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A Foundational Review of Ordinary Differential Equation Solution Methods and Their Inherent Symmetries

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ABSTRACT: This paper presents a focused pedagogical survey of fundamental solution methods for ordinary differential equations (ODEs), demonstrating the unifying and explanatory role of mathematical symmetry. The core of this study is that classical solution techniques are not mere algebraic manipulations but are inherently motivated by the equations' underlying invariant structure. We provide a targeted analysis showing that two critical classes of ODEs—the first-order homogeneous equation and the Cauchy-Euler equation are direct mathematical expressions of scale invariance. The substitution methods used to solve both types of equations are practical applications of exploiting this invariance property to transform the original non-separable equation into a simpler, solvable form. By explicitly reframing these core methods within a symmetry-based context, this work offers advanced practitioners and students a deeper conceptual foundation. This approach unifies seemingly disparate solution techniques under a single, powerful mathematical principle, thereby highlighting the significance of ODEs as mathematical models for analyzing physical systems that exhibit powerful geometrical and variational symmetries.

Keywords: Bernoulli equation, boundary value problem, Cauchy-Euler equation, initial value problem, mathematical model.

Contents

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1. Introduction

The ability to forecast and simulate the behavior of dynamic systems lies at the center of modern science and engineering. Ordinary differential equations (ODEs) provide the fundamental mathematical machinery for answering these questions, acting as a bridge from theoretical abstractions to measurable reality. The history of this powerful tool is inextricably tied to the history of calculus in the 17th century by visionaries like Sir Isaac Newton and Gottfried Wilhelm Leibniz. Newton's second law of motion, $F = ma$, for instance, expresses the rate of change of an object's momentum, which is fundamentally a differential equation [5,6]. Out of these beginnings, differential equations developed into the tool of greatest importance for analyzing dynamic systems [7].

Mathematical models of real-world phenomena are frequently expressed through equations involving functions and their derivatives; these are known as Differential Equations. In this context, the function typically represents a physical quantity, the derivatives represent its rate of change, and the differential equation describes the relationship between the two [5]. While differential equations are broadly categorized into Ordinary Differential Equations (ODEs) and Partial Differential Equations (PDEs), this paper focuses exclusively on ODEs—those involving functions of a single independent variable. ODEs occupy a pivotal position across diverse disciplines [5,7,8]. For example, they are essential in physics and astronomy (celestial mechanics) [9], meteorology (weather forecasting) [10], chemistry (rates of chemical reactions) [11], biology (epidemiology [12], genetic variation [13]), ecology and population modeling (competing populations) [14], and economics (stock behavior, interest rates, and market volatility) [15,16].

Crucially, the central thesis of this work is that the study of ordinary differential equations is deeply intertwined with the mathematical concept of symmetry. Many physical and engineering systems, when modeled by differential equations, exhibit intrinsic symmetrical properties that are critical to their analysis and solution [5,44]. This paper demonstrates how specific ODE types embody forms of symmetry, making them particularly relevant to system analysis. For instance, a homogeneous differential equation is characterized by a function that exhibits a specific scaling symmetry; this property is precisely what motivates and enables the transformation of the equation into a simpler, separable form.

This paper is intended as an analytical and pedagogical survey aimed at advanced undergraduate and beginning graduate students in mathematics, physics, and engineering. Our explicit goal is to unify the understanding of several classical solution methods by demonstrating how the principle of symmetry acts as a foundational, motivating concept.

We show that recognizing the inherent symmetry of a differential equation not only provides a powerful framework for solution but also offers profound insight into the physical system it models. Similarly, the Cauchy-Euler equation inherently demonstrates a crucial form of scale invariance.

After outlining fundamental concepts of ODEs and distinguishing them from partial differential equations, the main sections of this survey will proceed with a descriptive elaboration of some common forms of ODEs (including Bernoulli, Ricatti, Clairaut, and Cauchy-Euler equations) and major methods of

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solving first and second-order linear and non-linear equations. Through the examination of both theory and application, this paper highlights how ODEs, when viewed through the lens of symmetry, serve as fundamental mathematical models used to develop and solve complex real problems in a vast array of fields, including geometry and analytical mechanics [9], meteorology [10], and population modeling [14,46].

2. Preliminary Concepts

The construction and solution of differential equation models generally follows a three-step process: (1) defining the real-world problem, (2) formulating a simplified mathematical model by converting the problem into a system of differential equations, and (3) solving the resulting equations and interpreting the results in the original context [8,7,45]. The terminology below defines the core mathematical components necessary for this process.

Definition 1. *An ordinary differential equation (ODE) is an equation that contains derivatives of one or more dependent variables with respect to a single independent variable. If we are trying to find the function $y = f(x)$, its derivatives are taken only with respect to x [6,5]. An n-th order ODE with one dependent variable can be expressed in the general form:*

$$F(x, y, y', y'', \ldots, y^{(n)}) = 0, \tag{2.1}$$

where F is a mathematical function of x, y, and its derivatives, and $y^{(n)} = \frac{d^n y}{dx^n}$ is the n-th derivative [5,41].

Definition 2. *A partial differential equation (PDE) is a differential equation that involves two or more independent variables and their partial derivatives [5,43].*

Definition 3. *The order of a differential equation is the order of the highest derivative present in the equation [5,42].*

Definition 4. *The degree of a differential equation is the power to which the highest-order derivative is raised, after all terms have been cleared of fractions and radicals [5,6].*

Definition 5. *An n-th order ODE is considered linear if it can be written in the following form:*

$$y^{(n)} = a_0(x)y + a_1(x)y' + \ldots + a_{n-1}(x)y^{(n-1)} + r(x), \tag{2.2}$$

where the coefficient functions $a_i(x)$ for $i = 0, 1, \ldots, n-1$ and the source term $r(x)$ are continuous functions of the independent variable x [5].

(i) Homogeneous: If $r(x) = 0$, the equation is homogeneous. A trivial solution of $y = 0$ always exists. The general solution is a complementary function, here denoted by y_c.

(ii) Nonhomogeneous (or Inhomogeneous): If $r(x) \neq 0$, the equation is nonhomogeneous. The general solution is the sum of the complementary function and a particular integral, $y = y_c + y_p$ [5].

Definition 6. *A differential equation is non-linear if it cannot be expressed in the form of a linear combination of its dependent variable and its derivatives [6,7].*

Definition 7. *A singular solution of a differential equation is a solution that cannot be obtained from the general solution by assigning specific values to the arbitrary constants [5,6].*

Definition 8. *A function f is said to be analytic on an open set D in the real line if for every point $x_0 \in D$, it can be represented by a power series that converges to $f(x)$ in a neighborhood around x_0:*

$$f(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n, \tag{2.3}$$

where the coefficients a_n are real numbers [5].

Definition 9. *A singular point is a point $x_0 \in \mathbb{R}$ where a function $f(x)$ is not analytic [5,6].*

3. Bernoulli's Equation

In mathematics, an ordinary differential equation is called a Bernoulli differential equation if it can be written in the form:

$$\frac{dy}{dx} + P(x)y = Q(x)y^n, \quad n \in \mathbb{R}. \tag{3.1}$$

This equation is named after Jacob Bernoulli, who discussed it in a 1695 treatise [17]. The method for its solution, however, was first provided by Gottfried Leibniz in the same year, and his approach is still used today [17]. Bernoulli equations are noteworthy because they are a class of nonlinear differential equations for which an exact solution can be found [6,5]. A special case of the Bernoulli equation, the logistic differential equation, will be explored later in Section 11 [18].

To solve Equation (3.1), we must consider three distinct cases [6].

Case 1. If $n = 0$, Equation (3.1) becomes

$$\frac{dy}{dx} + P(x)y = Q(x),$$

which is a standard first-order linear differential equation.

Case 2. If $n = 1$, the equation simplifies to

$$\frac{dy}{dx} + P(x)y = Q(x)y.$$

This can be rearranged into the form

$$\frac{dy}{dx} + [P(x) - Q(x)]y = 0,$$

which is also a first-order linear DE.

Case 3. For the general case where $n \neq 0, 1$, we can transform Equation (3.1) into a linear DE. First, divide the equation by y^n to get:

$$y^{-n}\frac{dy}{dx} + P(x)y^{1-n} = Q(x),$$

We then introduce a substitution to linearize the equation. Let $u = y^{1-n}$. Differentiating u with respect to x yields:

$$\frac{du}{dx} = (1-n)y^{-n}\frac{dy}{dx} \quad \Longrightarrow \quad \frac{1}{1-n}\frac{du}{dx} = y^{-n}\frac{dy}{dx}.$$

Substituting this into the rearranged Bernoulli equation gives us a first-order linear DE in terms of u:

$$\frac{1}{1-n}\frac{du}{dx} + P(x)u = Q(x) \quad \Longrightarrow \quad \frac{du}{dx} + (1-n)P(x)u = (1-n)Q(x).$$

This resulting equation is now a first-order linear DE, which can be solved using an integrating factor [6].

