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Nonlinear Diophantine Rough Fuzzy Sets-Based Multicriteria Group Decision-Making in Uncertain Environments

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ABSTRACT: Several operational laws, Hamacher operations, and creative aggregation techniques for handling Nonlinear Diophantine Rough Fuzzy data are thoroughly examined in this work. We present NLDRFHWG, NLDRFHOWG, and NLDRFHHWG, three new geometric operators created especially to improve the accuracy and dependability of arithmetic aggregation. These operators are thoroughly examined, emphasizing their characteristics and connections to current aggregation techniques. We also illustrate how these operators are related to hamacher-type operations and how the suggested models broaden their use. For the suggested operators, three essential mathematical properties are established: idempotency, monotonicity, and boundedness. Additionally, the practical utility of the NLDRFHWG operator is demonstrated by applying it to multiple attribute decision-making situations employing NLDRF data. The results of this work provide sophisticated methods for combining and analyzing complicated, ambiguous, or inconsistent data, which adds insightful information. The usefulness and practicality of the suggested operators are further illustrated by a numerical example, which validates their possible importance in future research and decision-making.

Keywords: Rough fuzzy sets, Hamacher, decision making, fuzzy neural networking.

Contents

1. Introduction

Zadeh [1] proposed the idea of fuzzy sets, it led to a number of important extensions, such as Atanassov's intuitionistic fuzzy sets [3], interval-valued intuitionistic fuzzy sets [4], and interval-valued fuzzy sets [2]. Nonetheless, Atanassov's intuitionistic fuzzy sets was mathematically identical to interval-valued fuzzy sets, as mentioned by Zadeh [5] and others [6-7]. The main application of the AIFS framework is intuitionistic fuzzy information representation, a challenging field that necessitates a thorough comprehension of both fuzzy theory and aggregation operators [8]. Interval-valued intuitionistic fuzzy sets

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were created after the AIFS model was expanded to accommodate interval-valued membership and non-membership functions [9]. Intuitionistic fuzzy numbers were a crucial development to support decision-making applications. Xu [10] presented a number of aggregation operators for IFNs, including IFWA, IFOWA, and IFHA, and examined some of their salient characteristics. Later, Xu and Yager [11] developed novel geometric aggregation operators, such as IFWG, IFOGA, and IFHG, and showed their value in a number of real-world applications. Additional work was contributed by Zeng and Su [12], who created the intuitionistic fuzzy ordered weighted distance operator and used it to solve system selection and group decision-making problems, offering insightful information for study. Numerous important approaches have been created in the rapidly changing field of fuzzy decision-making to improve the management of imprecision and uncertainty in intricate assessment systems. In order to provide a methodical way to rank options under uncertainty, Gong et al. [15] suggested using goal programming to extract priority vectors from intuitionistic fuzzy preference relations. Li [16, 17] expanded on this basis by presenting a number of sophisticated techniques that improved multiattribute decision-making procedures in intuitionistic and interval-valued intuitionistic fuzzy environments. His contributions include a TOPSIS-based nonlinear programming framework for increased accuracy in interval-valued assessments Li [18], and generalized ordered weighted averaging operators for flexible aggregation Li [19]. The development of intuitionistic fuzzy multiple attribute decision-making techniques has played a crucial role in addressing uncertainty and hesitation in complex decision scenarios. Among the leading contributors in this area, Wei introduced several influential models that significantly enhanced the precision and flexibility of decision-making under intuitionistic fuzzy environments. In his early work, Wei [20] proposed the maximizing deviation method, providing a systematic approach to determine attribute weights objectively in uncertain contexts. Subsequently, Wei [21] developed a gray relational analysis method capable of handling incomplete weight information, expanding the applicability of intuitionistic fuzzy systems. His later research, such as Wei [22] and Wei and Zhao [23], introduced minimum deviation models and refined GRA-based methods, offering robust tools for optimizing decision outcomes. Wei et al [24] extended these concepts by applying correlation coefficients to interval-valued intuitionistic fuzzy data, demonstrating their effectiveness in managing incomplete and uncertain information. Collectively, these studies have established a strong theoretical foundation for IF-MADM, enabling more accurate, adaptable, and practical decision-making in uncertain environments. An integral component of contemporary decision-making under uncertainty is the study of intuitionistic fuzzy sets (IFSs) and their extensions. Grzegorzewski [25] established a distance measure based on the Hausdorff metric to quantify similarity and dissimilarity between fuzzy and intuitionistic fuzzy information, offering a fundamental framework for assessing relationships among fuzzy data. Wang and Liu [26, 27] built on this foundation by proposing novel aggregation methods for intuitionistic fuzzy information that are based on Einstein. Their approaches, such as the Einstein aggregation frameworks and intuitionistic fuzzy geometric aggregation operators, improved the accuracy and adaptability of information integration by nonlinearly taking membership and non-membership degrees into account. Later, by creating intuitionistic fuzzy Einstein hybrid aggregation operators, Zhao and Wei [28] extended these ideas and showed how they may be used practically in multiple attribute decision-making issues. When taken as a whole, these investigations have strengthened intuitionistic fuzzy aggregation's theoretical underpinnings and practical efficacy, providing strong instruments for intricate decision-making in unpredictable and changing contexts.

The break of this paper is systematized as follows: Section 2, define the basic idea. Section 3 Write a NonLDRFS and operational laws. Section 4 recommends a series of three aggregation operators. Section 5 introduces a NonLDRF Based Multi-Criteria Decision making for fuzzy neural networking and delivers a numerical instance to validate the rationality. Section 6 summarizes the conclusion.

2. Backgroud

Definition 2.1 (27) *Let $\Phi \neq X$ and a fuzzy set γ be defined as*

$$\gamma = \{\langle x, \mu_\gamma(x)\rangle \mid x \in X\},$$

where $\mu_\gamma(x) : X \to [0,1]$ is the membership function of element x in X.

Definition 2.2 (2) *Let C be a fixed set and A an intuitionistic fuzzy set (IFS) on C defined as*

$$A = \{\langle x, \varsigma_A(x), \chi_A(x)\rangle \mid x \in C\},$$

where $\varsigma_A(x)$ and $\chi_A(x)$ represent the membership (MED) and non-membership (NOMED) degrees, satisfying

$$0 \leq \varsigma_A(x), \chi_A(x) \leq 1, \quad 0 \leq \varsigma_A(x) + \chi_A(x) \leq 1.$$

The degree of indeterminacy is

$$\pi_A(x) = 1 - \varsigma_A(x) - \chi_A(x),$$

and the IFS is denoted by $A = \langle \varsigma_A, \chi_A\rangle$.

3. Operational laws on non-linear diophantie rough fuzzy set

In this section, we define the operational laws of non-linear diophantine rough fuzzy set, accuracy and score functions.

Definition 3.1 *Let* $\gamma_1 = \begin{bmatrix} \begin{pmatrix} \left\langle \overline{A_1^q}, \overline{B_1^q} \right\rangle, \\ \left\langle \overline{a_1}, \overline{b_1} \right\rangle \end{pmatrix}, \\ \begin{pmatrix} \left\langle \underline{A_1^q}, \underline{B_1^q} \right\rangle, \\ \left\langle \underline{a_1}, \underline{b_1} \right\rangle \end{pmatrix} \end{bmatrix}$ *and* $\gamma_2 = \begin{bmatrix} \begin{pmatrix} \left\langle \overline{A_2^q}, \overline{B_2^q} \right\rangle, \\ \left\langle \overline{a_2}, \overline{b_2} \right\rangle \end{pmatrix}, \\ \begin{pmatrix} \left\langle \underline{A_2^q}, \underline{B_2^q} \right\rangle, \\ \left\langle \underline{a_2}, \underline{b_2} \right\rangle \end{pmatrix} \end{bmatrix}$ *be any two NL-DRFSs, with $\gamma \succ 0, \lambda \succ 0$ and $q \succeq 1$, then define basic hamacher operations for NLDRFSs as follows:*

1) $\gamma_1 \oplus \gamma_2$

$$= \begin{bmatrix} \left(\left\langle \begin{array}{c} \frac{\overline{(A_1^q)}+\overline{(A_2^q)}-\overline{(A_1^q)(A_2^q)}-(1-\gamma)\overline{(A_1^q)(A_2^q)}}{1-(1-\gamma)\overline{(A_1^q)(A_2^q)}}, \\ \frac{\overline{(B_1^q)(B_2^q)}}{\gamma+(1-\gamma)\left(\overline{(B_1^q)}+\overline{(B_2^q)}-\overline{(B_1^q)(B_2^q)}\right)} \end{array} \right\rangle, \right) \\ \left(\left\langle \begin{array}{c} \frac{\sqrt[q]{\overline{(a_1)^q}+\overline{(a_2)^q}-\overline{(a_1)^q(a_2)^q}-(1-\gamma)\overline{(a_1)^q(a_2)^q}}}{1-(1-\gamma)\left(\overline{(a_1)^q(a_2)^q}\right)}, \\ \frac{\overline{(b_1)(b_2)}}{\sqrt[q]{\gamma+(1-\gamma)\left(\overline{(b_1)^q}+\overline{(b_2)^q}-\overline{(b_1)^q(b_2)^q}\right)}} \end{array} \right\rangle \right) \\ \left(\left\langle \begin{array}{c} \frac{\underline{(A_1^q)}+\underline{(A_2^q)}-\underline{(A_1^q)(A_2^q)}-(1-\gamma)\underline{(A_1^q)(A_2^q)}}{1-(1-\gamma)\underline{(A_1^q)(A_2^q)}}, \\ \frac{\underline{(B_1^q)(B_2^q)}}{\gamma+(1-\gamma)\left(\underline{(B_1^q)}+\underline{(B_2^q)}-\underline{(B_1^q)(B_2^q)}\right)} \end{array} \right\rangle, \right) \\ \left(\left\langle \begin{array}{c} \frac{\sqrt[q]{\underline{(a_1)^q}+\underline{(a_2)^q}-\underline{(a_1)^q(a_2)^q}-(1-\gamma)\underline{(a_1)^q(a_2)^q}}}{1-(1-\gamma)\left(\underline{(a_1)^q(a_2)^q}\right)}, \\ \frac{\underline{(b_1)(b_2)}}{\sqrt[q]{\gamma+(1-\gamma)\left(\underline{(b_1)^q}+\underline{(b_2)^q}-\underline{(b_1)^q(b_2)^q}\right)}} \end{array} \right\rangle \right) \end{bmatrix}$$

2) $\gamma_1 \otimes \gamma_2$

$$
= \left[\begin{array}{c}
\left(\left\langle \begin{array}{c}
\frac{\overline{\left(A_1^q\right)\left(A_2^q\right)}}{\gamma+(1-\gamma)\left(\overline{\left(A_1^q\right)}+\overline{\left(A_2^q\right)}-\overline{\left(A_1^q\right)\left(A_2^q\right)}\right)}, \\
\frac{\overline{\left(B_1^q\right)}+\overline{\left(B_2^q\right)}-\overline{\left(B_1^q\right)\left(B_2^q\right)}-(1-\gamma)\overline{\left(B_1^q\right)\left(B_2^q\right)}}{1-(1-\gamma)\overline{\left(B_1^q\right)\left(B_2^q\right)}}
\end{array} \right\rangle, \right) \\
\left(\left\langle \begin{array}{c}
\frac{\overline{(a_1)(a_2)}}{\sqrt[q]{\gamma+(1-\gamma)\left(\overline{(a_1)^q}+\overline{(a_2)^q}-\overline{(a_1)^q(a_2)^q}\right)}}, \\
\sqrt[q]{\frac{\overline{(b_1)^q}+\overline{(b_2)^q}-\overline{(b_1)^q(b_2)^q}-(1-\gamma)\overline{(b_1)^q(b_2)^q}}{1-(1-\gamma)\overline{(b_1)^q(b_2)^q}}}
\end{array} \right\rangle \right) \\
\left(\left\langle \begin{array}{c}
\frac{\underline{\left(A_1^q\right)\left(A_2^q\right)}}{\gamma+(1-\gamma)\left(\underline{\left(A_1^q\right)}+\underline{\left(A_2^q\right)}-\underline{\left(A_1^q\right)\left(A_2^q\right)}\right)}, \\
\frac{\underline{\left(B_1^q\right)}+\underline{\left(B_2^q\right)}-\underline{\left(B_1^q\right)\left(B_2^q\right)}-(1-\gamma)\underline{\left(B_1^q\right)\left(B_2^q\right)}}{1-(1-\gamma)\underline{\left(B_1^q\right)\left(B_2^q\right)}}
\end{array} \right\rangle, \right) \\
\left(\left\langle \begin{array}{c}
\frac{\underline{(a_1)(a_2)}}{\sqrt[q]{\gamma+(1-\gamma)\left(\underline{(a_1)^q}+\underline{(a_2)^q}-\underline{(a_1)^q(a_2)^q}\right)}}, \\
\sqrt[q]{\frac{\underline{(b_1)^q}+\underline{(b_2)^q}-\underline{(b_1)^q(b_2)^q}-(1-\gamma)\underline{(b_1)^q(b_2)^q}}{1-(1-\gamma)\underline{(b_1)^q(b_2)^q}}}
\end{array} \right\rangle \right)
\end{array} \right]
$$

