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On Ricci Pseudo-Symmetric Mixed Quasi-Einstein Hermitian Manifold

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ABSTRACT: In this paper, we investigate pseudo-symmetric mixed quasi-Einstein Hermitian (MQEH) manifolds under Bochner, holomorphically projective, and combined curvature structures. It is shown that a Bochner Ricci pseudo-symmetric MQEH manifold either reduces to an Einstein Hermitian manifold or satisfies a specific relation involving the Bochner tensor and the associated 1-forms. The co-directionality of the vector fields ρ and σ is characterized through the vanishing of the mixed components of the curvature tensors. For holomorphically projective and flat cases, corresponding classification results are obtained. A unified curvature condition is also introduced.

Keywords: Quasi-Einstein manifold, generalized quasi-Einstein manifold, mixed quasi-Einstein manifold, Bochner curvature tensor, holomorphically projective curvature tensor.

Contents

1. Introduction

An even dimensional differentiable manifold $\mathcal{M}^n$ is said to be a Hermitian manifold [17] if a complex structure $\mathcal{J}$ of type $(1,1)$ and a pseudo-Riemannian metric g of the manifold satisfy

$$\mathcal{J}^2 = -I \tag{1.1}$$

and

$$g(\mathcal{J}\Upsilon_1, \mathcal{J}\Upsilon_2) = g(\Upsilon_1, \Upsilon_2), \tag{1.2}$$

for all Υ_1, $\Upsilon_2 \in M^n$. An n-dimensional Riemannian manifold $\mathcal{M}$ is named an Einstein manifold [1] if the Ricci tensor $Ric(\neq 0)$ of type $(0,2)$ satisfies $Ric = \frac{scal}{n} g$, where $scal$ represents the scalar curvature. Since Einstein manifolds have important differential geometric properties and significant physical applications, they are studied by geometers in a broad perspective. Also, Einstein manifolds play a key role in Riemannian geometry, general theory of relativity, and mathematical physics.

An n-dimensional Riemannian manifold $\mathcal{M}$ is said to be a quasi-Einstein (QE) manifold [10,11] if its $Ric(\neq 0)$ satisfies

$$Ric(\Upsilon_1, \Upsilon_2) = a g(\Upsilon_1, \Upsilon_2) + b A(\Upsilon_1) A(\Upsilon_2), \tag{1.3}$$

where a, $b(\neq 0) \in \mathbb{R}$ and $A(\neq 0)$ is 1-form such that

$$g(\Upsilon_1, \rho) = A(\Upsilon_1), \quad g(\rho, \rho) = A(\rho) = 1, \tag{1.4}$$

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for all $\Upsilon_1 \in M^n$ and a unit vector field ρ named the generator of QE manifolds. Also, the 1-form A is called the associated 1-form. From (1.3) it is clear that for $b = 0$, QE manifolds reduces to an Einstein manifold.

Now contracting of (1.3) over Υ_1 and Υ_2 gives

$$scal = an + b. \tag{1.5}$$

From (1.2), (1.3) and (1.4), we have

$$\begin{aligned} &Ric(\Upsilon_1, \rho) = (a + b)A(\Upsilon_1),\ Ric(\rho, \rho) = (a + b),\\ &g(\mathcal{J}\rho, \rho) = 0,\ Ric(\mathcal{J}\rho, \rho) = 0. \end{aligned} \tag{1.6}$$

Some generalizations of Einstein manifolds have been defined and studied. One of them is generalized quasi-Einstein manifold is studied by Chaki [13]. An n-dimensional Riemannian manifold $\mathcal{M}$ is said to be a generalized quasi-Einstein (GQE) manifold [12,13,14] if its $Ric(\neq 0)$ satisfies

$$Ric(\Upsilon_1, \Upsilon_2) = ag(\Upsilon_1, \Upsilon_2) + bA(\Upsilon_1)A(\Upsilon_2) + cC(\Upsilon_1)C(\Upsilon_2), \tag{1.7}$$

where a, $b(\neq 0)$, $c(\neq 0) \in \mathbb{R}$ and $A(\neq 0)$, $C(\neq 0)$ are 1-forms such that

$$g(\Upsilon_1, \rho) = A(\Upsilon_1),\ g(\Upsilon_1, \sigma) = C(\Upsilon_1),\ g(\rho, \rho) = g(\sigma, \sigma) = 1, \tag{1.8}$$

for all Υ_1, $\Upsilon_2 \in M^n$, where ρ and σ are unit vector field generators of the GQE manifold.

Taking the contraction of (1.7) over the indices Υ_1 and Υ_2 yields

$$scal = an + b + c. \tag{1.9}$$

From (1.2), (1.7) and (1.8) we have

$$\begin{aligned} &Ric(\Upsilon_1, \rho) = (a + b)A(\Upsilon_1),\ Ric(\Upsilon_1, \sigma) = (a + c)C(\Upsilon_1),\ Ric(\sigma, \sigma) = a + c,\\ &Ric(\rho, \rho) = a + b,\ g(\mathcal{J}\rho, \rho) = g(\mathcal{J}\sigma, \sigma) = 0,\ Ric(\mathcal{J}\rho, \rho) = Ric(\mathcal{J}\sigma, \sigma) = 0. \end{aligned} \tag{1.10}$$

An n-dimensional Riemannian manifold $\mathcal{M}$ is said to be a nearly quasi-Einstein (NQE) manifold [20] if its $Ric(\neq 0)$ satisfies

$$Ric(\Upsilon_1, \Upsilon_2) = ag(\Upsilon_1, \Upsilon_2) + bE(\Upsilon_1, \Upsilon_2), \tag{1.11}$$

where a, $b(\neq 0) \in \mathbb{R}$ and $E(\neq 0)$ is symmetric tensor of type $(0, 2)$. There are many author works on this manifold like [5,16].