Remark 10. *Throughout the document, whenever integrating a term of the form $\frac{1}{y}$ (or $\frac{1}{u}$), mathematical rigor requires that the result be written using the absolute value, i.e., $\int \frac{1}{y}dy = \ln|y| + C$. We assume this practice is followed in all subsequent integration steps.*

4. Riccati's Equation

By its strict definition in mathematics, a Riccati equation is any first-order ordinary differential equation that is quadratic in the unknown function. Named after Jacob Riccati (1676-1754) [3], the equation has the general form:

$$\frac{dy}{dx} = P(x)y + Q(x)y^2 + R(x). \tag{4.1}$$

Unlike other types of first-order equations, Equation (4.1) cannot generally be solved by elementary methods. However, if one can find a known particular solution, say $y = \alpha(x)$, the general solution can be determined. This is accomplished by making the substitution $y = \alpha(x) + z$, where z is a new unknown function.

Substituting $y = \alpha(x) + z$ into Equation (4.1) and noting that $y' = \alpha' + z'$, we obtain:

$$\begin{aligned}\alpha' + z' &= P(x)(\alpha + z) + Q(x)(\alpha + z)^2 + R(x)\\ \alpha' + z' &= P\alpha + Pz + Q(\alpha^2 + 2\alpha z + z^2) + R\\ \alpha' + z' &= \underbrace{(P\alpha + Q\alpha^2 + R)}_{\text{Since } \alpha' = P\alpha + Q\alpha^2 + R} + Pz + 2Q\alpha z + Qz^2\end{aligned}$$

Since $\alpha(x)$ is the known particular solution, it satisfies the original Riccati equation, meaning $\alpha' = P\alpha + Q\alpha^2 + R$.

Substituting the expression for α' back into the previous line, we get the cancellation step:

$$\alpha' + z' = \alpha' + Pz + 2Q\alpha z + Qz^2$$

After canceling α' from both sides, the equation for z becomes:

$$z' = Pz + 2Q\alpha z + Qz^2.$$

Rearranging this equation, we arrive directly at the **Bernoulli equation for z**:

$$\frac{dz}{dx} - (P(x) + 2Q(x)\alpha(x))z = Q(x)z^2.$$

This is a Bernoulli equation (specifically, with $n = 2$), which can be solved for z. Substituting z back into the original assumption $y = \alpha(x) + z$ yields the general solution of the Riccati equation.

5. Clairaut's Equation

In mathematics, Clairaut's equation is a special type of differential equation with the form:

$$y(x) = x\frac{dy}{dx} + f\left(\frac{dy}{dx}\right), \tag{5.1}$$

where f is a continuously differentiable function. This equation is a specific case of the more general Lagrange differential equation [21,4]. Named after the French mathematician Alexis Clairaut, who first presented it in 1734 [4], the equation can be solved by differentiating it with respect to x.

Differentiating Equation (5.1) with respect to x yields:

$$\frac{dy}{dx} = \frac{dy}{dx} + x\frac{d^2y}{dx^2} + f'\left(\frac{dy}{dx}\right)\frac{d^2y}{dx^2}.$$

This simplifies to:

$$\left[x + f'\left(\frac{dy}{dx}\right)\right]\frac{d^2y}{dx^2} = 0.$$

From this, two possibilities arise: 1. The General Solution: If $\frac{d^2y}{dx^2} = 0$, then $\frac{dy}{dx}$ is a constant, which we can call C. Substituting C into Clairaut's equation gives the family of straight-line solutions, $y(x) = Cx + f(C)$. 2. The Singular Solution: The other possibility is that $x + f'\left(\frac{dy}{dx}\right) = 0$. This condition leads to a special solution that is not part of the general family of straight lines. This singular solution forms the envelope—a curve that is tangent to each member of the family of general solutions [4].

6. Cauchy-Euler Equation

In mathematics, a Cauchy-Euler equation (or Euler's equation) is a linear homogeneous ordinary differential equation with variable coefficients [5]. It is also known as an equidimensional equation because the power of the independent variable x in each term matches the order of its corresponding derivative. This structure is another manifestation of symmetry, specifically scale invariance, which allows for explicit solutions to be found through a systematic method of substitution. By recognizing and leveraging these symmetrical properties, mathematicians and scientists can gain deeper insights into the behavior of the dynamic systems being modeled. This simple structure allows the equation to be solved explicitly [5]. The general n-th order Cauchy-Euler equation is:

$$a_n x^n y^{(n)} + a_{n-1}x^{n-1}y^{(n-1)} + \ldots + a_1 x y' + a_0 y = 0, \tag{6.1}$$

where a_k are constants for $k = 0, 1, \ldots, n$. The most common form is the second-order equation, which is frequently encountered in physics and engineering problems, such as solving Laplace's equation in polar coordinates [22,6]:

$$a_2 x^2 y'' + a_1 x y' + a_0 y = 0. \tag{6.2}$$

We can assume a trial solution of the form $y = x^m$. By differentiating this assumption, we get $y' = mx^{m-1}$ and $y'' = m(m-1)x^{m-2}$. Substituting these derivatives into the second-order equation gives:

$$a_2 x^2 (m(m-1)x^{m-2}) + a_1 x (m x^{m-1}) + a_0 (x^m) = 0.$$

Dividing by x^m and rearranging, we obtain the characteristic equation (or indicial equation):

$$a_2 m(m-1) + a_1 m + a_0 = 0$$

Solving this quadratic equation for m gives us the roots, which determine the form of the general solution. There are three cases based on the nature of these roots [5]:
- Case 1: Two Distinct Real Roots (m_1, m_2). The general solution is $y = c_1 x^{m_1} + c_2 x^{m_2}$.
- Case 2: One Repeated Real Root $(m_1 = m_2 = m)$. The solution is $y = c_1 x^m + c_2 x^m \ln(x)$. The second linearly independent solution is found using the method of reduction of order.
- Case 3: Complex Conjugate Roots $(\alpha \pm i\beta)$. The solution is $y = x^\alpha[c_1 \cos(\beta \ln(x)) + c_2 \sin(\beta \ln(x))]$. This form is derived by letting $x = e^t$ (so $t = \ln x$) and using Euler's formula, $e^{\pm i\theta} = \cos\theta \pm i \sin\theta$ [5].

7. Methods of Solving ODEs

This section explores various methods for solving first and second-order differential equations.

7.1. First-Order Differential Equations

A first-order initial value problem is generally expressed as:

$$\frac{dy}{dx} = f(x, y), \tag{7.1}$$

subject to the initial condition $y(x_0) = y_0$ [5,6].

The function $y = \phi(x, c)$, which contains an arbitrary constant c, is known as the general solution of a first-order DE. For any given initial condition, a specific value c_0 can be found such that the function $y = \phi(x, c_0)$ satisfies the condition. When we solve a DE, we often obtain a relationship in the form $\phi(x, y, c) = 0$. While solving for y explicitly from this relationship yields the general solution, it is not always possible to express y as a combination of elementary functions. In such cases, the solution is left in its implicit form [5,6].

A particular solution is any function $y = \phi(x, c_0)$ derived from the general solution by substituting the arbitrary constant c with a specific value c_0. The expression $\phi(x, y, c_0) = 0$ is referred to as the particular integral [5,6].

Theorem 7.1 (Picard–Lindelöf Theorem: Existence and Uniqueness)**.** *Let $f(x, y)$ be a real-valued continuous function defined on the rectangle $R = \{(x, y) : |x - x_0| \leq a, |y - y_0| \leq b\}$. If $f(x, y)$ satisfies the Lipschitz condition with respect to y on R—that is, there exists a constant $L > 0$ such that:*

$$|f(x, y_1) - f(x, y_2)| \leq L|y_1 - y_2| \quad \text{for all } (x, y_1), (x, y_2) \in R,$$

then there exists an interval $I = [x_0 - h, x_0 + h]$ (where $h \leq a$) on which the initial value problem $y' = f(x, y)$, with $y(x_0) = y_0$, has a unique solution $y(x)$ [19,5,6].

Remark 11. *The condition that the partial derivative $\frac{\partial f}{\partial y}$ exists and is continuous on R is a sufficient, but stronger, hypothesis than the Lipschitz condition. If $\frac{\partial f}{\partial y}$ is continuous, then f is locally Lipschitz continuous, which guarantees existence and uniqueness [5,6]. The formal Lipschitz condition is generally preferred as it covers cases where $\frac{\partial f}{\partial y}$ may not exist (e.g., $f(x, y) = |y|$).*

We will now detail several common methods for solving Equation (7.1).

