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Approximation of Fuzzy Numbers by Modified Meyer-König and Zeller Operators: Implications for Medical Diagnosis Modeling

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ABSTRACT: In this work, the modified Meyer-König and Zeller operators are generalized to arbitrary compact intervals, and their approximation behavior and structural properties are thoroughly analyzed. In particular, we establish that the extended operators maintain key qualitative features such as monotonicity and shape preservation across any compact domain. To illustrate the practical utility of the framework, the proposed operators are applied to the fuzzy modeling of diagnostic indicators in cancer patients. The accompanying numerical experiments, supported by detailed graphical and tabular analyses, confirm that the modified Meyer-König and Zeller operators yield highly accurate approximations while preserving the interpretability and clinical relevance of fuzzy medical data.

Keywords: Linear Meyer-König and Zeller on $[a, b]$, fuzzy inference, applications.

Contents

1. Introduction

The approximation of continuous functions has been a fundamental theme in analysis since the classical theorem of Weierstrass (1885), which established that every continuous function on a compact interval may be uniformly approximated by polynomials to arbitrary precision. Bernstein's constructive formulation of this result through his celebrated polynomials provided the first explicit approximation scheme and subsequently inspired the systematic development of positive linear operators. Korovkin's theorem (1953) later refined this framework by offering concise and powerful criteria for the uniform convergence of operator sequences, thereby solidifying the foundations of modern approximation theory.

Among the numerous operator families studied within the Korovkin framework, the Meyer–König and Zeller (MKZ) operators, introduced in 1960 [19], hold a distinguished position due to their flexibility and strong approximation properties. For $\upsilon \in [0, 1]$, they are defined by

$$M_n(\upsilon; [0,1])(x) = \sum_{k=0}^{\infty} w_{n,k}(x) \cdot \upsilon\left(\frac{k}{n+k}\right)$$

where

$$w_{n,k}(x) = \binom{n+k}{k} x^k (1-x)^{n+1} \quad , n \in \mathbb{N}.$$

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In recent years, several modifications of the MKZ operators have been proposed to enhance their endpoint behavior, smoothness reproduction, and stability. A prominent example is the class of Modified Meyer–König and Zeller (M-MKZ) operators introduced by Rempulska and Tomczak [20]. These operators incorporate higher-order Taylor terms and are defined by

$$M_{n,r}\left(\upsilon;[0,1]\right)(x)=\sum_{k=0}^{\infty}w_{n,k}(x)\cdot\sum_{j=0}^{r}\frac{\upsilon^{(j)}\left(\xi_{n,k}\right)}{j!}\left(x-\xi_{n,k}\right)^{j}$$

where the weights and nodes are given by

$$w_{n,k}\left(x\right):=\binom{n+k}{k}x^{k}\left(1-x\right)^{n+1},\quad \xi_{n,k}=\frac{k}{n+k},\quad \textit{and}\quad \upsilon\in C^{r}[a,b]$$

Their construction ensures improved approximation accuracy as the smoothness of the target function increases, while maintaining essential structural qualities such as constant reproduction and endpoint preservation.

Parallel to these advancements, the emergence of fuzzy set theory, pioneered by Zadeh (1965), has transformed the mathematical treatment of imprecision and uncertainty. Fuzzy numbers—developed extensively by Dubois, Prade, and others [13]—provide a flexible representation of uncertain quantitative information. Because fuzzy membership functions can be analytically complex, a variety of approximation schemes have been developed to simplify their structure while preserving interpretability. These include interval-based approaches [10,15], triangular forms [4,5,8,14,22], trapezoidal models [6,7,16,21], L–U parameterizations [2,3,23], and max-product methods [1,9,17,18].

The interface between approximation operators and fuzzy number theory has therefore become a fertile area of research. By employing operator-theoretic techniques, one may obtain analytically tractable and computationally efficient representations of fuzzy membership functions, thereby merging the rigor of positive operator theory with the practical demands of uncertainty modeling. This synthesis has found applications across engineering, decision sciences, and biomedical diagnostics.

In this article, we pursue this line of inquiry by extending and employing the M-MKZ operators on general compact intervals and analyzing their approximation rates, uniform convergence properties, and preservation of essential shape characteristics. These theoretical results are complemented by a practical application to cancer diagnosis, where fuzzy representations of key oncological indicators—tumor size, biomarker concentration, and cell density—are approximated using the proposed operators. This dual theoretical–practical contribution highlights both the improved mathematical performance of the generalized M-MKZ operators and their effectiveness in modeling complex, uncertain biomedical data.