In 2011, Singh, Pandey, and Gautam [16] introduced a new class of NQE manifolds by specifying the tensor E in the following form

$$E(\Upsilon_1, \Upsilon_2) = A(\Upsilon_1)C(\Upsilon_2) + A(\Upsilon_2)C(\Upsilon_1). \tag{1.12}$$

Now, from (1.11) and (1.12), we have

$$Ric(\Upsilon_1, \Upsilon_2) = ag(\Upsilon_1, \Upsilon_2) + b[A(\Upsilon_1)C(\Upsilon_2) + A(\Upsilon_2)C(\Upsilon_1)], \tag{1.13}$$

where a, $b(\neq 0) \in \mathbb{R}$ and $A(\neq 0)$, $C(\neq 0)$ are 1-forms such that

$$g(\Upsilon_1, \rho) = A(\Upsilon_1),\ g(\Upsilon_1, \sigma) = C(\Upsilon_1),\ g(\rho, \rho) = g(\sigma, \sigma) = 1. \tag{1.14}$$

Contracting (1.13) over Υ_1 and Υ_2 yields

$$scal = na.$$

From (1.2), (1.13) and (1.14), we have

$$\begin{aligned} &Ric(\Upsilon_1, \rho) = aA(\Upsilon_1) + bC(\Upsilon_1),\ Ric(\Upsilon_1, \sigma) = aC(\Upsilon_1) + bA(\Upsilon_1),\\ &Ric(\sigma, \sigma) = Ric(\rho, \rho) = a,\ g(\mathcal{J}\rho, \rho) = g(\mathcal{J}\sigma, \sigma) = 0,\\ &Ric(\mathcal{J}\rho, \rho) = Ric(\mathcal{J}\sigma, \sigma) = 0. \end{aligned} \tag{1.15}$$

In 2010, Nagaraja introduced and investigated the concept of a Mixed Quasi-Einstein (MQE) manifold [6], along with several of its geometric properties.

Thus we can state the following proposition:

Proposition 1.1 *An n-dimensional MQE manifold is a special type of NQE manifold.*

The notion of Ricci pseudosymmetric manifold was introduced by Deszcz [15]. A geometrical interpretation of Ricci pseudosymmetric manifolds in the Riemannian case is given in [4]. A Riemannian manifold $\mathcal{M}$ is said to be Ricci pseudo-symmetric if the tensor $K.Ric$ and the Tachibana tensor $Q(g, Ric)$ are linearly dependent, where

$$(K(\Upsilon_1,\Upsilon_2).Ric)(\Upsilon_3,\Upsilon_4) = -Ric(K(\Upsilon_1,\Upsilon_2)\Upsilon_3,\Upsilon_4) - Ric(\Upsilon_3, K(\Upsilon_1,\Upsilon_2)\Upsilon_4), \tag{1.16}$$

$$Q(g, Ric)(\Upsilon_3,\Upsilon_4;\Upsilon_1,\Upsilon_2) = Ric((\Upsilon_1 \wedge_g \Upsilon_2)\Upsilon_3,\Upsilon_4) - Ric(\Upsilon_3,(\Upsilon_1 \wedge_g \Upsilon_2)\Upsilon_4) \tag{1.17}$$

and

$$(\Upsilon_1 \wedge_g \Upsilon_2)\Upsilon_3 = g(\Upsilon_2,\Upsilon_3)\Upsilon_1 - g(\Upsilon_1,\Upsilon_3)\Upsilon_2, \tag{1.18}$$

for all Υ_1, Υ_2, Υ_3, $\Upsilon_4 \in M^n$ and K denotes the curvature tensor. Then $\mathcal{M}$ is Ricci pseudo-symmetric if and only if

$$(K(\Upsilon_1,\Upsilon_2).Ric)(\Upsilon_3,\Upsilon_4) = L_s Q(g, Ric)(\Upsilon_3,\Upsilon_4;\Upsilon_1,\Upsilon_2), \tag{1.19}$$

holds on P, where $P = \{v \in \mathcal{M} : Ric \neq \frac{scal}{n} g \ at \ x\}$ and L_s is a certain function on P. Then by using (1.16)-(1.19), we can write $\mathcal{M}$ is Ricci pseudo-symmetric if and only if the equation

$$\begin{aligned} Ric(K(\Upsilon_1,\Upsilon_2)\Upsilon_3,\Upsilon_4) + Ric(\Upsilon_3, K(\Upsilon_1,\Upsilon_2)\Upsilon_4) &= L_s[g(\Upsilon_2,\Upsilon_3)Ric(\Upsilon_1,\Upsilon_4) \\ - g(\Upsilon_1,\Upsilon_3)Ric(\Upsilon_2,\Upsilon_4) + g(\Upsilon_2,\Upsilon_4)Ric(\Upsilon_3,\Upsilon_1) &- g(\Upsilon_1,\Upsilon_4)Ric(\Upsilon_2,\Upsilon_3), \end{aligned} \tag{1.20}$$

holds.

In 2012, De [21] studied some geometric and global properties of MQE manifolds. Also, the existence of a MQE manifold has been proved by two non-trivial examples. In 2017, Mallick, Yildiz and De [19] studied some geometric properties of MQE manifolds and also discussed the MQE spacetime with the space-matter tensor and some properties related to it. In 2018, Suh, Majhi and De [22] proved that every Z Ricci pseudosymmetric MQE spacetime is a Z Ricci semisymmetric spacetime. After that, they studied Z flat spacetimes. Zhiming, Huazhen and Weijun [7] continue the study of MQE manifolds that satisfy certain geometric properties such as Ricci semisymmetric, concircular Ricci pseudosymmetric and W_i Ricci pseudosymmetric. In 2020, Chaturvedi and Gupta [2] studied on Bochner Ricci pseudo-symmetric Hermitian manifold. In 2020 and 2021, Chaturvedi and Gupta [3] continue the study on Bochner Ricci pseudo-symmetric super quasi-Einstein Hermitian manifold and on Ricci pseudo-symmetric mixed generalized quasi-Einstein Hermitian manifold. In 2025, Vasiulla and Ali [9] studied extending the geometric analysis to Ricci semi-symmetric mixed quasi-Einstein Hermitian (MQEH) manifolds. Motivated by the above study, the authors continue their investigation of Ricci pseudo-symmetric mixed quasi-Einstein Hermitian (MQEH) manifolds.

2. Bochner Ricci Pseudo-Symmetric MQEH Manifold

The notion of a Bochner curvature tensor B has been introduced by S. Bochner [18] and is defined by