7.2. Separable First-Order Differential Equations

Definition 12. *A first-order differential equation is considered separable if it can be rearranged into one of the following forms [6,5]:*

$$\frac{dy}{dx} = \frac{f(x)}{g(y)}, \tag{7.2}$$

or

$$\frac{dy}{dx} = f(x)g(y). \tag{7.3}$$

Assuming in Equation (7.2) that $g(y) \neq 0$, we can separate the variables and write the equation as:

$$\frac{dy}{g(y)} = f(x)dx. \tag{7.4}$$

Integrating both sides of Equation (7.4) provides the general solution:

$$\int \frac{dy}{g(y)} = \int f(x)dx + c. \tag{7.5}$$

7.3. Integrating Factor

A first-order linear differential equation can be written in the form:

$$\frac{dy}{dx} + p(x)y = q(x),$$

where $p(x)$ and $q(x)$ are continuous functions over an interval containing the initial point x_0 [5,6].

To solve this, we first determine an integrating factor (IF), which is defined as $\mu(x) = e^{\int p(x)dx}$ [5,6]. By multiplying both sides of the differential equation by this integrating factor, the left-hand side becomes the derivative of the product $\mu(x)y$:

$$\mu(x)\frac{dy}{dx} + \mu(x)p(x)y = \mu(x)q(x) \implies \frac{d}{dx}\big(\mu(x)y\big) = \mu(x)q(x).$$

Integrating both sides with respect to x allows us to solve for y:

$$\mu(x)y = \int \mu(x)q(x)dx + C \implies y = \frac{1}{\mu(x)}\Big(\int \mu(x)q(x)dx + C\Big).$$

7.4. Exact Differential Equations

For a two-variable function, $u = u(x, y)$, its total differential is given by:

$$du = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy. \tag{7.6}$$

A differential equation of the form $M(x, y)dx + N(x, y)dy = 0$ is called an exact differential equation if there exists a function $u(x, y)$ such that $\frac{\partial u}{\partial x} = M(x, y)$ and $\frac{\partial u}{\partial y} = N(x, y)$ [5,6]. A necessary and sufficient condition for a DE to be exact is that the partial derivatives must satisfy $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$.

If these conditions are met, the DE can be written as $du = 0$, which implies that its general solution is simply $u(x, y) = C$, where C is a constant. Thus, solving an exact DE is equivalent to finding the function $u(x, y)$.

7.5. Integrating Factor for Non-Exact Equations

If a differential equation $M(x, y)dx+N(x, y)dy = 0$ is not exact (i.e., $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$), it may be possible to make it exact by multiplying it by a function μ, called an integrating factor [6,5]. The resulting equation, $\mu M dx + \mu N dy = 0$, will be exact, meaning it satisfies $\frac{\partial(\mu M)}{\partial y} = \frac{\partial(\mu N)}{\partial x}$.

Theorem 7.2. *Let M, N, and their partial derivatives be continuous on a domain D. If $\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N} = f(x)$, a function depending only on x, then $\mu(x) = e^{\int f(x)dx}$ is an integrating factor.*

Proof. Let μ be a function of x only, such that $\mu M dx + \mu N dy = 0$ is exact. The exactness condition gives $\frac{\partial(\mu M)}{\partial y} = \frac{\partial(\mu N)}{\partial x}$. Applying the product rule, we have $\mu\frac{\partial M}{\partial y} = \mu\frac{\partial N}{\partial x} + N\frac{d\mu}{dx}$. Rearranging this leads to $\frac{d\mu}{\mu} = \frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N}dx$. Since the right side is a function of x alone, we can integrate both sides to find $\ln \mu = \int f(x)dx$, which gives $\mu = e^{\int f(x)dx}$. □

Theorem 7.3. *Assuming M, N, and their partial derivatives are continuous on a domain D. If $\frac{\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}}{M} = g(y)$, a function depending only on y, then $\mu(y) = e^{\int g(y)dy}$ is an integrating factor.*

Proof. The proof is analogous to the previous theorem, but with μ being a function of y only. The exactness condition $\frac{\partial(\mu M)}{\partial y} = \frac{\partial(\mu N)}{\partial x}$ becomes $\mu\frac{\partial M}{\partial y} + M\frac{d\mu}{dy} = \mu\frac{\partial N}{\partial x}$. Rearranging gives $\frac{d\mu}{\mu} = \frac{\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}}{M}dy$, which upon integration yields $\mu = e^{\int g(y)dy}$. □

8. First-Order Homogeneous DEs

A function $f(x, y)$ is homogeneous of degree n if $f(\lambda x, \lambda y) = \lambda^n f(x, y)$ for any real number λ [5,6]. For a first-order DE $\frac{dy}{dx} = f(x, y)$, if $f(x, y)$ is homogeneous of degree 0, the equation is called a homogeneous DE. In this case, $f(x, y)$ can be rewritten as $f\left(1, \frac{y}{x}\right)$. This allows us to use the substitution $u = \frac{y}{x}$, which implies $y = ux$. Applying the product rule for differentiation gives the expression for $\frac{dy}{dx}$:

$$\frac{dy}{dx} = u + x\frac{du}{dx}.$$

The differential equation then becomes a separable equation:

$$u + x\frac{du}{dx} = f(1, u) \quad \Longrightarrow \quad x\frac{du}{dx} = f(1, u) - u.$$

The variables can now be separated and integrated:

$$\int \frac{du}{f(1, u) - u} = \int \frac{dx}{x} + C.$$

After integration, we substitute $u = \frac{y}{x}$ back into the solution to get the general integral.

Worked Example

To help novices, we provide an explicit worked example demonstrating the compact solution steps.
Example: Solve the differential equation

$$\frac{dy}{dx} = \frac{y^2 + xy}{x^2}.$$

Solution:

Step (i): Apply Substitution: We first rewrite the DE as $\frac{dy}{dx} = \left(\frac{y}{x}\right)^2 + \left(\frac{y}{x}\right)$. Using the substitution $u = \frac{y}{x}$ and $\frac{dy}{dx} = u + x\frac{du}{dx}$, the equation becomes:

$$u + x\frac{du}{dx} = u^2 + u$$

Step (ii): Separate and Integrate: Simplifying yields $x\frac{du}{dx} = u^2$. Separating the variables, we get $\frac{du}{u^2} = \frac{dx}{x}$. Integrating both sides gives:

$$\int u^{-2}\, du = \int \frac{1}{x}\, dx \quad \Longrightarrow \quad -\frac{1}{u} = \ln|x| + C.$$

Step (iii): Substitute Back: Replacing u with $\frac{y}{x}$ and simplifying provides the final general solution:

$$-\frac{1}{\frac{y}{x}} = \ln|x| + C \quad \Longrightarrow \quad -\frac{x}{y} = \ln|x| + C.$$

The solution written explicitly for y is $y = -\frac{x}{\ln|x|+C}$.

9. Second-Order Differential Equations

A second-order linear differential equation has the general form:

$$A(x)\frac{d^2y}{dx^2} + B(x)\frac{dy}{dx} + C(x)y = G(x), \tag{9.1}$$

where A, B, C, and G are continuous functions [5]. This section focuses on solving second-order linear equations with constant coefficients:

$$ay'' + by' + cy = G(x), \tag{9.2}$$

where a, b, and c are constants and $G(x)$ is a continuous function. The corresponding homogeneous equation, $ay'' + by' + cy = 0$, is known as the complementary equation and is crucial for finding the general solution to the non-homogeneous equation [5].

The general solution of the non-homogeneous equation (9.2) is the sum of any particular solution y_p and the general solution of the complementary equation y_c:

$$y(x) = y_c(x) + y_p(x).$$

This can be verified by substituting this form back into the original non-homogeneous equation.

Proof. Let y be any solution to (9.2). We want to show that $y - y_p$ is a solution to the complementary equation, where y_p is a particular solution.

$$a(y - y_p)'' + b(y - y_p)' + c(y - y_p) = (ay'' + by' + cy) - (ay_p'' + by_p' + cy_p) = G(x) - G(x) = 0.$$

This shows that $y - y_p$ is a solution to the homogeneous equation, so $y - y_p = y_c$. Thus, any solution is of the form $y = y_c + y_p$. □

Theorem 9.1 (Principle of Superposition). *If $y_1(x)$ and $y_2(x)$ are two linearly independent solutions of the homogeneous equation $ay'' + by' + cy = 0$, then any linear combination $y(x) = c_1y_1(x) + c_2y_2(x)$ is also a solution.*

Proof. By direct substitution and leveraging the linearity of derivatives:

$$a(c_1y_1 + c_2y_2)'' + b(c_1y_1 + c_2y_2)' + c(c_1y_1 + c_2y_2) = c_1(ay_1'' + by_1' + cy_1) + c_2(ay_2'' + by_2' + cy_2).$$

Since y_1 and y_2 are solutions, the terms in the parentheses are zero, proving that $y(x)$ is also a solution. □

10. The Wronskian

The Wronskian, credited to Józef Hoene-Wroński, is a determinant used to test the linear independence of solutions [6]. For two solutions y_1 and y_2 of a second-order homogeneous DE, the Wronskian is defined as:

$$W(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = y_1y_2' - y_2y_1'.$$

If y_1 and y_2 are linearly independent solutions, their Wronskian is non-zero [5].