2. Preliminaries

2.1. Basic concepts of fuzzy numbers

Definition 2.1 (fuzzy numbers) [11,13] A fuzzy number $\tilde{\mu}$ is a fuzzy subset of $\mathbb{R}$ with membership function $\mu_{\tilde{\mu}}:\mathbb{R}\to[0,1]$, if and only if $\mu_{\tilde{\mu}}$ satisfies the following conditions:

1. $\exists t_0\in\mathbb{R}$ such that $\mu_{\tilde{\mu}}\left(t_0\right)=1$ (normality);
2. $\mu_{\tilde{\mu}}\left(\alpha t_1+\left(1-\alpha\right)t_2\right)\geq\min\{\mu_{\tilde{\mu}}\left(t_1\right),\mu_{\tilde{\mu}}\left(t_2\right)\}$ for all $t_1,t_2\in\mathbb{R}$, $\alpha\in[0,1]$ (fuzzy convexity);
3. $\mu_{\tilde{\mu}}$ is upper semicontinuous;
4. the support of $\tilde{\mu}$ is compact.

Thus, for any fuzzy number $\tilde{\mu}$, the membership function $\mu_{\tilde{\mu}}$ can be expressed as:

$$\mu_{\tilde{\mu}}(t) = \begin{cases} 0, & t < t_1, \\ l_\mu(t), & t_1 \le t \le t_2, \\ 1, & t_2 \le t \le t_3, \\ r_\mu(t), & t_3 \le t \le t_4, \\ 0, & t > t_4, \end{cases}$$

where $l_\mu : [t_1, t_2] \to [0, 1]$ is non-decreasing (the left side of $\mu_{\tilde{\mu}}$), and $r_\mu : [t_3, t_4] \to [0, 1]$ is non-increasing (the right side of $\mu_{\tilde{\mu}}$). The collection of all fuzzy real numbers will be denoted by $\mathbb{R}_F$.

Definition 2.2 (α-cut) [12] The α-cut of $\tilde{\mu} \in \mathbb{R}_F$, for $\alpha \in (0, 1]$, is the crisp set

$$\tilde{\mu}_\alpha = \{t \in \mathbb{R} \mid \mu_{\tilde{\mu}}(t) \ge \alpha\}.$$

Clearly,

$$\tilde{\mu}_\alpha = [\tilde{\mu}^l(\alpha),\, \tilde{\mu}^r(\alpha)], \quad \alpha \in (0, 1],$$

where

$$\tilde{\mu}^l(\alpha) = \inf\{t \in \mathbb{R} \mid \mu_{\tilde{\mu}}(t) \ge \alpha\}, \qquad \tilde{\mu}^r(\alpha) = \sup\{t \in \mathbb{R} \mid \mu_{\tilde{\mu}}(t) \ge \alpha\}.$$

Remark 2.1

1. The *core* of $\tilde{\mu}$, denoted $\mathrm{core}(\tilde{\mu})$, corresponds to $\alpha = 1$, i.e., $\tilde{\mu}_1 = [\tilde{\mu}^l(1),\, \tilde{\mu}^r(1)]$.
2. The *support* of $\tilde{\mu}$, denoted $\mathrm{supp}(\tilde{\mu})$, corresponds to $\alpha = 0$, i.e., $\tilde{\mu}_0 = [\tilde{\mu}^l(0),\, \tilde{\mu}^r(0)]$.

2.2. Constructing Operators on Compact Intervals $[a, b]$ with Auxiliary Concepts.

In this part, we generalize the operator

$$M_{n,r}(\upsilon; [0, 1])\,(x)$$

to an arbitrary compact interval $[a, b]$. This extension, inspired by the classical Weierstrass approximation theorem, enables MKZ-type operators to be applied beyond the unit interval, thereby broadening their analytical and practical scope.

Definition 2.3 Let $\tilde{\mu} \in \mathbb{R}_F^+$ be a fuzzy-valued function continuous on $[a, b]$ with $\tilde{\mu} \in C^r[a, b]$. Suppose $a < c \le d < b$, where the *core* and *support* of $\tilde{\mu}$ satisfy

$$\mathrm{core}(\tilde{\mu}) = [c, d], \qquad \mathrm{supp}(\tilde{\mu}) = [a, b].$$

For any $t \in [a, b]$, we define the operator

$$\tilde{M}_{n,r}(\tilde{\mu}; [a, b])\,(t) = \sum_{k=0}^{\infty} w_{n,k}(t)\, \varphi_r^{(\tilde{\mu})}(\xi_{n,k}; t),$$

where the weights and nodes are given by

$$w_{n,k}(t) = \binom{n+k}{k} \left(\frac{t-a}{b-a}\right)^k \left(\frac{b-t}{b-a}\right)^{n+1}, \qquad \xi_{n,k} = \frac{k}{n+k}.$$