$$\begin{aligned} B(\Upsilon_1,\Upsilon_2,\Upsilon_3,\Upsilon_4) = \overline{K}(\Upsilon_1,\Upsilon_2,\Upsilon_3,\Upsilon_4) &- \frac{1}{2(n+2)}[Ric(\Upsilon_1,\Upsilon_4)g(\Upsilon_2,\Upsilon_3) \\ &- Ric(\Upsilon_1,\Upsilon_3)g(\Upsilon_2,\Upsilon_4) + g(\Upsilon_1,\Upsilon_4)Ric(\Upsilon_2,\Upsilon_3) \\ &- g(\Upsilon_1,\Upsilon_3)Ric(\Upsilon_2,\Upsilon_4) + Ric(\mathcal{J}\Upsilon_1,\Upsilon_4)g(\mathcal{J}\Upsilon_2,\Upsilon_3) \\ &- Ric(\mathcal{J}\Upsilon_1,\Upsilon_3)g(\mathcal{J}\Upsilon_2,\Upsilon_4) + Ric(\mathcal{J}\Upsilon_2,\Upsilon_3)g(\mathcal{J}\Upsilon_1,\Upsilon_4) \\ &- g(\mathcal{J}\Upsilon_1,\Upsilon_3)Ric(\mathcal{J}\Upsilon_2,\Upsilon_4) - 2Ric(\mathcal{J}\Upsilon_1,\Upsilon_2)g(\mathcal{J}\Upsilon_3,\Upsilon_4) \\ &- 2g(\mathcal{J}\Upsilon_1,\Upsilon_2)Ric(\mathcal{J}\Upsilon_3,\Upsilon_4)] + \frac{scal}{(2n+2)(2n+4)} \\ &[g(\Upsilon_2,\Upsilon_3)g(\Upsilon_1,\Upsilon_4) - g(\Upsilon_1,\Upsilon_3)g(\Upsilon_2,\Upsilon_4) \\ &+ g(\mathcal{J}\Upsilon_2,\Upsilon_3)g(\mathcal{J}\Upsilon_1,\Upsilon_4) - g(\mathcal{J}\Upsilon_1,\Upsilon_3)g(\mathcal{J}\Upsilon_2,\Upsilon_4) \\ &- 2g(\mathcal{J}\Upsilon_1,\Upsilon_2)g(\mathcal{J}\Upsilon_3,\Upsilon_4)], \end{aligned} \tag{2.1}$$

where *scal* is the scalar curvature of the manifold and $\overline{K}$ is the curvature tensor of type $(0,4)$.

In a Hermitian manifold, this tensor satisfies

$$B(\Upsilon_1,\Upsilon_2,\Upsilon_3,\Upsilon_4) = -B(\Upsilon_1,\Upsilon_2,\Upsilon_4,\Upsilon_3). \tag{2.2}$$

Now, we introduce the following.

Definition 2.1 *A Hermitian manifold is said to be a MQEH manifold if it satisfies* (1.13).

Definition 2.2 *An even dimensional Hermitian manifold $\mathcal{M}^n$ is said to be a Bochner Ricci pseudo-symmetric MQEH manifold if and only if the tensors $B.Ric$ and $Q(g,Ric)$ are linearly dependent, i.e.,*

$$Ric(B(\Upsilon_1,\Upsilon_2)\Upsilon_3,\Upsilon_4) + Ric(\Upsilon_3, B(\Upsilon_1,\Upsilon_2)\Upsilon_4) = L_s Q(g,Ric)(\Upsilon_3,\Upsilon_4;\Upsilon_1,\Upsilon_2) \tag{2.3}$$

holds on P, where $P = \{\upsilon \in M : Ric \neq \frac{scal}{n} g \text{ at } \upsilon\}$ and L_s is a certain function on P.

If we take a Bochner Ricci pseudo-symmetric MQEH manifold, then from (1.13) and (2.3), we get

$$\begin{aligned} & Ric(B(\Upsilon_1,\Upsilon_2)\Upsilon_3,\Upsilon_4) + Ric(\Upsilon_3, B(\Upsilon_1,\Upsilon_2)\Upsilon_4) = L_s[g(\Upsilon_2,\Upsilon_3)Ric(\Upsilon_1,\Upsilon_4) \\ - \; & g(\Upsilon_1,\Upsilon_3)Ric(\Upsilon_2,\Upsilon_4) + g(\Upsilon_2,\Upsilon_4)Ric(\Upsilon_3,\Upsilon_1) - g(\Upsilon_1,\Upsilon_4)Ric(\Upsilon_2,\Upsilon_3)]. \end{aligned} \tag{2.4}$$

Using (1.13) in (2.4), we have

$$\begin{aligned} & a[g(B(\Upsilon_1,\Upsilon_2)\Upsilon_3,\Upsilon_4) + g(\Upsilon_3, B(\Upsilon_1,\Upsilon_2)\Upsilon_4)] \\ + \; & b[A(B(\Upsilon_1,\Upsilon_2)\Upsilon_3)C(\Upsilon_4) + A(\Upsilon_4)C(B(\Upsilon_1,\Upsilon_2)\Upsilon_3) \\ + \; & A(B(\Upsilon_1,\Upsilon_2)\Upsilon_4)C(\Upsilon_3) + A(\Upsilon_3)C(B(\Upsilon_1,\Upsilon_2)\Upsilon_4)] \\ = \; & L_s\Big(b\Big\{g(\Upsilon_2,\Upsilon_3)[A(\Upsilon_1)C(\Upsilon_4) + A(\Upsilon_4)C(\Upsilon_1)] \\ - \; & g(\Upsilon_1,\Upsilon_3)[A(\Upsilon_2)C(\Upsilon_4) + A(\Upsilon_4)C(\Upsilon_2)] \\ + \; & g(\Upsilon_2,\Upsilon_4)[A(\Upsilon_1)C(\Upsilon_3) + A(\Upsilon_3)C(\Upsilon_1)] \\ - \; & g(\Upsilon_1,\Upsilon_4)[A(\Upsilon_2)C(\Upsilon_3) + A(\Upsilon_3)C(\Upsilon_2)]\Big\}\Big). \end{aligned} \tag{2.5}$$

From (2.2) and (2.5), we get

$$\begin{aligned} & b[A(B(\Upsilon_1,\Upsilon_2)\Upsilon_3)C(\Upsilon_4) + A(\Upsilon_4)C(B(\Upsilon_1,\Upsilon_2)\Upsilon_3) \\ + \; & A(B(\Upsilon_1,\Upsilon_2)\Upsilon_4)C(\Upsilon_3) + A(\Upsilon_3)C(B(\Upsilon_1,\Upsilon_2)\Upsilon_4)] \\ = \; & L_s\Big(b\Big\{g(\Upsilon_2,\Upsilon_3)[A(\Upsilon_1)C(\Upsilon_4) + A(\Upsilon_4)C(\Upsilon_1)] \\ - \; & g(\Upsilon_1,\Upsilon_3)[A(\Upsilon_2)C(\Upsilon_4) + A(\Upsilon_4)C(\Upsilon_2)] \\ + \; & g(\Upsilon_2,\Upsilon_4)[A(\Upsilon_1)C(\Upsilon_3) + A(\Upsilon_3)C(\Upsilon_1)] \\ - \; & g(\Upsilon_1,\Upsilon_4)[A(\Upsilon_2)C(\Upsilon_3) + A(\Upsilon_3)C(\Upsilon_2)]\Big\}\Big). \end{aligned} \tag{2.6}$$

Putting $\Upsilon_3 = \Upsilon_4 = \rho$ in (2.6) and using (1.14), we have

$$b\Big(B(\Upsilon_1,\Upsilon_2,\rho,\sigma) - L_s[A(\Upsilon_2)C(\Upsilon_1) - A(\Upsilon_1)C(\Upsilon_2)]\Big) = 0, \tag{2.7}$$

which implies that either $b = 0$ or $\Big(B(\Upsilon_1,\Upsilon_2,\rho,\sigma) - L_s[A(\Upsilon_2)C(\Upsilon_1) - A(\Upsilon_1)C(\Upsilon_2)]\Big) = 0$.
If $b = 0$, then from (1.13), we found that the manifold reduces to an Einstein manifold.