To solve the homogeneous equation $ay'' + by' + cy = 0$, we can assume a solution of the form $y = e^{rx}$ due to its property that derivatives are constant multiples of the original function. Substituting this into the equation yields the auxiliary (or characteristic) equation:

$$ar^2 + br + c = 0. \tag{10.1}$$

This is an algebraic equation whose roots determine the form of the general solution [5,6,2]. The roots are given by the quadratic formula $r = \frac{-b\pm\sqrt{b^2-4ac}}{2a}$. We consider three cases based on the discriminant, $b^2 - 4ac$.

i Distinct Real Roots ($b^2 - 4ac > 0$): The roots r_1 and r_2 are real and distinct, giving two linearly independent solutions $y_1 = e^{r_1x}$ and $y_2 = e^{r_2x}$. The general solution is $y = c_1e^{r_1x} + c_2e^{r_2x}$.

ii Repeated Real Root ($b^2 - 4ac = 0$): The quadratic equation has a single real root $r = -b/2a$. One solution is $y_1 = e^{rx}$. A second, linearly independent solution is found to be $y_2 = xe^{rx}$ by methods like reduction of order. The general solution is $y = c_1e^{rx} + c_2xe^{rx}$.

iii Complex Roots ($b^2 - 4ac < 0$): The roots are a pair of complex conjugates, $r = \alpha \pm i\beta$. Using Euler's formula, $e^{i\theta} = \cos\theta + i\sin\theta$, the real-valued general solution is $y = e^{\alpha x}(c_1\cos\beta x + c_2\sin\beta x)$.

10.1. Initial-Value and Boundary-Value Problems

An initial-value problem for a second-order DE requires finding a solution that satisfies given conditions at a single point, such as $y(x_0) = y_0$ and $y'(x_0) = y_1$ [5,6]. In contrast, a boundary-value problem requires finding a solution that satisfies conditions at two different points, for example, $y(x_0) = y_0$ and $y(x_1) = y_1$ [2,5,6]. A key difference is that a boundary-value problem may not have a solution, unlike an initial-value problem, which is guaranteed to have one under certain conditions.

For non-homogeneous equations, the particular solution y_p can be found using two main methods:

1. The Method of Undetermined Coefficients.

2. The Method of Variation of Parameters.

10.2. The Method of Undetermined Coefficients

This method is used when the non-homogeneous term $G(x)$ is a specific type of function (e.g., polynomial, exponential, sine/cosine, or a product of these). We assume a form for the particular solution y_p that is similar to $G(x)$ and contains undetermined coefficients [5,6,20]. We then substitute this assumed solution into the differential equation and solve for the coefficients by equating terms.

a If $G(x)$ is a polynomial of degree n, we assume y_p is a polynomial of the same degree.

b If $G(x) = ce^{kx}$, we assume $y_p = de^{kx}$.

c If $G(x)$ is a sine or cosine function, we assume y_p is a linear combination of both sine and cosine terms.

d If $G(x)$ is a product of these forms, we assume y_p is a corresponding product.

A special consideration is required if the assumed form for y_p is already a solution to the complementary equation. In such cases, the trial solution is multiplied by x (or x^n if necessary) to ensure linear independence [5,19].

10.3. The Method of Variation of Parameters

This is a more general method that works for any continuous non-homogeneous term $G(x)$. Assuming the homogeneous solution is $y_c = c_1 y_1 + c_2 y_2$, we seek a particular solution of the form $y_p(x) = u_1(x)y_1(x) + u_2(x)y_2(x)$, where u_1 and u_2 are functions, not constants [5,6,19]. The functions u_1' and u_2' are given by:

$$u_1'(x) = -\frac{y_2(x)G(x)}{W(y_1, y_2)}, \quad \text{and} \quad u_2'(x) = \frac{y_1(x)G(x)}{W(y_1, y_2)},$$

where W is the Wronskian of y_1 and y_2. We can then integrate to find u_1 and u_2.

10.4. Series Solution to Differential Equations

The power series method is used to find a solution to certain differential equations in the form of a power series, $y = \sum_{k=0}^{\infty} A_k x^k$. The method involves substituting this series into the DE to find a recurrence relation for the unknown coefficients A_k [5]. For a second-order homogeneous linear DE $a_2(x)y'' + a_1(x)y' + a_0(x)y = 0$, the power series method is applicable if the functions $\frac{a_1(x)}{a_2(x)}$ and $\frac{a_0(x)}{a_2(x)}$ are analytic [5]. If $a_2(x)$ is zero at some point, that point is a singular point, and a modified technique like the Frobenius method is required [5]. The method can be extended to higher-order equations and systems.

10.5. Frobenius Method

The Frobenius method, which is named after Ferdinand George Frobenius, is a method for finding an infinite series solution of the second-order ordinary differential equation of the form

$$x^2 y'' + xp(x)y' + q(x)y = 0, \tag{10.2}$$

in the vicinity of the regular singular point ($x = 0$). One can divide by x^2 to obtain a differential equation of the form

$$y'' + \frac{p(x)}{x}y' + \frac{q(x)}{x^2}y = 0,$$

which will not be solvable with the use of ordinary power series methods if either $\frac{p(x)}{x}$ or $\frac{q(x)}{x^2}$ are not analytic at $x = 0$. The Frobenius method enables one to develop a power series solution to such DE, if $p(x)$ and $q(x)$ are themselves analytic at $x = 0$ (or, if analytic elsewhere, both their limits at $x = 0$ exist and are finite). This method is to seek a power series solution of the form [1].

$$y = x^r \sum_{k=0}^{\infty} A_k x^k, \quad A_0 \neq 0.$$

Differentiating the above yields

$$y' = \sum_{k=0}^{\infty}(k+r)A_k x^{k+r-1}, \qquad y'' = \sum_{k=0}^{\infty}(k+r-1)(k+r)A_k x^{k+r-2}.$$

and substituting into (10.2) yields

$$\begin{aligned} x^2y'' + xp(x)y' + q(x)y &= \sum_{k=0}^{\infty}(k+r-1)(k+r)A_k x^{k+r} + p(x)\sum_{k=0}^{\infty}(k+r)A_k x^{k+r} + q(x)\sum_{k=0}^{\infty}A_k x^{k+r} \\ &= \sum_{k=0}^{\infty}\big[(k+r-1)(k+r) + p(x)(k+r) + q(x)\big]A_k x^{k+r}. \end{aligned}$$

Setting the entire series to zero, we extract the lowest power term ($k=0$):

$$[r(r-1) + p(x)r + q(x)]A_0 x^r + \sum_{k=1}^{\infty}\big[(k+r-1)(k+r) + p(x)(k+r) + q(x)\big]A_k x^{k+r} = 0.$$

For $x=0$ to be a regular singular point, $p(x)$ and $q(x)$ must be analytic at $x=0$, meaning they have power series expansions:

$$\begin{aligned} p(x) &= p_0 + p_1 x + p_2 x^2 + \dots, \\ q(x) &= q_0 + q_1 x + q_2 x^2 + \dots, \end{aligned}$$

where the constant coefficients $\mathbf{p_0}$ and $\mathbf{q_0}$ are defined by the limits:

$$\mathbf{p_0} = \lim_{x\to 0} p(x) \quad \text{and} \quad \mathbf{q_0} = \lim_{x\to 0} q(x).$$

Setting the coefficient of the lowest power term (x^r), for $k=0$, to zero and replacing the functions $p(x)$ and $q(x)$ with their constant limits p_0 and q_0 yields the equation:

$$r(r-1) + \mathbf{p_0}r + \mathbf{q_0} = 0.$$

This is called the **indicial equation** of the differential equation. The roots of the indicial equation may be real, equal, or complex. The solution of the DE depends on the nature of the roots of the indicial equation, [1]

1. **The Real Case:** When r_1 and r_2 are real and $r_1 > r_2$, then there are three cases, as follows; • Case 1: $r_1 - r_2$ is not an integer. Then we have

$$y_1 = \sum_{k=0}^{\infty}A_k x^{k+r_1}, \qquad y_2 = \sum_{k=0}^{\infty}B_k x^{k+r_2}.$$

   - Case 2: $r_1 = r_2$. Then we have

$$y_1 = \sum_{k=0}^{\infty}A_k x^{k+r_1}, \qquad y_2 = y_1 \ln|x| + \sum_{k=0}^{\infty}B_k x^{k+r_1}.$$

   - Case 3: $r_1 - r_2$ is a positive integer. Then we have

$$y_1 = \sum_{k=0}^{\infty}A_k x^{k+r_1}, \qquad y_2 = Ky_1 \ln|x| + \sum_{k=0}^{\infty}B_k x^{k+r_2}.$$

2. **The Complex Case:** If r_1 and r_2 are complex conjugates, the general solution is constructed from the real and imaginary parts of the series solution obtained for one of the roots, r.