Here $\varphi_r^{(\tilde{\mu})}(\xi_{n,k}; t)$ denotes the Taylor polynomial of degree r for $\tilde{\mu} \in C^r[a, b]$, expanded at

$$a + (b-a)\,\xi_{n,k},$$

and is given explicitly by

$$\varphi_r^{(\tilde{\mu})}(\xi_{n,k}; t) = \sum_{j=0}^{r} \frac{\tilde{\mu}^{(j)}(a + (b-a)\xi_{n,k})}{j!} \left(\frac{t-a}{b-a} - \xi_{n,k}\right)^j.$$

Note 1 *For brevity, we denote the general compact interval* $[a, b]$ *by* $\tilde{I}$ *and the unit interval* $[0, 1]$ *by* I *throughout the remainder of this paper.*

Theorem 2.1 *[20] Let* $r \in \mathbb{N}$ *be fixed. Then there exists a constant* $C(r) > 0$ *such that, for every* $\upsilon \in [0, 1]$ *and* $n \in \mathbb{N}$,

$$|M_{n,r}(\upsilon; [0,1])(x) - \upsilon(x)| \le C(r) \left(\frac{1}{\sqrt{n}}\right)^r \omega\left(\upsilon^{(r)}, \frac{1}{\sqrt{n}}\right)_I,$$

where $\omega(\cdot, \cdot)_I$ *denotes the modulus of continuity on the interval* I.

Remark 2.2 [17] Let $\tilde{\mu}$ be a $quasi-convex$ continuous function on $\tilde{I}$, then there is a point $c \in \tilde{I}$ such that $\tilde{\mu}$ is non-increasing on $[a, c]$ and non-decreasing on $[c, b]$.

3. Approximation and Preservation of Shape Properties

This section develops the approximation properties of the M–MKZ operators on general compact intervals and examines their ability to preserve essential shape characteristics. Subsection 3.1 begins with a key remark that extends a classical result attributed to Weierstrass and forms the analytical foundation for the results that follow.

3.1. Approximation by M–MKZ Operators on Compact Intervals $[a, b]$

Remark 3.1 Let $\tilde{\mu} : \tilde{I} \to \mathbb{R}^+$ be continuous on the compact interval $\tilde{I} = [a, b]$, and define $\upsilon : I \to \mathbb{R}^+$ on $I = [0, 1]$ by

$$\upsilon(x) = \tilde{\mu}\big(a + (b - a)x\big), \qquad a, b \in \mathbb{R}.$$

It follows directly that

$$\upsilon\left(\frac{k}{n+k}\right) = \tilde{\mu}\left(a + (b-a)\frac{k}{n+k}\right),$$

and, for every $j = 0, 1, \ldots, r$,

$$\upsilon^{(j)}\left(\frac{k}{n+k}\right) = \tilde{\mu}^{(j)}\left(a + (b-a)\frac{k}{n+k}\right).$$

Furthermore, introducing the transformation

$$x = \frac{t-a}{b-a}, \qquad t \in \tilde{I},$$

we obtain

$$\upsilon\left(\frac{t-a}{b-a}\right) = \tilde{\mu}(t), \qquad t = a + (b-a)x.$$

Combining these identities yields the correspondence

$$\tilde{M}_{n,r}(\tilde{\mu}; [a,b])(t) = M_{n,r}(\upsilon; [0,1])(x),$$

establishing a direct link between the operator on $[a, b]$ and its counterpart on the unit interval.

Theorem 3.1 *Let* $\tilde{\mu}$ *be a continuous fuzzy-valued function belonging to* $C^r[a, b]$, *and let* $r \in \mathbb{N}$ *be fixed. Then there exists a constant* $C(r) > 0$ *such that, for all* $t \in \tilde{I}$,

$$\left|\tilde{M}_{n,r}(\tilde{\mu}; [a,b])(t) - \tilde{\mu}(t)\right| \le C(r) \left(\frac{1}{\sqrt{n}}\right)^r \big([b-a] + 1\big)\, \omega\left(\tilde{\mu}^{(r)}, \frac{1}{\sqrt{n}}\right)_{\tilde{I}},$$

where $\omega(\cdot, \cdot)_{\tilde{I}}$ *denotes the modulus of continuity on the interval* $\tilde{I}$.

Proof: By Remark 3.1, we have

$$\tilde{M}_{n,r}(\tilde{\mu};[a,b])\,(t) = M_{n,r}(\upsilon;[0,1])\,(x),$$

with $x = (t-a)/(b-a)$.

