Bol. Soc. Paran. Mat.
©SPM – E-ISSN-2175-1188
SPM: www.spm.uem.br/bspm

(3s.) **v. 2026 (44) 7** : 1–16.
ISSN-0037-8712
doi:10.5269/bspm.80644

Analyzing the Limit Set of Rough Ideal $\lambda\gamma$-Statistical Convergence of Order ϱ in Lattice-Valued Fuzzy Normed Spaces

Ömer Kişi, Mehmet Gürdal and Selim Çetin

ABSTRACT: This study introduces the framework of rough $\mathcal{I}$-$\lambda\gamma$-statistical convergence of order ϱ within the setting of $\mathcal{L}$-fuzzy normed spaces (lattice-valued fuzzy normed spaces). This generalizes existing convergence notions by integrating ideal convergence ($\mathcal{I}$), generalized sequence transformations ($\lambda\gamma$), an arbitrary order (ϱ), and the concept of roughness (r). A primary focus is the characterization of the resulting rough limit set. We rigorously establish that, contrary to classical convergence, the limit is inherently a set. Furthermore, we prove that this limit set possesses key structural properties, specifically closure and convexity, under the topology induced by the $\mathcal{L}$-fuzzy norm. Finally, we define the corresponding notion of $\mathcal{I}$-$\lambda\gamma$-statistical cluster points of order ϱ and elucidate the relationship between this set of cluster points and the rough limit set.

Key Words: Lattice-valued fuzzy normed space ($\mathcal{L}$-fuzzy norm), rough convergence, ideal convergence, $\lambda\gamma$-statistical convergence, order of convergence (ϱ), limit set analysis, convexity.

Contents

1. Introduction

Classical convergence, while fundamental, often proves too restrictive for analyzing sequences exhibiting irregularities. Statistical convergence, introduced by Fast [9], provided a crucial generalization by utilizing the concept of asymptotic density, effectively ignoring sets of indices with density zero. This concept initiated a significant expansion in summability theory. A pivotal development in this trajectory was the introduction of ideal convergence ($\mathcal{I}$-convergence) by Kostyrko et al. [22], which generalized statistical convergence by replacing natural density with an arbitrary admissible ideal $\mathcal{I}$. This axiomatic approach has facilitated the extension of convergence theories across diverse mathematical structures [7,15,33,37,38,5,16,17,24,25,32].

Further generalizations have focused on modifying the manner and rate of convergence. Concepts such as λ-statistical convergence and, more broadly, $\alpha\beta$ (related to the $\lambda\gamma$ framework studied herein) statistical convergence of a specific order ϱ [1], have emerged. These methods offer refined control over the convergence process and have proven valuable in applications such as Korovkin-type approximation theorems [36] and the analysis of discrete operators [6,11,35]. Moreover, the investigation of operators on various function spaces remains a central theme in analysis, as seen in studies on Wiener algebras [14] and Englis algebras [18].

Parallel to these developments, the necessity of handling approximations and inherent errors in numerical analysis led to the concept of rough convergence, formalized by Phu [29]. Rough convergence acknowledges that a sequence may not converge to a single point but rather approach a vicinity defined by a "roughness degree" r. Aytar [4] integrated this notion with statistical convergence, introducing

2020 *Mathematics Subject Classification:* 40A35, 46S40, 54A40.
Submitted December 12, 2025. Published February 21, 2026

rough statistical convergence. This framework, where the limit is inherently a set rather than a point, has been extensively studied and adapted to various sequence spaces [8,20,21,27].

Concurrently, the mathematical framework for modeling uncertainty has evolved significantly since Zadeh's introduction of fuzzy sets [41] and Atanassov's intuitionistic fuzzy sets [3]. This led to the development of fuzzy metric spaces [23,10] and fuzzy normed spaces [31,28]. A critical limitation of classical fuzzy theory is its reliance on the unit interval $[0,1]$ for membership degrees. To address this, Goguen [13] introduced $\mathcal{L}$-fuzzy sets, replacing $[0,1]$ with a complete lattice $\mathcal{L}$. This generalization provides a substantially more flexible structure for handling complex, ordered systems of uncertainty. The fuzzy numbers have applications in computer programming [12], image segmentation [19] and many others.

The integration of these concepts led to the study of $\mathcal{L}$-fuzzy normed spaces, representing an intersection of functional analysis, lattice theory, and fuzzy mathematics. Yapalı [39] initiated the study of statistical convergence within this generalized lattice-valued framework. Subsequent research has rapidly adapted various convergence modes to $\mathcal{L}$-fuzzy normed spaces, including lacunary statistical convergence [40], rough statistical convergence [26], ideal convergence [20], and deferred statistical methods [30,2].

Building upon this extensive background, the present study introduces a highly generalized convergence framework: rough $\mathcal{I}$-$\lambda\gamma$-statistical convergence of order ϱ in $\mathcal{L}$-fuzzy normed spaces. This approach synthesizes multiple generalizations–ideal convergence, generalized sequence transformations ($\lambda\gamma$), arbitrary order (ϱ), and roughness (r)–within the flexible structure of lattice-valued norms. Our primary objective is to analyze the structure of the resulting rough limit set. We rigorously establish that this limit set is both closed and convex with respect to the $\mathcal{L}$-fuzzy norm. Furthermore, we define the corresponding set of $\mathcal{I}^r_{St^{\varrho,\mathcal{L}}_{\lambda\gamma}}$-cluster points and elucidate its relationship with the rough limit set.

2. Preliminaries

This section compiles the essential definitions and foundational concepts necessary for the subsequent analysis. Throughout this paper, $\mathbb{N}$ denotes the set of natural numbers.

2.1. Density, Ideals, and Convergence

We begin by recalling the concepts related to density and ideal convergence.

Definition 2.1 ([9]) *The natural density (or asymptotic density) $d(K)$ of a subset $K \subseteq \mathbb{N}$ is defined as $d(K) = \lim_{n\to\infty} \frac{1}{n}|\{k \le n : k \in K\}|$, provided the limit exists. Here, $|A|$ denotes the cardinality of set A.*

Definition 2.2 ([9]) *A sequence $\varpi = (\varpi_k)$ in a metric space (X,ρ) is statistically convergent to $\varpi_0 \in X$ if, for every $\epsilon > 0$, the set of indices $\{k \in \mathbb{N} : \rho(\varpi_k, \varpi_0) \ge \epsilon\}$ has natural density zero.*

Statistical convergence is generalized using the algebraic structure of ideals.

