



Periodicity of Transits and Oppositions by Continued Fractions

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ABSTRACT: The method of continued fractions offers a systematic approach to determining the periodic cycles of planetary phenomena such as transits and oppositions by approximating ratios of orbital periods through finite continued fractions known as convergents. By extending these convergents to larger values, the method refines the prediction accuracy of event timings. This approach involves analyzing a range of convergents for each planetary event, enabling the derivation of dates that are then cross-checked against modern ephemerides to validate their precision. The study concentrates on five key planets—Mercury, Venus, Mars, Jupiter, and Saturn—due to their prominence in observable celestial events and their significance in long-term astronomical calculations. Predicting planetary events like solar and lunar eclipses, occultations, transits, conjunctions, and oppositions requires accounting for multiple periodic influences, including gravitational perturbations from other celestial bodies and the elongation of planetary orbits. Although comprehensive calculations involving all parameters yield the most precise timing, in many practical scenarios, approximate methods that simplify these factors suffice to determine the event's date with reasonable accuracy. This balance between computational complexity and practical precision allows astronomers to forecast such phenomena years in advance efficiently, making the continued fraction method a valuable tool in both theoretical and applied astronomy.

Keywords: Phenomena, transit, opposition, periodicity, convergent, continued fraction.

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1. Introduction

The periodicity of eclipses is determined by the well-established cycle known as the *saros* which spans 6585.8 days or approximately eighteen years and eleven days. This cycle indicates that a similar eclipse will recur after this interval. Similarly the periodicity of the mean conjunctions of two celestial bodies remains constant.

If s_1 and s_2 are the sidereal periods of two bodies, the periodicity of their conjunction can be calculated accordingly

$$\frac{s_1 s_2}{|s_1 - s_2|}.$$

The sidereal and synodic periods are fundamental concepts in celestial mechanics that describe different aspects of orbital motion. The sidereal period measures the time a celestial object takes to complete one full orbit relative to the fixed stars, providing a frame of reference that is independent of the Earth's position. In contrast, the synodic period is the interval required for the celestial body to return to the same position relative to the Sun as observed from the Earth, incorporating the relative motion of both Earth and the object. These two periods are crucial for understanding the apparent motion of planets and other celestial bodies in the sky, particularly when studying phenomena such as planetary transits and oppositions.

This study leverages continued fractions as a mathematical tool to analyze the periodic cycles associated with the transits of inferior planets (those orbiting near to the Sun than Earth) and the oppositions of superior planets (those orbiting farther from the Sun). Continued fractions provide a powerful and elegant method for solving first-degree indeterminate equations of the form $ax + by = c$, which naturally arise in the relationships governing these orbital periods. By applying continued fractions, the research uncovers precise approximations and general solutions that clarify the interplay between sidereal and synodic periods. Subsequent sections elaborate on this connection, demonstrating how continued fractions facilitate the modeling and prediction of planetary cycles with high accuracy.

2. Periodicity of transits

The Transits of Mercury and Venus occur when these planets move directly between Earth and the Sun, creating the appearance of their silhouettes crossing the solar disk. This alignment, known as conjunction, is observable from Earth and provides valuable opportunities for astronomers to study planetary orbits and refine measurements of the solar system. While Mercury's transits are relatively frequent due to its close proximity to the Sun and smaller orbital radius, Venus transits are much rarer, reflecting the larger and slower orbit of Venus around the Sun.

Mercury's transit cycle is characterized by occurrences every 6 or 7 years, but more precisely, these intervals sum to a 13-year cycle during which the transit dates tend to recur with remarkable similarity. Although individual transits within this cycle may vary in exact timing, the date and month of occurrence generally align closely, differing by only a few minutes in recorded observations. This periodicity results from the interplay between Mercury's orbital period and the inclination of its orbit relative to Earth's, creating predictable windows for transit visibility that are essential for ongoing celestial mechanics studies and calibration of astronomical instruments. The periodicity of 13 years are listed below.⁽¹⁾

Table 1: Transit dates illustrating 13-year periodicity

Year	Date and time
1961	Feb 11,13:05
1974	Feb 11,13:11
1987	Feb 11,13:13
2000	Feb 11,13:12
2011	Feb 11,13:05
2024	Feb 11,13:11
2037	Feb 11,13:13

There is also a 33-year periodicity, with a more precise one being 46 years (the sum of 13 and 33). Below are some dates corresponding to the 46-year periodicity.

