Bol. Soc. Paran. Mat.
©SPM – E-ISSN-2175-1188
SPM: www.spm.uem.br/bspm

(3s.) **v. 2026 (44) 5** : 1–6.
ISSN-0037-8712
doi:10.5269/bspm.80588

Continuous Function by Using (1, 2)S_β-Open Sets in Bitopological Spaces

V. Subprabha, N. Durga Devi, S. Jafari and P. Jeyanthi

ABSTRACT: The aim of this paper is to introduce the concept of (1, 2)S_β-continuous function by using (1, 2)S_β-open sets in bitopological spaces and study some of their properties.

Keywords: (1, 2)semi-open set, (1, 2)semi-continuous function, (1, 2)β-closed set, (1, 2)S_β-open set, (1, 2)S_β-continuous function.

Contents

1 Introduction **1**

2 Preliminaries **2**

3 Main Results **3**

4 Conclusion **5**

1. Introduction

Topology is a branch of mathematics that deals with set theoretical definitions and constructions. It provides the foundational framework to understand the spaces and continuous functions. An open set is any member of the collection τ, that forms the topology. A function between topological spaces is continuous, if the preimage of every open set is open. This became the cornerstone of modern topology and analysis that enables the study of continuity through open sets, limit points and closures rather than through limits and distances. Also, it enriches the theory of bitopological spaces that helps to approach problems where interacting topological structures are present. A function is characterized as continuous in the bitopological sense if it preserves the structure of open sets with respect to both topologies. More specifically, a function f: (X, τ_1, τ_2) $\to$ (Y, σ_1, σ_2) is said to be continuous if for every open set V in either σ_1 or σ_2, the preimage under f is open in the corresponding topology in X.

In the year 1963, Kelly[6] initiated the systematic study of bitopology which is a triple (X, τ, σ), where X is a non-empty set together with two topologies namely τ, σ. Levine[7] initiated his study on semi-open sets and their properties in 1963. In 1983, Abd-El-monsef[1] proposed the notion of β-open sets and β-continuity in topological spaces. In 2013, Khalaf and Ahmed[3] have introduced and defined a new class of semi-open sets called S_β-open sets in topological spaces. Here we introduce a new class of open sets namely (1, 2)S_β-open set using (1, 2)semi-open set and β-closed set in bitopological spaces. The concept of (1, 2)S_β-open sets advances this study by establishing generalized open sets within a bitopological context allow for more nuanced continuity notions. This study addresses the concept of continuity for functions defined between bitopological spaces. When generalized by using the (1, 2)S_β-open sets, this notion of continuity is extended to accomodate sets that are open with respect to a synthesis of the underlying topologies, thus refining classical and modern forms of continuity.

In this paper, we define the concept of (1, 2)S_β-continuity by using (1, 2)S_β-open sets in bitopological spaces and exhibit some of its properties. We present our study as follows: In section 2, we recall the required definitions and known results which are used in the sequel. In section 3, we introduce the concept of $(1, 2)S_\beta$-continuous function in bitopological spaces and study its characterizations and properties. In the conclusion, we summarize the results and give a scope for further study.

2020 *Mathematics Subject Classification:* 54A05.
Submitted December 04, 2025. Published February 14, 2026

2. Preliminaries

In this section, we give some preliminary definitions which are used in the rest of the paper.

Definition 2.1 [8] Let A be a subset of a bitopologial space (X, τ_1, τ_2). Then A is said to be

(i) (1, 2)semi-open if $A \subseteq \tau_1\tau_2$-Cl(τ_1-Int(A)).

(ii) (1, 2)regular-open if $A = \tau_1$-Int($\tau_1\tau_2$-Cl(A)).

(iii) (1, 2)β-open if $A \subseteq \tau_1\tau_2$-Cl(τ_1-Int($\tau_1\tau_2$-Cl(A))), where τ_1-Int(A) is the interior of A with respect to topology τ_1 and $\tau_1\tau_2$-Cl(A) is the intersection of all $\tau_1\tau_2$-closed sets containing A.

(iv) (1, 2)β-Int(A) is the union of all (1, 2)β-open sets contained in A.

(v) (1, 2)β-Cl(A) is the intersection of all (1, 2)β-closed sets containing A.