Thus, we can state the following theorem.

Theorem 2.3 *A Bochner Ricci pseudosymmetric MQEH manifold is either a Bochner Ricci pseudosymmetric Einstein Hermitian manifold or $B(\Upsilon_1,\Upsilon_2,\rho,\sigma) = L_s[A(\Upsilon_2)C(\Upsilon_1) - A(\Upsilon_1)C(\Upsilon_2)]$ holds.*

Again, if $b \neq 0$ then from (2.7), we get

$$B(\Upsilon_1, \Upsilon_2, \rho, \sigma) - L_s[A(\Upsilon_2)C(\Upsilon_1) - A(\Upsilon_1)C(\Upsilon_2)] = 0. \tag{2.8}$$

Let us consider that $B(\Upsilon_1, \Upsilon_2, \rho, \sigma)$ vanishes and then from (2.8), we have

$$A(\Upsilon_2)C(\Upsilon_1) = A(\Upsilon_1)C(\Upsilon_2).$$

Thus, we can state the following conclusion.

Theorem 2.4 *In a Bochner Ricci pseudo symmetric MQEH manifold if $b \neq 0$ then $B(\Upsilon_1, \Upsilon_2, \rho, \sigma) = 0$ if and only if the 1-forms A and C are co-directional.*

3. Bochner Flat Ricci Pseudo Symmetric MQEH Manifold

If the Bochner curvature tensor is flat, then

$$B(\Upsilon_1, \Upsilon_2, \Upsilon_3, \Upsilon_4) = 0. \tag{3.1}$$

Using (3.1) in (2.1), we have

$$\begin{aligned}
\overline{K}(\Upsilon_1, \Upsilon_2, \Upsilon_3, \Upsilon_4) = \frac{1}{2(n+2)} \Big[& Ric(\Upsilon_1, \Upsilon_4) g(\Upsilon_2, \Upsilon_3) \\
& - Ric(\Upsilon_1, \Upsilon_3) g(\Upsilon_2, \Upsilon_4) + g(\Upsilon_1, \Upsilon_4) Ric(\Upsilon_2, \Upsilon_3) \\
& - g(\Upsilon_1, \Upsilon_3) Ric(\Upsilon_2, \Upsilon_4) + Ric(\mathcal{J}\Upsilon_1, \Upsilon_4) g(\mathcal{J}\Upsilon_2, \Upsilon_3) \\
& - Ric(\mathcal{J}\Upsilon_1, \Upsilon_3) g(\mathcal{J}\Upsilon_2, \Upsilon_4) + Ric(\mathcal{J}\Upsilon_2, \Upsilon_3) g(\mathcal{J}\Upsilon_1, \Upsilon_4) \\
& - g(\mathcal{J}\Upsilon_1, \Upsilon_3) Ric(\mathcal{J}\Upsilon_2, \Upsilon_4) - 2Ric(\mathcal{J}\Upsilon_1, \Upsilon_2) g(\mathcal{J}\Upsilon_3, \Upsilon_4) \\
& - 2g(\mathcal{J}\Upsilon_1, \Upsilon_2) Ric(\mathcal{J}\Upsilon_3, \Upsilon_4) \Big] - \frac{scal}{(2n+2)(2n+4)} \\
& \Big[g(\Upsilon_2, \Upsilon_3) g(\Upsilon_1, \Upsilon_4) - g(\Upsilon_1, \Upsilon_3) g(\Upsilon_2, \Upsilon_4) \\
& + g(\mathcal{J}\Upsilon_2, \Upsilon_3) g(\mathcal{J}\Upsilon_1, \Upsilon_4) - g(\mathcal{J}\Upsilon_1, \Upsilon_3) g(\mathcal{J}\Upsilon_2, \Upsilon_4) \\
& - 2g(\mathcal{J}\Upsilon_1, \Upsilon_2) g(\mathcal{J}\Upsilon_3, \Upsilon_4) \Big].
\end{aligned} \tag{3.2}$$

Using (2.4) in (3.2), we have

$$\begin{aligned}
& \frac{1}{2n+4} \Big[Ric(Q\Upsilon_1, \Upsilon_4) g(\Upsilon_2, \Upsilon_3) - g(\Upsilon_1, \Upsilon_3) Ric(Q\Upsilon_2, \Upsilon_4) \\
+ \;& Ric(Q\mathcal{J}\Upsilon_1, \Upsilon_4) g(\mathcal{J}\Upsilon_2, \Upsilon_3) - g(\mathcal{J}\Upsilon_1, \Upsilon_3) Ric(\mathcal{J}Q\Upsilon_2, \Upsilon_4) \\
+ \;& g(\Upsilon_2, \Upsilon_4) Ric(Q\Upsilon_1, \Upsilon_3) - g(\Upsilon_1, \Upsilon_4) Ric(Q\Upsilon_2, \Upsilon_3) \\
+ \;& Ric(Q\mathcal{J}\Upsilon_1, \Upsilon_3) g(\mathcal{J}\Upsilon_2, \Upsilon_4) - g(\mathcal{J}\Upsilon_1, \Upsilon_4) Ric(\mathcal{J}Q\Upsilon_2, \Upsilon_3) \\
- \;& 2g(\mathcal{J}\Upsilon_1, \Upsilon_2) Ric(\mathcal{J}Q\Upsilon_3, \Upsilon_4) - 2g(\mathcal{J}\Upsilon_1, \Upsilon_2) Ric(\mathcal{J}Q\Upsilon_4, \Upsilon_3) \Big] \\
- \;& \frac{scal}{(2n+2)(2n+4)} \Big[g(\Upsilon_2, \Upsilon_3) Ric(\Upsilon_1, \Upsilon_4) - g(\Upsilon_1, \Upsilon_3) Ric(\Upsilon_2, \Upsilon_4) \\
+ \;& g(\mathcal{J}\Upsilon_2, \Upsilon_3) Ric(\mathcal{J}\Upsilon_1, \Upsilon_4) - g(\mathcal{J}\Upsilon_1, \Upsilon_3) Ric(\mathcal{J}\Upsilon_2, \Upsilon_4) \\
+ \;& g(\Upsilon_2, \Upsilon_4) Ric(\Upsilon_1, \Upsilon_3) - g(\Upsilon_1, \Upsilon_4) Ric(\Upsilon_2, \Upsilon_3) \\
+ \;& g(\mathcal{J}\Upsilon_2, \Upsilon_4) Ric(\mathcal{J}\Upsilon_1, \Upsilon_3) - g(\mathcal{J}\Upsilon_1, \Upsilon_4) Ric(\mathcal{J}\Upsilon_2, \Upsilon_3) \Big] \\
= \;& L_s[g(\Upsilon_1, \Upsilon_2) Ric(\Upsilon_1, \Upsilon_4) - g(\Upsilon_1, \Upsilon_2) Ric(\Upsilon_1, \Upsilon_4) \\
+ \;& g(\Upsilon_1, \Upsilon_4) Ric(\Upsilon_2, \Upsilon_1) - g(\Upsilon_1, \Upsilon_4) Ric(\Upsilon_1, \Upsilon_2)].
\end{aligned} \tag{3.3}$$

