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Nilpotency of the Alternator Ideal of Binary (-1,1) Algebra

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ABSTRACT: In this paper we discuss the nilpotency of the alternator ideal of a finitely generated binary $(-1,1)$ algebra. If a binary $(-1,1)$ algebra R is generated by a finite set $X_n = x_1, x_2, \ldots\ldots.., x_n$ then the algebra $\triangle^*$ is nilpotent of $<4n^3$.

Keywords: Non associative rings, $(-1,1)$ rings, alternator ideal, nilpotent ideals, binary $(-1,1)$ algebra..

Contents

1. Introduction

Pchelintsev [1] studied some properties of central ideals of finitely generated binary $(-1,1)$ algebras. In [2] it is proved that a prime finitely generated binary $(-1,1)$ algebra is alternative. In this we discuss the nilpotency of the alternator ideal of a finitely generated binary $(-1,1)$ algebra.

In this we use the following notations: $aob = ab + ba$ is the Jordan product, (a,a,b) is the alternator, $S(a,b,c) = (a,b,c) + (b,c,a) + (c,a,b)$ is the Jacobi function of associators. A 2- and 3-divisible right alternative algebra is a binary $(-1,1)$ algebra if its elements satisfy the identiy.

$$((x,x,y),y) = 0. \tag{1.1}$$

By linearizing we have

$$((x,x,y),z) + ((x,x,z),y) = 0. \tag{1.2}$$

$$(a,x,y) + (a,y,x) = 0. \tag{1.3}$$

$$(a, x\circ y, z) + (a, y\circ z, x) + (a, z\circ x, y) = 0. \tag{1.4}$$

$$(a^2, b) = (a,b)\circ a + 2(a,a,b). \tag{1.5}$$

We denote $R(x)$ and $L(x)$ by the operators of right and left multiplications by an element x. For example, yL(x) = xy and let $T\epsilon R, L$. We introduce the following operators :
$H(x) = R(x) - L(x), R(x,y) = R(x)R(y) - R(xy),$
$L(x,y) = L(xy) - L(y)L(x), \Delta(x) = L(x^2) - L(x)^2$
$\Delta(x,y) = \Delta(x+y) - \Delta(x) - \Delta(y) = L(x,y) + L(y,x)$
Now by an algebra we mean a linear algebra over a 2-and 3-divisible field Φ. Also we use the following notations : F[x] is a free binary (-1,1) algebra with the set $X = x_1, x_2 \ldots.., x_n, \ldots.$of free generators ; Mon [x] is the set of monomials over X of F [X],
$X_n = x_1, x_2 \ldots.., x_n$ $(n \geqslant 2)$; R is a binary$(-1,1)$ algebra ;$K(R) = k/(k,R) = 0$ is the commutative center :
$V(R) = v/\forall a \in R, v\triangle(a) = 0$
is the weakly alternative center $\triangle$; is the set of operators of the form $\triangle$ (a, b); R (R) is the right multiplication algebra of R ; T (R) is the multiplication algebra of R ; $\triangle^*$ is the subalgebra of $T(R)$ generated by the set $\triangle$; $D_x = xUx\triangle^*$

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Lemma 1 : The algebra Δ^* satisfies the identity $(x, y)z = 0$.
Proof : We know that

$$(x, x, (y, y, z) - (y, y, (x, x, z) \in K(R). \tag{1.6}$$

Since $K(R) \subseteq V(R)$ by (1.5), the identity (1.6) implies the operator equality $(\Delta(a), (\Delta(b))\Delta(c) = 0$. Using the inner derivations of the associative algebra Δ^*, we obtain the identity $(x, y)z = 0$. This proves the lemma.
Lemma 2 : $x^2\Delta \in \Phi$ In particular, the element $(x \circ y)\Delta(a, b)$ is a linear combination of elements of the same content of the form $x\Delta(r, s)$ and $y\Delta(r, s)$
Proof : In the proof of the lemma we use the sign $\equiv$ to denote comparison modulo the space Φ, x, Δ . Also we note that $(y, z, x) + (z, y, x) \in \Phi, x, \Delta$. By applying (1.3) and (1.4), we obtain
$(x, a \circ x, a) = (x, x, a^2) = -(x, a^2, x) = -(x, a^2, x) - (a^2, x, x) \equiv 0.$
$(a \circ x, a, x) \equiv -(a, a \circ x, x) = (a, x, a \circ x).$
Since the identity $S(a, x, a \circ x) = 0.$ holds in R [3], by the above comparisons, we obtain $2(a, x, a \circ x) \equiv 0.$ Hence $(a, x, aox) \equiv 0$ and $(a, a, x^2) \equiv 0.$ which completes the proof of the lemma.
Monomials over X are of the form $x_i, x_ix_j(i < j)$ and $(x_ix_j)x_l$ which are known as regular. We denote $\Pi(x)$ as the set of regular monomials over $X_n(n \geqslant 2)$. Also we note that $P(n) = n+(n-1)/2+n^3 < 2n^3-1$ by the number of regular monomials over X_n. Also we note that . From [4] we have the following lemma.