The general solution of the DE is

$$y = C_1y_1 + C_2y_2.$$

11. Applications of ODEs

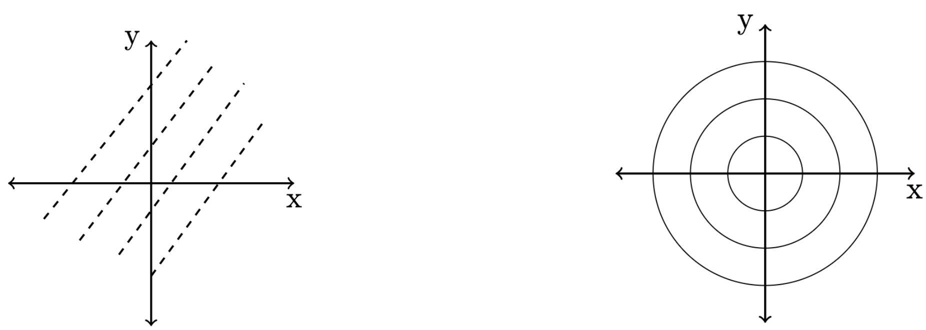
Ordinary differential equations (ODEs) find many uses in the domains of science and engineering. Essentially, ODEs constitute the mathematical language for expressing how some quantity changes over space or time, in order to convert intuitive assumptions into predictive equations. The following is the account of an investigation of common linear and non-linear models commonly seen in practice: population growth, for which exponential dynamics is corrected by the logistic equation to account for a finite carrying capacity; thermal phenomena governed by Newton's law of cooling; behavior of slight oscillations of a mass-spring system; and operating characteristics of series RLC electric circuits. Each example demonstrates a regular methodology—formulating a physical principle, translating it into an ODE, solving it subject to appropriate initial conditions, and interpreting parameters such that by the end of it, one can read off qualitative behavior.

11.1. Geometrical Application

If for each value of C, the equation
$$F(x, y, C) = 0, \tag{11.1}$$
is a curve in the xy-plane, then the totality of those curves is called a uni-parametric family of curves and C is called the parameter.

Example. *The equation*
$$f(x, y, C) = y - x - C = 0, \tag{11.2}$$
is a family of parallel lines having the y-intercept in Fig. 1

Example. *The equation*
$$g(x, y, C) = x^2 + y^2 - C^2 = 0, \; C \neq 0, \tag{11.3}$$
is a family of concentric circles of radius C and having center at the origin in Fig. 1.



(a) (b)

Figure 1: (a) one parameter family $y - x - C = 0$ (b) one parameter family $x^2 + y^2 - C^2 = 0$.

Figure 1(a) represents a one parameter family $y - x - C = 0$ (slope 1, intercept C), while Figure 1(b) is a one parameter family $x^2 + y^2 - C^2 = 0$ (circles of radius C centered at the origin). Differentiation removes the parameter and yields the governing first-order differential equation for each family.

We now find the DEs whose solutions are represented by the formulas (11.2) and (11.3) respectively. By taking the derivative of both sides of (11.2) with respect to x, we obtain

$$\frac{dy}{dx} - 1 = 0 \implies \frac{dy}{dx} = 1. \tag{11.4}$$

that corresponds to the family of lines represented by Fig. 1. Likewise, the DE of the family of circles (11.3) is given by

$$2x + 2y\frac{dy}{dx} = 0 \implies \frac{dy}{dx} = -\frac{x}{y}. \tag{11.5}$$

For every point with coordinates x and y, DEs (11.2) and (11.3) determine the value of the derivative $\frac{dy}{dx}$, that is, the angular coefficient of the tangent to the integral curve that passes through that point.

11.2. Orthogonal Trajectories

Let the slope of the base family of curves be

$$\frac{dy}{dx} = f(x, y) = \tan\theta. \tag{11.6}$$

Suppose that the solution of (11.6) is

$$F(x, y, C) = 0. \tag{11.7}$$

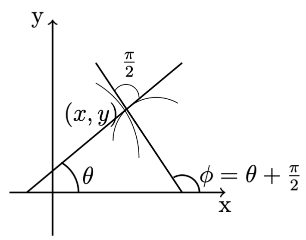


Figure 2: Orthogonal trajectories

Figure 2 is the orthogonal trajectories. Two curve families intersect at right angles: for a base family with tangent slope $y' = f(x, y) = \tan\theta$, the orthogonal family has slope $y' = -\cot\theta = -1/f(x, y)$. This relation defines the differential equation of the trajectories. Let us find another family

$$G(x, y, K) = 0 \tag{11.8}$$

so that these two families cut each other at right angles (Fig. 2). Under these conditions, the angular coefficient of the tangent to the integral curve (11.8) passing through point (x, y) is $\tan\phi$, where the angles are related by $\phi = \theta + \frac{\pi}{2}$.

The slope of the orthogonal trajectory is:

$$\frac{dy}{dx}\Big|_{\text{orthogonal}} = \tan\phi = \tan\Big(\theta + \frac{\pi}{2}\Big) = -\cot\theta = -\frac{1}{\tan\theta} = -\frac{1}{f(x, y)}.$$

Thus, the differential equation (DE) corresponding to the family of orthogonal curves to (11.7) has the form

$$\frac{dy}{dx} = -\frac{1}{f(x, y)}. \tag{11.9}$$

This is clearly a first-order DE which can be solved using standard methods.

11.3. Elementary Mechanics: Falling Object With Air Resistance

An object of mass m is dropped from a certain height. We will determine the law according to which the falling speed v changes if on the object. In addition to the force of gravity, the force of air resistance acts proportionally to the speed v (the coefficient of proportionality is k) [23,25,24], that is, we will find $v = f(t)$.

That is, force of air resistance, $F_1 \propto v \Rightarrow F_1 = kv$, force of gravity, $F_2 = mg$. So that, the resultant of the two forces becomes $F = F_2 - F_1$, [24].

$$F = mg - kv. \tag{11.10}$$

By Newton's second law of motion, the time rate of change of the momentum of a body is equal in both magnitude and direction to the force imposed on it [26]. That is,

$$F = ma. \tag{11.11}$$

Without air resistance, $\frac{dv}{dt} = a = g$. So that (11.11) becomes

$$F = m\frac{dv}{dt} \tag{11.12}$$

Equating (11.10) and (11.12) yields

$$m\frac{dv}{dt} = mg - kv.$$

So that we obtain

$$\frac{dv}{dt} + \frac{k}{m}v = g. \tag{11.13}$$

The integrating factor corresponding to (11.13) is given by,

$$\mu = e^{\int p(t)dt} = e^{\int \frac{k}{m}dt} = e^{\frac{k}{m}t}.$$

Using this factor, we obtain

$$\frac{d}{dt}\left[ve^{\frac{k}{m}t}\right] = ge^{\frac{k}{m}t} \implies \int \frac{d}{dt}\left[ve^{\frac{k}{m}t}\right] = \int ge^{\frac{k}{m}t}dt \implies ve^{\frac{k}{m}} = \frac{mg}{k}e^{\frac{k}{m}t} + C,$$

which gives

$$v(t) = \frac{mg}{k} + Ce^{-\frac{k}{m}t}. \tag{11.14}$$

The initial condition may now be written as $v(t_0) = v_0$. From (11.14), $t = 0, v = 0$

$$0 = \frac{mg}{k} + C \Rightarrow C = -\frac{mg}{k}.$$

We obtain from (11.14)

$$v(t) = \frac{mg}{k}\left[1 - e^{-\frac{k}{m}t}\right], \tag{11.15}$$

which is the speed at time, t.

11.4. Population Growth and Decay

To illustrate the use of differential equations to population problems, we consider the most basic mathematical model presented to control the population dynamics of a given species, [8,31].