Since $\upsilon \in C^r[0,1]$, Theorem 2.1 yields

$$\left|\tilde{M}_{n,r}(\tilde{\mu};[a,b])\,(t) - \tilde{\mu}(t)\right| = |M_{n,r}(\upsilon;[0,1])\,(x) - \upsilon(x)| \le C(r)\left(\frac{1}{\sqrt{n}}\right)^r \omega\left(\upsilon^{(r)}, \frac{1}{\sqrt{n}}\right)_I .$$

Using the relation between $\upsilon^{(r)}$ and $\tilde{\mu}^{(r)}$ established in Remark 3.1, and applying standard scaling properties of the modulus of continuity, we obtain

$$\omega\left(\upsilon^{(r)}, \frac{1}{\sqrt{n}}\right)_I \le \omega\left(\tilde{\mu}^{(r)}, \frac{b-a}{\sqrt{n}}\right)_{\tilde{I}} \le \big([b-a]+1\big)\omega\left(\tilde{\mu}^{(r)}, \frac{1}{\sqrt{n}}\right)_{\tilde{I}} .$$

The stated estimate follows immediately. □

Corollary 3.1 *Let $\tilde{\mu} \in C^r[a,b]$ for some $r \in \mathbb{N}$, and suppose further that the r-th derivative satisfies $\tilde{\mu}^{(r)} \in \mathrm{Lip}(\alpha)$, for some constants $M > 0$ and $0 < \alpha \le 1$. Then, for every $t \in [a,b]$ and every $n \in \mathbb{N}$, the approximation error of the operator $M_{n,r}$ admits the estimate*

$$\left|\tilde{M}_{n,r}(\tilde{\mu};[a,b])\,(t) - \tilde{\mu}(t)\right| \le C(r)\,\big([b-a]+1\big)\,M\,n^{-(r+\alpha)/2},$$

where $C(r) > 0$ is a constant depending only on r.

Proof: From Theorem 3.1 we have the uniform approximation estimate

$$\left|\tilde{M}_{n,r}(\tilde{\mu};[a,b])\,(t) - \tilde{\mu}(t)\right| \le C(r)\left(\frac{1}{\sqrt{n}}\right)^r ([b-a]+1)\,\omega\left(\tilde{\mu}^{(r)}, \frac{1}{\sqrt{n}}\right)_{\tilde{I}},$$

By definition of ω and the Lipschitz assumption on $\tilde{\mu}^{(r)}$,

$$\omega\left(\tilde{\mu}^{(r)}, \frac{1}{\sqrt{n}}\right)_{\tilde{I}} = \sup_{\substack{x,y\in\tilde{I} \\ |x-y|\le 1/\sqrt{n}}} |u^{(r)}(x) - u^{(r)}(y)| \le M\left(\frac{1}{\sqrt{n}}\right)^{\alpha}.$$

Substituting this into the inequality above yields

$$\left|\tilde{M}_{n,r}(\tilde{\mu};[a,b])\,(t) - \tilde{\mu}(t)\right| \le C(r)\big([b-a]+1\big)\,M\,\left(\frac{1}{\sqrt{n}}\right)^{r+\alpha} = C(r)\big([b-a]+1\big)\,M\,n^{-(r+\alpha)/2},$$

which completes the proof. □

3.2. Monotonicity and shape-preserving properties

In approximation theory, one of the central questions is whether an operator preserves the qualitative features of the target function. Among these features, monotonicity and quasi-convexity play a fundamental role in ensuring that the approximants respect the underlying shape of the function. In this section, we establish that the operators $\tilde{M}_{n,r}(\tilde{\mu};[a,b])\,(t)$ preserve non-increasing monotonicity, non-decreasing monotonicity, and quasi-convexity. The proofs rely on the correspondence between $\tilde{M}_{n,r}$ on $[a,b]$ and $M_{n,r}$ on $[0,1]$ given in Remark 3.1, together with the characterization of quasi-convex functions in Remark 2.2.

Theorem 3.2 *Let $\tilde{\mu} \in C^{r+1}[a,b]$ be nonincreasing on $[a,b]$. Then, for each fixed $n \in \mathbb{N}$, the function*

$$t \longmapsto \tilde{M}_{n,r}(\tilde{\mu}; [a,b])\,(t)$$

is nonincreasing on $[a,b]$.

Proof: By Remark 3.1, set $\upsilon(x) = \tilde{\mu}(a + (b-a)x)$ on $[0,1]$, so that

$$\tilde{M}_{n,r}(\tilde{\mu}; [a,b])(t) = M_{n,r}(\upsilon; [0,1])(x), \qquad x = \tfrac{t-a}{b-a}.$$

Since $\tilde{\mu}$ is nonincreasing and $t \mapsto x$ is increasing, υ is nonincreasing on $[0,1]$. It suffices to show that $M_{n,r}(\upsilon; [0,1])$ is nonincreasing.