Definition 2.2 [8] A subset A of X is said to be

(i) (1, 2)semi-open if $A \subseteq \tau_1\tau_2$-Cl(τ_1-Int(A)).

(ii) (1, 2)regular-open if $A = \tau_1$-Int($\tau_1\tau_2$-Cl(A)).

(iii) (1, 2)β-open if $A \subseteq \tau_1\tau_2$-Cl(τ_1-Int($\tau_1\tau_2$-Cl(A))).

The set of all (1, 2)semi-open, (1, 2)regular-open and (1, 2)β-open are denoted by (1, 2)SO(X, τ_1, τ_2), (1, 2)RO(X, τ_1, τ_2), (1, 2)βO(X, τ_1, τ_2) or simply, (1, 2)SO(X), (1, 2)RO(X), (1, 2)β-O(X) respectively.

Definition 2.3 [8] A subset A of X is said to be

(i) (1, 2)semi-closed if $\tau_1\tau_2$-Int(τ_1-Cl(A)) $\subseteq A$.

(ii) (1, 2)regular-closed if $A = \tau_1$-Cl($\tau_1\tau_2$-Int(A)).

(iii) (1, 2)β- closed if $\tau_1\tau_2$- Int(τ_1-Cl($\tau_1\tau_2$-Int(A))) $\subseteq A$.

The set of all (1, 2)semi-closed, (1, 2)regular-closed and (1, 2)β-closed are denoted by (1, 2)SCL(X, τ_1, τ_2), (1, 2)RCL(X, τ_1, τ_2), (1, 2)βCL(X, τ_1, τ_2) or simply, (1, 2)SCL(X), (1, 2)RCL(X), (1, 2)β-CL(X) respectively.

Definition 2.4 [8] For any subset A of X,

(i) τ_1-Int(A) $\subseteq \tau_1\tau_2$-Int(A) and τ_2-Int(A) $\subseteq \tau_1\tau_2$-Int(A).

(ii) $\tau_1\tau_2$-Cl(A) $\subseteq \tau_1$-Cl(A) and $\tau_1\tau_2$-Cl(A) $\subseteq \tau_2$-Cl(A).

(iii) $\tau_1\tau_2$-Cl($A \cap B$) $\subseteq \tau_1\tau_2$-Cl(A) $\cap$ $\tau_1\tau_2$-Cl(B).

(iv) $\tau_1\tau_2$-Int(A) $\cup$ $\tau_1\tau_2$-Int(B) $\subseteq \tau_1\tau_2$-Int($A \cup B$).

Definition 2.5 [3] A semi-open subset A of a topological space (X, τ) is said to be S_β-open if for each $x \in A$ there exists a β-closed set F such that $x \in F \subseteq A$. A subset B of a topological space (X, τ) is S_β-closed if $X-B$ is S_β-open.

Definition 2.6 [8] A function $f : X \to Y$ is said to be semi-continuous if the inverse image of each open subset of Y is semi-open in X.

Definition 2.7 [8] A function f:(X, τ_1, τ_2) $\to$ (Y, σ_1, σ_2) is (1, 2)semi-continuous if $f^{-1}(B)$ is (1, 2)semi-open in X, for every (1, 2)semi-open set B in Y.

Definition 2.8 [3] A bitopological space (X, τ_1, τ_2) is said to be τ_1 -locally indiscrete if every τ_1-open subset of X is τ_1-closed.

Lemma 2.9 [3] If B is τ_1-clopen subset of a space (X, τ) and A is S_β-open set in X, then $A \cap B \in$ S_β-O(X).

Definition 2.10 [9] A (1, 2)semi-open subset A of a bitopological space (X, τ_1, τ_2) is said to be (1, 2)S_β-open set if for each $x \in A$ there exists a (1, 2)β-closed set F such that $x \in F \subseteq A$.

The complement of a (1, 2)S_β-open set is (1, 2)S_β-closed set and the family of all (1, 2)S_β-open ((1, 2)S_β-closed) subsets of X, is denoted by (1, 2)S_β-O(X) ((1, 2)S_β-CL(X)) respectively.