$3)\lambda\gamma_1$

$$
= \left[\begin{array}{c}
\left(\left\langle \begin{array}{c}
\frac{\left(1+(\gamma-1)\overline{A_1^q}\right)^\lambda-\overline{\left(1-A_1^q\right)^\lambda}}{\left(1+(\gamma-1)\overline{A_1^q}\right)^\lambda+(\gamma-1)\overline{\left(1-A_1^q\right)^\lambda}}, \\
\frac{\gamma\left(B_1^q\right)^\lambda}{\left(1+(\gamma-1)\overline{\left(1-B_1^q\right)}\right)^\lambda+(\gamma-1)\overline{\left(B_1^q\right)^\lambda}}
\end{array} \right\rangle, \right) \\
\left(\left\langle \begin{array}{c}
\sqrt[q]{\frac{\left(1+(\gamma-1)\overline{(a_1)^q}\right)^\lambda-\left(1-\overline{(a_1)^q}\right)^\lambda}{\left(1-\overline{(\gamma-1)(a_1)^q}\right)^\lambda+\overline{(\gamma-1)(1-(a_1)^q)^\lambda}}}, \\
\frac{\overline{\sqrt[q]{\gamma}(b_1)^\lambda}}{\sqrt[q]{\left(1+(\gamma-1)\overline{(1-(b_1)^q)}\right)^\lambda+(\gamma-1)\overline{((b_1)^q)^\lambda}}}
\end{array} \right\rangle \right) \\
\left(\left\langle \begin{array}{c}
\frac{\left(1+(\gamma-1)\underline{A_1^q}\right)^\lambda-\underline{\left(1-A_1^q\right)^\lambda}}{\left(1+(\gamma-1)\underline{A_1^q}\right)^\lambda+(\gamma-1)\underline{\left(1-A_1^q\right)^\lambda}}, \\
\frac{\underline{\gamma\left(B_1^q\right)^\lambda}}{\left(1+(\gamma-1)\underline{\left(1-B_1^q\right)}\right)^\lambda+(\gamma-1)\underline{\left(B_1^q\right)^\lambda}}
\end{array} \right\rangle, \right) \\
\left(\left\langle \begin{array}{c}
\sqrt[q]{\frac{\left(1+(\gamma-1)\underline{(a_1)^q}\right)^\lambda-\underline{(1-(a_1)^q)^\lambda}}{\left(1-(\gamma-1)\underline{(a_1)^q}\right)^\lambda+(\gamma-1)\underline{(1-(a_1)^q)^\lambda}}}, \\
\frac{\underline{\sqrt[q]{\gamma}(b_1)^\lambda}}{\sqrt[q]{\left(1+(\gamma-1)\underline{(1-(b_1)^q)}\right)^\lambda+(\gamma-1)\underline{((b_1)^q)^\lambda}}}
\end{array} \right\rangle \right)
\end{array} \right]
$$

$$4)\ \gamma_1^{\lambda}=\left[\begin{array}{c}\left(\left\langle\begin{array}{c}\frac{\overline{\gamma\left(A_1^q\right)^{\lambda}}}{\left(1+(\gamma-1)\overline{\left(1-A_1^q\right)}\right)^{\lambda}+(\gamma-1)\overline{\left(A_1^q\right)^{\lambda}}},\\ \frac{\left(1+(\gamma-1)\overline{\left(B_1^q\right)}\right)^{\lambda}-\overline{\left(1-B_1^q\right)^{\lambda}}}{\left(1+(\gamma-1)\overline{B_1^q}\right)^{\lambda}+(\gamma-1)\overline{\left(1-B_1^q\right)^{\lambda}}}\end{array}\right\rangle,\right)\\ \left(\left\langle\begin{array}{c}\frac{\overline{\sqrt[q]{\gamma}(a_1)^{\lambda}}}{\sqrt[q]{\left(1+(\gamma-1)\overline{(1-(a_1)^q)}\right)^{\lambda}+\gamma-1\overline{((a_1)^q)^{\lambda}}}},\\ \sqrt[q]{\frac{\left(1+(\gamma-1)\overline{(b_1)^q}\right)^{\lambda}-\overline{(1-(b_1)^q)^{\lambda}}}{\left(1+(\gamma-1)\overline{(b_1)^q}\right)^{\lambda}+(\gamma-1)\overline{(1-(b_1)^q)^{\lambda}}}}\end{array}\right\rangle\right)\\ \left(\left\langle\begin{array}{c}\frac{\underline{\gamma\left(A_1^q\right)^{\lambda}}}{\left(1+(\gamma-1)\underline{\left(1-A_1^q\right)}\right)^{\lambda}+(\gamma-1)\underline{\left(A_1^q\right)^{\lambda}}},\\ \frac{\left(1+(\gamma-1)\underline{\left(B_1^q\right)}\right)^{\lambda}-\underline{\left(1-B_1^q\right)^{\lambda}}}{\left(1+(\gamma-1)\underline{B_1^q}\right)^{\lambda}+(\gamma-1)\underline{\left(1-B_1^q\right)^{\lambda}}}\end{array}\right\rangle,\right)\\ \left(\left\langle\begin{array}{c}\frac{\underline{\sqrt[q]{\gamma}(a_1)^{\lambda}}}{\sqrt[q]{\left(1+(\gamma-1)\underline{(1-(a_1)^q)}\right)^{\lambda}+(\gamma-1)\underline{((a_1)^q)^{\lambda}}}},\\ \sqrt[q]{\frac{\left(1+(\gamma-1)\underline{(b_1)^q}\right)^{\lambda}-\underline{(1-(b_1)^q)^{\lambda}}}{\left(1+(\gamma-1)\underline{(b_1)^q}\right)^{\lambda}+(\gamma-1)\underline{(1-(b_1)^q)^{\lambda}}}}\end{array}\right\rangle\right)\end{array}\right]$$

Definition 3.2 *Let* $H=\left[\begin{array}{c}\left(\left\langle\overline{A^q},\overline{B^q}\right\rangle,\left\langle\overline{a},\overline{b}\right\rangle\right),\\ \left(\left\langle\underline{A^q},\underline{B^q}\right\rangle,\left\langle\underline{a},\underline{b}\right\rangle\right)\end{array}\right]$ *be a NLDRFS, then score function is defined as:* $H=\left[\frac{\left\langle\overline{A^q}-\overline{B^q}\right\rangle+\left\langle\overline{a}-\overline{b}\right\rangle+\left\langle\underline{A^q}-\underline{B^q}\right\rangle+\left\langle\underline{a}-\underline{b}\right\rangle}{4}\right]$.

Definition 3.3 *Let* $L=\left[\begin{array}{c}\left(\left\langle\overline{A^q},\overline{B^q}\right\rangle,\left\langle\overline{a},\overline{b}\right\rangle\right),\\ \left(\left\langle\underline{A^q},\underline{B^q}\right\rangle,\left\langle\underline{a},\underline{b}\right\rangle\right)\end{array}\right]$ *be a NLDRFS, then accuracy function is defined as:* $L=\left[\frac{\left\langle\overline{A^q}+\overline{B^q}\right\rangle+\left\langle\overline{a}+\overline{b}\right\rangle+\left\langle\underline{A^q}+\underline{B^q}\right\rangle+\left\langle\underline{a}+\underline{b}\right\rangle}{4}\right]$.

4. Aggregation operators on non-linear diophantine rough fuzzy set

In this section, we define the NLDRFWG, NLDRFOWG and NLDRFHWG operators.

4.1. NLDRFHWG operator

Definition 4.1 *Let* $\gamma_{dq\Psi}=\left\{\begin{array}{c}\left(\left\langle\overline{A_{dq}^{\Psi}},\overline{B_{dq}^{\Psi}}\right\rangle,\left\langle\overline{a},\overline{b}\right\rangle\right),\\ \left(\left\langle\underline{A_{dq}^{\Psi}},\underline{B_{dq}^{\Psi}}\right\rangle,\left\langle\underline{a},\underline{b}\right\rangle\right):\Psi\epsilon\mathbb{Z}\end{array}\right\}$ *be a collection of NLDRFSs over the fixed sey* $\mathbb{Z}$ *and* $\Omega=(\Omega_1,\Omega_2,...,\Omega_n)^T$ *is the weights with* $\sum_{\Psi=1}^{n}\Omega_{\Psi}=1,q\geq 1$;*then define NLDRF Hamacher weighted geometric operator as follows and let the transformation* $\theta:NLDRF(\mathbb{Z})\rightarrow NLDRFN(\mathbb{Z})$

$$NLDRFHWG(\gamma_{dq1},\gamma_{dq2},...,\gamma_{dqn})$$
$$=\bigotimes_{\Psi=1}^{n}\left(\gamma_{dq\Psi}^{\Omega_{\Psi}}\right)=\left(\gamma_{dq1}^{\Omega_1}\bigotimes\gamma_{dq2}^{\Omega_2}\bigotimes,...,\bigotimes\gamma_{dqn}^{\Omega_n}\right).$$

Theorem 4.1 *Let* $\gamma_{dq\Psi}=\left\{\begin{array}{c}\left(\left\langle\overline{A_{dq}^{\Psi}},\overline{B_{dq}^{\Psi}}\right\rangle,\left\langle\overline{a},\overline{b}\right\rangle\right),\\ \left(\left\langle\underline{A_{dq}^{\Psi}},\underline{B_{dq}^{\Psi}}\right\rangle,\left\langle\underline{a},\underline{b}\right\rangle\right):\Psi\epsilon\mathbb{Z}\end{array}\right\}$ *be an assembling of NLDRFSs over the fixed set* $\mathbb{Z}$ *and* $\Omega=(\Omega_1,\Omega_2,...,\Omega_n)^T$ *is the weights with* $\sum_{\Psi=1}^{n}\Omega_{\Psi}=1$,*and* $q\geq 1,\gamma>0$;*then by applying the NLDRFHWG operator their aggregated values is also an NLDRFS,and the transformation* $\theta:NLDRF(\mathbb{Z})\rightarrow NLDRFN(\mathbb{Z})$ *is called NLDRF Hamacher weighted geometric operator and define as follows:*$NLDRFHWG(\gamma_{dq1},\gamma_{dq2},...,\gamma_{dqn})$

$$
= \left[\begin{array}{c}
\left(\left\langle \dfrac{\gamma\prod_{\Psi=1}^{n}\overline{\left(A_{dq}^{\Psi}\right)}^{\Omega_\Psi}}{\prod_{\Psi=1}^{n}\left(1+(\gamma-1)\left(1-\overline{A_{dq}^{\Psi}}\right)\right)^{\Omega_\Psi}+(\gamma-1)\prod_{\Psi=1}^{n}\overline{\left(A_{dq}^{\Psi}\right)}^{\Omega_\Psi}}, \dfrac{\prod_{\Psi=1}^{n}\left(1+(\gamma-1)\overline{B_{dq}^{\Psi}}\right)^{\Omega_\Psi}-\prod_{\Psi=1}^{n}\left(1-\overline{B_{dq}^{\Psi}}\right)^{\Omega_\Psi}}{\prod_{\Psi=1}^{n}\left(1+(\gamma-1)\overline{B_{dq}^{\Psi}}\right)^{\Omega_\Psi}+(\gamma-1)\prod_{\Psi=1}^{n}\left(1-\overline{B_{dq}^{\Psi}}\right)^{\Omega_\Psi}} \right\rangle\right), \\
\left(\left\langle \dfrac{\sqrt[q]{\gamma}\prod_{\Psi=1}^{n}\overline{(a)}^{\Omega_\Psi}}{\sqrt[q]{\prod_{\Psi=1}^{n}\left(1+(\gamma-1)\left(1-\overline{(a)^q}\right)\right)^{\Omega_\Psi}+(\gamma-1)\prod_{\Psi=1}^{n}\left(\overline{(a)^q}\right)^{\Omega_\Psi}}}, \sqrt[q]{\dfrac{\prod_{\Psi=1}^{n}\left(1+(\gamma-1)\overline{(b)^q}\right)^{\Omega_\Psi}-\prod_{\Psi=1}^{n}\left(1-\overline{(b)^q}\right)^{\Omega_\Psi}}{\prod_{\Psi=1}^{n}\left(1+(\gamma-1)\overline{(b)^q}\right)^{\Omega_\Psi}+(\gamma-1)\prod_{\Psi=1}^{n}\left(1-\overline{(b)^q}\right)^{\Omega_\Psi}}} \right\rangle\right). \\
\left(\left\langle \dfrac{\gamma\prod_{\Psi=1}^{n}\underline{\left(A_{dq}^{\Psi}\right)}^{\Omega_\Psi}}{\prod_{\Psi=1}^{n}\left(1+(\gamma-1)\left(1-\underline{A_{dq}^{\Psi}}\right)\right)^{\Omega_\Psi}+(\gamma-1)\prod_{\Psi=1}^{n}\underline{\left(A_{dq}^{\Psi}\right)}^{\Omega_\Psi}}, \dfrac{\prod_{\Psi=1}^{n}\left(1+(\gamma-1)\underline{B_{dq}^{\Psi}}\right)^{\Omega_\Psi}-\prod_{\Psi=1}^{n}\left(1-\underline{B_{dq}^{\Psi}}\right)^{\Omega_\Psi}}{\prod_{\Psi=1}^{n}\left(1+(\gamma-1)\underline{B_{dq}^{\Psi}}\right)^{\Omega_\Psi}+(\gamma-1)\prod_{\Psi=1}^{n}\left(1-\underline{B_{dq}^{\Psi}}\right)^{\Omega_\Psi}} \right\rangle\right), \\
\left(\left\langle \dfrac{\sqrt[q]{\gamma}\prod_{\Psi=1}^{n}\underline{(a)}^{\Omega_\Psi}}{\sqrt[q]{\prod_{\Psi=1}^{n}\left(1+(\gamma-1)\left(1-\underline{(a)^q}\right)\right)^{\Omega_\Psi}+(\gamma-1)\prod_{\Psi=1}^{n}\left(\underline{(a)^q}\right)^{\Omega_\Psi}}}, \sqrt[q]{\dfrac{\prod_{\Psi=1}^{n}\left(1+(\gamma-1)\underline{(b)^q}\right)^{\Omega_\Psi}-\prod_{\Psi=1}^{n}\left(1-\underline{(b)^q}\right)^{\Omega_\Psi}}{\prod_{\Psi=1}^{n}\left(1+(\gamma-1)\underline{(b)^q}\right)^{\Omega_\Psi}+(\gamma-1)\prod_{\Psi=1}^{n}\left(1-\underline{(b)^q}\right)^{\Omega_\Psi}}} \right\rangle\right).
\end{array}\right]
$$