Let λ be an eigenvalue of Q and $\mathcal{J}Q$ corresponding to the eigenvectors Υ_1 and $\mathcal{J}\Upsilon_1$, then $Q\Upsilon_1 = \lambda\Upsilon_1$ and $Q\mathcal{J}\Upsilon_1 = \lambda\mathcal{J}\Upsilon_1$, i.e., $Ric(\Upsilon_1, \Upsilon_3) = \lambda g(\Upsilon_1, \Upsilon_3)$ (where the manifold is not Einstein), and therefore

$$Ric(Q\Upsilon_1, \Upsilon_3) = \lambda Ric(\Upsilon_1, \Upsilon_3),\ Ric(Q\mathcal{J}\Upsilon_1, \Upsilon_3) = \lambda Ric(\mathcal{J}\Upsilon_1, \Upsilon_3). \tag{3.4}$$

From (3.3) and (3.4), we get

$$\begin{aligned}
& \Big(\frac{\lambda}{2n+4} - \frac{scal}{(2n+2)(2n+4)}\Big)[g(\Upsilon_2, \Upsilon_3)Ric(\Upsilon_1, \Upsilon_4 \\
- \quad & g(\Upsilon_1, \Upsilon_3)Ric(\Upsilon_2, \Upsilon_4) + g(\Upsilon_2, \Upsilon_4)Ric(\Upsilon_1, \Upsilon_3) \\
- \quad & g(\Upsilon_1, \Upsilon_4)Ric(\Upsilon_2, \Upsilon_3) + g(\mathcal{J}\Upsilon_2, \Upsilon_3)Ric(\mathcal{J}\Upsilon_1, \Upsilon_4) \\
- \quad & g(\mathcal{J}\Upsilon_1, \Upsilon_3)Ric(\mathcal{J}\Upsilon_2, \Upsilon_4) + g(\mathcal{J}\Upsilon_2, \Upsilon_4)Ric(\mathcal{J}\Upsilon_1, \Upsilon_3) \\
- \quad & g(\mathcal{J}\Upsilon_1, \Upsilon_4)Ric(\mathcal{J}\Upsilon_2, \Upsilon_3)] \\
= \quad & L_s[g(\Upsilon_1, \Upsilon_2)Ric(\Upsilon_1, \Upsilon_4) - g(\Upsilon_1, \Upsilon_2)Ric(\Upsilon_1, \Upsilon_4) \\
+ \quad & g(\Upsilon_1, \Upsilon_4)Ric(\Upsilon_2, \Upsilon_1) - g(\Upsilon_1, \Upsilon_4)Ric(\Upsilon_1, \Upsilon_2)].
\end{aligned} \tag{3.5}$$

Putting $\Upsilon_4 = \Upsilon_3 = \rho$ in (3.5), we have

$$\begin{aligned}
& \Big(\frac{\lambda}{2n+4} - \frac{scal}{(2n+2)(2n+4)}\Big)[Ric(\Upsilon_1, \rho)g(\Upsilon_2, \rho) \\
- \quad & g(\Upsilon_1, \rho)Ric(\Upsilon_2, \rho) + Ric(\mathcal{J}\Upsilon_1, \rho)g(\mathcal{J}\Upsilon_2, \rho) \\
- \quad & Ric(\mathcal{J}\Upsilon_2, \rho)g(\mathcal{J}\Upsilon_1, \rho)] \\
= \quad & L_s[g(\Upsilon_2, \rho)Ric(\Upsilon_1, \rho) - g(\Upsilon_1, \rho)Ric(\Upsilon_2, \rho)],
\end{aligned} \tag{3.6}$$

which in view of (1.14) and (1.15) the relation (3.6) reduces to

$$\begin{aligned}
& b\Big(\frac{\lambda}{2n+4} - \frac{scal}{(2n+2)(2n+4)} - L_s\Big)[C(\Upsilon_1)A(\Upsilon_2) - C(\Upsilon_2)A(\Upsilon_1)] \\
= \quad & b\Big(\frac{\lambda}{2n+4} - \frac{scal}{(2n+2)(2n+4)}\Big)[A(\mathcal{J}\Upsilon_1)C(\mathcal{J}\Upsilon_2) - A(\mathcal{J}\Upsilon_2)C(\mathcal{J}\Upsilon_1)].
\end{aligned} \tag{3.7}$$

If we take $\lambda = \frac{scal}{2n+2}$ and $b \neq 0$, then from (1.14) and (3.7), we obtain

$$g(\Upsilon_2, \rho)g(\Upsilon_1, \sigma) = g(\Upsilon_1, \rho)g(\Upsilon_2, \sigma).$$

This shows that the vector fields ρ and σ corresponding to the 1-forms A and C respectively are co-directional.

Thus, we can state the following theorem.

