then, (a)* $l_{\infty} \cap N_q^{\theta}(\mathcal{I}^{\alpha})_p \subseteq N_q^{\vartheta}(\mathcal{I}^{\alpha})_p$ *for each* $0 < \alpha \leq 1$,
(b) $l_{\infty} \cap N_q^{\theta}(\mathcal{I}^{\alpha})_p \subseteq N_q^{\vartheta}(\mathcal{I})_p$ *for each* $0 < \alpha \leq 1$,
(c) $l_{\infty} \cap N_q^{\theta}(\mathcal{I})_p \subseteq N_q^{\vartheta}(\mathcal{I})_p$.

Theorem 2.8 *Let $\theta = (v_r)$ and $\vartheta = (s_r)$ be two lacunary sequences such that $I_r \subset J_r$ for all $r \in \mathbb{N}$ and let α and β be positive real numbers such that $0 < \alpha \leq \beta \leq 1$, and $0 < p < \infty$. Then*

(i) Let (8) *holds, if a sequence is $N_q^{\vartheta}(\mathcal{I}^{\beta})_p-$summable to γ, then it is $S_q^{\theta}(\mathcal{I}^{\alpha})-$ statistically convergent to γ,*

(ii) Let (9) *holds and $\xi = (\xi_k)$ be a bounded sequence, if a sequence is $S_q^{\theta}(\mathcal{I}^{\alpha})-$statistically convergent to γ then it is $N_q^{\vartheta}(\mathcal{I}^{\beta})_p-$ summable to ξ.*

Proof:

For any sequence $\xi = (\xi_k)$, and $\epsilon > 0$ we have

$$\begin{aligned}
\sum_{k \in I_r} q(\xi_k - \xi)^p &= \sum_{k \in I_r, q(\xi_k - \gamma) \geq \epsilon} q(\xi_k - \gamma)^p + \sum_{k \in I_r, q(\xi_k - \gamma) < \epsilon} q(\xi_k - \gamma)^p \\
\geq & \sum_{k \in I_r, q(\xi_k - \gamma) \geq \epsilon} q(\xi_k - \gamma)^p \geq |\{k \in I_r : \ q(\xi_k - \gamma) \geq \epsilon\}| \varepsilon^p
\end{aligned}$$

and so $$\frac{1}{\phi_r^{\beta}} \sum_{k \in I_r} q(\xi_k - \gamma)^p \geq \frac{1}{\phi_r^{\beta}} |\{k \in I_r : \ q(\xi_k - \gamma) \geq \epsilon\}| \varepsilon^p \geq \frac{\phi_r^{\alpha}}{\ell_r^{\beta}} \frac{1}{\phi_r^{\alpha}} |\{k \in I_r : \ q(\xi_k - \gamma) \geq \epsilon\}| \varepsilon^p.$$

Using (8) we obtain that $S_q^{\theta}(\mathcal{I}^{\alpha}) - \lim \xi_k = \gamma$. whenever $N_q^{\vartheta}(\mathcal{I}^{\beta})_p - \lim \xi_k = \gamma$, Which proves the result.

(ii) Suppose that $S_q^\theta(\mathcal{I}^\alpha) - \lim \xi_k = \gamma$ and $\xi = (\xi_k) \in \ell_\infty$. Then there exists some $M > 0$ such that $q(\xi_k - \gamma) \le M$ for all k, then for every $\varepsilon > 0$ we may write

$$\begin{aligned}
\frac{1}{\ell_r^\beta} \sum_{k \in J_r} q(\xi_k - \gamma)^p &= \frac{1}{\ell_r^\beta} \sum_{k \in J_r - I_r} q(\xi_k - \gamma)^p + \frac{1}{\ell_4^\beta} \sum_{k \in I_r} q(\xi_k - \gamma)^p \\
&\le \left(\frac{\ell_r - \phi_r}{\ell_r^\beta} \right) M^p + \frac{1}{\ell_r^\beta} \sum_{k \in I_r} q(\xi_k - \gamma)^p \\
&\le \left(\frac{\ell_r - \phi_r^\beta}{\ell_r^\beta} \right) M^p + \frac{1}{\ell_r^\beta} \sum_{k \in I_n} q(\xi_k - \gamma)^p \\
&= \left(\frac{\ell_r}{\phi_r^\beta} - 1 \right) M^p + \frac{1}{h_r^\beta} \sum_{\substack{k \in I_r \\ q(\xi_k - \gamma) \ge \varepsilon}} q(\xi_k - \gamma)^p + \frac{1}{\phi_r^\beta} \sum_{\substack{k \in I_r \\ q(\xi_k - \gamma) < \varepsilon}} q(\xi_k - \gamma)^p \\
&\le \left(\frac{\ell_r}{\phi_r^\beta} - 1 \right) M^p + \frac{M^p}{\phi_r^\alpha} |\{k \in I_n : q(\xi_k - \gamma) \ge \varepsilon\}| + \frac{\ell_r}{\phi_r^\beta} \varepsilon^p
\end{aligned}$$

for all $r \in \mathbb{N}$.