Proof: By induction method to prove this theorem. Now we put $n = 2$.

So for NLDRFSs based on hamacher product, obtained the associated result.

The left side of the equation become NLDRFHWG $(\gamma_{dq1}, \gamma_{dq2}) = \gamma_{dq1}^{\Omega_1} \bigotimes \gamma_{dq2}^{\Omega_2}$.

$$
= \left[\begin{array}{c}
\left(\left\langle \frac{\gamma\left(\overline{A^1_{dq}}\right)^{\Omega_1}}{\left(1+(\gamma-1)\left(1-\overline{A^1_{dq}}\right)\right)^{\Omega_1}+(\gamma-1)\left(\overline{A^1_{dq}}\right)^{\Omega_1}}, \frac{\left(1+(\gamma-1)\overline{B^1_{dq}}\right)^{\Omega_1}-\left(1-\overline{B^1_{dq}}\right)^{\Omega_1}}{\left(1+(\gamma-1)\overline{B^1_{dq}}\right)^{\Omega_1}+(\gamma-1)\left(1-\overline{B^1_{dq}}\right)^{\Omega_1}} \right\rangle\right), \\
\left(\left\langle \frac{\sqrt[q]{\gamma}\,\overline{(a)}^{\Omega_1}}{\sqrt[q]{\left(1+(\gamma-1)\left(1-\overline{(a)^q}\right)\right)^{\Omega_1}+(\gamma-1)\left(\overline{(a)^q}\right)^{\Omega_1}}}, \sqrt[q]{\frac{\left(1+(\gamma-1)\overline{(b)^q}\right)^{\Omega_1}-\left(1-\overline{(b)^q}\right)^{\Omega_1}}{\left(1+(\gamma-1)\overline{(b)^q}\right)^{\Omega_1}+(\gamma-1)\left(1-\overline{(b)^q}\right)^{\Omega_1}}} \right\rangle\right) \\
\left(\left\langle \frac{\gamma\left(\underline{A^1_{dq}}\right)^{\Omega_1}}{\left(1+(\gamma-1)\left(1-\underline{A^1_{dq}}\right)\right)^{\Omega_1}+(\gamma-1)\left(\underline{A^1_{dq}}\right)^{\Omega_1}}, \frac{\left(1+(\gamma-1)\underline{B^1_{dq}}\right)^{\Omega_1}-\left(1-\underline{B^1_{dq}}\right)^{\Omega_1}}{\left(1+(\gamma-1)\underline{B^1_{dq}}\right)^{\Omega_1}+(\gamma-1)\left(1-\underline{B^1_{dq}}\right)^{\Omega_1}} \right\rangle\right), \\
\left(\left\langle \frac{\sqrt[q]{\gamma}\,\underline{(a)}^{\Omega_1}}{\sqrt[q]{\left(1+(\gamma-1)\left(1-\underline{(a)}^q\right)\right)^{\Omega_1}+(\gamma-1)\left(\underline{(a)}^q\right)^{\Omega_1}}}, \sqrt[q]{\frac{\left(1+(\gamma-1)\underline{(b)}^q\right)^{\Omega_1}-\left(1-\underline{(b)}^q\right)^{\Omega_1}}{\left(1+(\gamma-1)\underline{(b)}^q\right)^{\Omega_1}+(\gamma-1)\left(1-\underline{(b)}^q\right)^{\Omega_1}}} \right\rangle\right)
\end{array}\right] \otimes
$$

$$
\left[\begin{array}{c}
\left(\left\langle \frac{\gamma\left(\overline{A^2_{dq}}\right)^{\Omega_2}}{\left(1+(\gamma-1)\left(1-\overline{A^2_{dq}}\right)\right)^{\Omega_2}+(\gamma-1)\left(\overline{A^2_{dq}}\right)^{\Omega_2}}, \frac{\left(1+(\gamma-1)\overline{B^2_{dq}}\right)^{\Omega_2}-\left(1-\overline{B^2_{dq}}\right)^{\Omega_2}}{\left(1+(\gamma-1)\overline{B^2_{dq}}\right)^{\Omega_2}+(\gamma-1)\left(1-\overline{B^2_{dq}}\right)^{\Omega_2}} \right\rangle\right), \\
\left(\left\langle \frac{\sqrt[q]{\gamma}\,\overline{(a)}^{\Omega_2}}{\sqrt[q]{\left(1+(\gamma-1)\left(1-\overline{(a)^q}\right)\right)^{\Omega_2}+(\gamma-1)\left(\overline{(a)^q}\right)^{\Omega_2}}}, \sqrt[q]{\frac{\left(1+(\gamma-1)\overline{(b)^q}\right)^{\Omega_2}-\left(1-\overline{(b)^q}\right)^{\Omega_2}}{\left(1+(\gamma-1)\overline{(b)^q}\right)^{\Omega_2}+(\gamma-1)\left(1-\overline{(b)^q}\right)^{\Omega_2}}} \right\rangle\right) \\
\left(\left\langle \frac{\gamma\left(\underline{A^2_{dq}}\right)^{\Omega_2}}{\left(1+(\gamma-1)\left(1-\underline{A^2_{dq}}\right)\right)^{\Omega_2}+(\gamma-1)\left(\underline{A^2_{dq}}\right)^{\Omega_2}}, \frac{\left(1+(\gamma-1)\underline{B^2_{dq}}\right)^{\Omega_2}-\left(1-\underline{B^2_{dq}}\right)^{\Omega_2}}{\left(1+(\gamma-1)\underline{B^2_{dq}}\right)^{\Omega_2}+(\gamma-1)\left(1-\underline{B^2_{dq}}\right)^{\Omega_2}} \right\rangle\right), \\
\left(\left\langle \frac{\sqrt[q]{\gamma}\,\underline{(a)}^{\Omega_2}}{\sqrt[q]{\left(1+(\gamma-1)\left(1-\underline{(a)}^q\right)\right)^{\Omega_2}+(\gamma-1)\left(\underline{(a)}^q\right)^{\Omega_2}}}, \sqrt[q]{\frac{\left(1+(\gamma-1)\underline{(b)}^q\right)^{\Omega_2}-\left(1-\underline{(b)}^q\right)^{\Omega_2}}{\left(1+(\gamma-1)\underline{(b)}^q\right)^{\Omega_2}+(\gamma-1)\left(1-\underline{(b)}^q\right)^{\Omega_2}}} \right\rangle\right)
\end{array}\right].
$$

$$\left[\begin{array}{c}
\left(\left\langle \dfrac{\gamma\prod\limits_{\Psi=1}^{2}\overline{\left(A_{dq}^{\Psi}\right)}^{\Omega_\Psi}}{\prod\limits_{\Psi=1}^{2}\left(1+(\gamma-1)\left(1-\overline{A_{dq}^{\Psi}}\right)\right)^{\Omega_\Psi}+(\gamma-1)\prod\limits_{\Psi=1}^{2}\overline{\left(A_{dq}^{\Psi}\right)}^{\Omega_\Psi}}, \dfrac{\prod\limits_{\Psi=1}^{2}\left(1+(\gamma-1)\overline{B_{dq}^{\Psi}}\right)^{\Omega_\Psi}-\prod\limits_{\Psi=1}^{2}\left(1-\overline{B_{dq}^{\Psi}}\right)^{\Omega_\Psi}}{\prod\limits_{\Psi=1}^{2}\left(1+(\gamma-1)\overline{B_{dq}^{\Psi}}\right)^{\Omega_\Psi}+(\gamma-1)\prod\limits_{\Psi=1}^{2}\left(1-\overline{B_{dq}^{\Psi}}\right)^{\Omega_\Psi}} \right\rangle\right), \\
\left(\left\langle \dfrac{\sqrt[q]{\gamma}\prod\limits_{\Psi=1}^{2}\overline{(a)}^{\Omega_\Psi}}{\sqrt[q]{\prod\limits_{\Psi=1}^{2}\left(1+(\gamma-1)\left(1-\overline{(a)^{q}}\right)\right)^{\Omega_\Psi}+(\gamma-1)\prod\limits_{\Psi=1}^{2}\left(\overline{(a)^{q}}\right)^{\Omega_\Psi}}}, \sqrt[q]{\dfrac{\prod\limits_{\Psi=1}^{2}\left(1+(\gamma-1)\overline{(b)^{q}}\right)^{\Omega_\Psi}-\prod\limits_{\Psi=1}^{2}\left(1-\overline{(b)^{q}}\right)^{\Omega_\Psi}}{\prod\limits_{\Psi=1}^{2}\left(1+(\gamma-1)\overline{(b)^{q}}\right)^{\Omega_\Psi}+(\gamma-1)\prod\limits_{\Psi=1}^{2}\left(1-\overline{(b)^{q}}\right)^{\Omega_\Psi}}} \right\rangle\right). \\
\left(\left\langle \dfrac{\gamma\prod\limits_{\Psi=1}^{2}\underline{\left(A_{dq}^{\Psi}\right)}^{\Omega_\Psi}}{\prod\limits_{\Psi=1}^{2}\left(1+(\gamma-1)\left(1-\underline{A_{dq}^{\Psi}}\right)\right)^{\Omega_\Psi}+(\gamma-1)\prod\limits_{\Psi=1}^{2}\underline{\left(A_{dq}^{\Psi}\right)}^{\Omega_\Psi}}, \dfrac{\prod\limits_{\Psi=1}^{2}\left(1+(\gamma-1)\underline{B_{dq}^{\Psi}}\right)^{\Omega_\Psi}-\prod\limits_{\Psi=1}^{2}\left(1-\underline{B_{dq}^{\Psi}}\right)^{\Omega_\Psi}}{\prod\limits_{\Psi=1}^{2}\left(1+(\gamma-1)\underline{B_{dq}^{\Psi}}\right)^{\Omega_\Psi}+(\gamma-1)\prod\limits_{\Psi=1}^{2}\left(1-\underline{B_{dq}^{\Psi}}\right)^{\Omega_\Psi}} \right\rangle\right), \\
\left(\left\langle \dfrac{\sqrt[q]{\gamma}\prod\limits_{\Psi=1}^{2}\underline{(a)}^{\Omega_\Psi}}{\sqrt[q]{\prod\limits_{\Psi=1}^{2}\left(1+(\gamma-1)\left(1-\underline{(a)^{q}}\right)\right)^{\Omega_\Psi}+(\gamma-1)\prod\limits_{\Psi=1}^{2}\left(\underline{(a)^{q}}\right)^{\Omega_\Psi}}}, \sqrt[q]{\dfrac{\prod\limits_{\Psi=1}^{2}\left(1+(\gamma-1)\underline{(b)^{q}}\right)^{\Omega_\Psi}-\prod\limits_{\Psi=1}^{2}\left(1-\underline{(b)^{q}}\right)^{\Omega_\Psi}}{\prod\limits_{\Psi=1}^{2}\left(1+(\gamma-1)\underline{(b)^{q}}\right)^{\Omega_\Psi}+(\gamma-1)\prod\limits_{\Psi=1}^{2}\left(1-\underline{(b)^{q}}\right)^{\Omega_\Psi}}} \right\rangle\right).
\end{array}\right]$$

It is true when put $n = 2$.