Proposition 2.11 [9] A subset A of a bitopological space (X, τ_1, τ_2) is (1, 2)S_β-open set if and only if A is (1, 2)semi-open and it is the union of (1, 2)β-closed sets.

Proposition 2.12 [9] A function f : (X, τ_1, τ_2) $\rightarrow$ (Y, σ_1, σ_2) is (1, 2)S_β-continuous if and only if the inverse image of every σ_1 - open set in Y is (1, 2)S_β-open in X.

3. Main Results

In this section, we define (1, 2)S_β-continuous function and study its characterizations.

Definition 3.1 A function f : (X, τ_1, τ_2) $\rightarrow$ (Y, σ_1, σ_2) is called (1, 2)S_β-continuous at a point $x \in X$, if for each (1, 2)S_β-open set V of Y containing $f(x)$, there exists (1, 2)S_β-open set U in X containing x, such that $f(U) \subseteq V$. If f is (1, 2)S_β-continuous at every point x of X, then it is called a (1, 2)S_β-continuous function.

Example 3.2 Let X ={a, b, c} and Y = {1, 2, 3}. Let f : (X, τ_1, τ_2) $\rightarrow$ (Y, σ_1, σ_2) be a bijective mapping function which is defined by f(a) = 1, f(b)= 2, f(c)=3. Then (1, 2)S_β-O(X) = {ϕ, X, {a}, {b, c}} and (1, 2)S_β-O(Y) = {ϕ, Y, {2, 3}}. Thus, f is a (1,2)S_β-continuous function.

Proposition 3.3 A function $f : (X, \tau_1, \tau_2) \rightarrow$ (Y, σ_1, σ_2) is (1, 2)S_β-continuous if and only if the inverse image of every (1, 2)S_β-open set in Y is (1, 2)S_β-open in X.

Proof: Let f be a (1, 2)S_β-continuous function and U be any (1, 2)S_β-open set in Y. We have to prove that $f^{-1}(U)$ is (1, 2)S_β-open set in X. If $f^{-1}(U)$ is empty, then there is nothing to prove. Suppose $f^{-1}(U) \neq \phi$. Then there exists $x \in f^{-1}(U)$ which implies $f(x) \in U$. Since f is a (1, 2)S_β-continuous function, there exists a (1, 2)S_β-open set V in X containing x such that $f(V) \subseteq U$ that implies $x \in V \subseteq f^{-1}(U)$. Hence $f^{-1}(U)$ is a (1, 2)S_β-open set in X.

Conversely, let U be a (1, 2)S_β-open set in Y and the inverse image of (1, 2)S_β-open set in Y which is (1, 2)S_β-open in X. Since $f(x) \in U$, $x \in f^{-1}(U)$ and by hypothesis, $f^{-1}(U)$ is a (1, 2)S_β-open set in X containing x, then $f(f^{-1}(U)) \subseteq U$. Hence f is a (1, 2)S_β-continuous function. □

Remark 3.4 Every (1, 2)S_β-continuous function is (1, 2)semi-continuous.

Example 3.5 Let f : $(X, \tau_1, \tau_2) \rightarrow$ (Y, σ_1, σ_2) be a function with two bitopological spaces and τ_1 = {ϕ, X, {a}, {b}, {d}, {a, b}, {a, d}, {b, d}, {a, b, d} }, τ_2 = {ϕ, X } as topologies on X and σ_1 = {ϕ, Y, {1}, {2}, {1, 2}}, σ_2 = {ϕ, Y, {1, 2, 4} } as topologies on Y and it is defined by f(a)=1, f(b)=2, f(c)=3, f(d)=2. Here, we consider f is a (1,2)S_β-continuous function. Then (1, 2)SO(X) = (1, 2)S_β-O(X) = {ϕ, X, {a}, {b}, {d}, {a, b}, {a, c}, {a, d}, {b, c}, {b, d}, {c, d}, {a, b, c}, {a, b, d}, {a, c, d}, {b, c, d}} and (1, 2)SO(Y) = (1, 2)S_β-O(Y) = {ϕ, Y, {1}, { 2}, {1, 2}, {1, 3}, {1, 4}, {2, 3}, {2, 4}, {1, 2, 3}, {1, 2, 4}, {1, 3, 4}, {2, 3, 4}}. Therefore, f is a (1, 2)semi-continuous function.