Definition 2.3 ([22]) *Let U be a non-empty set. A collection of subsets $\mathcal{I} \subset P(U)$ (the power set of U) is an ideal in U if it satisfies the additive property ($A, B \in \mathcal{I} \implies A \cup B \in \mathcal{I}$) and the hereditary property ($A \in \mathcal{I}, B \subset A \implies B \in \mathcal{I}$).*

An ideal $\mathcal{I}$ is non-trivial if $\mathcal{I} \ne P(U)$ (i.e., $U \notin \mathcal{I}$) and admissible if it is non-trivial and contains all finite subsets of U. Throughout this paper, $\mathcal{I}$ denotes a non-trivial admissible ideal in $\mathbb{N}$.

Definition 2.4 ([22]) *Associated with a non-trivial ideal $\mathcal{I}$ is the corresponding filter $\mathcal{F}(\mathcal{I})$, defined as $\mathcal{F}(\mathcal{I}) = \{K \subseteq U : U \setminus K \in \mathcal{I}\}$.*

Definition 2.5 ([22]) *A sequence $\varpi = (\varpi_k)$ in a metric space (X,ρ) is ideal convergent ($\mathcal{I}$-convergent) to $\varpi_0 \in X$ if, for every $\epsilon > 0$, the set $\{k \in \mathbb{N} : \rho(\varpi_k, \varpi_0) \ge \epsilon\}$ belongs to $\mathcal{I}$.*

2.2. Rough Convergence

Rough convergence addresses convergence up to a certain degree of approximation.

Definition 2.6 ([29]) *Let $(U, \|.\|)$ be a normed linear space and let $r \geq 0$ be a fixed real number (the roughness degree). A sequence $\varpi = (\varpi_k)$ in U is rough convergent (r-convergent) to $\varpi_0 \in U$ if for every $\epsilon > 0$, there exists $k_0 \in \mathbb{N}$ such that $\|\varpi_k - \varpi_0\| < r + \epsilon$ for all $k \geq k_0$. This is denoted by $r - \lim_k \varpi = \varpi_0$.*

Definition 2.7 ([27]) *A sequence $\varpi = (\varpi_k)$ in a normed space U is rough ideal convergent ($r - \mathcal{I}$-convergent) to ϖ_0, denoted $r - \mathcal{I} - \lim_k \varpi = \varpi_0$, if for any $\epsilon > 0$, the set $\{k \in \mathbb{N} : \|\varpi_k - \varpi_0\| \geq r + \epsilon\}$ belongs to $\mathcal{I}$.*

2.3. Lattice Structures and $\mathcal{L}$-Fuzzy Normed Spaces

We now introduce the framework of lattice-valued fuzzy normed spaces. Let $\mathcal{L} = (L, \preccurlyeq)$ be a complete lattice, with 0_L and 1_L denoting the least and greatest elements, respectively. The strict order relation is denoted by $\prec$ (and $\succ$ for its inverse).

Definition 2.8 ([34]) *An $\mathcal{L}$-fuzzy set on a non-empty set Q is a mapping $X : Q \to L$. The collection of all $\mathcal{L}$-fuzzy sets on Q is denoted by L^Q. Operations such as union ($\cup$) and intersection ($\cap$) are defined pointwise using the lattice join ($\vee$) and meet ($\wedge$) operations.*

Definition 2.9 ([34]) *A function $\Xi : L \times L \to L$ is a t-norm on $\mathcal{L}$ if it is commutative, associative, monotonic in both arguments, and satisfies the boundary condition $\Xi(a, 1_L) = a$ for all $a \in L$.*

Definition 2.10 ([34]) *A function $\mathcal{N} : L \longrightarrow L$ is a negator on $\mathcal{L}$ if it is order-reversing (i.e., $a \preccurlyeq b \implies \mathcal{N}(b) \preccurlyeq \mathcal{N}(a)$) and satisfies $\mathcal{N}(0_L) = 1_L$ and $\mathcal{N}(1_L) = 0_L$. It is involutive if $\mathcal{N}(\mathcal{N}(a)) = a$ for all $a \in L$.*

We now define the main structure utilized in this study.

Definition 2.11 ([34]) *Let T be a real vector space, $\mathcal{L} = (L, \preccurlyeq)$ a complete lattice, and Ξ a continuous t-norm on L. The triplet (T, ρ, Ξ) is called an $\mathcal{L}$-fuzzy normed space ($\mathcal{L}$-FNS) if ρ is an $\mathcal{L}$-fuzzy set on $T \times (0, \infty)$ satisfying the following conditions for all $q, r \in T$ and $\sigma, t > 0$:*

(i) $\rho(q, \sigma) \succ 0_L$.

(ii) $\rho(q, \sigma) = 1_L$ *if and only if* $q = \theta$ *(the zero vector).*

(iii) $\rho(\beta q, \sigma) = \rho\left(q, \frac{\sigma}{|\beta|}\right)$ *for* $\beta \in \mathbb{R} \setminus \{0\}$.

(iv) (Fuzzy triangle inequality) $\rho(q + r, \sigma + t) \succeq \Xi(\rho(q, \sigma), \rho(r, t))$.

(v) The function $f_q : (0, \infty) \to L$ *defined by* $f_q(\sigma) = \rho(q, \sigma)$ *is continuous.*

(vi) $\lim_{\sigma \to \infty} \rho(q, \sigma) = 1_L$ *and* $\lim_{\sigma \to 0} \rho(q, \sigma) = 0_L$ *for* $q \neq \theta$.

Example 2.1 ([26]) *Let $L = [0, 1]$ with the usual order. Let $(T, \|.\|)$ be a classical normed space. Define $\Xi(a, b) = ab$ (product t-norm). Define $\rho(q, \sigma) = \frac{\sigma}{\sigma + \|q\|}$. Then (T, ρ, Ξ) is an $\mathcal{L}$-FNS.*

Convergence in $\mathcal{L}$-FNS is defined using a negator to express the notion of closeness.

Definition 2.12 ([34]) *Let (T, ρ, Ξ) be an $\mathcal{L}$-FNS with a negator $\mathcal{N}$. A sequence $\varpi = (\varpi_k)$ in T converges to $\varpi_0 \in T$ if for every $t \in L \setminus \{0_L\}$ and $\sigma > 0$, there exists $k_0 \in \mathbb{N}$ such that $\rho(\varpi_k - \varpi_0; \sigma) \succ \mathcal{N}(t)$ for all $k \geq k_0$.*

We conclude this section by recalling the definitions of statistical, ideal, and rough convergence adapted to the $\mathcal{L}$-FNS setting. (Note: We use ϖ_k notation for consistency).