Malthusian Model Among the very first attempts at mathematically describing human population growth was made by the English economist Thomas Malthus in 1798 [27]. The idea behind the Malthusian model is essentially the presumption that the rate at which a population of a country grows at some time is in proportion to the population of the country at that time [27,8]. Mathematically, if $P(t)$ is the total population at time t, then this assumption is expressed as

$$\frac{dP}{dt} \propto P(t)$$

Such that we have

$$\frac{dP}{dt} = kP, \tag{11.16}$$

with the initial condition $P(t_0) = P_0$, where k is the proportionality constant, $k = \beta - \delta$ where β represents birth rate, δ represents death rate. Observe that (11.16) is a first-order linear DE. Solution by separable,

$$\frac{dp}{p} = kdt \Rightarrow \int \frac{dp}{p} = \int kdt \Rightarrow \ln|p| = kt + C_1 \Rightarrow P = e^{kt+C_1} \Rightarrow P = e^{kt}e^{C_1} = Ce^{kt},\ C = e^{C_1}.$$

Thus, we have

$$P(t) = Ce^{kt} \tag{11.17}$$

From (11.17), at $t = 0$, $P = P_0$

$$\Rightarrow P_0 = Ce^0 \Rightarrow C = P_0.$$

we obtain

$$P(t) = P_0 e^{kt} \tag{11.18}$$

is the population after time, t, with k as the growth constant or constant of proportionality. The equation (11.18) will return the population at any time t in the future. This simplistic model (11.16) that does not take into consideration many factors (immigration and emigration, say) that have the possibility of influencing human populations to increase or decline, nevertheless was found to be reasonably close in predicting the population [30]. The DE (11.16) is an insect population growth model, animal and human population at a certain location and time interval [8]. Therefore, we have the following conclusion:

- If $k > 0$, population grows and continues to grow to infinity [31]. That is, $\lim_{t\to\infty} P(t) = \infty$
- When $k < 0$, the population will decline and converge to 0. That is, $\lim_{t\to\infty} P(t) = 0$. In short, we are headed for extinction [31].

Clearly, the first case, $k > 0$, is not enough, and the model can be thrown away. The issue is that population growth eventually gets limited by some factor, most often one of several basic resources [8].

Logistic Model Another model had been given by the Belgian mathematician Pierre Verhulst in 1838, to fix the deficiency of the exponential model and to provide a more realistic model of population growth [28]. It is known as the logistic model (also known as the Verhulst-Pearl model). It includes the carrying capacity K, which is the point at which the rate of birth and the rate of death are equal; the population stabilizes after some time [29,30]. When $P > K$, the rate of growth, k, is negative and the population decreases. When $P = K$, the rate of growth, k = 0. When $P < K$, the growth rate, k is positive. So, the assumption for the logistic growth model is that the maximum sustainable population is K. The rate of growth thus can be represented by.

$$k = k_0\left(1 - \frac{P}{K}\right), \tag{11.19}$$

where K, the upper limit of the population, is the carrying capacity. The resulting differential equation is

$$\frac{dp}{dt} = kP\left(1 - \frac{P}{K}\right), \tag{11.20}$$

subject to $P(t_0) = P_0$, which is a non-linear first-order DE. Solving separably,

$$\frac{dp}{P\left(1 - \frac{P}{K}\right)} = kdt \Rightarrow \left(\frac{A}{P} + \frac{B}{1 - \frac{P}{K}}\right) dp = kdt \Rightarrow \int \left(\frac{1}{P} + \frac{1}{K}\frac{1}{1 - \frac{P}{K}}\right) dp = \int kdt$$

$$\Rightarrow \ln|P| - \ln\left|1 - \frac{P}{K}\right| = kt + C.$$

$$\Rightarrow \frac{P}{1 - \frac{P}{K}} = e^{kt+C} = e^{kt}e^C = Ce^{kt},\ C = e^C \Rightarrow \frac{P}{1 - \frac{P}{K}} = Ce^{kt}.$$

So that we have

$$P(t) = \frac{KCe^{kt}}{K + Ce^{kt}} \tag{11.21}$$

At $t = 0$, $P = P_0$. Thus, $C = \dfrac{P_0 K}{K - P_0}$.

Substituting C in (11.21)

$$P(t) = \frac{KP_0}{P_0 + (K - P_0)e^{-kt}}. \tag{11.22}$$

Let's take a closer look at this solution to see what happens to the population as time goes on, specifically as $t \to \infty$.

- If $P_0 = 0$, then $\lim_{t\to\infty} P(t) = 0$. In this case, there simply isn't any population [31].
- If $P_0 = K$, then $\lim_{t\to\infty} P(t) = K$. Here, since K represents the population level, we see that there's no growth or decline [31].
- If $0 < P_0 < K$, then $\lim_{t\to\infty} P(t) = K$. This means that if we start with a small population (less than K), it will grow until it reaches the equilibrium population $P = K$ [30,31].
- If $P_0 > K$, then $\lim_{t\to\infty} P(t) = K$. In this scenario, if we begin with a population that exceeds what the available resources can support, the population will decrease until it stabilizes at the equilibrium level $P = K$ [30,31].

11.5. Newton's Law Of Cooling

The transfer of thermal energy is a fundamental law in thermodynamics, which governs the change in temperature of a system. Heat transfer can be to a system from its surroundings, leading to an increase in temperature, or from a system to its surroundings, leading to a decrease in temperature [33].

This is a universal principle of very significant implications in many areas. To illustrate, consider the cooling of a warm body to the ambient temperatures. This is mathematically precisely rendered by Newton's Law of Cooling, which states that the rate of heat transfer from a body is proportional to the differential of the temperatures of the body and the environment.

The law has been of significant use in the resolution of common problems. One common prediction is in food science to predict the time when a baked good cools from a starting oven temperature to a temperature safe for eating. A more applied prediction is in forensic science, where Newton's Law of Cooling is applied theoretically to make a prediction of the time of death based on how fast a body will cool [34,35]. This illustrates the extensive variety of applications of the law as a quantitative device for describing and predicting heat behavior. *Newton's law of cooling* states that the rate of loss of heat of a body is equal to the difference between the body and its surroundings, and is proportional to the difference. Mathematically, this can be represented by

$$\frac{dT}{dt} \propto T(t) - T_a, \tag{11.23}$$

which can be written as

$$\frac{dT}{dt} = -k[T(t) - T_a], \tag{11.24}$$

where $T(t)$ is the body temperature, T_a is the constant temperature of the surrounding medium or the ambient medium, and k is a proportionality constant. Newton's Law of Cooling leads to the standard equation of exponential decay with time, which may be used to model many science and engineering processes, such as radioactivity decay [32,33].

Equation (11.24) can also be written as

$$\frac{dT}{dt} + kT(t) = kT_a, \tag{11.25}$$

which is a first-order linear DE. So the body temperature at time t becomes

$$T(t) = T_a + Ce^{-kt}, \tag{11.26}$$

which means that the body temperature approaches that of its ambient as time increases. The IVP can be stated as $T(t_0) = T_0$, so that (11.26) becomes

$$T(t) = T_a + (T_0 - T_a)e^{-kt},$$

where T_0 is initial body temperature at $t = 0$. Cooling Curve is the name given to the curve drawn with respect to body temperature and time.

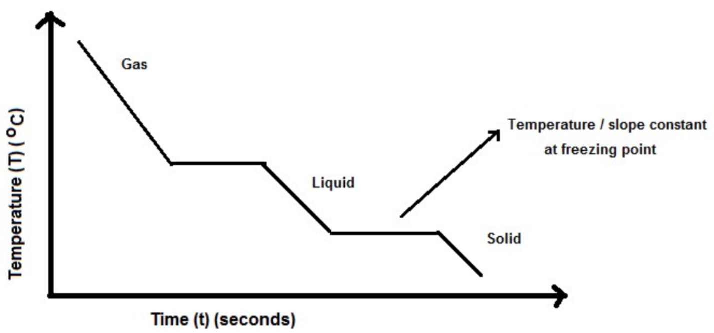


Figure 3: Cooling curve with phase changes

Figure 3 is the graph of Temperature T versus time t shows: (i) monotonic decrease within single phases (gas, then liquid, then solid); (ii) near horizontal plateaus at the phase transitions where latent heat is released and T is approximately constant (condensation and freezing, slope ≈ 0); (iii) further cooling of the solid after solidification. The annotation marks the isothermal plateau at the freezing point.