(ii): Assume that equation holds for $n = k$,

NLDRFHWG$(\gamma_{dq1}, \gamma_{dq2}, ..., \gamma_{dqk}) =$

$$
= \left[\begin{array}{c}
\left(\left\langle \dfrac{\gamma \prod\limits_{\Psi=1}^{k} \overline{\left(A_{dq}^{\Psi}\right)}^{\Omega_\Psi}}{\prod\limits_{\Psi=1}^{k} \left(1+(\gamma-1)\left(1-\overline{A_{dq}^{\Psi}}\right)\right)^{\Omega_\Psi} + (\gamma-1) \prod\limits_{\Psi=1}^{k} \overline{\left(A_{dq}^{\Psi}\right)}^{\Omega_\Psi}}, \dfrac{\prod\limits_{\Psi=1}^{k} \left(1+(\gamma-1)\overline{B_{dq}^{\Psi}}\right)^{\Omega_\Psi} - \prod\limits_{\Psi=1}^{k} \left(1-\overline{B_{dq}^{\Psi}}\right)^{\Omega_\Psi}}{\prod\limits_{\Psi=1}^{k} \left(1+(\gamma-1)\overline{B_{dq}^{\Psi}}\right)^{\Omega_\Psi} + (\gamma-1) \prod\limits_{\Psi=1}^{k} \left(1-\overline{B_{dq}^{\Psi}}\right)^{\Omega_\Psi}} \right\rangle \right), \\
\left(\left\langle \dfrac{\sqrt[q]{\gamma} \prod\limits_{\Psi=1}^{k} \overline{(a)}^{\Omega_\Psi}}{\sqrt[q]{\prod\limits_{\Psi=1}^{k} \left(1+(\gamma-1)\left(1-\overline{(a)^q}\right)\right)^{\Omega_\Psi} + (\gamma-1) \prod\limits_{\Psi=1}^{k} \left(\overline{(a)^q}\right)^{\Omega_\Psi}}}, \sqrt[q]{\dfrac{\prod\limits_{\Psi=1}^{k} \left(1+(\gamma-1)\overline{(b)^q}\right)^{\Omega_\Psi} - \prod\limits_{\Psi=1}^{k} (1-\overline{(b)^q})^{\Omega_\Psi}}{\prod\limits_{\Psi=1}^{k} \left(1+(\gamma-1)\overline{(b)^q}\right)^{\Omega_\Psi} + (\gamma-1) \prod\limits_{\Psi=1}^{k} \left(1-\overline{(b)^q}\right)^{\Omega_\Psi}}} \right\rangle \right). \\
\left(\left\langle \dfrac{\gamma \prod\limits_{\Psi=1}^{k} \underline{\left(A_{dq}^{\Psi}\right)}^{\Omega_\Psi}}{\prod\limits_{\Psi=1}^{k} \left(1+(\gamma-1)\left(1-\underline{A_{dq}^{\Psi}}\right)\right)^{\Omega_\Psi} + (\gamma-1) \prod\limits_{\Psi=1}^{k} \underline{\left(A_{dq}^{\Psi}\right)}^{\Omega_\Psi}}, \dfrac{\prod\limits_{\Psi=1}^{k} \left(1+(\gamma-1)\underline{B_{dq}^{\Psi}}\right)^{\Omega_\Psi} - \prod\limits_{\Psi=1}^{k} \left(1-\underline{B_{dq}^{\Psi}}\right)^{\Omega_\Psi}}{\prod\limits_{\Psi=1}^{k} \left(1+(\gamma-1)\underline{B_{dq}^{\Psi}}\right)^{\Omega_\Psi} + (\gamma-1) \prod\limits_{\Psi=1}^{k} \left(1-\underline{B_{dq}^{\Psi}}\right)^{\Omega_\Psi}} \right\rangle \right), \\
\left(\left\langle \dfrac{\sqrt[q]{\gamma} \prod\limits_{\Psi=1}^{k} \underline{(a)}^{\Omega_\Psi}}{\sqrt[q]{\prod\limits_{\Psi=1}^{k} \left(1+(\gamma-1)\left(1-\underline{(a)}^q\right)\right)^{\Omega_\Psi} + (\gamma-1) \prod\limits_{\Psi=1}^{k} \left(\underline{(a)}^q\right)^{\Omega_\Psi}}}, \sqrt[q]{\dfrac{\prod\limits_{\Psi=1}^{k} \left(1+(\gamma-1)\underline{(b)}^q\right)^{\Omega_\Psi} - \prod\limits_{\Psi=1}^{k} \left(1-\underline{(b)}^q\right)^{\Omega_\Psi}}{\prod\limits_{\Psi=1}^{k} \left(1+(\gamma-1)\underline{(b)}^q\right)^{\Omega_\Psi} + (\gamma-1) \prod\limits_{\Psi=1}^{k} (1-\underline{(b)}^q)^{\Omega_\Psi}}} \right\rangle \right).
\end{array} \right]
$$

(iii): To prove that equation holds for $n = k + 1$,

Let NLDRFHWG $(\gamma_{dq1}, \gamma_{dq2}, ..., \gamma_{dqk+1}) = \prod\limits_{\Psi=1}^{k} \gamma_{dqk}^{\Omega_k} \bigotimes \gamma_{dqk+1}^{\Omega_{k+1}} =$

$$\left[\begin{array}{c}
\left(\left\langle \dfrac{\gamma\prod\limits_{\Psi=1}^{k}\overline{\left(A^{\Psi}_{dq}\right)}^{\Omega_\Psi}}{\prod\limits_{\Psi=1}^{k}\left(1+(\gamma-1)\left(1-\overline{A^{\Psi}_{dq}}\right)\right)^{\Omega_\Psi}+(\gamma-1)\prod\limits_{\Psi=1}^{k}\overline{\left(A^{\Psi}_{dq}\right)}^{\Omega_\Psi}}, \dfrac{\prod\limits_{\Psi=1}^{k}\left(1+(\gamma-1)\overline{B^{\Psi}_{dq}}\right)^{\Omega_\Psi}-\prod\limits_{\Psi=1}^{k}\left(1-\overline{B^{\Psi}_{dq}}\right)^{\Omega_\Psi}}{\prod\limits_{\Psi=1}^{k}\left(1+(\gamma-1)\overline{B^{\Psi}_{dq}}\right)^{\Omega_\Psi}+(\gamma-1)\prod\limits_{\Psi=1}^{k}\left(1-\overline{B^{\Psi}_{dq}}\right)^{\Omega_\Psi}} \right\rangle\right), \\
\left(\left\langle \dfrac{\sqrt[q]{\gamma}\prod\limits_{\Psi=1}^{k}\overline{(a)}^{\Omega_\Psi}}{\sqrt[q]{\prod\limits_{\Psi=1}^{k}\left(1+(\gamma-1)\left(1-\overline{(a)^{q}}\right)\right)^{\Omega_\Psi}+(\gamma-1)\prod\limits_{\Psi=1}^{k}\left(\overline{(a)^{q}}\right)^{\Omega_\Psi}}}, \sqrt[q]{\dfrac{\prod\limits_{\Psi=1}^{k}\left(1+(\gamma-1)\overline{(b)^{q}}\right)^{\Omega_\Psi}-\prod\limits_{\Psi=1}^{k}(1-\overline{(b)^{q}})^{\Omega_\Psi}}{\prod\limits_{\Psi=1}^{k}\left(1+(\gamma-1)\overline{(b)^{q}}\right)^{\Omega_\Psi}+(\gamma-1)\prod\limits_{\Psi=1}^{k}\left(1-\overline{(b)^{q}}\right)^{\Omega_\Psi}}} \right\rangle\right). \\
\left(\left\langle \dfrac{\gamma\prod\limits_{\Psi=1}^{k}\underline{\left(A^{\Psi}_{dq}\right)}^{\Omega_\Psi}}{\prod\limits_{\Psi=1}^{k}\left(1+(\gamma-1)\left(1-\underline{A^{\Psi}_{dq}}\right)\right)^{\Omega_\Psi}+(\gamma-1)\prod\limits_{\Psi=1}^{k}\underline{\left(A^{\Psi}_{dq}\right)}^{\Omega_\Psi}}, \dfrac{\prod\limits_{\Psi=1}^{k}\left(1+(\gamma-1)\underline{B^{\Psi}_{dq}}\right)^{\Omega_\Psi}-\prod\limits_{\Psi=1}^{k}\left(1-\underline{B^{\Psi}_{dq}}\right)^{\Omega_\Psi}}{\prod\limits_{\Psi=1}^{k}\left(1+(\gamma-1)\underline{B^{\Psi}_{dq}}\right)^{\Omega_\Psi}+(\gamma-1)\prod\limits_{\Psi=1}^{k}\left(1-\underline{B^{\Psi}_{dq}}\right)^{\Omega_\Psi}} \right\rangle\right), \\
\left(\left\langle \dfrac{\sqrt[q]{\gamma}\prod\limits_{\Psi=1}^{k}\underline{(a)}^{\Omega_\Psi}}{\sqrt[q]{\prod\limits_{\Psi=1}^{k}\left(1+(\gamma-1)\left(1-\underline{(a)}^{q}\right)\right)^{\Omega_\Psi}+(\gamma-1)\prod\limits_{\Psi=1}^{k}\left(\underline{(a)}^{q}\right)^{\Omega_\Psi}}}, \sqrt[q]{\dfrac{\prod\limits_{\Psi=1}^{k}\left(1+(\gamma-1)\underline{(b)}^{q}\right)^{\Omega_\Psi}-\prod\limits_{\Psi=1}^{k}(1-\underline{(b)}^{q})^{\Omega_\Psi}}{\prod\limits_{\Psi=1}^{k}\left(1+(\gamma-1)\underline{(b)}^{q}\right)^{\Omega_\Psi}+(\gamma-1)\prod\limits_{\Psi=1}^{k}\left(1-\underline{(b)}^{q}\right)^{\Omega_\Psi}}} \right\rangle\right)
\end{array}\right] \bigotimes$$

$$\left[\begin{array}{c}
\left(\left\langle \begin{array}{c} \dfrac{\gamma\overline{\left(A_{dq}^{k+1}\right)}^{\Omega_{k+1}}}{\left(1+(\gamma-1)\left(1-\overline{A_{dq}^{k+1}}\right)\right)^{\Omega_{k+1}}+(\gamma-1)\overline{\left(A_{dq}^{k+1}\right)}^{\Omega k+1}}, \\ \dfrac{\left(1+(\gamma-1)\overline{B_{dq}^{k+1}}\right)^{\Omega_{k+1}}-\left(1-\overline{B_{dq}^{k+1}}\right)^{\Omega_{k+1}}}{\left(1+(\gamma-1)\overline{B_{dq}^{k+1}}\right)^{\Omega_{k+1}}+(\gamma-1)\left(1-\overline{B_{dq}^{k+1}}\right)^{\Omega_{k+1}}} \end{array} \right\rangle\right), \\
\left(\left\langle \begin{array}{c} \dfrac{\sqrt[q]{\gamma}\overline{(a)}^{\Omega_{k+1}}}{\sqrt[q]{\left(1+(\gamma-1)\left(1-\overline{(a)}^{q}\right)\right)^{\Omega_{k+1}}+(\gamma-1)\left(\overline{(a)}^{q}\right)^{\Omega_{k+1}}}}, \\ \sqrt[q]{\dfrac{\left(1+(\gamma-1)\overline{(b)}^{q}\right)^{\Omega_{k+1}}-\left(1-\overline{(b)}^{q}\right)^{\Omega_{k+1}}}{\left(1+(\gamma-1)\overline{(b)}^{q}\right)^{\Omega k+1}+(\gamma-1)\left(1-\overline{(b)}^{q}\right)^{\Omega_{k+1}}}} \end{array} \right\rangle\right). \\
\left(\left\langle \begin{array}{c} \dfrac{\gamma\underline{\left(A_{dq}^{k+1}\right)}^{\Omega_{k+1}}}{\left(1+(\gamma-1)\left(1-\underline{A_{dq}^{k+1}}\right)\right)^{\Omega_{k+1}}+(\gamma-1)\underline{\left(A_{dq}^{k+1}\right)}^{\Omega k+1}}, \\ \dfrac{\left(1+(\gamma-1)\underline{B_{dq}^{k+1}}\right)^{\Omega_{k+1}}-\left(1-\underline{B_{dq}^{k+1}}\right)^{\Omega_{k+1}}}{\left(1+(\gamma-1)\underline{B_{dq}^{k+1}}\right)^{\Omega_{k+1}}+(\gamma-1)\left(1-\underline{B_{dq}^{k+1}}\right)^{\Omega_{k+1}}} \end{array} \right\rangle\right), \\
\left(\left\langle \begin{array}{c} \dfrac{\sqrt[q]{\gamma}\underline{(a)}^{\Omega_{k+1}}}{\sqrt[q]{\left(1+(\gamma-1)\left(1-\underline{(a)}^{q}\right)\right)^{\Omega_{k+1}}+(\gamma-1)\left(\underline{(a)}^{q}\right)^{\Omega_{k+1}}}}, \\ \sqrt[q]{\dfrac{\left(1+(\gamma-1)\underline{(b)}^{q}\right)^{\Omega_{k+1}}-\left(1-\underline{(b)}^{q}\right)^{\Omega_{k+1}}}{\left(1+(\gamma-1)\underline{(b)}^{q}\right)^{\Omega k+1}+(\gamma-1)\left(1-\underline{(b)}^{q}\right)^{\Omega_{k+1}}}} \end{array} \right\rangle\right).
\end{array}\right]$$