Table 2: Transit dates illustrating 46-year periodicity

Year	Date
1927	Nov 10
1973	Nov 10
2019	Nov 11
2065	Nov 11
2111	Nov 12

Mercury demonstrates a more precise periodicity of 217 years. Various periodicities, including 6, 7, 13, 33, 46, and 217, can be derived from the sidereal periods of Earth and Mercury.

The sidereal periods of Earth and Mercury are 365.256363 and 87.969256, respectively. Dividing Earth's sidereal period by Mercury's results in a quotient of 4.152091. This value, 4.152091, can be expressed as a series of fractions with integers in both the numerator and denominator, which progressively

Table 3: Transits that fall within the same 46-year periodicity cycle

Year	Date
1913	Aug 22
1959	Aug 23
2005	Aug 23
2051	Aug 25
2097	Aug 25

approximate the value 4.152091 more accurately through the method of continued fractions as

$$\frac{4}{1}, \frac{25}{6}, \frac{29}{7}, \frac{54}{13}, \frac{137}{33}, \frac{191}{46}, \frac{901}{217}, \dots$$

The denominators in these fractions represent the frequency of Mercury transits, but interpreting them as simple yearly occurrences can be misleading. Mercury’s greatest western elongation—the point at which it appears farthest from the Sun in the morning sky—does not happen annually. Instead, it occurs only after four sidereal cycles or equivalently three synodic periods of Mercury. The sidereal cycle refers to the time Mercury takes to complete one orbit relative to the fixed stars, while the synodic period is the interval between successive similar alignments of Mercury with the Sun and Earth. This means that the timing of Mercury’s transits is governed by a more complex orbital resonance rather than a straight forward yearly pattern, making the use of a one-year fraction an over simplification.

Table 1 demonstrates how rational numbers, such as these fractional frequencies, can be effectively expressed through finite continued fractions, providing a more precise mathematical representation of periodic phenomena like planetary transits. Continued fractions allow for the approximation of these rational values in a way that captures the underlying cyclical relationships between orbital periods. By understanding Mercury’s transit frequency through the lens of synodic periods and continued fractions, astronomers can better predict transit events and appreciate the intricate timing dictated by celestial mechanics, rather than relying on simplified annual estimates.

Table 4: Expressing the rational numbers as finite continued fractions

$\frac{4}{1}$	4
$\frac{25}{6}$	$4 + \frac{1}{6}$
$\frac{29}{7}$	$4 + \frac{1}{6 + \frac{1}{7}}$
$\frac{54}{13}$	$4 + \frac{1}{6 + \frac{1}{1 + \frac{1}{13}}}$
$\frac{137}{33}$	$4 + \frac{1}{6 + \frac{1}{1 + \frac{1}{1 + \frac{1}{33}}}}$
$\frac{191}{46}$	$4 + \frac{1}{6 + \frac{1}{1 + \frac{1}{1 + \frac{1}{1 + \frac{1}{46}}}}}$

These continued fractions are of the form

$$a + \frac{1}{b + \frac{1}{c + \frac{1}{d + \dots}}}$$

and the fractions referred to as "convergents" arise from the process of approximating irrational numbers through continued fractions. Each convergent represents a rational approximation that progressively

improves in accuracy as more terms are included. This refinement occurs because convergents are constructed to minimize the difference between the actual value and the approximation, thereby reducing the error significantly with each additional convergent. Importantly, it is a well-established mathematical fact that every rational number can be represented exactly by a finite number of convergents, which provides a systematic way to express rationals through this method, as illustrated in the referenced table.

In the specific application to planetary phenomena, the practical utility of convergents depends on the precision of the parameters involved, often limited by the number of decimal places used in calculations. Since planetary parameters such as orbital periods or distances are measured and expressed with finite precision, the range and effectiveness of convergents in modeling or predicting these phenomena are constrained accordingly. Thus, while the convergent approach offers a powerful tool for rational approximation, its application in planetary science must consider the inherent limitations imposed by measurement accuracy and numerical precision. This relationship underscores the balance between mathematical theory and empirical data in the study of celestial mechanics.