Theorem 3.1 *In a Bochner flat Ricci pseudo-symmetric MQEH manifold, if* $\frac{scal}{2n+2}$ *is an eigenvalue of the Ricci operator* Q *and* $\mathcal{J}Q$ *and* $b \neq 0$*, then the vector fields* ρ *and* σ *corresponding to the* 1*-forms* A *and* C*, respectively, are co-directional.*

4. Holomorphically Projective Ricci Pseudo-Symmetric MQEH Manifold

The holomorphically projective curvature tensor P is given by [8]

$$\begin{aligned}
P(\Upsilon_1, \Upsilon_2, \Upsilon_3, \Upsilon_4) = \overline{K}(\Upsilon_1, \Upsilon_2, \Upsilon_3, \Upsilon_4) - \frac{1}{n-2}&[Ric(\Upsilon_2, \Upsilon_3)g(\Upsilon_1, \Upsilon_4) \\
&- Ric(\Upsilon_1, \Upsilon_3)g(\Upsilon_2, \Upsilon_4) + Ric(\mathcal{J}\Upsilon_1, \Upsilon_3)g(\mathcal{J}\Upsilon_2, \Upsilon_4) \\
&- Ric(\mathcal{J}\Upsilon_2, \Upsilon_3)g(\mathcal{J}\Upsilon_1, \Upsilon_4)].
\end{aligned} \tag{4.1}$$

In a Hermitian manifold, this tensor satisfies

$$P(\Upsilon_1, \Upsilon_2, \Upsilon_3, \Upsilon_4) = -P(\Upsilon_2, \Upsilon_1, \Upsilon_3, \Upsilon_4),\ P(\mathcal{J}\Upsilon_1, \mathcal{J}\Upsilon_2, \Upsilon_3, \Upsilon_4) = P(\Upsilon_1, \Upsilon_2, \Upsilon_3, \Upsilon_4).$$

Now we introduce the following.

Definition 4.1 *An even dimensional Hermitian manifold M^n is said to be a holomorphically projective Ricci pseudo-symmetric MQEH manifold if the holomorphically projective curvature tensor of the manifold satisfies $P.Ric = 0$, i.e.,*

$$(P(\Upsilon_1, \Upsilon_2).Ric)(\Upsilon_3, \Upsilon_4) = L_s Q(g, Ric)(\Upsilon_3, \Upsilon_4; \Upsilon_1, \Upsilon_2), \tag{4.2}$$

for all $\Upsilon_1,\ \Upsilon_2,\ \Upsilon_3,\ \Upsilon_4 \in M^n$

Let us consider a holomorphically projective Ricci pseudo-symmetric MQEH manifold, then from (1.13) and (4.2), we have

$$\begin{aligned} & a[g(P(\Upsilon_1, \Upsilon_2)\Upsilon_3, \Upsilon_4) + g(\Upsilon_3, P(\Upsilon_1, \Upsilon_2)\Upsilon_4)] \\ + \quad & b[A(P(\Upsilon_1, \Upsilon_2)\Upsilon_3)C(\Upsilon_4) + A(\Upsilon_4)C(P(\Upsilon_1, \Upsilon_2)\Upsilon_3) \\ + \quad & A(\Upsilon_3)C(P(\Upsilon_1, \Upsilon_2)\Upsilon_4) + C(\Upsilon_3)A(P(\Upsilon_1, \Upsilon_2)\Upsilon_4)] \\ = \quad & L_s[g(\Upsilon_2, \Upsilon_3)Ric(\Upsilon_1, \Upsilon_4) - g(\Upsilon_1, \Upsilon_3)Ric(\Upsilon_2, \Upsilon_4) \\ + \quad & g(\Upsilon_2, \Upsilon_4)Ric(\Upsilon_3, \Upsilon_1) - g(\Upsilon_1, \Upsilon_4)Ric(\Upsilon_2, \Upsilon_3)]. \end{aligned} \tag{4.3}$$

Putting $\Upsilon_3 = \Upsilon_4 = \rho$ in (4.3), we get

$$\begin{aligned} aP(\Upsilon_1, \Upsilon_2, \rho, \rho) + bP(\Upsilon_1, \Upsilon_2, \rho, \sigma) = bL_s[& g(\Upsilon_2, \rho)Ric(\Upsilon_1, \rho) - g(\Upsilon_1, \rho)Ric(\Upsilon_2, \rho) \\ & + g(\Upsilon_2, \rho)Ric(\Upsilon_3, \rho) - g(\Upsilon_1, \rho)Ric(\Upsilon_2, \rho)]. \end{aligned} \tag{4.4}$$

By virtue of (1.15) the relation (4.4) yields

$$aP(\Upsilon_1, \Upsilon_2, \rho, \rho) + bP(\Upsilon_1, \Upsilon_2, \rho, \sigma) = bL_s[A(\Upsilon_2)C(\Upsilon_1) - A(\Upsilon_1)C(\Upsilon_2)]. \tag{4.5}$$

Putting $\Upsilon_3 = \Upsilon_4 = \rho$ in (4.1), then we have

$$\begin{aligned} P(\Upsilon_1, \Upsilon_2, \rho, \rho) = -\frac{b}{n-2}[& A(\Upsilon_1)C(\Upsilon_2) - A(\Upsilon_2)C(\Upsilon_1) \\ & + A(\mathcal{J}\Upsilon_2)C(\mathcal{J}\Upsilon_1) - A(\mathcal{J}\Upsilon_1)C(\mathcal{J}\Upsilon_2)]. \end{aligned} \tag{4.6}$$

Again, putting $\Upsilon_3 = \rho$ and $\Upsilon_4 = \sigma$ in (4.1), we have

$$\begin{aligned} P(\Upsilon_1, \Upsilon_2, \rho, \sigma) = \overline{K}(\Upsilon_1, \Upsilon_2, \rho, \sigma) - \frac{1}{n-2}[& A(\Upsilon_2)C(\Upsilon_1) - C(\Upsilon_2)A(\Upsilon_1) \\ + A(\mathcal{J}\Upsilon_1)C(\mathcal{J}\Upsilon_2) - A(\mathcal{J}\Upsilon_2)C(\mathcal{J}\Upsilon_1)], \end{aligned} \tag{4.7}$$

which in view of (4.5) and (4.6) the relation (4.7) reduces to

$$b\Big(\overline{K}(\Upsilon_1, \Upsilon_2, \rho, \sigma) - L_s[A(\Upsilon_2)C(\Upsilon_1) - A(\Upsilon_1)C(\Upsilon_2)]\Big) = 0, \tag{4.8}$$

which implies that either $b = 0$ or $\overline{K}(\Upsilon_1, \Upsilon_2, \rho, \sigma) = L_s[A(\Upsilon_1)C(\Upsilon_2) - A(\Upsilon_2)C(\Upsilon_1)]$.
If $b = 0$, then the (1.13) manifold reduces to an Einstein manifold.

Thus, we can state the following theorem.

Theorem 4.2 *A holomorphically projective Ricci pseudo-symmetric MQEH manifold is either a holomorphically projective Ricci pseudo-symmetric Einstein Hermitian manifold or $\overline{K}(\Upsilon_1, \Upsilon_2, \rho, \sigma) = L_s[A(\Upsilon_1)C(\Upsilon_2) - A(\Upsilon_2)C(\Upsilon_1)]$ holds.*

Also, we can conclude.