$$
= \left[\begin{array}{c}
\left(\left\langle \frac{\gamma \prod_{\Psi=1}^{k+1} \overline{\left(A_{dq}^{\Psi}\right)}^{\Omega_\Psi}}{\prod_{\Psi=1}^{k+1} \left(1+(\gamma-1)\left(1-\overline{A_{dq}^{\Psi}}\right)\right)^{\Omega_\Psi} + (\gamma-1) \prod_{\Psi=1}^{k+1} \overline{\left(A_{dq}^{\Psi}\right)}^{\Omega_\Psi}}, \frac{\prod_{\Psi=1}^{k+1} \left(1+(\gamma-1)\overline{B_{dq}^{\Psi}}\right)^{\Omega_\Psi} - \prod_{\Psi=1}^{k+1} \left(1-\overline{B_{dq}^{\Psi}}\right)^{\Omega_\Psi}}{\prod_{\Psi=1}^{k+1} \left(1+(\gamma-1)\overline{B_{dq}^{\Psi}}\right)^{\Omega_\Psi} + (\gamma-1) \prod_{\Psi=1}^{k+1} \left(1-\overline{B_{dq}^{\Psi}}\right)^{\Omega_\Psi}} \right\rangle, \right. \\
\left. \left\langle \frac{\sqrt[q]{\gamma} \prod_{\Psi=1}^{k+1} \overline{(a)}^{\Omega_\Psi}}{\sqrt[q]{\prod_{\Psi=1}^{k+1} \left(1+(\gamma-1)\left(1-\overline{(a)^q}\right)\right)^{\Omega_\Psi} + (\gamma-1) \prod_{\Psi=1}^{k+1} \left(\overline{(a)^q}\right)^{\Omega_\Psi}}}, \sqrt[q]{\frac{\prod_{\Psi=1}^{k+1} \left(1+(\gamma-1)\overline{(b)^q}\right)^{\Omega_\Psi} - \prod_{\Psi=1}^{k+1} (1-\overline{(b)^q})^{\Omega_\Psi}}{\prod_{\Psi=1}^{k+1} \left(1+(\gamma-1)\overline{(b)^q}\right)^{\Omega_\Psi} + (\gamma-1) \prod_{\Psi=1}^{k+1} \left(1-\overline{(b)^q}\right)^{\Omega_\Psi}}} \right\rangle \right). \\
\left(\left\langle \frac{\gamma \prod_{\Psi=1}^{k+1} \underline{\left(A_{dq}^{\Psi}\right)}^{\Omega_\Psi}}{\prod_{\Psi=1}^{k+1} \left(1+(\gamma-1)\left(1-\underline{A_{dq}^{\Psi}}\right)\right)^{\Omega_\Psi} + (\gamma-1) \prod_{\Psi=1}^{k+1} \underline{\left(A_{dq}^{\Psi}\right)}^{\Omega_\Psi}}, \frac{\prod_{\Psi=1}^{k+1} \left(1+(\gamma-1)\underline{B_{dq}^{\Psi}}\right)^{\Omega_\Psi} - \prod_{\Psi=1}^{k+1} \left(1-\underline{B_{dq}^{\Psi}}\right)^{\Omega_\Psi}}{\prod_{\Psi=1}^{k+1} \left(1+(\gamma-1)\underline{B_{dq}^{\Psi}}\right)^{\Omega_\Psi} + (\gamma-1) \prod_{\Psi=1}^{k+1} \left(1-\underline{B_{dq}^{\Psi}}\right)^{\Omega_\Psi}} \right\rangle, \right. \\
\left. \left\langle \frac{\sqrt[q]{\gamma} \prod_{\Psi=1}^{k+1} \underline{(a)}^{\Omega_\Psi}}{\sqrt[q]{\prod_{\Psi=1}^{k+1} \left(1+(\gamma-1)\left(1-\underline{(a)}^q\right)\right)^{\Omega_\Psi} + (\gamma-1) \prod_{\Psi=1}^{k+1} \left(\underline{(a)}^q\right)^{\Omega_\Psi}}}, \sqrt[q]{\frac{\prod_{\Psi=1}^{k+1} \left(1+(\gamma-1)\underline{(b)}^q\right)^{\Omega_\Psi} - \prod_{\Psi=1}^{k+1} \left(1-\underline{(b)}^q\right)^{\Omega_\Psi}}{\prod_{\Psi=1}^{k+1} \left(1+(\gamma-1)\underline{(b)}^q\right)^{\Omega_\Psi} + (\gamma-1) \prod_{\Psi=1}^{k+1} (1-\underline{(b)}^q)^{\Omega_\Psi}}} \right\rangle \right).
\end{array} \right]
$$

Hence equation is true for $n = k + 1$. Which proved the theorem. □

Theorem 4.2 ***(Idempotency):*** *If* $\gamma_{dq\Psi} = \left\{ \begin{array}{c} \left(\begin{array}{c} \left\langle \overline{A_{dq}^{\Psi}}, \overline{B_{dq}^{\Psi}} \right\rangle, \\ \langle \overline{a}, \overline{b} \rangle \end{array} \right), \\ \left(\begin{array}{c} \left\langle \underline{A_{dq}^{\Psi}}, \underline{B_{dq}^{\Psi}} \right\rangle, \\ \langle \underline{a}, \underline{b} \rangle \end{array} \right) : \\ \Psi \epsilon \mathbb{N} \end{array} \right\}$ *for all* $(dq = 1, 2, 3, ..., m)$, *then*

$$NLDRFHWG(\gamma, \gamma, \gamma, ..., \gamma) = \gamma.$$

Theorem 4.3 ***(Commutativity) :****If $(bg_1^{'}, bg_2^{'}, ..., bg_n^{'})$ is any permutation of $(bg_1, bg_2, ..., bg_n)$, then NLDRFHWG$(bg_1^{'}, bg_2^{'}, ..., bg_n^{'}) = NLDRFHWG(bg_1, bg_2, ..., bg_n)$.*

Theorem 4.4 ***(Boundedness):****If $Z^- = min(gf_1, gf_2, ..., gf_n)$, $Z^+ = \max(gf_1, gf_2, ..., gf_n)$, then*

$$Z^- \leq NLDRFHWG(gf_1, gf_2, ..., gf_n) \leq Z^+.$$

4.2. NLDRFHOWG operator

Definition 4.2 *Let $\gamma_{dq\Psi} = \left\{ \begin{array}{c} \left(\begin{array}{c} \left\langle \overline{A_{dq}^{\Psi}}, \overline{B_{dq}^{\Psi}} \right\rangle, \\ \langle \overline{a}, \overline{b} \rangle \end{array} \right), \\ \left(\begin{array}{c} \left\langle \underline{A_{dq}^{\Psi}}, \underline{B_{dq}^{\Psi}} \right\rangle, \\ \langle \underline{a}, \underline{b} \rangle \end{array} \right) \\ : \Psi \epsilon \mathbb{N} \end{array} \right\}$ be a collection of NLDRFSs over the fixed sey $\mathbb{Z}$ and $\Omega = (\Omega_1, \Omega_2, ..., \Omega_n)^T$ is the weights with $\sum_{\Psi=1}^{n} \Omega_\Psi = 1, q \geq 1$;then define NLDRF hamacher ordered weighted geometric operator as follows and let the transformation $\theta : NLDRF(\mathbb{Z}) \rightarrow NLDRFN(\mathbb{Z})$*

$$\begin{aligned} & NLDRFHOWG(\gamma_{dq1}, \gamma_{dq2}, ..., \gamma_{dqn}) \\ & = \bigotimes_{\Psi=1}^{n} \left(\gamma_{dq\Psi}^{\Omega_\Psi} \right) = \gamma_{dq1}^{\Omega_1} \bigotimes \gamma_{dq2}^{\Omega_2} \bigotimes ... \bigotimes \gamma_{dqn}^{\Omega_n}. \end{aligned}$$

Theorem 4.5 *Let $\gamma_{dq\Psi} = \left\{ \begin{array}{c} \left(\left\langle \overline{A_{dq}^{\Psi}}, \overline{B_{dq}^{\Psi}} \right\rangle, \langle \overline{a}, \overline{b} \rangle \right), \\ \left(\left\langle \underline{A_{dq}^{\Psi}}, \underline{B_{dq}^{\Psi}} \right\rangle, \langle \underline{a}, \underline{b} \rangle \right) : \Psi \epsilon \mathbb{N} \end{array} \right\}$ be an assembling of NLDRFSs over the fixed set $\mathbb{Z}$ and $\Omega = (\Omega_1, \Omega_2, ..., \Omega_n)^T$ is the weights with $\sum_{\Psi=1}^{n} \Omega_\Psi = 1$,and $q \geq 1, \gamma > 0$;then by applying the NLDRFHOWG operator their aggregated values is also an NLDRFS, and the transformation $\theta : NLDRF(\mathbb{Z}) \rightarrow NLDRFN(\mathbb{Z})$ is called NLDRF hamacher ordered weighted geometric operator and define as follows:NLDRFHOWG$(\gamma_{dq1}, \gamma_{dq2}, ..., \gamma_{dqn}) = \bigotimes_{\Psi=1}^{n} (\gamma_{dq\Psi})^{\Omega_\Psi}$*

$$
= \left[\begin{array}{c}
\left(\left\langle \dfrac{\gamma \prod_{\Psi=1}^{n} \overline{\left(A_{dq}^{\Psi}\right)}^{\Omega_\Psi}}{\prod_{\Psi=1}^{n}\left(1+(\gamma-1)\left(1-\overline{A_{dq}^{\Psi}}\right)\right)^{\Omega_\Psi}+(\gamma-1)\prod_{\Psi=1}^{n}\overline{\left(A_{dq}^{\Psi}\right)}^{\Omega_\Psi}}, \dfrac{\prod_{\Psi=1}^{n}\left(1+(\gamma-1)\overline{B_{dq}^{\Psi}}\right)^{\Omega_\Psi}-\prod_{\Psi=1}^{n}\left(1-\overline{B_{dq}^{\Psi}}\right)^{\Omega_\Psi}}{\prod_{\Psi=1}^{n}\left(1+(\gamma-1)\overline{B_{dq}^{\Psi}}\right)^{\Omega_\Psi}+(\gamma-1)\prod_{\Psi=1}^{n}\left(1-\overline{B_{dq}^{\Psi}}\right)^{\Omega_\Psi}} \right\rangle\right), \\
\left(\left\langle \dfrac{\sqrt[q]{\gamma} \prod_{\Psi=1}^{n} \overline{(a)}^{\Omega_\Psi}}{\sqrt[q]{\prod_{\Psi=1}^{n}\left(1+(\gamma-1)\left(1-\overline{(a)^{q}}\right)\right)^{\Omega_\Psi}+(\gamma-1)\prod_{\Psi=1}^{n}\left(\overline{(a)^{q}}\right)^{\Omega_\Psi}}}, \sqrt[q]{\dfrac{\prod_{\Psi=1}^{n}\left(1+(\gamma-1)\overline{(b)^{q}}\right)^{\Omega_\Psi}-\prod_{\Psi=1}^{n}\left(1-\overline{(b)^{q}}\right)^{\Omega_\Psi}}{\prod_{\Psi=1}^{n}\left(1+(\gamma-1)\overline{(b)^{q}}\right)^{\Omega_\Psi}+(\gamma-1)\prod_{\Psi=1}^{n}\left(1-\overline{(b)^{q}}\right)^{\Omega_\Psi}}} \right\rangle\right). \\
\left(\left\langle \dfrac{\gamma \prod_{\Psi=1}^{n} \underline{\left(A_{dq}^{\Psi}\right)}^{\Omega_\Psi}}{\prod_{\Psi=1}^{n}\left(1+(\gamma-1)\left(1-\underline{A_{dq}^{\Psi}}\right)\right)^{\Omega_\Psi}+(\gamma-1)\prod_{\Psi=1}^{n}\underline{\left(A_{dq}^{\Psi}\right)}^{\Omega_\Psi}}, \dfrac{\prod_{\Psi=1}^{n}\left(1+(\gamma-1)\underline{B_{dq}^{\Psi}}\right)^{\Omega_\Psi}-\prod_{\Psi=1}^{n}\left(1-\underline{B_{dq}^{\Psi}}\right)^{\Omega_\Psi}}{\prod_{\Psi=1}^{n}\left(1+(\gamma-1)\underline{B_{dq}^{\Psi}}\right)^{\Omega_\Psi}+(\gamma-1)\prod_{\Psi=1}^{n}\left(1-\underline{B_{dq}^{\Psi}}\right)^{\Omega_\Psi}} \right\rangle\right), \\
\left(\left\langle \dfrac{\sqrt[q]{\gamma} \prod_{\Psi=1}^{n} \underline{(a)}^{\Omega_\Psi}}{\sqrt[q]{\prod_{\Psi=1}^{n}\left(1+(\gamma-1)\left(1-\underline{(a)^{q}}\right)\right)^{\Omega_\Psi}+(\gamma-1)\prod_{\Psi=1}^{n}\left(\underline{(a)^{q}}\right)^{\Omega_\Psi}}}, \sqrt[q]{\dfrac{\prod_{\Psi=1}^{n}\left(1+(\gamma-1)\underline{(b)^{q}}\right)^{\Omega_\Psi}-\prod_{\Psi=1}^{n}\left(1-\underline{(b)^{q}}\right)^{\Omega_\Psi}}{\prod_{\Psi=1}^{n}\left(1+(\gamma-1)\underline{(b)^{q}}\right)^{\Omega_\Psi}+(\gamma-1)\prod_{\Psi=1}^{n}\left(1-\underline{(b)^{q}}\right)^{\Omega_\Psi}}} \right\rangle\right).
\end{array}\right]
$$