But the converse is not true and it is shown in the following example.

Example 3.6 Let f : $(X, \tau_1, \tau_2) \rightarrow$ (Y, σ_1, σ_2) be a bijective mapping function with two bitopological spaces and τ_1 = {ϕ, X, {a}, {b}, {a, b} }, τ_2 = {ϕ, X, {a, b, d}} as topologies on X and σ_1 = {ϕ, Y, {1}, {2}, {1, 3}, {1, 4}, {1, 2, 3}, {1, 2, 4}, {1, 3, 4}}, σ_2 = {ϕ, Y } as topologies on Y. Then (1, 2)SO(X) = (1, 2)S_β-O(X) = {ϕ, X, {a}, {b}, {a, b}, {a, c}, {a, d}, {b, c}, {b, d}, {a, b, c}, {a, b, d}, {a, c, d}, {b, c, d}} and (1, 2)SO(Y) = (1, 2)S_β-O(Y) = {ϕ, Y, {1}, {1, 2}, {1, 3}, {1, 4}, {1, 2, 3}, {1, 2, 4}, {1, 3, 4}}. Here, f is a (1, 2)semi-continuous function but not a (1,2)S_β-continuous function.

Corollary 3.7 If f:(X, τ_1, τ_2) $\rightarrow$ (Y, σ_1, σ_2) is (1, 2)semi-continuous function and (X, τ_1, τ_2) is τ_1-locally indiscrete, then f is a (1, 2)S_β-continuous function.

Proof: Let f be a (1, 2)semi-continuous function, (X, τ_1, τ_2) is τ_1-locally indiscrete and U be any (1, 2)S_β-open subset in Y. Observe that, $f^{-1}(U)$ is a (1, 2)semi-open subset in X. Also, since X is τ_1-locally indiscrete space, $f^{-1}(U) \in$ (1, 2)S_β-O(X). Thus, by Proposition 2.11, f is (1, 2)S_β-continuous function. □

Theorem 3.8 Let f:(X, τ_1, τ_2) $\rightarrow$ (Y, σ_1, σ_2) be a function, then the following statements are equivalent.

(i) f is a (1, 2)S_β-continuous function.

(ii) The inverse image of every (1, 2)S_β-open set in Y is (1, 2)S_β-open set in X.

(iii) The inverse image of every (1, 2)S_β-closed set in Y is (1, 2)S_β-closed set in X.

(iv) For each $B \subseteq X$, f((1, 2)$S_\beta - cl(B)) \subseteq \tau_1\tau_2 - cl(f(B))$.

(v) For each $B \subseteq X$, τ_1-int(f(B)) $\subseteq$ f((1, 2)S_β-int(B)).

(vi) For each $E \subseteq Y$, (1, 2)$S_\beta - cl(f^{-1}(E)) \subseteq f^{-1}(\tau_1\tau_2$-cl(E)).

(vii) For each $E \subseteq Y$, $f^{-1}(\tau_1$-int(E)) $\subseteq$ (1, 2)S_β-int($f^{-1}(E)$).

Proof: **(i)$\Rightarrow$ (ii)** follows from Proposition 2.11.

(ii)$\Rightarrow$ (iii) Let E be any (1, 2)S_β-closed subset of Y, then Y–E is an open subset in Y and $f^{-1}(Y$–$E)$ = X–$f^{-1}(E)$ is a (1, 2)S_β-open set in X. Thus, $f^{-1}(E)$ is a (1, 2)S_β-closed set in X.

(iii)$\Rightarrow$ (iv) Let $B \subseteq X$, then $f(B) \subseteq Y$. Since $f(B) \subseteq \tau_1\tau_2$-cl(f(B)) and by hypothesis $f^{-1}(\tau_1\tau_2$-cl(B)) is a (1, 2)S_β-closed set in X, therefore $B \subseteq f^{-1}(\tau_1\tau_2$-cl(f(B))) that implies (1, 2)S_β-cl(B) $\subseteq$ $f^{-1}(\tau_1\tau_2$-cl(f(B))). Therefore we have, $f((1,2)S_\beta$-cl(B)) $\subseteq$ ($\tau_1\tau_2$-cl(f(B))).