Definition 2.13 ([39]) *A sequence $\varpi=(\varpi_k)$ in an $\mathcal{L}$-FNS (T,ρ,Ξ) is statistically convergent to ϖ_0 if for any $t\in L\setminus\{0_L\}$ and $\sigma>0$, $d(\{k\in\mathbb{N}:\rho(\varpi_k-\varpi_0;\sigma)\nsucc\mathcal{N}(t)\})=0$. This is denoted by $St_{\mathcal{L}}-\lim_k\varpi=\varpi_0$.*

Definition 2.14 ([20]) *A sequence $\varpi=(\varpi_k)$ in an $\mathcal{L}$-FNS (T,ρ,Ξ) is ideal convergent to ϖ_0 if for any $t\in L\setminus\{0_L\}$ and $\sigma>0$, $\{k\in\mathbb{N}:\rho(\varpi_k-\varpi_0;\sigma)\nsucc\mathcal{N}(t)\}\in\mathcal{I}$. This is denoted by $\mathcal{I}_{\mathcal{L}}-\lim_k\varpi=\varpi_0$.*

Definition 2.15 ([26]) *Let $r\geq 0$. A sequence $\varpi=(\varpi_k)$ in an $\mathcal{L}$-FNS (T,ρ,Ξ) is rough convergent to ϖ_0 if for any $t\in L\setminus\{0_L\}$ and $\sigma>0$, there exists k_0 such that $\rho(\varpi_k-\varpi_0;r+\sigma)\succ\mathcal{N}(t)$ for all $k\geq k_0$.*

Definition 2.16 ([26]) *A sequence $\varpi=(\varpi_k)$ in an $\mathcal{L}$-FNS (T,ρ,Ξ) is rough statistically convergent to ϖ_0 if for any $t\in L\setminus\{0_L\}$ and $\sigma>0$, $d(\{k\in\mathbb{N}:\rho(\varpi_k-\varpi_0;r+\sigma)\nsucc\mathcal{N}(t)\})=0$. This is denoted by $r-St_{\mathcal{L}}-\lim_k\varpi=\varpi_0$.*

3. Main results

This section is dedicated to the introduction and analysis of rough $\mathcal{I}$-$\lambda\gamma$-statistical convergence of order ϱ within $\mathcal{L}$-fuzzy normed spaces. We explore its core characteristics and the topological structure of the associated limit set.

We fix the setting for this section as follows: (T,ρ,Ξ) denotes an $\mathcal{L}$-FNS equipped with a negator $\mathcal{N}$. We consider sequences $\lambda=(\lambda_u)$ and $\gamma=(\gamma_u)$ of non-negative integers satisfying $\lambda_u\leq\gamma_u$ for all $u\in\mathbb{N}$ and $\lim_{u\to\infty}(\gamma_u-\lambda_u)=\infty$. The interval of indices is denoted by $P_u^{\lambda,\gamma}=[\lambda_u,\gamma_u]$, and the order of convergence is given by $\varrho\in(0,1]$.

Definition 3.1 *We say that a sequence $\varpi=(\varpi_\varsigma)\in\mathrm{T}$ is $\lambda\gamma$-statistically convergent of order ϱ to ϖ_0 with respect to the fuzzy norm ρ, provided that for every $t\in\mathcal{L}\setminus\{0_{\mathcal{L}}\}$ and $\sigma\in\mathbb{R}^+$*

$$\lim_{u\to\infty}\frac{1}{(\gamma_u+1-\lambda_u)^{\varrho}}\left|\left\{\varsigma\in P_u^{\lambda,\gamma}:\rho(\varpi_\varsigma-\varpi_0,\sigma)\nsucc\mathcal{N}(t)\right\}\right|=0 \tag{3.1}$$

is satisfied. We denote this convergence by $St_{\lambda\gamma}^{\varrho,\mathcal{L}}-\lim_\varsigma\varpi_\varsigma=\varpi_0$.

It is evident from Definition 3.1 that convergence of order ϱ w.r.t. the fuzzy norm ρ implies $\lambda\gamma$-statistical convergence of order ϱ w.r.t. the same norm.

Remark 3.1 *If we set $\varrho=1$ in (3.1), the sequence (ϖ_ς) is referred to as $\lambda\gamma$-statistically convergent to ϖ_0 w.r.t. ρ. Moreover, by choosing $\lambda_u=1$ and $\gamma_u=u$, Definition 3.1 reduces to the standard statistical convergence of (ϖ_ς) w.r.t. ρ.*

Definition 3.2 *Let (T,ρ,Ξ) be an $\mathcal{L}$-FNS, and let $\varpi=(\varpi_\varsigma)\in\mathrm{T}$. Given a roughness degree $r\geq 0$, the sequence (ϖ_ς) is said to exhibit rough $\mathcal{I}$-$\lambda\gamma$-statistical convergence of order ϱ to ϖ_0 w.r.t. ρ, if for any $t\in\mathcal{L}\setminus\{0_{\mathcal{L}}\}$, $\xi>0$, and $\sigma\in\mathbb{R}^+$, the set*

$$\left\{u\in\mathbb{N}:\frac{1}{(\gamma_u+1-\lambda_u)^{\varrho}}\left|\left\{\varsigma\in P_u^{\lambda,\gamma}:\rho(\varpi_\varsigma-\varpi_0,r+\sigma)\nsucc\mathcal{N}(t)\right\}\right|\geq\xi\right\} \tag{3.2}$$

belongs to the ideal $\mathcal{I}$. In this case, we write $\mathcal{I}^r_{St_{\lambda\gamma}^{\varrho,\mathcal{L}}}-\lim_\varsigma\varpi_\varsigma=\varpi_0$.

Following Definition 3.2, it is clear that $\mathcal{I}$-statistical convergence of order ϱ implies rough $\mathcal{I}$-$\lambda\gamma$-statistical convergence of order ϱ w.r.t. the fuzzy norm ρ.

Remark 3.2 *Let (T,ρ,Ξ) be an $\mathcal{L}$-FNS and $\varpi=(\varpi_\varsigma)\in\mathrm{T}$.*
(1) If $\mathcal{I}=\mathcal{I}_f$ (the ideal of finite sets) in (3.2), (ϖ_ς) is called rough-$\lambda\gamma$-statistically convergent of order ϱ to ϖ_0 w.r.t. ρ.
(2) If $r=0$ in (3.2), (ϖ_ς) is termed $\mathcal{I}$-$\lambda\gamma$-statistically convergent of order ϱ to ϖ_0 w.r.t. ρ, denoted by $\mathcal{I}_{St_{\lambda\gamma}^{\varrho,\mathcal{L}}}-\lim_\varsigma\varpi_\varsigma=\varpi_0$.