Theorem 4.6 ***(Idempotency):*** *If* $\gamma_{dq\Psi} = \left\{ \begin{array}{c} \left(\left\langle \overline{A_{dq}^{\Psi}}, \overline{B_{dq}^{\Psi}} \right\rangle, \left\langle \overline{a}, \overline{b} \right\rangle\right), \\ \left(\left\langle \underline{A_{dq}^{\Psi}}, \underline{B_{dq}^{\Psi}} \right\rangle, \left\langle \underline{a}, \underline{b} \right\rangle\right) : \Psi \epsilon \mathbb{N} \end{array} \right\}$ *for all* $(dq = 1, 2, 3, ..., m)$, *then*

$$NLDRFHOWG(\gamma, \gamma, \gamma, ..., \gamma) = \gamma.$$

Theorem 4.7 ***(Commutativity)*** ***:****If* $(bg_1^{'}, bg_2^{'}, ..., bg_n^{'})$ *is any permutation of* $(bg_1, bg_2, ..., bg_n)$, *then* $NLDRFHOWG(bg_1^{'}, bg_2^{'}, ..., bg_n^{'}) = NLDRFHOWG(bg_1, bg_2, ..., bg_n)$.

Theorem 4.8 ***(Boundedness):****If* $Z^- = min(gf_1, gf_2, ..., gf_n)$, $Z^+ = \max(gf_1, gf_2, ..., gf_n)$, *then*

$$Z^- \leq NLDRFHOWG(gf_1, gf_2, ..., gf_n) \leq Z^+.$$

4.3. NLDRFHHWG operator

Definition 4.3 *Let* $\gamma_{dq\Psi} = \left\{ \begin{array}{c} \left(\left\langle \overline{A^{\Psi}_{dq}}, \overline{B^{\Psi}_{dq}} \right\rangle, \langle \overline{a}, \overline{b} \rangle \right), \\ \left(\left\langle \underline{A^{\Psi}_{dq}}, \underline{B^{\Psi}_{dq}} \right\rangle, \langle \underline{a}, \underline{b} \rangle \right) : \Psi \epsilon \mathbb{N} \end{array} \right\}$ *be a collection of NLDRFSs over the fixed sey* $\mathbb{Z}$ *and* $\Omega = (\Omega_1, \Omega_2, ..., \Omega_n)^T$ *is the weights with* $\sum_{\Psi=1}^{n} \Omega_\Psi = 1, q \geq 1$; *then define NLDRF hamacher hybrid weighted geometric operator as follows and let the transformation* $\theta : NLDRF(\mathbb{Z}) \to NLDRFN(\mathbb{Z})$

$$NLDRFHHWG(\gamma_{dq1}, \gamma_{dq2}, ..., \gamma_{dqn}) = \bigotimes_{\Psi=1}^{n} \left(\gamma_{dq\Psi}^{\Omega_\Psi}\right) = \gamma_{dq1}^{\Omega_1} \bigotimes \gamma_{dq2}^{\Omega_2} \bigotimes ... \bigotimes \gamma_{dqn}^{\Omega_n}.$$

Theorem 4.9 *Let* $\gamma_{dq\Psi} = \left\{ \begin{array}{c} \left(\left\langle \overline{A^{\Psi}_{dq}}, \overline{B^{\Psi}_{dq}} \right\rangle, \langle \overline{a}, \overline{b} \rangle \right), \\ \left(\left\langle \underline{A^{\Psi}_{dq}}, \underline{B^{\Psi}_{dq}} \right\rangle, \langle \underline{a}, \underline{b} \rangle \right) : \Psi \epsilon \mathbb{N} \end{array} \right\}$ *be an assembling of NLDRFSs over the fixed set* $\mathbb{Z}$ *and* $\Omega = (\Omega_1, \Omega_2, ..., \Omega_n)^T$ *is the weights with* $\sum_{\Psi=1}^{n} \Omega_\Psi = 1$,*and* $q \geq 1, \gamma > 0$; *then by applying the NLDRFHHWG operator their aggregated values is also an NLDRFS, and the transformation* $\theta : NLDRF(\mathbb{Z}) \to NLDRFN(\mathbb{Z})$ *is called NLDRF hamacher hybrid weighted geometric operator and define as follows:*$NLDRFHOWG(\gamma_{dq1}, \gamma_{dq2}, ..., \gamma_{dqn}) = \bigotimes_{\Psi=1}^{n} (\gamma_{dq\Psi})^{\Omega_\Psi}$

$$
= \left[\begin{array}{c}
\left(\left\langle \frac{\gamma \prod_{\Psi=1}^{n} \overline{\left(A_{dq}^{\Psi}\right)}^{\Omega_\Psi}}{\prod_{\Psi=1}^{n} \left(1+(\gamma-1)\left(1-\overline{A_{dq}^{\Psi}}\right)\right)^{\Omega_\Psi} + (\gamma-1) \prod_{\Psi=1}^{n} \overline{\left(A_{dq}^{\Psi}\right)}^{\Omega_\Psi}}, \frac{\prod_{\Psi=1}^{n} \left(1+(\gamma-1)\overline{B_{dq}^{\Psi}}\right)^{\Omega_\Psi} - \prod_{\Psi=1}^{n} \left(1-\overline{B_{dq}^{\Psi}}\right)^{\Omega_\Psi}}{\prod_{\Psi=1}^{n} \left(1+(\gamma-1)\overline{B_{dq}^{\Psi}}\right)^{\Omega_\Psi} + (\gamma-1) \prod_{\Psi=1}^{n} \left(1-\overline{B_{dq}^{\Psi}}\right)^{\Omega_\Psi}} \right\rangle \right), \\
\left(\left\langle \frac{\sqrt[q]{\gamma} \prod_{\Psi=1}^{n} \overline{(a)}^{\Omega_\Psi}}{\sqrt[q]{\prod_{\Psi=1}^{n} \left(1+(\gamma-1)\left(1-\overline{(a)^q}\right)\right)^{\Omega_\Psi} + (\gamma-1) \prod_{\Psi=1}^{n} \left(\overline{(a)^q}\right)^{\Omega_\Psi}}}, \sqrt[q]{\frac{\prod_{\Psi=1}^{n} \left(1+(\gamma-1)\overline{(b)^q}\right)^{\Omega_\Psi} - \prod_{\Psi=1}^{n} \left(1-\overline{(b)^q}\right)^{\Omega_\Psi}}{\prod_{\Psi=1}^{n} \left(1+(\gamma-1)\overline{(b)^q}\right)^{\Omega_\Psi} + (\gamma-1) \prod_{\Psi=1}^{n} \left(1-\overline{(b)^q}\right)^{\Omega_\Psi}}} \right\rangle \right). \\
\left(\left\langle \frac{\gamma \prod_{\Psi=1}^{n} \underline{\left(A_{dq}^{\Psi}\right)}^{\Omega_\Psi}}{\prod_{\Psi=1}^{n} \left(1+(\gamma-1)\left(1-\underline{A_{dq}^{\Psi}}\right)\right)^{\Omega_\Psi} + (\gamma-1) \prod_{\Psi=1}^{n} \underline{\left(A_{dq}^{\Psi}\right)}^{\Omega_\Psi}}, \frac{\prod_{\Psi=1}^{n} \left(1+(\gamma-1)\underline{B_{dq}^{\Psi}}\right)^{\Omega_\Psi} - \prod_{\Psi=1}^{n} \left(1-\underline{B_{dq}^{\Psi}}\right)^{\Omega_\Psi}}{\prod_{\Psi=1}^{n} \left(1+(\gamma-1)\underline{B_{dq}^{\Psi}}\right)^{\Omega_\Psi} + (\gamma-1) \prod_{\Psi=1}^{n} \left(1-\underline{B_{dq}^{\Psi}}\right)^{\Omega_\Psi}} \right\rangle \right), \\
\left(\left\langle \frac{\sqrt[q]{\gamma} \prod_{\Psi=1}^{n} \underline{(a)}^{\Omega_\Psi}}{\sqrt[q]{\prod_{\Psi=1}^{n} \left(1+(\gamma-1)\left(1-\underline{(a)^q}\right)\right)^{\Omega_\Psi} + (\gamma-1) \prod_{\Psi=1}^{n} \left(\underline{(a)^q}\right)^{\Omega_\Psi}}}, \sqrt[q]{\frac{\prod_{\Psi=1}^{n} \left(1+(\gamma-1)\underline{(b)^q}\right)^{\Omega_\Psi} - \prod_{\Psi=1}^{n} \left(1-\underline{(b)^q}\right)^{\Omega_\Psi}}{\prod_{\Psi=1}^{n} \left(1+(\gamma-1)\underline{(b)^q}\right)^{\Omega_\Psi} + (\gamma-1) \prod_{\Psi=1}^{n} \left(1-\underline{(b)^q}\right)^{\Omega_\Psi}}} \right\rangle \right).
\end{array} \right]
$$

Theorem 4.10 ***(Idempotency):****If* $\gamma_{dq\Psi} = \left\{ \begin{array}{c} \left(\left\langle \overline{A_{dq}^{\Psi}}, \overline{B_{dq}^{\Psi}} \right\rangle, \langle \overline{a}, \overline{b} \rangle\right), \\ \left(\left\langle \underline{A_{dq}^{\Psi}}, \underline{B_{dq}^{\Psi}} \right\rangle, \langle \underline{a}, \underline{b} \rangle\right) : \Psi \epsilon \mathbb{N} \end{array} \right\}$ *for all* $(dq = 1, 2, 3, ..., m)$, *then*

$$NLDRFHHWG(\gamma, \gamma, \gamma, ..., \gamma) = \gamma.$$

Theorem 4.11 ***(Commutativity)*** ***:****If* $(bg_1^{'}, bg_2^{'}, ..., bg_n^{'})$ *is any permutation of* $(bg_1, bg_2, ..., bg_n)$, *then* $NLDRFHHWG(bg_1^{'}, bg_2^{'}, ..., bg_n^{'}) = NLDRFHHWG(bg_1, bg_2, ..., bg_n)$.