11.6. Vibrating Springs

We consider the motion of a mass m attached at the end of the spring on a horizontal plane, as in Fig. 4. Hooke's law states that if the spring is stretched (compressed) x units away (from) from its natural length, then it is subjected to a restoring force proportional to x (displacement), restoring force $= -kx$, where k is a positive constant (spring constant). In case we ignore any external forces (due to resistance or friction) then, by Newton's second law of motion, ([36,37]) (force equals mass times acceleration), we have;

$$m\frac{d^2x}{dt^2} = -kx \implies m\frac{d^2x}{dt^2} + kx = 0, \tag{11.27}$$

which is a second-order linear DE. Its auxiliary equation is

$$mr^2 + k = 0,$$

with roots $r = \pm i\omega$, where $\omega = \sqrt{\frac{k}{m}}$. The general solution (which is the position of the mass at any time t) becomes,

$$x(t) = c_1 \cos \omega t + c_2 \sin \omega t, c_1, c_2 \in \mathbb{R}.$$

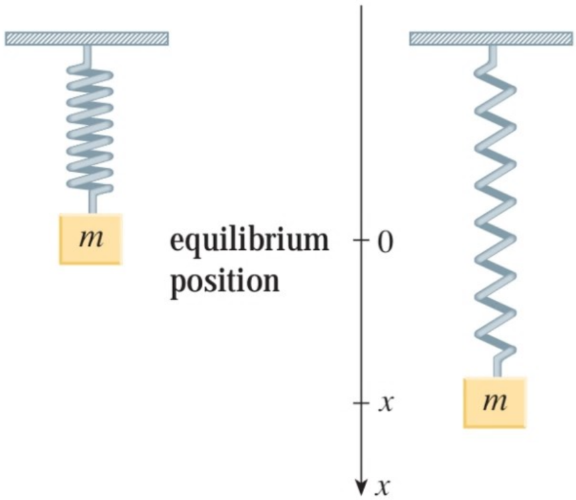


Figure 4: Mass spring system for undamped free oscillation

In Figure 4, a mass m attached to a spring with constant k moves on a horizontal surface; displacement $x(t)$ is measured from equilibrium. At equilibrium, the spring's static extension balances the weight. Small motions about equilibrium are modeled by $m\,x'' + k\,x = 0$.

11.7. RLC Circuits And Flow Of Electricity

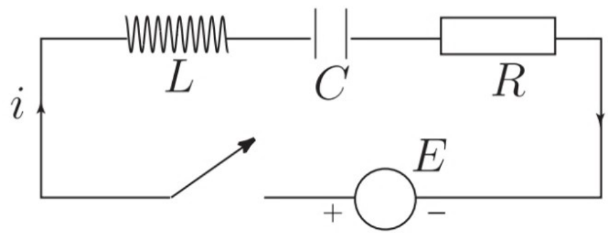


Figure 5: RLC Circuit

The circuit of Figure 5 consists of an electromotive force E (from a battery or generator), a resistor R, an inductor L, and a capacitor C, all connected in series. If $Q = Q(t)$ denotes the charge on the capacitor at time t, then current is the rate of change of Q with respect to t (i.e., $I = \frac{dQ}{dt}$). The voltages across the resistor, inductor, and capacitor are RI, $L\dfrac{dI}{dt}$, $\frac{Q}{C}$ respectively [38,39].

Kirchhoff's voltage law implies that the sum of these voltage drops is the applied voltage [40,38]:

$$L\frac{dI}{dt} + RI + \frac{Q}{C} = E(t). \tag{11.28}$$

Since $I = \frac{dQ}{dt}$, equation (11.28) becomes

$$L\frac{d^2Q}{dt^2} + R\frac{dQ}{dt} + \frac{1}{C}Q = E(t). \tag{11.29}$$

which is a second-order linear DE with constant coefficients. If the charge Q_0 and the current I_0 are known at time 0, then we have the initial conditions,

$$Q(0) = Q_0,\ Q'(0) = I_0. \tag{11.30}$$

We provide solution to equation (11.29)-(11.30) when the total voltage drop equals zero (i.e., $E = 0$). Equation (11.29) is also equal to

$$Q'' + \frac{R}{L}Q' + \frac{1}{LC}Q = 0.$$

Then, the characteristic polynomial is

$$\lambda^2 + \frac{R}{L}\lambda + \frac{1}{LC} = 0.$$

Solving for lambda in the quadratic formula:

$$\lambda = -\frac{R}{L} \pm \sqrt{\left(\frac{R}{L}\right)^2 - \frac{1}{LC}} = -\frac{R}{L} \pm \sqrt{D}. \tag{11.31}$$

We now analyze the discriminant D in (11.31), in order to determine the behavior of the circuit [38,39].

- **Case 1:** $D > 0$, here the circuit is over damped. This means that the circuit will decay.
- **Case 2:** $D = 0$, here the circuit is critically damped. This means that there will be no oscillations whatsoever.
- **Case 3:** $D < 0$, here the circuit is under damped. This means that there will be oscillation until the equilibrium amplitude is reached.

12. Practical Examples

Example 1. *Consider a lumped thermal enclosure with ambient temperature T_a, thermal capacitance $C > 0$, and thermal conductance $k > 0$. A resistive heater of constant power $P > 0$ is controlled by a hysteresis thermostat with thresholds $T_{\text{low}} < T_{\text{high}}$. The heater is ON when $T \le T_{\text{low}}$ and OFF when $T \ge T_{\text{high}}$. Between switchings, the temperature $T(t)$ satisfies*

$$C\,\dot{T} = -k\,(T - T_a) + u(t)\,P, \tag{12.1}$$

with control law given by

$$u(t) = \begin{cases} 1, & T(t) \le T_{\text{low}}, \\ 0, & T(t) \ge T_{\text{high}}. \end{cases}$$

Assume the room temperature T_a and heater power P are constant over a cycle and that the lumped model is valid. In what follows, we derive closed-form expressions for $T(t)$ during heater-ON and heater-OFF segments, compute the corresponding switching times $t_\uparrow$ and $t_\downarrow$ for the transitions $T_{\text{low}} \to T_{\text{high}}$ and $T_{\text{high}} \to T_{\text{low}}$, and determine the total period $\mathfrak{T}$ of the temperature cycle. We also identify the natural thermal time constant, construct a symmetry-preserving nondimensionalization of the model, and clarify the feasibility conditions under which a periodic ON–OFF cycling regime exists.

Let τ be the natural thermal time constant denoted by $\tau := C/k$. Dividing by C gives the linear ODE

$$\dot{T} + \frac{1}{\tau}(T - T_a) = \frac{uP}{C}.$$

On any interval where u is constant, the integrating-factor method yields

$$T(t) = T_a + \left(T(0) - T_a - \frac{uP}{k}\right) e^{-t/\tau} + \frac{uP}{k}. \tag{12.2}$$

Case 1: $u = 1$ (Heater ON segment)
Suppose a heating phase begins at $t = 0$ with $T(0) = T_{\text{low}}$. From (12.2),

$$T(t) = T_a + \left(T_{\text{low}} - T_a - \frac{P}{k}\right) e^{-t/\tau} + \frac{P}{k}. \tag{12.3}$$

Next, observe that

$$T_{\text{high}} - T_a - \frac{P}{k} = \left(T_{\text{low}} - T_a - \frac{P}{k}\right) e^{-t_\uparrow/\tau} \implies t_\uparrow = \tau \ln\left(\frac{T_{\text{low}} - T_a - \frac{P}{k}}{T_{\text{high}} - T_a - \frac{P}{k}}\right), \tag{12.4}$$

which is the time $t_\uparrow$ to reach T_{high}. Feasibility requires $T_a + \frac{P}{k} > T_{\text{high}}$.
Case 2: $u = 0$ (Heater OFF segment)
Suppose a cooling phase begins at $t = 0$ with $T(0) = T_{\text{high}}$. From (12.2),

$$T(t) = T_a + \big(T_{\text{high}} - T_a\big)e^{-t/\tau}. \tag{12.5}$$

Next, we observe that

$$T_{\text{low}} - T_a = \big(T_{\text{high}} - T_a\big)e^{-t_\downarrow/\tau} \implies t_\downarrow = \tau \ln\left(\frac{T_{\text{high}} - T_a}{T_{\text{low}} - T_a}\right), \tag{12.6}$$

which is the time $t_\downarrow$ to reach T_{low}.
Limit cycle and period
Assume the feasibility conditions $T_a < T_{\text{low}} < T_{\text{high}} < T_a + \frac{P}{k}$. Then the thermostat alternates between (12.4) and (12.6), producing a two-phase limit cycle with

$$\mathcal{T} = t_\uparrow + t_\downarrow.$$

During ON phases, $T(t)$ increases monotonically toward the ON asymptote $T_a + \frac{P}{k}$; during OFF phases, $T(t)$ decreases monotonically toward T_a. For the entire cycle, we have that $T(t) \in [T_{low}, T_{high}]$. Consequently, the cycle mean

$$\overline{T} := \frac{1}{\mathcal{T}} \int_0^{\mathcal{T}} T(t)\, dt$$

satisfies $\overline{T} \in [T_{low}, T_{high}]$.
Symmetry and nondimensionalization
We introduce dimensionless variables

$$\theta := \frac{T - T_a}{\Delta T}, \qquad s := \frac{t}{\tau}. \tag{12.7}$$

Then from (12.1), we obtain

$$\frac{d\theta}{ds} = \frac{d}{ds}\Big(\frac{T - T_a}{\Delta T}\Big) = \frac{\tau}{\Delta T}\dot{T} = \frac{\tau}{\Delta T}\Big(-\frac{k}{C}(T - T_a) + \frac{uP}{C}\Big) = -\frac{T - T_a}{\Delta T} + u\,\frac{\tau P}{C\,\Delta T}.$$