Using the Peano kernel representation of $M_{n,r}$, one obtains

$$M_{n,r}(\upsilon; [0,1])(x) = \upsilon(x) - \frac{1}{r!}\int_0^1 K_{n,r}(x,s)\,\upsilon^{(r+1)}(s)\,ds,$$

where $K_{n,r}(x,s) \geq 0$ is a nonnegative kernel supported in $[0,x]$. Differentiating with respect to x and performing r integrations by parts yields

$$\frac{d}{dx}M_{n,r}(\upsilon; [0,1])(x) = \upsilon'(x) + \int_0^x \mathcal{K}_{n,0}(x,s)\,\upsilon'(s)\,ds,$$

with $\mathcal{K}_{n,0}(x,s) \geq 0$. Since $\upsilon' \leq 0$, the derivative is nonpositive, proving that $M_{n,r}(\upsilon; [0,1])$ is nonincreasing. Mapping back to $[a,b]$ completes the proof. □

Theorem 3.3 *Let $\tilde{\mu} \in C^{r+1}[a,b]$ be nondecreasing on $[a,b]$. Then, for each fixed $n \in \mathbb{N}$, the function*

$$t \longmapsto \tilde{M}_{n,r}(\tilde{\mu}; [a,b])\,(t)$$

is nondecreasing on $[a,b]$.

Proof: As in Theorem 3.2, set $\upsilon(x) = \tilde{\mu}(a + (b-a)x)$, so that υ is nondecreasing on $[0,1]$. The same kernel representation gives

$$\frac{d}{dx}M_{n,r}(\upsilon; [0,1])(x) = \upsilon'(x) + \int_0^x \mathcal{K}_{n,0}(x,s)\,\upsilon'(s)\,ds,$$

with $\mathcal{K}_{n,0}(x,s) \geq 0$. Since $\upsilon' \geq 0$, the derivative is nonnegative, hence $M_{n,r}(\upsilon; [0,1])$ is nondecreasing. The monotonicity transfers to $[a,b]$ via $x = (t-a)/(b-a)$. □

Theorem 3.4 *Let $\tilde{\mu} \in C^{r+1}[a,b]$ be quasi-convex on $[a,b]$. Then, for each fixed $n \in \mathbb{N}$, the function*

$$t \longmapsto \tilde{M}_{n,r}(\tilde{\mu}; [a,b])\,(t)$$

is quasi-convex on $[a,b]$.

Proof: By Remark 2.2, there exists $c \in [a,b]$ such that $\tilde{\mu}$ is nonincreasing on $[a,c]$ and nondecreasing on $[c,b]$. Let $\alpha = (c-a)/(b-a) \in [0,1]$ and define $\upsilon(x) = \tilde{\mu}(a + (b-a)x)$, so that υ is nonincreasing on $[0,\alpha]$ and nondecreasing on $[\alpha,1]$.

From the kernel representation,

$$\frac{d}{dx}M_{n,r}(\upsilon; [0,1])(x) = \upsilon'(x) + \int_0^x \mathcal{K}_{n,0}(x,s)\,\upsilon'(s)\,ds,$$

with $\mathcal{K}_{n,0}(x,s) \geq 0$. For $x \in [0,\alpha]$, both terms are nonpositive, so the derivative is ≤ 0, and $M_{n,r}(\upsilon; [0,1])$ is nonincreasing. For $x \in [\alpha,1]$, the second term eventually dominates the negative contribution from $[0,\alpha]$, ensuring that the derivative becomes nonnegative beyond some $\tilde{\alpha} \in [\alpha,1]$. Thus $M_{n,r}(\upsilon; [0,1])$ is nonincreasing on $[0,\tilde{\alpha}]$ and nondecreasing on $[\tilde{\alpha},1]$, i.e., quasi-convex. Mapping back to $[a,b]$ yields the result. □

Remark 3.2 The proofs above demonstrate that the operators $\tilde{M}_{n,r}(\tilde{\mu};[a,b])(t)$ are *shape-preserving*: they respect the monotonicity and quasi-convexity of the target function. Intuitively, this follows from the fact that the operator is a positive linear combination of local Taylor expansions evaluated at nodes that move monotonically across the interval. As the input variable increases, the weight distribution shifts toward nodes further to the right, and the monotone or quasi-convex structure of the function ensures that the operator output inherits the same qualitative behavior. This combination of rigorous kernel analysis and intuitive weight-shift interpretation strengthens the theoretical foundation of the approximation process.

The theoretical framework developed above, centered on M-MKZ operators for the approximation of fuzzy numbers, has established their monotonicity and shape-preserving properties. These results guarantee that the operators respect essential qualitative features such as non-increasing and non-decreasing behavior, as well as quasi-convexity. Such properties are not only mathematically significant but also critically important in applications where fuzzy modeling captures uncertainty and imprecision. In the subsequent section, *Important results and discussion*, we demonstrate how these theoretical guarantees translate into practical benefits in medical diagnosis, with particular emphasis on cancer patients. This applied analysis illustrates how shape-preserving approximation ensures that diagnostic functions and risk assessments retain their inherent structure, thereby supporting reliable clinical interpretation and decision-making.