Notation. Consider (T, ρ, Ξ) as an $\mathcal{L}$-FNS. Let (ϖ_ς) be a sequence in T and $r \geq 0$.
(a) The limit $\mathcal{I}^r_{St^{\varrho,\mathcal{L}}_{\lambda\gamma}} - \lim_\varsigma \varpi_\varsigma$ is generally not unique. We define the set of all rough $\mathcal{I}$-$\lambda\gamma$-statistical limits of order ϱ as:

$$\mathcal{I}_{St^{\varrho,\mathcal{L}}_{\lambda\gamma}} - \mathrm{LIM}^r \varpi_\varsigma = \left\{ \varpi_0 \in T : \mathcal{I}^r_{St^{\varrho,\mathcal{L}}_{\lambda\gamma}} - \lim_\varsigma \varpi_\varsigma = \varpi_0 \right\}.$$

The sequence (ϖ_ς) is deemed rough $\mathcal{I}$-$\lambda\gamma$-statistically convergent of order ϱ if this limit set is non-empty for some $r \geq 0$.
According to Definition 3.2, we observe the following monotonicity properties:
(1) If $0 \leq r_1 \leq r_2$ for a fixed $\varrho \in (0, 1]$, then

$$\mathcal{I}^{r_1}_{St^{\varrho,\mathcal{L}}_{\lambda\gamma}} - \lim_\varsigma \varpi_\varsigma \subseteq \mathcal{I}^{r_2}_{St^{\varrho,\mathcal{L}}_{\lambda\gamma}} - \lim_\varsigma \varpi_\varsigma.$$

(2) If $0 < \varrho \leq \tau \leq 1$ for a fixed $r \geq 0$, then $\mathcal{I}^r_{St^{\varrho}_{(\lambda,\gamma)_{\mathcal{L}}}} - \lim_\varsigma \varpi_\varsigma = \varpi_0$ implies

$$\mathcal{I}^r_{St^{\tau,\mathcal{L}}_{\lambda\gamma}} - \lim_\varsigma \varpi_\varsigma = \varpi_0.$$

Example 3.1 *Consider the normed space* $(\mathbb{R}, \|.\|)$. *Let* Ξ *be the continuous t-norm defined by* $\Xi(\varsigma_1, \varsigma_2) = \varsigma_1\varsigma_2$ *for* $\varsigma_1, \varsigma_2 \in \mathcal{L}$. *Furthermore, let* ρ *be the* $\mathcal{L}$*-fuzzy set on* $\mathbb{R} \times (0, \infty)$ *given by* $\rho(q, \sigma) = \frac{\sigma}{\sigma + |q|}$ *for all* $\sigma > 0$ *and* $q \in \mathbb{R}$. *The triple* (T, ρ, Ξ) *constitutes an* $\mathcal{L}$*-FNS. Now, define the sequences* (ϖ_ς) *and* (φ_ς) *as follows:*

$$\varpi_\varsigma = \begin{cases} \varsigma, & \text{if } \varsigma = j^2 \\ 0, & \text{otherwise,} \end{cases}$$

and

$$\varphi_\varsigma = \begin{cases} 0, & \text{if } \varsigma = j^2 \\ -1, & \text{if } 2^{2j} + 1 \leq \varsigma \leq 2^{2j} + 2^j \\ 1, & \text{otherwise.} \end{cases}$$

Let $\lambda_u = u^2 + 1$ *and* $\gamma_u = u^2 + u$. *For any* $\alpha \in (0, 1)$, $\varrho \in (0, 1]$ *and* $\sigma > 0$,

$$\begin{aligned} &\tfrac{1}{(\gamma_u + 1 - \lambda_u)^\varrho} \left| \left\{ \varsigma \in P^{\lambda,\gamma}_u : \rho(\varpi_\varsigma, \sigma) \not\succ \mathcal{N}(\alpha) \right\} \right| \\ &= \tfrac{1}{u^\varrho} \left| \left\{ \varsigma : \varsigma \in \left[u^2 + 1, u^2 + u\right]; \|\varpi_\varsigma\| \geq \tfrac{\sigma\alpha}{1-\alpha} > 0 \right\} \right| = 0. \end{aligned}$$

Hence, $St^{\varrho,\mathcal{L}}_{\lambda\gamma} - \lim_\varsigma \varpi_\varsigma = 0$. *For* $\varpi_0 \in \mathbb{R}$ *and* $r \geq 0$, *consider*

$$\begin{aligned} &\tfrac{1}{(\gamma_u + 1 - \lambda_u)^\varrho} \left| \left\{ \varsigma \in P^{\lambda,\gamma}_u : \rho(\varphi_\varsigma - \varpi_0, \sigma + r) \not\succ \mathcal{N}(\alpha) \right\} \right| \\ &= \tfrac{1}{u^\varrho} \left| \left\{ \varsigma : \varsigma \in \left[u^2 + 1, u^2 + u\right]; \|\varphi_\varsigma - \varpi_0\| \geq \tfrac{\alpha}{1-\alpha}(\sigma + r) \right\} \right| = 0. \end{aligned} \tag{3.3}$$

Setting $r_1 = \frac{\alpha r}{1-\alpha} \geq 0$ *and* $0 < \varrho = \frac{\sigma\alpha}{1-\alpha}$ *infinitesimally small, for* $\varpi_0 = 1$, *the righthand side of (3.3) reduces to*

$$\frac{1}{u^\varrho} \left| \left\{ \varsigma : \varsigma \in \left[u^2 + 1, u^2 + u\right]; \|\varphi_\varsigma - 1\| \geq r_1 + \varrho \right\} \right| = z_u.$$

Then

$$\begin{aligned} z_u &= \tfrac{1}{u^\varrho} \left| \left\{ \varsigma : \varsigma \in \left[u^2 + 1, u^2 + u\right]; \varphi_\varsigma \geq 1 + r_1 + \varrho \right\} \right| \\ &= \begin{cases} \frac{u}{u^\varrho}, & \text{if } u = 2^j,\ j = 1, 2, \ldots \\ 0, & \text{otherwise.} \end{cases} \end{aligned}$$

Take $\mathcal{I} = \mathcal{I}_d = \{W \subset \mathbb{N} : d(W) = 0\}$. *For any arbitrarily small* $t \in \mathcal{L} \setminus \{0_{\mathcal{L}}\}$,

$$\{u \in \mathbb{N} : z_u \succ \mathcal{N}(t)\} = \left\{ u \in \mathbb{N} : u = 2^j,\ j = 1, 2, 3, \ldots \right\} \in \mathcal{I}_d.$$