Corollary 4.3 *In a holomorphically projective Ricci pseudo-symmetric MQEH manifold if $b \neq 0$ and $\overline{K}(\Upsilon_1, \Upsilon_2, \rho, \sigma) = 0$ if and only if the vector fields ρ and σ corresponding to 1-forms A and C, respectively, are co-directional.*

5. Holomorphically Projective Flat Ricci Pseudo-Symmetric MQEH Manifold

If the holomorphically projective curvature tensor is flat, then

$$P(\Upsilon_1, \Upsilon_2, \Upsilon_3, \Upsilon_4) = 0. \tag{5.1}$$

Using (5.1) in (4.1), we have

$$\begin{aligned} K(\Upsilon_1, \Upsilon_2, \Upsilon_3, \Upsilon_4) = \frac{1}{n-2}\Big[&\mathrm{Ric}(\Upsilon_2, \Upsilon_3)g(\Upsilon_1, \Upsilon_4) - \mathrm{Ric}(\Upsilon_1, \Upsilon_3)g(\Upsilon_2, \Upsilon_4) \\ &+ \mathrm{Ric}(J\Upsilon_1, \Upsilon_3)g(J\Upsilon_2, \Upsilon_4) - \mathrm{Ric}(J\Upsilon_2, \Upsilon_3)g(J\Upsilon_1, \Upsilon_4)\Big]. \end{aligned} \tag{5.2}$$

Again, using (4.2) in (5.2), we have

$$\begin{aligned} &\frac{1}{n-2}\Big[\mathrm{Ric}(Q\Upsilon_2, \Upsilon_3)g(\Upsilon_1, \Upsilon_4) - \mathrm{Ric}(Q\Upsilon_1, \Upsilon_3)g(\Upsilon_2, \Upsilon_4) \\ &+ \mathrm{Ric}(QJ\Upsilon_1, \Upsilon_3)g(J\Upsilon_2, \Upsilon_4) - \mathrm{Ric}(QJ\Upsilon_2, \Upsilon_3)g(J\Upsilon_1, \Upsilon_4)\Big] \\ &= L_s\Big[g(\Upsilon_2, \Upsilon_3)\mathrm{Ric}(\Upsilon_1, \Upsilon_4) - g(\Upsilon_1, \Upsilon_3)\mathrm{Ric}(\Upsilon_2, \Upsilon_4) \\ &+ g(\Upsilon_2, \Upsilon_4)\mathrm{Ric}(\Upsilon_3, \Upsilon_1) - g(\Upsilon_1, \Upsilon_4)\mathrm{Ric}(\Upsilon_2, \Upsilon_3)\Big]. \end{aligned} \tag{5.3}$$

Let λ be an eigenvalue of Q and JQ corresponding to the eigenvectors Υ_1 and $J\Upsilon_1$, then $Q\Upsilon_1 = \lambda\Upsilon_1$ and $QJ\Upsilon_1 = \lambda J\Upsilon_1$, i.e., $\mathrm{Ric}(\Upsilon_1, \Upsilon_3) = \lambda g(\Upsilon_1, \Upsilon_3)$ (where the manifold is not Einstein), and therefore

$$\mathrm{Ric}(Q\Upsilon_1, \Upsilon_3) = \lambda\mathrm{Ric}(\Upsilon_1, \Upsilon_3), \quad \mathrm{Ric}(QJ\Upsilon_1, \Upsilon_3) = \lambda\mathrm{Ric}(J\Upsilon_1, \Upsilon_3). \tag{5.4}$$

From (5.3) and (5.4), we get

$$\begin{aligned} &\frac{\lambda}{n-2}\Big[g(\Upsilon_2, \Upsilon_3)g(\Upsilon_1, \Upsilon_4) - g(\Upsilon_1, \Upsilon_3)g(\Upsilon_2, \Upsilon_4) \\ &+ g(J\Upsilon_1, \Upsilon_3)g(J\Upsilon_2, \Upsilon_4) - g(J\Upsilon_2, \Upsilon_3)g(J\Upsilon_1, \Upsilon_4)\Big] \\ &= L_s\Big[g(\Upsilon_2, \Upsilon_3)\mathrm{Ric}(\Upsilon_1, \Upsilon_4) - g(\Upsilon_1, \Upsilon_3)\mathrm{Ric}(\Upsilon_2, \Upsilon_4) \\ &+ g(\Upsilon_2, \Upsilon_4)\mathrm{Ric}(\Upsilon_3, \Upsilon_1) - g(\Upsilon_1, \Upsilon_4)\mathrm{Ric}(\Upsilon_2, \Upsilon_3)\Big]. \end{aligned} \tag{5.5}$$

Putting $\Upsilon_4 = \Upsilon_3 = \rho$ in (5.5), we have

$$\begin{aligned} &\frac{\lambda}{n-2}\Big[g(\Upsilon_2, \rho)g(\Upsilon_1, \rho) - g(\Upsilon_1, \rho)g(\Upsilon_2, \rho) \\ &+ g(J\Upsilon_1, \rho)g(J\Upsilon_2, \rho) - g(J\Upsilon_2, \rho)g(J\Upsilon_1, \rho)\Big] \\ &= L_s\Big[g(\Upsilon_2, \rho)\mathrm{Ric}(\Upsilon_1, \rho) - g(\Upsilon_1, \rho)\mathrm{Ric}(\Upsilon_2, \rho)\Big], \end{aligned} \tag{5.6}$$

which in view of (1.14) and (1.15) the above relation reduces to

$$\begin{aligned} &b\left(\frac{\lambda}{n-2} - L_s\right)[C(\Upsilon_1)A(\Upsilon_2) - C(\Upsilon_2)A(\Upsilon_1)] \\ &= b\left(\frac{\lambda}{n-2}\right)[A(J\Upsilon_1)C(J\Upsilon_2) - A(J\Upsilon_2)C(J\Upsilon_1)]. \end{aligned} \tag{5.7}$$

If we take $\lambda = \frac{\mathrm{scal}}{n}$ and $b \neq 0$, then from (1.14) and (5.7), we obtain

$$g(\Upsilon_2,\rho)g(\Upsilon_1,\sigma)=g(\Upsilon_1,\rho)g(\Upsilon_2,\sigma). \tag{5.8}$$

This shows that the vector fields ρ and σ corresponding to the 1-forms A and C respectively are co-directional.

Thus, we can state the following theorem.

Theorem 5.1 *In a holomorphically projective flat Ricci pseudo-symmetric MQEH manifold if $\frac{scal}{n}$ is an eigenvalue of the Ricci operator Q and JQ and $b\neq 0$ then the vector fields ρ and σ corresponding to 1-forms A and C, respectively, are co-directional.*

6. Bochner-Holomorphically Projective Ricci Pseudo-Symmetric MQEH Manifold

In this section, We introduce a novel curvature condition that integrates the Bochner and holomorphically projective tensors within the Ricci pseudo-symmetry framework.