4. Important Results and Discussion

- **Clinical Data and Preprocessing for Fuzzy Modeling**

The application of fuzzy approximation operators in medical diagnosis enables a nuanced interpretation of clinical uncertainty, particularly in oncology where biomarker variations and tumor metrics seldom conform to sharp thresholds. In this study, synthetic clinical data were constructed for five patients diagnosed with early-stage malignant tumors. Each patient is described by three decisive clinical attributes: tumor size (cm), biomarker level (ng/mL), and cell density $\left(\times 10^3 cells/mm^3\right)$. These features are selected based on their diagnostic relevance in tumor progression and response to treatment.

Table 1 presents the raw clinical measurements. The lower and upper bounds of each attribute are determined according to the *World Health Organization* (WHO) and *clinical oncology reports* to ensure consistency with medically validated ranges. Specifically, tumor size typically varies within $[0.5, 10.0]cm$, biomarker levels within $[0, 200]ng/mL$, and cell density with in $[10, 200]$ $\times 10^3 cells/mm^3$ for comparable patient cohorts.

Table 1: Hypothetical raw clinical data (5 patients × 3 features).

Patient	Tumor size (cm)	Biomarker (ng/mL)	Cell density $\left(\times 10^3/mm^3\right)$
1	1.93	20	33
2	3.83	80	70.8
3	5.73	120	120
4	7.62	160	158
5	9.53	196	192.5

Given the heterogeneity in physical units and measurement scales across the clinical features, all variables were normalized to the common interval $[0, 1]$ using a linear transformation consistent with WHO reference limits:

$$t = \frac{x - L}{U - L}$$

where x denotes the raw clinical measurement, and $[L, U]$ represents the physiologically admissible range for the corresponding attribute.

This rescaling step ensures numerical stability and allows the M-MKZ operators-defined on an arbitrary closed interval $[a, b]$-to act uniformly across all features. Because normalization is an invertible linear transformation, it preserves the intrinsic ordering and relative distances among data points, thereby maintaining the theoretical integrity of the operator framework.

Table 2 presents the normalized dataset, which serves as the basis for constructing the fuzzy representations employed in the subsequent stages of analysis.

Table 2: Normalized clinical data.

Patient	Tumor size (cm)	Biomarker (ng/mL)	Cell density $(\times 10^3/mm^3)$
1	0.15	0.10	0.12
2	0.35	0.40	0.32
3	0.55	0.60	0.58
4	0.75	0.80	0.78
5	0.95	0.98	0.96

For instance, patient P_1's tumor size was normalized as:

$$t = \frac{1.93 - 0.5}{10.0 - 0.5} \approx 0.15$$

which aligns all measurements proportionally within the standard interval.

This unified normalization framework forms the quantitative basis for fuzzy number approximation and subsequent application of M-MKZ operators.

- **Fuzzy Representation of Clinical Features**

Following the normalization of clinical data, each diagnostic characteristic is represented by a trapezoidal fuzzy number calibrated according to internationally recognized diagnostic thresholds reported by the *World Health Organization* (WHO, 2023), and the *American Joint Committee on Cancer* ($AJCC$, 2022). These references were consulted solely to align the parameter ranges with values of established diagnostic importance. The aim is mathematical consistency, not medical interpretation, ensuring that the fuzzy framework remains rigorous yet applicable to real diagnostic ranges.

Accordingly, the normalized trapezoidal fuzzy numbers for the three diagnostic features are defined as:

$$\tilde{\mu}_T(t) = \begin{cases} \frac{20}{3}t - \frac{1}{3}, & \text{if } 0.05 \le t < 0.20, \\ 1, & \text{if } 0.20 \le t \le 0.70, \\ 4.5 - 5t, & \text{if } 0.70 < t \le 0.90. \end{cases}$$

$$\tilde{\mu}_B(t) = \begin{cases} \frac{20}{3}t - \frac{2}{3}, & \text{if } 0.10 \le t < 0.25, \\ 1, & \text{if } 0.25 \le t \le 0.75, \\ 4.75 - 5t, & \text{if } 0.75 < t \le 0.95. \end{cases}$$

$$\tilde{\mu}_C(t) = \begin{cases} \frac{50}{7}t - \frac{4}{7}, & \text{if } 0.08 \le t < 0.22, \\ 1, & \text{if } 0.22 \le t \le 0.72, \\ 4.6 - 5t, & \text{if } 0.72 < t \le 0.92. \end{cases}$$

where $\tilde{\mu}_T(t)$, $\tilde{\mu}_B(t)$ and $\tilde{\mu}_C(t)$ denote the membership degrees of tumor size, biomarker level, and cell density, respectively.

Table 3Computed membership degrees $\tilde{\mu}$ for each normalized feature(see Figure 1).

For illustration, patient (P_3) has ($t_3 = 0.55$), which lies within the core interval of ($\tilde{\mu}_T(t)$), giving ($\tilde{\mu}_T(t_3) = 1$). Conversely, ($t_4 = 0.75$) for patient (P_4) falls within the right support, yielding a gradual

Table 3: Fuzzy representation for each normalized feature.