For $\varpi_0 = -1$, *we have*

$$\begin{aligned} q_u &= \tfrac{1}{u^\varrho} \left| \left\{ \varsigma : \varsigma \in \left[u^2 + 1, u^2 + u\right]; \varphi_\varsigma \geq -1 + r_1 + \varrho \right\} \right| \\ &= \begin{cases} \frac{u}{u^\varrho}, & \text{if } u \neq 2^j,\ j = 1, 2, \ldots \\ 0, & \text{otherwise.} \end{cases} \end{aligned}$$

$$\{u \in \mathbb{N} : q_u \succ \mathcal{N}(t)\} = \{u \in \mathbb{N} : u \neq 2^j,\ j = 1, 2, 3, \dots\} \notin \mathcal{I}_d.$$

Similarly for $\varpi_0 = 0$,

$$\left\{ u \in \mathbb{N} : \frac{1}{u^{\varrho}} \left| \left\{ \varsigma : \varsigma \in \left[u^2+1, u^2+u\right] ; \varphi_\varsigma > r_1 + \varrho \right\} \right| > \xi \right\} = \mathbb{N} \notin \mathcal{I}_d.$$

So,

$$\mathcal{I}^r_{St^{\varrho,\mathcal{L}}_{\lambda\gamma}} - \lim \varphi_\varsigma = \begin{cases} [1-r, 1+r], & \text{if } r \geq 0 \\ \emptyset, & \text{otherwise.} \end{cases}$$

It is important to note that neither sequence (ϖ_ς) *nor* (φ_ς) *converges in the classical sense w.r.t. the fuzzy norm* ρ.

In standard analysis, every subsequence of a convergent sequence in an $\mathcal{L}$-FNS also converges w.r.t. ρ. However, this property does not generally hold for rough $\mathcal{I}$-$\lambda\gamma$-statistical convergence of order ϱ. Specifically, the existence of a rough limit, $\mathcal{I}^r_{St^{\varrho,\mathcal{L}}_{\lambda\gamma}} - \lim_\varsigma \varpi_\varsigma \neq \emptyset$, does not ensure that every subsequence possesses a corresponding rough limit.

To illustrate, consider the sequence (ϖ_ς) from Example 3.1. We have $\mathcal{I}^r_{St^{\varrho,\mathcal{L}}_{\lambda\gamma}} - \lim_\varsigma \varpi_\varsigma = [-r, r]$ for any nontrivial admissible ideal $\mathcal{I}$ and $r \geq 0$. However, the subsequence $(q_\varsigma) = (\varsigma)$ satisfies $\mathcal{I}^r_{St^{\varrho,\mathcal{L}}_{\lambda\gamma}} - \lim_\varsigma q_\varsigma = \emptyset$ for any $r \geq 0$.

Lemma 3.1 *If* (ϖ_ς) *is a sequence in* (T, ρ, Ξ) *such that* $St^{\varrho,\mathcal{L}}_{\lambda\gamma} - \lim_\varsigma \varpi_\varsigma = \varpi_0$, *then for any* $r \geq 0$, *we have* $\mathcal{I}^r_{St^{\varrho,\mathcal{L}}_{\lambda\gamma}} - \lim_\varsigma \varpi_\varsigma = \varpi_0$.

Proof: Let $r = 0$ and $\mathcal{I} = \mathcal{I}_f$. Under these conditions, Equation (3.2) reduces to Equation (3.1). Therefore, the result immediately follows. □

The converse of Lemma 3.1 is generally not true, as shown in the following example.

Example 3.2 *Let* (T, ρ, Ξ) *be defined as in Example 3.1. Consider the sequence* (ϖ_ς) *defined by*

$$\varpi_\varsigma = \begin{cases} 1, & \text{if } \varsigma \text{ is even} \\ -1, & \text{if } \varsigma \text{ is odd.} \end{cases}$$

Let $\lambda_u = 1$ *and* $\gamma_u = u^{\frac{1}{\varrho}}$, *with* $\varrho = \frac{1}{2}$. *In this case,* $St^{\varrho,\mathcal{L}}_{\lambda\gamma} - \lim_\varsigma \varpi_\varsigma$ *does not exist. However, the rough* $\mathcal{I}$-$\lambda\gamma$*-statistical limit of order* ϱ *is given by*

$$\mathcal{I}^r_{St^{\varrho,\mathcal{L}}_{\lambda\gamma}} - \lim_\varsigma \varpi_\varsigma = \begin{cases} [1-r, -1+r], & \text{if } r \geq 1 \\ \emptyset, & \text{otherwise} \end{cases}$$

for any admissible ideal $\mathcal{I}$.

Lemma 3.2 *Suppose* (T, ρ, Ξ) *is an* $\mathcal{L}$*-FNS and* $(\varpi_\varsigma) \in \mathrm{T}$. *Let* $r \geq 0$. *For any* $t \in \mathcal{L} \setminus \{0_\mathcal{L}\}$, $\sigma \in \mathbb{R}^+$, *the following statements are equivalent:*

(a) $\mathcal{I}^r_{St^{\varrho,\mathcal{L}}_{\lambda\gamma}} - \lim_\varsigma \varpi_\varsigma = \varpi_0$,

(b)

$$\left\{ u \in \mathbb{N} : \frac{1}{(\gamma_u + 1 - \lambda_u)^{\varrho}} \left| \left\{ \varsigma \in P^{\lambda,\gamma}_u : \rho\left(\varpi_\varsigma - \varpi_0, r + \sigma\right) \nsucc \mathcal{N}(t) \right\} \right| \geq \xi \right\} \in \mathcal{I};$$

(c)

$$\left\{ u \in \mathbb{N} : \frac{1}{(\gamma_u + 1 - \lambda_u)^{\varrho}} \left| \left\{ \varsigma \in P^{\lambda,\gamma}_u : \rho\left(\varpi_\varsigma - \varpi_0, r + \sigma\right) \nsucc \mathcal{N}(t) \right\} \right| < \xi \right\} \in F(\mathcal{I});$$

(d)

$$\mathcal{I}^r_{St^{\varrho,\mathcal{L}}_{\lambda\gamma}} - \lim_u \frac{1}{(\gamma_u + 1 - \lambda_u)^{\varrho}} \left| \left\{ \varsigma \in P^{\lambda,\gamma}_u : \rho\left(\varpi_\varsigma - \varpi_0, r + \sigma\right) \nsucc \mathcal{N}(t) \right\} \right| = 0.$$

Definition 3.3 *A sequence $\varpi = (\varpi_\varsigma)$ in an $\mathcal{L}$-FNS is defined as $\mathcal{I}_{St^{\varrho,\mathcal{L}}_{\lambda\gamma}}$-bounded if for any $t \in \mathcal{L} \setminus \{0_\mathcal{L}\}$ and $\xi > 0$, there exists $H > 0$ such that*

$$\left\{u \in \mathbb{N} : \frac{1}{(\gamma_u + 1 - \lambda_u)^\varrho} \left|\left\{\varsigma \in P_u^{\lambda,\gamma} : \rho(\varpi_\varsigma, H) \nsucc \mathcal{N}(t)\right\}\right| \geq \xi\right\} \in \mathcal{I}.$$

We now present fundamental properties related to $\mathcal{I}^r_{St^{\varrho,\mathcal{L}}_{\lambda\gamma}}$-convergence.