Using $\theta = (T - T_a)/\Delta T$ *and* $\tau = C/k$ *gives*

$$\frac{d\theta}{ds} = -\theta + u\,\Pi, \qquad \Pi := \frac{P}{k\,\Delta T}.$$

The hysteresis thresholds transform to

$$\theta_{\text{low}} = \frac{T_{\text{low}} - T_a}{\Delta T}, \qquad \theta_{\text{high}} = \frac{T_{\text{high}} - T_a}{\Delta T},$$

and the control law becomes

$$u = \begin{cases} 1, & \theta \le \theta_{\text{low}}, \\ 0, & \theta \ge \theta_{\text{high}}. \end{cases}$$

Thus the original model (12.1) *is exactly equivalent to the dimensionless system*

$$\frac{d\theta}{ds} = -\theta + u\,\Pi, \qquad u \in \{0, 1\}. \tag{12.8}$$

From (12.7)*, the solutions correspond one-to-one via*

$$T(t) = T_a + \Delta T\,\theta\left(\frac{t}{\tau}\right).$$

Closed forms map directly. Indeed, for any interval with constant u*, solving* (12.8) *yields*

$$\theta(s) = (\theta_0 - \Pi u)e^{-s} + \Pi u \quad \Longrightarrow \quad T(t) = T_a + \Delta T\Big((\theta_0 - \Pi u)e^{-t/\tau} + \Pi u\Big),$$

which recovers the ON (12.3) *and OFF* (12.5) *expressions.*

Switching times scale linearly with τ*. If* $s_\uparrow$ *and* $s_\downarrow$ *are the dimensionless times to traverse* $\theta_{\text{low}} \to \theta_{\text{high}}$ *(ON) and* $\theta_{\text{high}} \to \theta_{\text{low}}$ *(OFF), then*

$$t_\uparrow = \tau s_\uparrow, \qquad t_\downarrow = \tau s_\downarrow, \qquad \mathcal{T} = \tau(s_\uparrow + s_\downarrow).$$

Feasibility carries over without change. In physical variables, heating must be able to reach the upper threshold: $T_a + \frac{P}{k} > T_{\text{high}}$*. Dividing by* ΔT *and using* $\theta = (T - T_a)/\Delta T$ *yields* $\Pi > \theta_{\text{high}}$ *in dimensionless form. Therefore holding* $(\Pi, \theta_{\text{low}}, \theta_{\text{high}})$ *fixed produces dynamically similar behavior. This is the symmetry (scale invariance) preserved by the nondimensionalization.*

Example 2. *Consider the Hermite differential equation given by*

$$y'' - 2xy' + \lambda y = 0,\ \lambda = 1. \tag{12.9}$$

We can construct a series solution

$$y = \sum_{k=0}^{\infty} A_k x^k$$

$$y' = \sum_{k=1}^{\infty} k A_k x^{k-1}$$

$$y'' = \sum_{k=2}^{\infty} k(k-1) A_k x^{k-2}.$$

Substituting this in the differential equation

$$\begin{aligned} y'' - 2xy' + \lambda y &= \sum_{k=2}^{\infty} k(k-1)A_k x^{k-2} - 2x\sum_{k=1}^{\infty} kA_k x^{k-1} + \sum_{k=0}^{\infty} A_k x^k \\ &= \sum_{k=2}^{\infty} k(k-1)A_k x^{k-2} - \sum_{k=1}^{\infty} 2kA_k x^k + \sum_{k=0}^{\infty} A_k x^k. \end{aligned}$$

Making a shift on the first sum (that is, $k \to k+2$)

$$\begin{aligned}
&= \sum_{k=0}^{\infty}(k+2)(k+1)A_{k+2}x^k - \sum_{k=1}^{\infty} 2kA_kx^k + \sum_{k=0}^{\infty} A_kx^k \\
&= 2A_2 + \sum_{k=1}^{\infty}(k+2)(k+1)A_{k+2}x^k - \sum_{k=1}^{\infty} 2kA_kx^k + A_0 + \sum_{k=1}^{\infty} A_kx^k \\
&= 2A_2 + A_0 + \sum_{k=1}^{\infty}\Big((k+2)(k+1)A_{k+2} + (-2k+1)A_k\Big)x^k = 0.
\end{aligned}$$

If this series is a solution, then all these coefficients must be zero, so for both $k = 0$ and $k > 0$:

$$(k+2)(k+1)A_{k+2} + (-2k+1)A_k = 0$$

We can rearrange this to get a recurrence relation for A_{k+2}.

$$(k+2)(k+1)A_{k+2} = -(-2k+1)A_k \quad \Rightarrow \quad A_{k+2} = \frac{(2k-1)}{(k+2)(k+1)}A_k.$$

Now, we have

$$A_2 = -\frac{1}{2}A_0, \quad A_3 = \frac{1}{6}A_1, \quad A_4 = -\frac{1}{8}A_0, \quad A_5 = \frac{1}{24}A_1, \ldots$$

So that the series solution becomes

$$\begin{aligned}
y = \sum_{k=0}^{\infty} A_kx^k &= A_0x^0 + A_1x^1 + A_2x^2 + A_3x^3 + A_4x^4 + A_5x^5 + \ldots \\
&= A_0x^0 + A_1x^1 - \frac{1}{2}A_0x^2 + \frac{1}{6}A_1x^3 - \frac{1}{8}A_0x^4 + \frac{1}{24}A_1x^5 + \ldots \\
&= A_0x^0 - \frac{1}{2}A_0x^2 - \frac{1}{8}A_0x^4 + \frac{1}{6}A_1x^3 + \frac{1}{24}A_1x^5 + \ldots,
\end{aligned}$$

Which we can break up into the sum of two linearly independent series solutions:

$$y = A_0\left(1 - \frac{1}{2}x^2 - \frac{1}{8}x^4 + \ldots\right) + A_1\left(x + \frac{1}{6}x^3 + \frac{1}{24}x^5 + \ldots\right).$$

13. Conclusion

This study set out to demystify how ordinary differential equations (ODEs) turn simple, testable assumptions into predictions we can use. We began with a small but powerful toolbox: separation of variables, integrating factors, exactness, homogeneous substitutions, constant coefficient theory for second-order equations, variation of parameters, and series methods; and we showed how each technique connects to interpretation rather than mere algebra. From this discussion, we can see how differential equations are deeply intertwined with real-world applications and how various natural laws across different scientific fields are expressed through these equations.

The relevance of this work is both practical and pedagogical. The models are intentionally minimal, yet they already support estimation, control, and design decisions: cooling curves calibrate thermal processes; logistic fits inform resource planning; drag-based fall models benchmark experimental data; and second-order templates guide damping and resonance choices in mechanical and electrical systems [33,31,24,38]. Our aim is that readers can reuse and apply these modeling templates quickly, with clear assumptions and units, and adapt them to their own domains.

13.1. Limitations

Despite the breadth of techniques covered, this work is necessarily limited in scope. Specifically, the analysis is restricted to linear ODEs and a handful of nonlinear forms (separable, exact, homogeneous) with analytic solutions. This focus means we do not address common complexities like strong nonlinearities (e.g., population models with Allee effects), systems of equations, or the treatment of stochastic forcing often encountered in real-world data. Furthermore, we rely exclusively on exact, closed-form solutions, bypassing the essential role of numerical methods for systems with complex or non-elementary right-hand sides.

13.2. Future Work

Looking ahead, the same modeling cycle (assumption → differential law → solution → inference) extends naturally to richer problems. To increase the utility of this work, especially for students, we propose the following concrete next steps:

(i) Computational Notebook: Develop a companion computational notebook (e.g., in Python/Jupyter or MATLAB LiveScript) that illustrates every model and solution discussed. This notebook must include short proposals for numerical illustrations, such as using the Runge-Kutta 4th order method to solve the standard second-order damping equation with non-sinusoidal forcing.

(ii) Parameter Estimation Examples: Add concrete parameter estimation examples. For instance, use least squares regression on synthetic or real cooling data to estimate the heat transfer coefficient (k) in Newton's Law of Cooling model, quantifying the uncertainty of the estimate.

(iii) Qualitative Analysis: Extend the discussion to nonlinear oscillators and phase plane methods to capture self-excited dynamics and bifurcations beyond the linear regime (e.g., a basic Lotka-Volterra predator-prey system).

(iv) Foundation for PDEs: Carry these ideas into partial differential equations (PDEs) for spatially distributed systems (diffusion, waves, potential theory), explicitly showing how the separation of variables and Fourier/Laplace transform methods presented here build directly on the ODE foundations developed in this manuscript [8,5].

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