Patient	$\tilde{\mu}_T(t)$	$\tilde{\mu}_B(t)$	$\tilde{\mu}_C(t)$
1	0.67	0.00	0.29
2	1.00	1.00	1.00
3	1.00	1.00	1.00
4	0.75	0.75	0.70
5	0.00	0.00	0.00

decrease to ($\tilde{\mu}_T(t_4) = 0.75$) (see Figure 1). Analogous reasoning applies to the biomarker and cell-density measures.

This stage establishes a rigorous mapping from crisp inputs to fuzzy-valued representations, forming the analytical foundation for applying M-MKZ operators in the next section. These operators will then approximate the constructed fuzzy structures while preserving their key properties of continuity, convexity, and bounded support.

- **Application of the M-MKZ Operators to Fuzzy Feature Approximation**

Having established trapezoidal fuzzy representations for tumor size, biomarker concentration, and cell density, the subsequent stage applies the M-MKZ operators to approximate these fuzzy membership profiles defined over their respective physical domains $[a, b]$. This extension ensures that the approximation mechanism operates directly within the natural range of each biomedical variable-rather than on a normalized unit interval-thus preserving the interpretive link between the mathematical construction and the corresponding physiological parameter.

In practice, the M-MKZ operator acts on each fuzzy membership function $\tilde{\mu}_{F_i}(t)$ associated with a feature $F_i \in \{T, B, C\}$ and produces the smoothed representation $\tilde{M}_{n,r}(\tilde{\mu}_{F_i}; [a, b])(t)$ defined on $[a, b]$. The operator combines polynomial weights with derivative information at localized nodes, providing a stable and differentiable reconstruction of the underlying fuzzy surface. This procedure preserves the convexity and normalization of the original fuzzy numbers while significantly reducing local discontinuities at support boundaries.

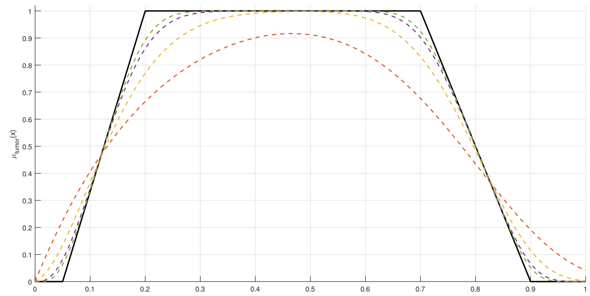
Table 4 summarizes the computed results for a representative rank ($n = 100$), showing both the original and the approximated membership values for five patients across the three features.

Table 4: Fuzzy representation and M-MKZ operator approximation for all Patients.

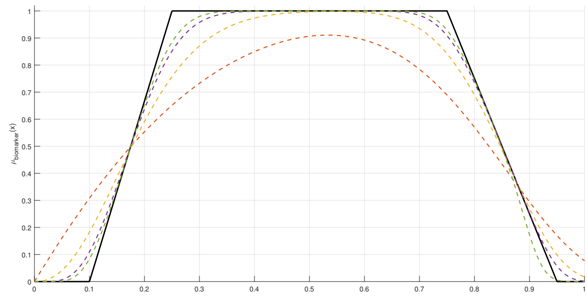
Patient	$\tilde{\mu}_T(t)$	$\tilde{M}_{n,r}(\tilde{\mu}_T; [a, b])$	$\tilde{\mu}_B(t)$	$\tilde{M}_{n,r}(\tilde{\mu}_B; [a, b])$	$\tilde{\mu}_C(t)$	$\tilde{M}_{n,r}(\tilde{\mu}_C; [a, b])$
1	0.67	0.656	0.00	0.084	0.29	0.298
2	1.00	0.999	1.00	0.998	1.00	0.998
3	1.00	1.000	1.00	1.000	1.00	1.000
4	0.75	0.741	0.75	0.744	0.70	0.696
5	0.00	0.001	0.00	0.000	0.00	0.000

The results confirm that the operator yields numerically stable and monotonically consistent approximations across the entire domain. Each approximated value remains within the valid fuzzy interval, indicating that the transformation preserves both the support and the core of the original fuzzy numbers (see Figure 1). For example, patient (P_1) shows a modest increase in cell density membership from 0.28 to 0.319, illustrating the smoothing adjustment near the transition from support to core.