(iv)$\Rightarrow$(v). Let $B \subseteq$ X, then X –$B \subseteq$ X and by (iv), f((1,2)S_β-cl(X–B)) $\subseteq \tau_1\tau_2$-cl(f(X–B). Therefore, $f(X$– (1, 2)S_β-int(B)) $\subseteq \tau_1\tau_2$-cl(Y– f(B)) which implies Y – $f((1,2)S_\beta$-int(B)) $\subseteq$ Y – τ_1-int(f(B)).Thus, τ_1-int(f(B)) $\subseteq f((1,2)S_\beta$-int(B)).

(v)$\Rightarrow$(vi) Let $E \subseteq$ Y, then $f^{-1}(E) \subseteq X$ implies X–$f^{-1}(E) \subseteq X$. By hypothesis (v), τ_1-int(f(X–$f^{-1}(E)$)) $\subseteq$ f((1, 2)S_β-int(X– $f^{-1}(E)$))), then τ_1-int(Y–$f(f^{-1}(E))$) $\subseteq$ f(X–((1, 2)S_β-cl$f^{-1}(E)$))), this implies τ_1-int(Y–B) $\subseteq$ Y – f((1, 2)S_β-cl$f^{-1}(B)$). Hence, Y – $\tau_1\tau_2$-cl(E) $\subseteq$ Y – f((1, 2)S_β-cl$f^{-1}(E)$), that is f((1, 2)S_β-cl$f^{-1}(E)$)$\subseteq \tau_1\tau_2$-cl(E). It follows that (1, 2)S_β-cl($f^{-1}(E)$) $\subseteq f^{-1}(\tau_1\tau_2$-cl(E)).

(vi) $\Rightarrow$ (vii) Let $E \subseteq$ Y, then $(Y$–$E) \subseteq$ Y. By hypothesis (vi), (1, 2)$S_\beta - clf^{-1}(Y$–$E) \subseteq f^{-1}(\tau_1\tau_2 - cl(Y$–$E))$, which implies that (1, 2)$S_\beta - cl(X$–$f^{-1}(E)) \subseteq f^{-1}(Y$–$\tau_1 - int(E))$. Thus, X– (1, 2)S_β-int($f^{-1}(E)$) $\subseteq X$–$f^{-1}(\tau_1$-int E). Hence, $f^{-1}(\tau_1$-int(E)) $\subseteq$ (1, 2)$S_\beta(f^{-1}(E))$.

(vii) $\Rightarrow$ (i) Let $x \in X$ and V be any open subset of Y containing $f(x)$, then by (vii), $f^{-1}(\tau_1$-int (V)) $\subseteq$ (1, 2)S_β-int($f^{-1}(V)$),which implies ($f^{-1}(V)$) $\subseteq$ (1, 2)$S_\beta(f^{-1}(V))$. Hence, $f^{-1}(V)$ is a (1, 2)S_β-open set in X containing x such that $f(f^{-1}(V)) \subseteq V$. Thus, f is a (1, 2)S_β-continuous function. □

Theorem 3.9 Let f:(X, τ_1, τ_2) $\rightarrow$ (Y, σ_1, σ_2) be a function and τ_1 be any basis for σ in Y. Then f is a (1, 2)S_β-continuous function if and only if for each $B \in \tau_1$, $f^{-1}(B)$ is a (1, 2)S_β- open subset of X.

Proof: Suppose f is a (1, 2)S_β-continuous function and each $U \in \tau_1$ is an open subset of Y then by Proposition 2.11, $f^{-1}(B)$ is a (1, 2)S_β- open subset of X.