3. Computation of successive convergents by the method of continued fractions

Let us consider the value of number α in the form of fraction $\alpha = 4.152091 = \frac{4152091}{1000000} = \frac{A}{B}$

Four constants a, b, c and r are required to compute continued fractions. Where $a = A$, $b = B$, $c = \text{Int}(a/b)$, and $r = a - bc$, then the convergents are of the form $\frac{p_n}{q_n}$. These p_n and q_n are calculated by using:

$$p_1 = c_1, \quad q_1 = 1$$

But in further steps:

$$p_2 = p_1c_2 + 1, \quad q_2 = c_2$$

and

$$p_n = c_n p_{n-1} + p_{n-2}, \quad q_n = c_n q_{n-1} + q_{n-2}$$

Table 5: Continued fractions for the transit of Mercury

n	1	2	3	4	5	6	7	8
a	4152091	1000000	152091	87454	64637	22817	19003	3814
b	1000000	152091	87454	64637	22817	19003	3814	3747
$c = \text{int}(a/b)$	4	6	1	1	2	1	4	1
$r = a - bc$	152091	87454	64637	22817	19003	3814	3747	67
p_n	4	25	29	54	137	191	901	1092
q_n	1	6	7	13	33	46	217	263

The above table gives the convergents

$$\frac{4}{1}, \frac{25}{6}, \frac{29}{7}, \frac{54}{13}, \frac{137}{33}, \frac{191}{46}, \frac{901}{217}, \frac{1092}{263} \dots$$

From this analysis, it is apparent that an increased number of convergents results in minimal error when compared to the value of α . The fraction closely approximates $\alpha = 4.152091$, indicating that the convergents are finite. An intriguing pattern emerges in the periodicities presented in the last row of Table 5, where each subsequent periodicity value is either the sum of its two preceding values or the sum of its preceding values multiplied by a positive integer. For instance, $13 = 6 + 7$, $46 = 13 + 33$, and $217 = 46 \times 4 + 33$. Therefore, successive periodicities can be expressed as the sum of the previous ones.

Similarly we can obtain the periodicity of transit of Venus as

$$\frac{2}{1}, \frac{5}{3}, \frac{13}{8}, \frac{382}{235}, \frac{395}{243} \dots$$

The sidereal periods of the Earth and the Venus, measured as approximately 365.25636 days and 224.70080 days respectively, exhibit a near-rational ratio when the Earth’s period is divided by that of Venus, yielding a value close to 1.62552. The above ratios closely aligns with several fractional approximations such as 2, 1.666666, 1.625, 1.6255319, and 1.625514, where higher-order convergents provide increasingly precise approximations of this value. These convergents demonstrate how continued fraction expansions can effectively approximate irrational ratios in celestial mechanics, offering insight into orbital resonances and periodicities between planetary bodies.

Venus’s well-documented periodicities further illustrate this relationship, with an 8-year cycle and a longer 243-year cycle governing its transits relative to Earth. Within the 243-year cycle, transits occur following a pattern of intervals spanning 8 years, 121.5 years, 8 years, and 105.5 years, whose sum equals to full 243-year period. This cyclical pattern confirms the finiteness of the convergents involved and highlights the predictability of Venus transits based on orbital resonance. Such periodicities are critical for understanding the timing of rare astronomical events and underscore the mathematical harmony inherent in planetary motion.

4. Periodicity of opposition in case of exterior planets

The phenomena of conjunction and opposition are significant observational events involving the exterior planets Mars, Jupiter, and Saturn. Conjunction occurs when a planet aligns with the Sun as seen from Earth, while opposition happens when the planet is positioned directly opposite the Sun in the sky relative to Earth. These configurations are crucial for astronomers as they influence the planet’s visibility and apparent brightness. The frequency of opposition, in particular, can be quantitatively analyzed by examining the ratio of Earth’s sidereal to the sidereal period of the planet in question. This ratio provides insight into the relative orbital speeds and synodic periods, which determine how often these oppositions occur.