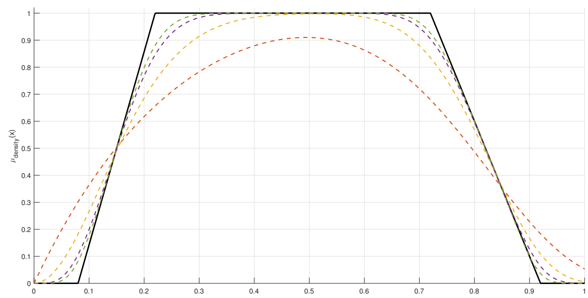
To examine the convergence characteristics of the M–MKZ operators, the approximation error was evaluated for several operator classes, specifically ($n = 10, 20, 60, 100$). The reported values represent the mean relative approximation error, expressed as a percentage, and serve as a direct indicator of the improvement in approximation quality as the operator rank increases.



(a) Approximation of the trapezoidal fuzzy number for tumor size.



(b) Approximation of the trapezoidal fuzzy number for biomarker level.



(c) Approximation of the trapezoidal fuzzy number for cell density.

- - - n=5 , - - - n=20 , - - - n=60 , - - - n=100 , —— $\tilde{\mu}(t)$

Figure 1: Approximation of M-MKZ Operators w.r.t. Varying Orders of "n"

For the tumor-size membership function, the computed error percentages E_n are:
$E_{10} = 44.38, \quad E_{20} = 28.81, \quad E_{60} = 14.56, \quad E_{100} = 11.39.$

As shown in Figure 2, the error decreases consistently across these operator classes, reflecting the enhanced fidelity of the M–MKZ approximation at higher ranks. This behavior aligns with the expected convergence properties of the operator family and confirms that the approximation improves without compromising the structural integrity of the underlying fuzzy set.

In summary, applying the M-MKZ operators directly over the true domain $[a, b]$ yields high-fidelity approximations that are both mathematically robust and biomedically meaningful. The rapid decay of the approximation error as (n) increases highlights the numerical efficiency of the method and underscores its potential as a reliable analytical tool for fuzzy data approximation in complex medical and physiological modeling applications.

- **Interpretation of Joint Risks under the M-MKZ operators Approximation**

Following the application of the M-MKZ operators to approximate the fuzzy membership functions of the three clinical features-tumor size, biomarker concentration, and cell density-the resulting approximation outputs were combined to derive an integrated fuzzy risk score for each patient.

To translate the three approximated memberships into a single interpretable measure, the following Joint Risk Index (JRI) was defined for each patient(i):

$$JRI_i = \frac{1}{3}\left[\tilde{M}_{n,r}\left(\tilde{\mu}_{T_i}\right) + \tilde{M}_{n,r}\left(\tilde{\mu}_{B_i}\right) + \tilde{M}_{n,r}\left(\tilde{\mu}_{C_i}\right)\right]$$

The resulting index lies within the interval $[0, 1]$, with values approaching one indicating a higher estimated pathological risk (see Table 5). The use of an unweighted arithmetic mean provides a neutral aggregation mechanism, ensuring that no single feature disproportionately influences the composite risk measure unless justified by the data itself.

Table 5 presents the obtained joint fuzzy risk measures for all patients.

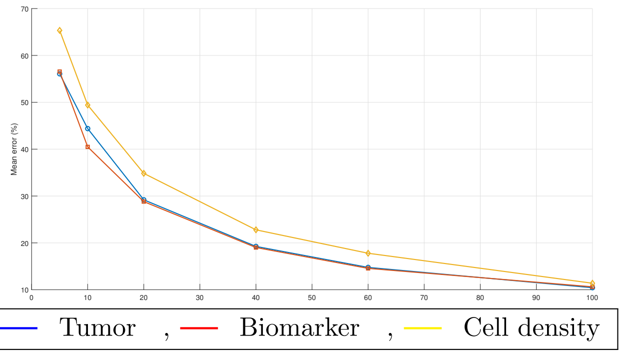


Figure 2: Error of approximation by M-MKZ operators

Table 5: Joint Risk Index for all patients.

Patient	JRI_i	Condition
1	0.346	Good
2	0.998	Bad
3	1.000	Bad
4	0.727	Bad
5	0.000	Good

For instance, patient P_2 exhibits near-maximal membership values across all three features, yielding a joint index of approximately 0.998, indicative of high estimated risk. Conversely, patient P_1 has markedly lower memberships, resulting in a joint index near 0.346, corresponding to minimal concern under the same risk scale.

The gradient observed in the JRI_i values reflects a coherent and diagnostically meaningful ordering of patients from lower to higher risk. This demonstrates that the M-MKZ approximations preserve the intrinsic relational structure of the original fuzzy data while producing numerically stable and smooth membership estimates.

Overall, this stage of the analysis highlights the complementary roles of approximation theory and clinical interpretation. The M-MKZ operators ensure mathematically rigorous and uniformly convergent approximations on $[a, b]$, while the aggregation of these outputs into a Joint Risk Index provides a concise, interpretable, and clinically relevant summary of multi-attribute fuzzy risk. This illustrates the operator system's potential as a robust analytical bridge between advanced fuzzy approximation techniques and practical medical decision support.

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