Theorem 4.12 ***(Boundedness):****If* $Z^{-} = min(gf_1, gf_2, ..., gf_n)$, $Z^{+} = \max(gf_1, gf_2, ..., gf_n)$, *then*

$$Z^{-} \leq NLDRFHHWG(gf_1, gf_2, ..., gf_n) \leq Z^{+}.$$

5. MCDM METHOD based on fuzzy neural networking

In this subsection, we have developed an algorithm based on the NLDRFHWG aggregation operators. The algorithm is applied to a multi-attribute decision making problem to identify and select the most suitable near-earth asteroid detector technologies. This selection process is carried out using NLDRF data in combination with hamacher operators. Consider a set of alternatives $R = \{R_1, R_2, R_3, ..., R_m\}$ and $\check{N} = \left\{\check{N}_1, \check{N}_2, \check{N}_3, ..., \check{N}_n\right\}$ be a set of criteria. Suppose the weight of criteria $\Omega_\Psi\ (\Psi = 1, 2, ..., n)$ are $\Omega = (\Omega_1, \Omega_2, ..., \Omega_n)^T$ with $\Omega_\Psi > 0$, $\sum_{\Psi=1}^{n} \Omega_\Psi = 1$. Suppose the NLDRF $DM = \begin{bmatrix} \left\langle \left(\overline{A^{h\Psi}_{dqK}}, \overline{B^{h\Psi}_{dqK}}\right), (\overline{a}, \overline{b}) \right\rangle, \\ \left\langle \left(\underline{A^{h\Psi}_{dqK}}, \underline{B^{h\Psi}_{dqK}}\right), (\underline{a}, \underline{b}) \right\rangle \end{bmatrix}_{m\times n}$

where $A^{h\Psi}_{dqK}$ is the MG, $B^{h\Psi}_{dqK}$ is the NMG and a, b are the RPs for which the alternatives (R_Ψ) fulfill the criteria $\left(\check{N}_\Psi\right)$, where $A^{h\Psi}_{dqK}, B^{h\Psi}_{dqK}, a, b \subset [0, 1]$ such that $0 \leq ((a)^q A_{dq})^{h\Psi} + ((b)^q B_{dq})^{h\Psi} \leq 1$, $(h = 1, 2, ..., m)$, based on information to solve MADM problem with NLDRFNs based on Hamacher operators by mean of NLDRFHWG aggregation operators.

Input:

Step 1 : For an acceptable number of alternatives and criteria, a decision-makers group is constructed using NLDRF information. The group of decision-makers is represented as $DM = \{DM_1, DM_2, ..., DM_u, DM_l\}$, along with a corresponding weight vector Ω. Each decision-maker is evaluated using NLDRFSs based on Hamacher operators.

Step 2 : The weights for each decision-maker's opinion are selected.

Calculations:

Step 3 : By applying equation of the NLDRF hamacher aggregation operators with weights $\Omega_\Psi\ (\Psi = 1, 2, 3)$ assigned to the criteria $\check{N}_j$, the decision information presented in the matrices DM_k $(k = 1, 2, 3)$ is combined into a collective NLDRFDM.

$$\text{NLDRFHWG}\left(\gamma^1_{dq(h\Psi)}, \gamma^2_{dq(h\Psi)}, ..., \gamma^n_{dq(h\Psi)}\right)$$

$$
= \left[
\begin{array}{l}
\left(\left\langle \dfrac{\gamma \prod_{\Psi=1}^{n} \left(\overline{A_{dq}^{h\Psi}}\right)^{\Omega_\Psi}}{\prod_{\Psi=1}^{n} \left(1+(\gamma-1)\left(1-\overline{A_{dq}^{h\Psi}}\right)\right)^{\Omega_\Psi} + (\gamma-1) \prod_{\Psi=1}^{n} \left(\overline{A_{dq}^{h\Psi}}\right)^{\Omega_\Psi}}, \dfrac{\prod_{\Psi=1}^{n} \left(1+(\gamma-1)\left(\overline{B_{dq}^{h\Psi}}\right)\right)^{\Omega_\Psi} - \prod_{\Psi=1}^{n} \left(1-\overline{B_{dq}^{h\Psi}}\right)^{\Omega_\Psi}}{\prod_{\Psi=1}^{n} \left(1+(\gamma-1)\left(\overline{B_{dq}^{h\Psi}}\right)\right)^{\Omega_\Psi} + (\gamma-1) \prod_{\Psi=1}^{n} \left(1-\overline{B_{dq}^{h\Psi}}\right)^{\Omega_\Psi}} \right\rangle \right), \\
\left(\left\langle \dfrac{\sqrt[q]{\gamma} \prod_{\Psi=1}^{n} \overline{(a)}^{\Omega_\Psi}}{\sqrt[q]{\prod_{\Psi=1}^{n} \left(1+(\gamma-1)\left(1-\overline{(a)}^{q}\right)\right)^{\Omega_\Psi} + (\gamma-1) \prod_{\Psi=1}^{n} \left(\overline{(a)}^{q}\right)^{\Omega_\Psi}}}, \sqrt[q]{\dfrac{\prod_{\Psi=1}^{n} \left(1+(\gamma-1)\overline{(b)}^{q}\right)^{\Omega_\Psi} - \prod_{\Psi=1}^{n} \left(1-\overline{(b)}^{q}\right)^{\Omega_\Psi}}{\prod_{\Psi=1}^{n} \left(1+(\gamma-1)\overline{(b)}^{q}\right)^{\Omega_\Psi} + (\gamma-1) \prod_{\Psi=1}^{n} \left(1-\overline{(b)}^{q}\right)^{\Omega_\Psi}}} \right\rangle \right). \\
\left(\left\langle \dfrac{\gamma \prod_{\Psi=1}^{n} \left(\underline{A_{dq}^{h\Psi}}\right)^{\Omega_\Psi}}{\prod_{\Psi=1}^{n} \left(1+(\gamma-1)\left(1-\underline{A_{dq}^{h\Psi}}\right)\right)^{\Omega_\Psi} + (\gamma-1) \prod_{\Psi=1}^{n} \left(\underline{A_{dq}^{h\Psi}}\right)^{\Omega_\Psi}}, \dfrac{\prod_{\Psi=1}^{n} \left(1+(\gamma-1)\left(\underline{B_{dq}^{h\Psi}}\right)\right)^{\Omega_\Psi} - \prod_{\Psi=1}^{n} \left(1-\underline{B_{dq}^{h\Psi}}\right)^{\Omega_\Psi}}{\prod_{\Psi=1}^{n} \left(1+(\gamma-1)\left(\underline{B_{dq}^{h\Psi}}\right)\right)^{\Omega_\Psi} + (\gamma-1) \prod_{\Psi=1}^{n} \left(1-\underline{B_{dq}^{h\Psi}}\right)^{\Omega_\Psi}} \right\rangle \right), \\
\left(\left\langle \dfrac{\sqrt[q]{\gamma} \prod_{\Psi=1}^{n} \underline{(a)}^{\Omega_\Psi}}{\sqrt[q]{\prod_{\Psi=1}^{n} \left(1+(\gamma-1)\left(1-\underline{(a)}^{q}\right)\right)^{\Omega_\Psi} + (\gamma-1) \prod_{\Psi=1}^{n} \left(\underline{(a)}^{q}\right)^{\Omega_\Psi}}}, \sqrt[q]{\dfrac{\prod_{\Psi=1}^{n} \left(1+(\gamma-1)\underline{(b)}^{q}\right)^{\Omega_\Psi} - \prod_{\Psi=1}^{n} \left(1-\underline{(b)}^{q}\right)^{\Omega_\Psi}}{\prod_{\Psi=1}^{n} \left(1+(\gamma-1)\underline{(b)}^{q}\right)^{\Omega_\Psi} + (\gamma-1) \prod_{\Psi=1}^{n} \left(1-\underline{(b)}^{q}\right)^{\Omega_\Psi}}} \right\rangle \right).
\end{array}
\right]
$$

Step 4:

The scores of each alternative are determined by applying the above definitions of the Standard Function (S.F), the Qualified Standard Function (Q.S.F), and the Enhanced Standard Function (E.S.F).

Output

Step 5:

Based on the values of S.F , Q.S.F , and E.S.F, rank the alternatives.

Step 6:

The alternative with the greatest score holds the highest rank. Therefore, it must be selected for the final decision.

5.1. Numerical example

The area of smart systems is developing fast with the emergence of fuzzy neural networking, a blending paradigm that combines the explainability of fuzzy logic with the learning adaptability of neural networks. Beyond being one of the most important breakthroughs in artificial intelligence, FNN stands as a critical facilitator of the UAE Vision 2031, the country's digital transformation agenda. The UAE aims to utilize FNN to propel innovation in manufacturing, healthcare, transport, education, and smart city development through its ongoing investments in cutting-edge technological infrastructure. Classic machine learning and rule-based decision models typically have difficulty coping with the vagueness, ambiguity, and uncertainty of real-world data. Paralleling its early embrace of cutting-edge telecom infrastructure, the UAE is aware of the importance of adaptive and interpretable AI models. The momentum for FNN was further driven by the expansion of data-hungry applications like AI-powered financial forecasting, logistics optimization, and adaptive learning in education that highlighted the shortcomings of traditional models. Ever since 2018, institutions of higher learning and research institutions have been testing fuzzy-neural integration with IT firms to keep pace with the UAE's vision to become a world-leading hub for Industry 4.0 innovation. By 2019, the UAE was among the first Middle Eastern countries to implement pilot programs implementing FNN across various sectors. In transportation, FNN has allowed self-driving cars to manage noisy sensor data, making vehicle-to-infrastructure (V2I) interactions safer. In the healthcare sector, hospitals have employed FNN-driven diagnostic support systems and remote surgery platforms to provide low-latency adaptive decision-making. The retail and tourism industries, on the other hand, are streamlining customer experiences with AR/VR personalization and FNN-driven recommendation systems.

Cost-Effectiveness Criteria for Evaluation

ensures scalability and long-term sustainability by weighing implementation costs against expected research findings and financial returns.

Accessibility & Coverage

evaluates the reach and inclusivity of FNN apps across various geographies and social groupings.

The availability of computer capacity, cloud platforms, big data infrastructure, and qualified personnel required for successful implementation are all evaluated by technological readiness.

Impact on the Economy and Society Assesses contributions to digital inclusion, job creation, industrial transformation, and general quality of life.

Step 1 : For an acceptable number of alternatives and criteria, a decision-makers (DMs) group is constructed using N-LDRF

Decision matrix table 1

	Expert 1	Expert 2	Expert 3
G_1	$\left[\left(\left\langle \overline{0.1^2}, \overline{0.3^2}\right\rangle, \left\langle \overline{0.5}, \overline{0.6}\right\rangle\right), \left(\left\langle \underline{0.05^2}, \underline{0.03^2}\right\rangle, \left\langle \underline{0.01}, \underline{0.02}\right\rangle\right)\right]$	$\left[\left(\left\langle \overline{0.2^2}, \overline{0.4^2}\right\rangle, \left\langle \overline{0.8}, \overline{0.3}\right\rangle\right), \left(\left\langle \underline{0.02^2}, \underline{0.04^2}\right\rangle, \left\langle \underline{0.05}, \underline{0.01}\right\rangle\right)\right]$	$\left[\left(\left\langle \overline{0.5^2}, \overline{0.3^2}\right\rangle, \left\langle \overline{0.8}, \overline{0.4}\right\rangle\right), \left(\left\langle \underline{0.01^2}, \underline{0.03^2}\right\rangle, \left\langle \underline{0.04}, \underline{0.07}\right\rangle\right)\right]$
G_2	$\left[\left(\left\langle \overline{0.3^2}, \overline{0.4^2}\right\rangle, \left\langle \overline{0.3}, \overline{0.2}\right\rangle\right), \left(\left\langle \underline{0.03^2}, \underline{0.02^2}\right\rangle, \left\langle \underline{0.03}, \underline{0.06}\right\rangle\right)\right]$	$\left[\left(\left\langle \overline{0.5^2}, \overline{0.8^2}\right\rangle, \left\langle \overline{0.6}, \overline{0.2}\right\rangle\right), \left(\left\langle \underline{0.06^2}, \underline{0.03^2}\right\rangle, \left\langle \underline{0.01}, \underline{0.06}\right\rangle\right)\right]$	$\left[\left(\left\langle \overline{0.4^2}, \overline{0.3^2}\right\rangle, \left\langle \overline{0.6}, \overline{0.3}\right\rangle\right), \left(\left\langle \underline{0.05^2}, \underline{0.02^2}\right\rangle, \left\langle \underline{0.06}, \underline{0.04}\right\rangle\right)\right]$
G_3	$\left[\left(\left\langle \overline{0.2^2}, \overline{0.6^2}\right\rangle, \left\langle \overline{0.7}, \overline{0.3}\right\rangle\right), \left(\left\langle \underline{0.04^2}, \underline{0.05^2}\right\rangle, \left\langle \underline{0.02}, \underline{0.03}\right\rangle\right)\right]$	$\left[\left(\left\langle \overline{0.1^2}, \overline{0.7^2}\right\rangle, \left\langle \overline{0.4}, \overline{0.6}\right\rangle\right), \left(\left\langle \underline{0.03^2}, \underline{0.08^2}\right\rangle, \left\langle \underline{0.02}, \underline{0.04}\right\rangle\right)\right]$	$\left[\left(\left\langle \overline{0.5^2}, \overline{0.2^2}\right\rangle, \left\langle \overline{0.4}, \overline{0.2}\right\rangle\right), \left(\left\langle \underline{0.01^2}, \underline{0.04^2}\right\rangle, \left\langle \underline{0.08}, \underline{0.03}\right\rangle\right)\right]$