Using (9) we obtain that $N_q^\vartheta(\mathcal{I}^\beta)_p - \lim \xi_k = \gamma$, whenever $S_q^\theta(\mathcal{I}^\alpha) - \lim \xi_k = \gamma$. □

Corollary 2.4 *Let α and β be fixed real numbers such that $0 < \alpha \le \beta \le 1$ and $0 < p < \infty$ and let $\theta = (v_r)$ and $\vartheta = (s_r)$ be two lacunary sequences such that $I_r \subset J_r$ for all $r \in \mathbb{N}$.If (8) holds then,*

(i) If a sequence is $N_q^\vartheta(\mathcal{I}^\alpha)_p-$summable to γ, then it is $S_q^\theta(\mathcal{I}^\alpha)-$statistically convergent to γ,

(ii) If a sequence is $N_q^\vartheta(\mathcal{I})_p-$summable to γ, then it is $S_q^\theta(\mathcal{I}^\alpha)-$statistically convergent to γ,

(iii) If a sequence is $N_q^\vartheta(\mathcal{I})_p-$summable to γ, then it is $S_q^\theta(\mathcal{I})-$statistically convergent to γ.

If (9) holds and let $\alpha \in (0, 1]$ then,

(i) If a bounded sequence $x = (x_k)$ is $S_q^\theta(\mathcal{I}^\alpha)-$statistically convergent to γ then it is $N_q^\vartheta(\mathcal{I}^\alpha)_p-$summable to γ,

(ii) If a bounded sequence $x = (x_k)$ is $S_q^\theta(\mathcal{I}^\alpha)-$ statistically convergent to γ then it is $N_q^\vartheta(\mathcal{I})_p-$summable to γ,

(iii) If a sequence is $S_q^\theta(\mathcal{I})-$statistically convergent to ξ then it is $N_q^\vartheta(\mathcal{I})_p-$summable to γ.

Theorem 2.9 *Suppose (X, q) is a seminormed space with the seminorm q and let a sequence $\xi = (\xi_k)$ where $\xi_k \in X$. If g is a continuous function on X then it preserves $\mathcal{I}_q$-lacunary statistically convergent of order α in X*

Proof: For any sequence ξ_k, ${\mathcal{I}^\alpha}_q$- lacunary statistically convergent to γ. Since g is continuous then for each $\varepsilon_1 > 0$ the exists $\varepsilon_2 > 0$ such that if $\xi \in B(\gamma, \varepsilon_1)$, then $g(\xi) \in B(g(\gamma), \varepsilon_2)$. Also we write

$$C(\varepsilon_1, \tau) = \{r \in \mathbb{N} : \frac{1}{\phi_r^\alpha} |\{k \in I_r : q(\xi_k - \gamma) \ge \epsilon_1\}| < \tau\} \in F(\mathcal{I}).$$

Now

$$\{k \in I_r : q(\xi_k - \gamma) \ge \epsilon_1\} \supseteq \{k \in I_r : q(\xi_k - \gamma) \ge \epsilon_2\},$$

so

$$\frac{1}{\phi_r^\alpha} |\{k \in I_r : q(\xi_k - \gamma) \ge \epsilon_1\}| \ge \frac{1}{\phi_r^\alpha} |\{k \in I_r : q(\xi_k - \gamma) \ge \epsilon_2\}|,$$

for $\tau > 0$

$$\{r \in \mathbb{N} : \frac{1}{\phi_r^\alpha}|\{k \in I_r : q(\xi_k - \gamma) \geq \epsilon_1\}| < \tau\} \supseteq \{r \in \mathbb{N} : \frac{1}{\phi_r^\alpha}|\{k \in I_r : q(g(\xi_k) - g(\gamma)) \geq \epsilon_2\}| < \tau\} \in F(\mathcal{I}).$$