For Mars, with a sidereal period of approximately 686.979852 days, the ratio of Earth’s sidereal period (about 365.256 days) to Mars’s sidereal period is roughly 0.53168424. This ratio is instrumental in calculating the synodic period—the time interval between successive oppositions or conjunctions—using methods such as continued fractions. By analyzing this ratio, astronomers can predict the timing and frequency of oppositions, which are particularly important for observational campaigns and mission planning. The continued fractions method refines this ratio into a series of approximations that yield precise periodicities, enabling a deeper understanding of the orbital dynamics of Mars relative to Earth. Similar analyses apply to other superior planets like Jupiter and Saturn, each with distinct sidereal periods influencing their opposition frequencies.

Table 6: Continued fractions for the opposition of Mars

n	-1	0	1	2	3	4	5	6
a	53168424	100000000	53168424	46831576	6336848	2473640	1389568	1084072
b	100000000	53168424	46831576	6336848	2473640	1389568	1084072	305496
$c = \text{int}(a/b)$	0	1	1	7	2	1	1	3
$r = a - bc$	53168424	46831576	6336848	2473640	1389568	1084072	305496	167584
p	0	1	1	8	17	25	42	151
q	1	1	2	15	32	47	79	284

From the above table 6, the sequence of convergents for the planet Mars is

$$\frac{1}{2}, \frac{8}{15}, \frac{17}{32}, \frac{25}{47}, \frac{42}{79}, \frac{151}{284}, \dots$$

In this context, the value 2 signifies an approximate periodicity corresponding to a single synodic cycle of Mars. However, the periods of 15, 32, 47, 79, and 284 provide a more precise depiction of the phenomenon. The sequence of convergents demonstrates its convergence, as evidenced by their minimal differences, which are detailed in Table 7.

Table 7: The difference of the fractional values converging to $\alpha = 0.53168424$

Convergents (c)	Difference ($\alpha-c$)
$\frac{1}{2} = 0.5$	0.03168424
$\frac{8}{15} = 0.5333333$	0.00164909
$\frac{17}{32} = 0.53125$	0.00043424
$\frac{25}{47} = 0.531914893$	0.00023065
$\frac{42}{79} = 0.531645561$	0.00003868
$\frac{151}{284} = 0.53169014$	0.0000059
$\frac{193}{363} = 0.53168044$	0.0000038

In case of Jupiter, the sidereal periods of the Earth and the Jupiter are respectively 365.256363 and 4332.589. The ratio of them results to 0.0843044108. The approximated value $\alpha = 0.084304$ can be expressed as successive fractions as

$$\frac{1}{11}, \frac{7}{83}, \frac{29}{344}, \frac{268}{3179}, \dots$$

as computed from the above method. Out of these fractions the periodicities of 12 and 83 years are the ones that are happening because the sidereal revolution of Jupiter is 11.86 years.

Similarly, the ratio of Earth's sidereal / Saturn's sidereal is considered with the sidereal period of the planet Saturn as 10759.23 years, then the ratio is 0.033948188, which is approximated as 0.033948 for the calculation purpose, then the convergents obtained by above method of continued fractions are

$$\frac{1}{29}, \frac{5}{59}, \frac{11}{324}, \frac{35}{1031}, \frac{81}{2386}, \dots$$

Therefore the periodicities are 29, 59, 324, 1031, etc. The period of 29 years is neglected because Saturn's revolution period is 29.46 years. Hence the periodicity of 59 years is the better one to list them. For example, there was an opposition of Saturn on 1960 July 07, again the planet was in opposition with the Sun on 2019 July 09.

5. Conclusion

Investigating and calculating the precise timing of celestial phenomena requires a detailed consideration of various physical factors beyond initial approximations. While continued fractions provide a useful method for estimating approximate dates, achieving high accuracy necessitates incorporating the gravitational constant, which governs the strength of gravitational interactions. This constant plays a critical role in modeling the forces acting between celestial bodies, influencing their orbital motions and, consequently, the timing of observable events such as eclipses, transits, or conjunctions.

Furthermore, to refine these calculations, it is essential to include all perturbation terms and small deviations in the orbits caused by gravitational influences from other bodies, non-uniform mass distributions, and relativistic effects. These perturbations can accumulate over time, causing shifts in the predicted occurrence of events. By systematically accounting for these factors through advanced celestial mechanics and numerical methods, astronomers can achieve precise predictions that align closely with observed phenomena, enabling more reliable planning for observations and enhancing our understanding of dynamic celestial systems.

Acknowledgments

The authors thank the management of REVA University for their support in publishing the research article.

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