Theorem 3.1 *Let (T, ρ, Ξ) be an $\mathcal{L}$-FNS. A sequence $\varpi = (\varpi_\varsigma)$ in T is $\mathcal{I}_{St^{\varrho,\mathcal{L}}_{\lambda\gamma}}$-bounded if and only if $\mathcal{I}^r_{St^{\varrho,\mathcal{L}}_{\lambda\gamma}} - \lim_\varsigma \varpi_\varsigma \neq \emptyset$ for some $r > 0$.*

Proof: Necessary Part: Let (ϖ_ς) be a sequence that is $\mathcal{I}_{St^{\varrho,\mathcal{L}}_{\lambda\gamma}}$-bounded. Then, for any $t \in \mathcal{L} \setminus \{0_\mathcal{L}\}$ and $\xi > 0$, there exists a real number $\sigma > 0$ such that the set

$$\left\{u \in \mathbb{N} : \frac{1}{(\gamma_u + 1 - \lambda_u)^\varrho} \left|\left\{\varsigma \in P_u^{\lambda,\gamma} : \rho(\varpi_\varsigma, \sigma) \nsucc \mathcal{N}(t)\right\}\right| \geq \xi\right\} \in \mathcal{I}.$$

Since $\mathcal{I}$ is an admissible ideal, the complement of this set in $\mathbb{N}$, namely $W = \mathbb{N} \setminus U$ is non-empty, where

$$U = \left\{u \in \mathbb{N} : \frac{1}{(\gamma_u + 1 - \lambda_u)^\varrho} \left|\left\{\varsigma \in P_u^{\lambda,\gamma} : \rho(\varpi_\varsigma, \sigma) \nsucc \mathcal{N}(t)\right\}\right| \geq \xi\right\}.$$

Now, let us choose any $\varsigma \in W$. Then, we have

$$\begin{aligned} &\left\{u \in \mathbb{N} : \tfrac{1}{(\gamma_u+1-\lambda_u)^\varrho} \left|\left\{\varsigma \in P_u^{\lambda,\gamma} : \rho(\varpi_\varsigma, \sigma) \nsucc \mathcal{N}(t)\right\}\right| < \xi\right\} \\ \Rightarrow &\left\{u \in \mathbb{N} : \tfrac{1}{(\gamma_u+1-\lambda_u)^\varrho} \left|\left\{\varsigma \in P_u^{\lambda,\gamma} : \rho(\varpi_\varsigma, \sigma) \succ \mathcal{N}(t)\right\}\right| \geq 1 - \xi\right\}. \end{aligned} \tag{3.4}$$

Define the set

$$K = \left\{\varsigma \in P_u^{\lambda,\gamma} : \rho(\varpi_\varsigma, \sigma) \succ \mathcal{N}(t)\right\}.$$

Now, applying the triangular property of the fuzzy norm, we obtain

$$\begin{aligned} \rho(\varpi_\varsigma, r + \sigma) &\succcurlyeq \Xi(\rho(0, r), \rho(\varpi_\varsigma, \sigma)) \\ &= \Xi(1_\mathcal{L}, \rho(\varpi_\varsigma, \sigma)) \succ \mathcal{N}(t), \end{aligned}$$

which implies that

$$K \subset \left\{\varsigma \in P_u^{\lambda,\gamma} : \rho(\varpi_\varsigma, \sigma) \succ \mathcal{N}(t)\right\}.$$

Using (3.4), we get

$$\begin{aligned} 1 - \xi &\leq \tfrac{|K|}{(\gamma_u+1-\lambda_u)^\varrho} \\ &\leq \tfrac{1}{(\gamma_u+1-\lambda_u)^\varrho} \left|\left\{\varsigma \in P_u^{\lambda,\gamma} : \rho(\varpi_\varsigma, r + \sigma) \succ \mathcal{N}(t)\right\}\right|. \end{aligned}$$

Hence, we conclude that

$$\frac{1}{(\gamma_u + 1 - \lambda_u)^\varrho} \left|\left\{\varsigma \in P_u^{\lambda,\gamma} : \rho(\varpi_\varsigma, r + \sigma) \nsucc \mathcal{N}(t)\right\}\right| < \xi.$$

That is,

$$\left\{u \in \mathbb{N} : \frac{1}{(\gamma_u + 1 - \lambda_u)^\varrho} \left|\left\{\varsigma \in P_u^{\lambda,\gamma} : \rho(\varpi_\varsigma, r + \sigma) \nsucc \mathcal{N}(t)\right\}\right| < \xi\right\} \subset U \in \mathcal{I}.$$

Therefore, we have $0 \in \mathcal{I}^r_{St^{\varrho,\mathcal{L}}_{\lambda\gamma}} - \lim_\varsigma \varpi_\varsigma$, which implies that the rough $\mathcal{I}^r_{St^{\varrho,\mathcal{L}}_{\lambda\gamma}}$-statistical limit of the sequence is non-empty for some $r > 0$.

Sufficient Part: Assume that $\mathcal{I}^r_{St^{\varrho,\mathcal{L}}_{\lambda\gamma}} - \lim_\varsigma \varpi_\varsigma \neq \emptyset$ for some $r > 0$. Then, there exists an element $\beta \in T$ such that $\beta \in \mathcal{I}^r_{St^{\varrho,\mathcal{L}}_{\lambda\gamma}} - \lim_\varsigma \varpi_\varsigma$. This implies that for any $t \in \mathcal{L} \setminus \{0_\mathcal{L}\}$, $\xi > 0$, and $\sigma \in \mathbb{R}^+$, the following set

$$\left\{u \in \mathbb{N} : \frac{1}{(\gamma_u + 1 - \lambda_u)^\varrho} \left|\left\{\varsigma \in P_u^{\lambda,\gamma} : \rho\left(\varpi_\varsigma - \beta, \sigma\right) \nsucc \mathcal{N}(t)\right\}\right| \geq \xi\right\}$$

belongs to the ideal.