Since $C(\varepsilon_1, \tau) \in F(\mathcal{I})$. □

References

1. Colak, R., *Statistical convergence of order* α, Modern methods in Analysis and its Applications, New Delhi, India, Anamaya Pub., (2010), 121-129.
2. Colak, R., Bektas, C. A., λ*-statistical convergence of order* α, Acta Math. Scientia, 31B (3) (2011), 953-959.
3. R. Colak, B. C. Tripathy and M. Et, Lacunary strongly summable sequences and $q-$ lacunary almost statistical convergence, Vietnam J. Math. 34(2),(2006), 129-138.
4. Das, P., Savas, E., Ghosal, S. K., *On generalizations of certain summability methods using ideals*, Appl. Math. Letters, 24 (2011), 1509 - 1514.
5. Fast, H., *Sur la convergence statistique*, Colloq. Math., 2 (1951), 241-244.
6. A. R. Freedman, A. R., Sember, J. J., Rapnael, M., *Some Cesaro type summability spaces*, Proc. London. Math. Soc., 37 (1978), 508-520.
7. Fridy, J. A. *On statistical convergence*, Analysis, 5 (1985), 301-313.
8. Fridy, J. A., Orhan, C., *Lacunary statistical convergence*, Pacific J. Math., 160 (1993) 43-51.
9. Kostyrko, p., Šalát, T., Wilczynki, W., *I-convergence*, Real Anal. Exchange, 26 (2) (2000/2001), 669-685.
10. Li, J., *Lacunary statistical convergence and inclusion properties between lacunary methods*, Int. J. Math. Math. Sc., 23(3) (2000), 175-180.
11. Mohuiddine, S.A, Alotaibi, A., Alsulami, S. M., *ideal convergence of double sequences in random 2-normed spaces*, Adv. Difference Equ. 149 (2012).
12. Mursaleen, M., Mohiuddine, S. A., Edely, O. H., *On the ideal convergence of double sequences in intuitionistic fuzzy normed spaces*, Comput. Math. Appl. 59(2) (2010), 603-611.
13. Šalát, T., *On Statistically convergent sequences of real numbers*, Math. Slovaca, 30 (1980), 139-150.
14. Savas, E., *On lacunary strong* σ *-convergence*, Indian J. Pure Appl. Math., 21 (4)(1990), 359 - 365.
15. Savas, E., Patterson, R. F., *Double* σ*-convergence lacunary statistical sequences*, J. Comput. Anal. Appl., 11 (4)(2009), 610 - 615.
16. Savas, E., Das, P., *A generalized statistical convergence via ideals*, Appl. Math. Letters, 24 (2011), 826 - 830.
17. Savas, E., Das, P., Dutta, S., *A note on strong matrix summability via ideals*, Appl. Math Letters, 25 (4) (2012), 733 - 738.
18. Savas, E., Gürdal, M., *I-statistical convergence in probabilistic normed spaces*, UPB Sci. Bull. Ser. A Appl. Math. Phys., 77(4) (2015), 195-204.
19. E. Savaş and M. Et, On $\Delta_\lambda^m, I)$ statistical convergence of order α, Period. Math. Hungar.71(2)(2015), 1-10.
20. Savaş, R., *On double almost lacunary summable sequences of order* α *defined via orlicz function*, Romanian J. Math and Computer Sci., 10 (2)(2020), 82-87.
21. Savaş, R., *Lacunary statistical boundedness of measurable functions*,. Math. in Natural Sci., 6, (2020), 15-19.
22. Savaş R., Gürdal, M., *On lacunary statistical convergence of double sequences in credibility theory*, Inter. J. General Systems, 52(3) (2023), 1-18.
23. Savaş, R., Patterson, R. F., *I2-Lacunary Strongly Summability for Multidimensional Measurable Functions*, Publications de l'Institut Math., 107 (121) (2020),93-107.
24. Savaş, R., *I- Almost Lacunary Vector Valued Sequence Spaces of Order* γ *in 2-Normed Spaces*, Maltepe J. Math., Volume II Issue 2, (2020), Pages 76-81.
25. Savaş, R., *On some multiple statistical convergence of order* θ *defined by almost lacunary sequences*, J. Anal. Number Theor., 8 (1) (2020), 7-10.
26. Savaş, R., Patterson, R. F., *Multidimensional Matrix Characterization of Asymptotic I2- Equivalent and Ideal for Double Sequences*, Numer. Func. Anal. and Optimization, 2019, DOI: 10.1080/01630563.2018.1557204.
27. Savas, R., *New summability methods via* $\widetilde{\phi}$ *functions*, Communications Faculty of Sciences University of Ankara Series A1: Mathematics and Statistics, 72 (2) (2023), 449-461. Doi: 10.31801/cfsuasmas.1114983.

28. Savas, R., *On the Bivariate Lacunary Statistical Convergence of Order $\overline{\omega}$ in Rough Set Theory*, Suleyman Demirel University Journal of Natural and Applied Sciences, 29(3):621-629, 2025.

29. Schoenberg, I. J., *The integrability of certain functions and related summability methods*, Amer. Math. Monthly, 66 (1959) 361-375.

30. Yamanci, U., Gürdal, M., *On lacunary ideal convergence in random n-normed space*, J. Math., 2013; 868457, 1-8.

Ekrem SAVAŞ,

Department of Mathematics, Usak University, Usak, Turkey

E-mail address: ekremsavas@yahoo.com