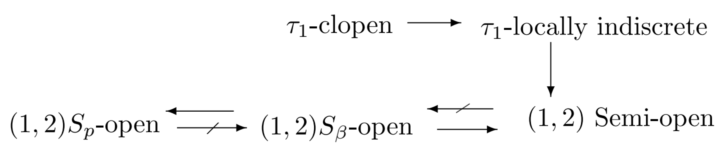
Conversely, for each $U \in \tau_1$ is a (1, 2)S_β-open subset of X. Let V be any (1, 2)S_β-open set in Y, $V = \{\cup U_i : i \in \Delta\}$ where U_i is a member of τ_1. Then $f^{-1}(V) = f^{-1}\{\cup U_i : i \in \Delta\} = \cup f^{-1}(U_i); i \in \Delta$. Since $f^{-1}(U_i)$ is a (1, 2)S_β-open subset in X for each $i \in \Delta$, therefore $f^{-1}(V)$ is the union of a family of (1, 2)S_β- open sets of X and also a (1, 2)S_β-open set of X. Hence, by Proposition 2.11, f is a (1, 2)S_β-continuous function. □

Theorem 3.10 Let f:(X, τ_1, τ_2) $\to$ (Y, σ_1, σ_2) be a (1, 2)S_β-continuous function and $B \subseteq X$ such that B is τ_1 - clopen, then by the restriction function $f|B : B \to Y$ is a (1, 2)S_β-continuous function.

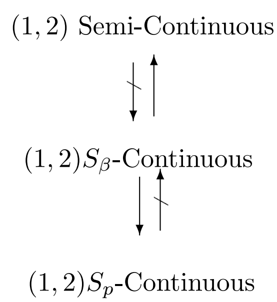
Proof: Let B be any (1, 2)S_β-open subset of Y. Since f is a (1, 2)S_β-continuous function, by Proposition 2.11, $f^{-1}(V) \in (1, 2)S_\beta - O(X)$, but B is τ_1 - clopen, then $B \in$ (1, 2)β O(X). By Lemma 2.8, $f^{-1}(V) \cap A \in$(1, 2)β-O(X), then $f^{-1}(V) \cap A = (f|A)^{-1}(V) \in$ (1, 2)S_β-O(B). Hence, $f|B$ is a (1, 2)S_β-continuous function. □

From the above theorems, we obtain the following diagrams for open sets and continuous functions respectively.

(*i*)



(*ii*)



4. Conclusion

In this work, we define continuous function by using (1, 2)S_β-open sets in bitopological spaces and study some of its properties. We believe that this work will lead to further generalizations such as $(1,2)^*$ - α^*-continuous functions in bitopological properties.

References

1. M. E. Abd El-Monsef, S. N. El-Deeb, R. A. Mahmoud, β-open sets and β-continuity mappings, Bull. Fac. Sci. Assiut Univ., Vol. 12 (1983), 77-90.
2. N. K. Ahmed, On Some types of separation axioms, M. Sc., Thesis, College of Science, Salahaddin Univ., (1990).
3. A. B.Khalaf, N. K.Ahmed, S_β-open sets and S_β-continuity in topological spaces, Thai Journal of Mathematics, Vol. 11 (2) (2013), 319-335.
4. A. C. S. Raj, S. Jackson, T. G. Denosha, Some Continuous functions in Bitopological spaces, OUTREACH Vol. XII, (2019), 60-67

5. D. Andrijevic, Semi-pre open sets, Mat. Vesnik Vol.38 (1986), 24-32.

6. J.C, Kelly, Bitopological spaces Proc.London Math. Soc.,13 Vol.3 (1963), 71-89.

7. N. Levine, Semi-open Sets and Semi- Continuity in topological spaces, The American Mathematical Monthly, Vol. 70 (1)(1963), 36-41.

8. S. Bose, Semi-open sets, semi-continuity and semi-open mapping in bitopological spaces, Bull. Calcutta Math. Soc., Vol.73 (1981), 237-246.

9. V. Subprabha, N. Durgadevi, A New approach to (1, 2)semi-open sets in Bitopological spaces, International Journal of Mathematics Trends and Technology, Special Issue (2019), 58-61.

V. Subprabha, P. Jeyanthi,
Department of Mathematics,
Govindammal Aditanar College for Women, Tiruchendur-628215
Tamilnadu, India.
E-mail address: suvekafamily@gmail.com, jeyajeyanthi@rediffmail.com

and

N. Durga Devi,
Department of Mathematics,
Sri Parasakthi College for Women (Autonomous), Courtallam, Tenkasi-627 802
Tamilnadu, India.
E-mail address: durgadevin681@gmail.com

and

S. Jafari,
Mathematical and Physical Science Foundation,
4200 *Slagelse, Denmark.*
E-mail address: jafaripersia@gmail.com