In other words, the majority of the terms ϖ_ς are located within a fuzzy neighborhood (or fuzzy ball) centered at β, according to the $\mathcal{L}$-fuzzy norm. This directly leads to the conclusion that the sequence $\varpi = (\varpi_\varsigma)$ is $\mathcal{I}_{St^{\varrho,\mathcal{L}}_{\lambda\gamma}}$-bounded in $\mathcal{L}$-FNS. □

It is established that the sum of two rough statistically convergent sequences, and the scalar multiple of such a sequence, remain rough statistically convergent in an $\mathcal{L}$-FNS for a fixed roughness parameter $r \geq 0$. However, this algebraic property does not universally extend to rough $\mathcal{I}$-$\lambda\gamma$-statistical convergence of order ϱ. We present the following proposition to address this.

Proposition 3.1 *Let* (T, ρ, Ξ) *be an* $\mathcal{L}$*-FNS. Suppose* (ϖ_ς),$(\varphi_\varsigma) \in \mathrm{T}$. *For* $r_1, r_2 \geq 0$, *the following properties hold:*
(1) If $\mathcal{I}^{r_1}_{St^{\varrho,\mathcal{L}}_{\lambda\gamma}} - \lim_\varsigma \varpi_\varsigma = \varpi_0$ *and* $\mathcal{I}^{r_2}_{St^{\varrho,\mathcal{L}}_{\lambda\gamma}} - \lim_\varsigma \varphi_\varsigma = \varphi_0$, *then* $\mathcal{I}^{(r_1+r_2)}_{St^{\varrho,\mathcal{L}}_{\lambda\gamma}} - \lim_\varsigma [\varpi_\varsigma + \varphi_\varsigma] = \varpi_0 + \varphi_0$.
(2) If $\mathcal{I}^{r_1}_{St^{\varrho,\mathcal{L}}_{\lambda\gamma}} - \lim_\varsigma \varpi_\varsigma = \varpi_0$, *then* $\mathcal{I}^{|q|r_1}_{St^{\varrho,\mathcal{L}}_{\lambda\gamma}} - \lim_\varsigma q\varpi_\varsigma = q\varpi_0$ *for any scalar* $q \in \mathbb{R}$.

Proof: The result follows directly from the definitions and standard arguments; hence, the proof is omitted. □

In the subsequent theorem, we establish the closedness of the limit set $\mathcal{I}_{St^{\varrho,\mathcal{L}}_{\lambda\gamma}} - \lim_\varsigma \varpi_\varsigma$.

Theorem 3.2 *Let* $\varpi = (\varpi_\varsigma)$ *be a sequence in an* $\mathcal{L}$*-FNS* (T, ρ, Ξ) *and* $r \geq 0$. *Then, the set of rough* $\mathcal{I}$*-*$\lambda\gamma$*-limit points,* $\mathcal{I}^r_{St^{\varrho,\mathcal{L}}_{\lambda\gamma}} - \mathrm{LIM}^r \varpi_\varsigma$, *is a closed set.*

Proof: If $r = 0$ the result is straightforward since $\mathcal{I}_{St^{\varrho,\mathcal{L}}_{\lambda\gamma}} - \mathrm{LIM}^r \varpi_\varsigma$ is either empty or a singleton set. Now, suppose that $\mathcal{I}_{St^{\varrho,\mathcal{L}}_{\lambda\gamma}} - \mathrm{LIM}^r \varpi_\varsigma \neq \emptyset$ for some $r > 0$. Consider a sequence $\varphi = (\varphi_\varsigma) \subset \mathcal{I}^r_{St^{\varrho,\mathcal{L}}_{\lambda\gamma}} - \mathrm{LIM}^r \varpi_\varsigma$ in (T, ρ, Ξ), that converges to $\varphi_0 \in T$ w.r.t. fuzzy norm ρ. For any $t \in \mathcal{L} \setminus \{0_\mathcal{L}\}$ and $\sigma > 0$, there exists $\varsigma_0 \in \mathbb{N}$ such that $\rho\left(\varphi_\varsigma - \varphi_0, \frac{\sigma}{2}\right) \succ \mathcal{N}(t)$ for all $\varsigma > \varsigma_0$. Now, take an element $\varphi_{\varsigma_1} \in \mathcal{I}_{St^{\varrho,\mathcal{L}}_{\lambda\gamma}} - \mathrm{LIM}^r \varpi_\varsigma$ and $\xi > 0$ then

$$U = \left\{u \in \mathbb{N} : \frac{1}{(\gamma_u + 1 - \lambda_u)^\varrho} \left|\left\{\varsigma \in P_u^{\lambda,\gamma} : \rho\left(\varpi_\varsigma - \varphi_{\varsigma_1}, r + \frac{\sigma}{2}\right) \succ \mathcal{N}(t)\right\}\right| \geq \xi\right\} \in \mathcal{I}.$$

Since $\mathcal{I}$ is an admissible ideal, the complement set $Y = \mathbb{N} \setminus U$ is non-empty. Choose any $\varsigma \in Y$, then

$$\begin{aligned} &\tfrac{1}{(\gamma_u+1-\lambda_u)^\varrho} \left|\left\{\varsigma \in P_u^{\lambda,\gamma} : \rho\left(\varpi_\varsigma - \varphi_{\varsigma_1}, r + \tfrac{\sigma}{2}\right) \nsucc \mathcal{N}(t)\right\}\right| < \xi \\ \Rightarrow &\tfrac{1}{(\gamma_u+1-\lambda_u)^\varrho} \left|\left\{\varsigma \in P_u^{\lambda,\gamma} : \rho\left(\varpi_\varsigma - \varphi_{\varsigma_1}, r + \tfrac{\sigma}{2}\right) \succ \mathcal{N}(t)\right\}\right| \geq 1 - \xi. \end{aligned}$$

Let

$$Q = \left\{\varsigma \in P_u^{\lambda,\gamma} : \rho\left(\varpi_\varsigma - \varphi_{\varsigma_1}, r + \frac{\sigma}{2}\right) \succ \mathcal{N}(t)\right\}.$$

For $m \in Q$ with $m \geq \varsigma_0$, we have

$$\begin{aligned} \rho\left(\varpi_m - \varphi_0, r + \sigma\right) &\succeq \Xi\left(\rho\left(\varpi_m - \varphi_{\varsigma_1}, r + \tfrac{\sigma}{2}\right), \rho\left(\varphi_{\varsigma_1} - \varphi_0, \tfrac{\sigma}{2}\right)\right) \\ &\succ \mathcal{N}(t). \end{aligned}$$

Therefore,

$$m \in \left\{\varsigma \in P_u^{\lambda,\gamma} : \rho\left(\varpi_\varsigma - \varphi_0, r + \sigma\right) \succ \mathcal{N}(t)\right\}.$$