Lemma 3: If $R = F(x)$ then $R^4 \subseteq \Phi.\ a \circ b, v\Delta(a, b)v \in \Pi(x)$

Proof : An element of the form $v\triangle(a, b)$, where $v \in \Pi[x]$ and a, b $\in$ Mon[x]. is a rank 1 called as regular alternator. We denote the set of rank 1 regular alternators by $\Pi^1_a(x)$. Also we note that the set $\Pi^1_a(x_n)$.) is countable unlike the finite set $\pi(x_n)$ of regular monomials. The polynomials of R of the form

$$a\triangle(b_1, c_1)\triangle(b_2, c_2)....\triangle(b_n, c_n)\triangle(r, s) \tag{1.7}$$

are called the iterated alternators and the number of the operators $\triangle$ in this polynomial is called its $\triangle$ degee.
The iterated alternators are $\triangle$ -normal if all monomials b_i are regular, i.e., $b_i \in \pi(x)$

Theorem 1 : If a binary $(-1, 1)$ algebra R is generated by a finite set $X_n = x_1, x_2, ..., x_n$ then the algebra $\triangle^*$ is nilpotent of index ¡$4n^3$
Proof :- First we may assume that $n \geqslant 2$. From [5,6], we know that the operator $\triangle(x)$ is nilpotent of index, 2, since a binary $(-1, 1)$ algebra satisfies the identity $(x, x, (x, x, y)) = 0$. We define $U \equiv W$ in $\triangle^*$ if $(U - W)\triangle = 0$.
Linearizing the identity $\triangle(x)^2 = 0$ and using the relation $(\triangle, \triangle) \equiv 0$ (from Lemma 1), we obtain
$\triangle(x)\triangle(x, z) \equiv 0,$
$\triangle(x, y)\triangle(x, z) + \triangle(x)\triangle(y, z) \equiv 0$
Hence, $\triangle(x, y)\triangle(x, z)\triangle(x, t) \equiv -\triangle(x)\triangle(y, z)\triangle(x, t) \equiv -\triangle(x)\triangle(x, t)\triangle(y, z) = 0$

$$\triangle(x, y)\triangle(x, z)\triangle(x, t) \tag{1.8}$$

Now we consider the homogeneous polynomial $w = (a \circ b)\triangle(c, d)$. By lemma 2, the element w is linear comibation of words of the form $t\triangle(p, q)(witht = aort = b)$ of the same content as the polynomial w. Now by lemma 3, every iterated alternator of the form (1.7) is representable as elements of the same form and content starting with regular monomials over X_n and have the $\triangle$- degree at least N+1. Thus, without loss of generality we may assume that the iterated alternator of the form (1.7) starts with a regular monomial a.
Since the linearization of the identity $(x, x, x) = 0$ has the form
$x\triangle(a, b) + a\triangle(x, b) + b\triangle(x, a) = 0$, we can assume that the monomial b_1 in the iterated alternator of the form (1.7) is regular. Using lemma 1, we can shift the first operator $\triangle(b_1, c_1)$ to the last but one place and assume that the monomial b_N in (1.7) is regular. Executing the indicated we can arrive at a linear representation for (1.7) in terms of $\triangle-$ normal polynomials. Now we assume that procedure,

we have a $\triangle-$ normal polynomial of the form (1.7) and $N = 2p(n) + 1$. Then (7) contains the three operators $\triangle(b, q_1)\triangle(b, q_2) and \triangle(b, q_3)$ with a common element b. Since the operators $\triangle$, commute by (1.8), a polynomial of the form (1.7) of $\triangle-$ degree atleast $2p(n) + 2$ equal to zero. Hence, the algebra $\triangle^*$is nilpotent of index at most

$2p(n) + 2 < 2(2n^3 - 1) + 2 = 4n^3$, as required.

2. Conclusion

In this paper we discussed the nilpotency of the alternator ideal of a finitely generated binary $(-1, 1)$ algebra. In our future work we can try for the nilpotency of the radical of a free finitely generated binay $(-1, 1)$ algebra.

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