The Bochner-holomorphically projective curvature tensor H is defined as

$$H(\Upsilon_1,\Upsilon_2,\Upsilon_3,\Upsilon_4)=B(\Upsilon_1,\Upsilon_2,\Upsilon_3,\Upsilon_4)+\alpha P(\Upsilon_1,\Upsilon_2,\Upsilon_3,\Upsilon_4), \tag{6.1}$$

where α is a scalar parameter, B is the Bochner curvature tensor given by (2.1), and P is the holomorphically projective curvature tensor given by (4.1). In a Hermitian manifold, H inherits the symmetries of both B and P, including

$$\begin{aligned}&H(\Upsilon_1,\Upsilon_2,\Upsilon_3,\Upsilon_4)=-H(\Upsilon_1,\Upsilon_2,\Upsilon_4,\Upsilon_3),\\&H(J\Upsilon_1,J\Upsilon_2,\Upsilon_3,\Upsilon_4)=H(\Upsilon_1,\Upsilon_2,\Upsilon_3,\Upsilon_4).\end{aligned} \tag{6.2}$$

Now, we introduce the following.

Definition 8.1. An even-dimensional Hermitian manifold M^n is said to be a Bochner-holomorphically projective Ricci pseudo-symmetric MQEH manifold if the tensor
$H\cdot\mathrm{Ric}=L_sQ(g,\mathrm{Ric})$, i.e.,

$$\begin{aligned}&\mathrm{Ric}(H(\Upsilon_1,\Upsilon_2)\Upsilon_3,\Upsilon_4)+\mathrm{Ric}(\Upsilon_3,H(\Upsilon_1,\Upsilon_2)\Upsilon_4)\\&=L_s[g(\Upsilon_2,\Upsilon_3)\mathrm{Ric}(\Upsilon_1,\Upsilon_4)-g(\Upsilon_1,\Upsilon_3)\mathrm{Ric}(\Upsilon_2,\Upsilon_4)\\&+g(\Upsilon_2,\Upsilon_4)\mathrm{Ric}(\Upsilon_3,\Upsilon_1)-g(\Upsilon_1,\Upsilon_4)\mathrm{Ric}(\Upsilon_2,\Upsilon_3)],\end{aligned} \tag{6.3}$$

for all $\Upsilon_1,\Upsilon_2,\Upsilon_3,\Upsilon_4\in M^n$, where L_s is a function on $P=\{\upsilon\in M:\mathrm{Ric}\neq\frac{\mathrm{scal}}{n}g \text{ at } \upsilon\}$.

Let us consider a Bochner-holomorphically projective Ricci pseudo-symmetric MQEH manifold. Then, from (1.13) and the definition, we have

$$\begin{aligned}&a[g(H(\Upsilon_1,\Upsilon_2)\Upsilon_3,\Upsilon_4)+g(\Upsilon_3,H(\Upsilon_1,\Upsilon_2)\Upsilon_4)]\\&+b[A(H(\Upsilon_1,\Upsilon_2)\Upsilon_3)C(\Upsilon_4)+A(\Upsilon_4)C(H(\Upsilon_1,\Upsilon_2)\Upsilon_3)\\&+A(\Upsilon_3)C(H(\Upsilon_1,\Upsilon_2)\Upsilon_4)+C(\Upsilon_3)A(H(\Upsilon_1,\Upsilon_2)\Upsilon_4)]\\&=L_s[g(\Upsilon_2,\Upsilon_3)\mathrm{Ric}(\Upsilon_1,\Upsilon_4)-g(\Upsilon_1,\Upsilon_3)\mathrm{Ric}(\Upsilon_2,\Upsilon_4)\\&+g(\Upsilon_2,\Upsilon_4)\mathrm{Ric}(\Upsilon_3,\Upsilon_1)-g(\Upsilon_1,\Upsilon_4)\mathrm{Ric}(\Upsilon_2,\Upsilon_3)].\end{aligned} \tag{6.4}$$

Putting $\Upsilon_3=\Upsilon_4=\rho$ in (6.4), we get

$$aH(\Upsilon_1,\Upsilon_2,\rho,\rho)+bH(\Upsilon_1,\Upsilon_2,\rho,\sigma)=bL_s[A(\Upsilon_2)C(\Upsilon_1)-A(\Upsilon_1)C(\Upsilon_2)]. \tag{6.5}$$

Substituting the expressions for B and P, and using (1.14) and (1.15), this yields

$$\begin{aligned}&b\left[(1+\alpha)K(\Upsilon_1,\Upsilon_2,\rho,\sigma)-L_s[A(\Upsilon_2)C(\Upsilon_1)-A(\Upsilon_1)C(\Upsilon_2)]\right]\\&+\frac{b\alpha}{n-2}[A(J\Upsilon_1)C(J\Upsilon_2)-A(J\Upsilon_2)C(J\Upsilon_1)]=0.\end{aligned} \tag{6.6}$$

This implies that either $b = 0$ or the expression vanishes for specific α. If $b = 0$, then from (1.13) the manifold reduces to an Einstein manifold.

Thus, we can state the following theorem.

Theorem 6.1 *A Bochner-holomorphically projective Ricci pseudo-symmetric MQEH manifold with* $\alpha \neq -1$ *is a Bochner-holomorphically projective Ricci pseudo-symmetric Einstein Hermitian manifold or*

$$\begin{aligned}(1+\alpha)K(\Upsilon_1, \Upsilon_2, \rho, \sigma) &= L_s[A(\Upsilon_2)C(\Upsilon_1) - A(\Upsilon_1)C(\Upsilon_2)] \\ &\quad - \frac{\alpha}{n-2}[A(J\Upsilon_1)C(J\Upsilon_2) - A(J\Upsilon_2)C(J\Upsilon_1)]\end{aligned} \tag{6.7}$$

holds.

Discussion

The results obtained in this paper provide a unified understanding of Ricci pseudo-symmetric structures on mixed quasi-Einstein Hermitian (MQEH) manifolds under different curvature tensors. The behavior of the Bochner and holomorphically projective tensors reveals that pseudo-symmetry imposes strong restrictions on the associated vector fields and their 1-forms. In particular, the co-directionality of the generators is shown to be enforced by the vanishing of specific mixed components, which indicates a deeper geometric compatibility within the MQEH framework.

The classification results derived for the Bochner-flat and holomorphically projective flat cases demonstrate that the MQEH structure is highly rigid under curvature degeneracies. Moreover, under the combined curvature condition, the manifold is compelled to approach an Einstein structure, except in explicitly determined exceptional cases. These findings offer a foundation for further investigation into Hermitian manifolds with hybrid curvature constraints.

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