Decision matrix table 2

	Expert 1	Expert 2	Expert 3
G_1	$\left[\left(\left\langle \overline{0.4^2}, \overline{0.2^2}\right\rangle, \left\langle \overline{0.2}, \overline{0.3}\right\rangle\right), \left(\left\langle \underline{0.02^2}, \underline{0.01^2}\right\rangle, \left\langle \underline{0.03}, \underline{0.01}\right\rangle\right)\right]$	$\left[\left(\left\langle \overline{0.2^2}, \overline{0.5^2}\right\rangle, \left\langle \overline{0.8}, \overline{0.3}\right\rangle\right), \left(\left\langle \underline{0.01^2}, \underline{0.04^2}\right\rangle, \left\langle \underline{0.05}, \underline{0.01}\right\rangle\right)\right]$	$\left[\left(\left\langle \overline{0.4^2}, \overline{0.2^2}\right\rangle, \left\langle \overline{0.7}, \overline{0.1}\right\rangle\right), \left(\left\langle \underline{0.03^2}, \underline{0.02^2}\right\rangle, \left\langle \underline{0.02}, \underline{0.05}\right\rangle\right)\right]$
G_2	$\left[\left(\left\langle \overline{0.3^2}, \overline{0.2^2}\right\rangle, \left\langle \overline{0.2}, \overline{0.1}\right\rangle\right), \left(\left\langle \underline{0.04^2}, \underline{0.01^2}\right\rangle, \left\langle \underline{0.03}, \underline{0.06}\right\rangle\right)\right]$	$\left[\left(\left\langle \overline{0.5^2}, \overline{0.4^2}\right\rangle, \left\langle \overline{0.3}, \overline{0.2}\right\rangle\right), \left(\left\langle \underline{0.02^2}, \underline{0.01^2}\right\rangle, \left\langle \underline{0.01}, \underline{0.06}\right\rangle\right)\right]$	$\left[\left(\left\langle \overline{0.2^2}, \overline{0.6^2}\right\rangle, \left\langle \overline{0.2}, \overline{0.4}\right\rangle\right), \left(\left\langle \underline{0.05^2}, \underline{0.02^2}\right\rangle, \left\langle \underline{0.03}, \underline{0.05}\right\rangle\right)\right]$
G_3	$\left[\left(\left\langle \overline{0.2^2}, \overline{0.5^2}\right\rangle, \left\langle \overline{0.4}, \overline{0.3}\right\rangle\right), \left(\left\langle \underline{0.02^2}, \underline{0.01^2}\right\rangle, \left\langle \underline{0.01}, \underline{0.05}\right\rangle\right)\right]$	$\left[\left(\left\langle \overline{0.1^2}, \overline{0.6^2}\right\rangle, \left\langle \overline{0.4}, \overline{0.6}\right\rangle\right), \left(\left\langle \underline{0.03^2}, \underline{0.05^2}\right\rangle, \left\langle \underline{0.02}, \underline{0.05}\right\rangle\right)\right]$	$\left[\left(\left\langle \overline{0.5^2}, \overline{0.2^2}\right\rangle, \left\langle \overline{0.1}, \overline{0.7}\right\rangle\right), \left(\left\langle \underline{0.01^2}, \underline{0.06^2}\right\rangle, \left\langle \underline{0.08}, \underline{0.02}\right\rangle\right)\right]$

Step 2 : The weights for each decision-maker's opinion are selected in table 3.

	Expert 1	Expert 2	Expert 3
G_1	$\left[\left(\langle 0.246, 0.324\rangle, \langle 0.398, 0.709\rangle\right), \left(\langle 0.063, 0.019\rangle, \langle 0.038, 0.181\rangle\right)\right]$	$\left[\left(\langle 0.144, 0.380\rangle, \langle 0.765, 0.485\rangle\right), \left(\langle 0.006, 0.021\rangle, \langle 0.027, 0.062\rangle\right)\right]$	$\left[\left(\langle 0.166, 0.130\rangle, \langle 0.002, 0.2\rangle\right), \left(\langle 0.0004, 0.0006\rangle, \langle 0.0009, 0.0591\rangle\right)\right]$
G_2	$\left[\left(\langle 0.381, 0.254\rangle, \langle 0.324, 0.415\rangle\right), \left(\langle 0.067, 0.033\rangle, \langle 0.060, 0.324\rangle\right)\right]$	$\left[\left(\langle 0.435, 0.504\rangle, \langle 0.357, 0.380\rangle\right), \left(\langle 0.017, 0.007\rangle, \langle 0.003, 0.184\rangle\right)\right]$	$\left[\left(\langle 0.08, 0.18\rangle, \langle 0.119, 0.346\rangle\right), \left(\langle 0.0025, 0.0004\rangle, \langle 0.0017, 0.04\rangle\right)\right]$
G_3	$\left[\left(\langle 0.275, 0.617\rangle, \langle 0.601, 0.439\rangle\right), \left(\langle 0.057, 0.047\rangle, \langle 0.032, 0.272\rangle\right)\right]$	$\left[\left(\langle 0.064, 0.594\rangle, \langle 0.333, 0.736\rangle\right), \left(\langle 0.014, 0.036\rangle, \langle 0.009, 0.095\rangle\right)\right]$	$\left[\left(\langle 0.25, 0.04\rangle, \langle 0.039, 0.374\rangle\right), \left(\langle 0.0001, 0.0024\rangle, \langle 0.0063, 0.0244\rangle\right)\right]$

Step 3 : By applying Equation of the NLDRF hamacher aggregation operators with weights assigned to the criteria, the decision information presented in the matrices is combined into a collective NLDRFDM.

	Expert table 4
G_1	$\left[\left(\langle 1.9999, 0.2152, \rangle, \langle 0.0342, 0.5796\rangle\right), \left(\langle 0.0024, 0.0034\rangle, \langle 0.0052, 0.273\rangle\right)\right]$
G_2	$\left[\left(\langle 0.1816, 0.2626\rangle, \langle 0.2021, 0.6075\rangle\right), \left(\langle 0.0085, 0.0037\rangle, \langle 0.0041, 0.3099\rangle\right)\right]$
G_3	$\left[\left(\langle 0.1693, 0.1441\rangle, \langle 0.1282, 0.6785\rangle\right), \left(\langle 0.0186, 0.0098\rangle, \langle 0.0096, 0.2438\rangle\right)\right]$

Step 4: The scores of each alternative are determined by applying the above definitions of the Standard Function.

Find the score function

$R_1 = 0.242, R_2 = -0.195, R_3 = -0.187$

Step 5:Find the quadratic score function

$R_1 = 0.8859, R_2 = -0.1150, R_3 = -0.1238$

Step 6:Find the expectation score function

$R_1 = 0.6213, R_2 = 0.4015, R_3 = 0.4061.$

5.2. Sensitivity analysis

In this subsection, we define the sensitive study and written below in table 5.

Methods	Score function	Ranking
$q = 0.1$	$\left\{ \begin{array}{c} R_1 = 0.0112, \\ R_2 = 0.8011, \\ R_3 = 0.3661 \end{array} \right\}$	$\left\{ \begin{array}{c} R_2 > \\ R_3 > \\ R_1 \end{array} \right\}$
0.2	$\left\{ \begin{array}{c} R_1 = 0.0122, \\ R_2 = 0.8185, \\ R_3 = 0.3989 \end{array} \right\}$	$\left\{ \begin{array}{c} R_2 > \\ R_3 > \\ R_1 \end{array} \right\}$
0.3	$\left\{ \begin{array}{c} R_1 = 0.0856, \\ R_2 = 0.7845, \\ R_3 = 0.5985 \end{array} \right\}$	$\left\{ \begin{array}{c} R_2 > \\ R_3 > \\ R_1 \end{array} \right\}$
0.4	$\left\{ \begin{array}{c} R_1 = 0.0555, \\ R_2 = 0.4969, \\ R_3 = 0.4858 \end{array} \right\}$	$\left\{ \begin{array}{c} R_2 > \\ R_3 > \\ R_1 \end{array} \right\}$
0.5	$\left\{ \begin{array}{c} R_1 = 0.0036, \\ R_2 = 0.7625, \\ R_3 = 0.3968 \end{array} \right\}$	$\left\{ \begin{array}{c} R_2 > \\ R_3 > \\ R_1 \end{array} \right\}$
0.6	$\left\{ \begin{array}{c} R_1 = 0.0441, \\ R_2 = 0.8817, \\ R_3 = 0.4896 \end{array} \right\}$	$\left\{ \begin{array}{c} R_2 > \\ R_3 > \\ R_1 \end{array} \right\}$
0.7	$\left\{ \begin{array}{c} R_1 = 0.0478, \\ R_2 = 0.2987, \\ R_3 = 0.2358 \end{array} \right\}$	$\left\{ \begin{array}{c} R_2 > \\ R_3 > \\ R_1 \end{array} \right\}$
0.8	$\left\{ \begin{array}{c} R_1 = 0.0459, \\ R_2 = 0.5263, \\ R_3 = 0.3975 \end{array} \right\}$	$\left\{ \begin{array}{c} R_2 > \\ R_3 > \\ R_1 \end{array} \right\}$
0.9	$\left\{ \begin{array}{c} R_1 = 0.1099, \\ R_2 = 0.8978, \\ R_3 = 0.3911 \end{array} \right\}$	$\left\{ \begin{array}{c} R_2 > \\ R_3 > \\ R_1 \end{array} \right\}$
1	$\left\{ \begin{array}{c} R_1 = 0.0015, \\ R_2 = 0.6611, \\ R_3 = 0.5062 \end{array} \right\}$	$\left\{ \begin{array}{c} R_2 > \\ R_3 > \\ R_1 \end{array} \right\}$

5.3. Analysis of Aggregation Operators in fuzzy Network Evaluation

The NLDRFS method in the fuzzy neural networking framework. Several distance measures were used in the FNN model to calculate dominance scores under various parameter settings in order to guarantee a thorough study. These dominance values and the ensuing rankings are shown in table 6, which also shows how various aggregation methods affect the final assessment of alternatives. Incorporating NLDRF data into the FNN structure resulted in significantly higher dominance ratings, suggesting that the framework conserved more relational information than alternative distance measures in table 6.

NLDRFS method table 6.

Alternative	R_1	R_2	R_3
NLDRFHWG operator	0.8532	0.6606	0.7578
NLDRFHOWG operator	0.1008	0.7866	0.0199
NLDRFHHWG operator	0.0211	0.2229	0.0086

5.4. Superiority of their proposed method

An effective adaptable tool in Nonlinear diophantine rough fuzzy settings is the hamacher geometric aggregation operator. It skillfully combines membership and non-membership grades, offering a fair portrayal of both upper and lower values. Several crucial elements should be taken into account when comparing the sensitivity of NLDRF hamacher geometric aggregation operators to alternative aggregation techniques:

To combine input data of different aggregation operators use a variety of weighting values. While other operators might prioritize some inputs differently, the NLDRF framework uses a weighting method, which could produce various results.

Boundedness, Commutativity, and idempotency are some of the properties that affect aggregation performance. The final results are affected by the NLDRF operators' maintenance of particular mathematical behaviors that may be different from those of other aggregation methods. When making decisions, the capacity to understand combined results is essential. Other operators might produce findings that are less obvious or more difficult to read, but the NLDRF framework provides an organized and interpretable method. An operator's efficacy differs depending on the application domain. Alternative operators might be more useful in some decision-making situations, even while the NLDRF hamacher operator can remarkably well in others.

5.5. Limitation

High computational complexity results from the multiple calculations required to compute nonlinear diophantine rough fuzzy hamacher geometric aggregation operators, and efficiency decreases as the computing burden increases in tandem with the number of input pieces. It can be difficult to evaluate and convey the combined results. These operators frequently provide complex outcomes that can be challenging for non-experts to comprehend or articulate intelligibly. As a result, in real-world decision-making situations, it may be difficult to derive useful interpretations and insights from the combined values. Notwithstanding these limitations, nonlinear diophantine rough fuzzy hamacher geometric aggregation operators are nonetheless useful instruments in some situations, providing important new information and improved analytical power when used properly.

6. Conclusion

For multicriteria group decision-making in the context of nonlinear diophantine Rough Fuzzy sets, we presented Hamacher aggregation operators in this paper. The suggested operators include uncertain and nonlinear data, offering a more accurate and adaptable method for handling challenging decision-making issues. To examined some novel operators, including the NLDRFHWG, NLDRFHOWG, and NLDRFH-HWG, determining their essential characteristics, such as boundedness, monotonicity, and idempotency. These operators of aggregation techniques in handling imprecise, ambiguous, as well as inconsistent data, as shown by a real-world application. According to the results, the Hamacher based NLDRF operators provide a strong mathematical instrument for making decisions in ambiguous situations. These operators may be extended to other complex NLDRF models in future studies, and their potential in fields including risk assessment, optimization, and intelligent decision support systems may be investigated. Funding
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Contributions
All authors equally contributed to this manuscript.

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