Hence,

$$Q \subset \left\{ \varsigma \in P_u^{\lambda,\gamma} : \rho\left(\varpi_\varsigma - \varphi_0, r+\sigma\right) \succ \mathcal{N}\left(t\right) \right\}$$

that provides

$$\begin{array}{ll} 1-\xi & \leq \frac{|Q_u|}{(\gamma_u+1-\lambda_u)^\varrho} \\ & \leq \frac{1}{(\gamma_u+1-\lambda_u)^\varrho} \left| \left\{ \varsigma \in P_u^{\lambda,\gamma} : \rho\left(\varpi_\varsigma - \varphi_0, r+\sigma\right) \succ \mathcal{N}\left(t\right) \right\} \right| . \end{array}$$

Therefore

$$\frac{1}{(\gamma_u+1-\lambda_u)^\varrho} \left| \left\{ \varsigma \in P_u^{\lambda,\gamma} : \rho\left(\varpi_\varsigma - \varphi_0, r+\sigma\right) \not\succ \mathcal{N}\left(t\right) \right\} \right| < \xi.$$

Then

$$\begin{array}{l} \left\{ u \in \mathbb{N} : \frac{1}{(\gamma_u+1-\lambda_u)^\varrho} \left| \left\{ \varsigma \in P_u^{\lambda,\gamma} : \rho\left(\varpi_\varsigma - \varphi_0, r+\sigma\right) \not\succ \mathcal{N}\left(t\right) \right\} \right| \geq \xi \right\} \\ \subset U \in \mathcal{I}. \end{array}$$

Hence, $\varphi_0 \in \mathcal{I}^r_{St^{\varrho,\mathcal{L}}_{\lambda\gamma}} - \mathrm{LIM}^r \varpi_\varsigma$ in an $\mathcal{L}$-FNS (T, ρ, Ξ). □

We now address the convexity of the limit set $\mathcal{I}_{St^{\varrho,\mathcal{L}}_{\lambda\gamma}} - \mathrm{LIM}^r \varpi_\varsigma$.

Theorem 3.3 *Let $\varpi = (\varpi_\varsigma)$ be a sequence in an $\mathcal{L}$-FNS (T, ρ, Ξ). For any fixed $r > 0$, the set of rough $\mathcal{I}$-$\lambda\gamma$-limit points of order ϱ, $\mathcal{I}_{St^{\varrho,\mathcal{L}}_{\lambda\gamma}} - \mathrm{LIM}^r \varpi_\varsigma$, is a convex set in T.*

Proof: Let $\beta_1, \beta_2 \in \mathcal{I}_{St^{\varrho,\mathcal{L}}_{\lambda\gamma}} - \mathrm{LIM}^r \varpi_\varsigma$. To show convexity, we must demonstrate that the convex combination $(1-\zeta)\beta_1 + \zeta\beta_2 \in \mathcal{I}_{St^{\varrho,\mathcal{L}}_{\lambda\gamma}} - \mathrm{LIM}^r \varpi_\varsigma$ for any $\zeta \in (0,1)$. Since $\beta_1, \beta_2 \in \mathcal{I}_{St^{\varrho,\mathcal{L}}_{\lambda\gamma}} - \mathrm{LIM}^r \varpi_\varsigma$, it follows that for any $t \in \mathcal{L} \setminus \{0_\mathcal{L}\}$ and $\sigma \in \mathbb{R}^+$, the following sets are in the ideal.

Let us define

$$U_0 := \left\{ \varsigma \in P_u^{\lambda,\gamma} : \rho\left(\varpi_\varsigma - \beta_1, \frac{r+\sigma}{2(1-\zeta)} \right) \not\succ \mathcal{N}\left(t\right) \right\},$$

and

$$U_1 := \left\{ \varsigma \in P_u^{\lambda,\gamma} : \rho\left(\varpi_\varsigma - \beta_2, \frac{r+\sigma}{2\zeta} \right) \not\succ \mathcal{N}\left(t\right) \right\}.$$

For any $\xi > 0$,

$$\left\{ u \in \mathbb{N} : \frac{1}{(\gamma_u+1-\lambda_u)^\varrho} \left| \left\{ \varsigma \in P_u^{\lambda,\gamma} : \varsigma \in U_0 \cup U_1 \right\} \right| \geq \xi \right\} \in \mathcal{I}.$$

Choose $0 < \xi_1 < 1$ such that $0 < 1 - \xi_1 < \xi$, and define

$$U = \left\{ u \in \mathbb{N} : \frac{1}{(\gamma_u+1-\lambda_u)^\varrho} \left| \left\{ \varsigma \in P_u^{\lambda,\gamma} : \varsigma \in U_0 \cup U_1 \right\} \right| \geq \xi_1 \right\} \in \mathcal{I}.$$

Then, for all $u \notin U$, we have

$$\begin{array}{l} \frac{1}{(\gamma_u+1-\lambda_u)^\varrho} \left| \left\{ \varsigma \in P_u^{\lambda,\gamma} : \varsigma \in U_0 \cup U_1 \right\} \right| < 1 - \xi_1 \\ \Rightarrow \frac{1}{(\gamma_u+1-\lambda_u)^\varrho} \left| \left\{ \varsigma \in P_u^{\lambda,\gamma} : \varsigma \notin U_0 \cup U_1 \right\} \right| \geq 1 - (1-\xi_1) = \xi_1, \end{array}$$

which implies that the set $\left\{ \varsigma \in P_u^{\lambda,\gamma} : \varsigma \notin U_0 \cup U_1 \right\}$ is non-empty. Let $\varsigma \in (U_0 \cup U_1)^c = U_0^c \cap U_1^c$. Then,

$$\begin{array}{l} \rho\left(\varpi_\varsigma - \left[(1-\zeta)\beta_1 + \zeta\beta_2\right], r+\sigma\right) \\ = \rho\left((1-\zeta)(\varpi_\varsigma - \beta_1) + \zeta(\varpi_\varsigma - \beta_2), r+\sigma\right) \\ \succeq \Xi\left(\rho\left((1-\zeta)(\varpi_\varsigma - \beta_1), \frac{r+\sigma}{2}, \rho\left(\zeta(\varpi_\varsigma - \beta_2), \frac{r+\sigma}{2}\right)\right)\right) \\ = \Xi\left(\rho\left((\varpi_\varsigma - \beta_1), \frac{r+\sigma}{2(1-\zeta)}, \rho\left((\varpi_\varsigma - \beta_2), \frac{r+\sigma}{2\zeta}\right)\right)\right) \\ \succ \mathcal{N}\left(t\right). \end{array}$$

This implies $U_0^c \cap U_1^c \subset B_\varsigma^c$, where

$$B_\varsigma := \left\{ \varsigma \in P_u^{\lambda,\gamma} : \rho\left(\varpi_\varsigma - \left[(1-\zeta)\beta_1 + \zeta\beta_2\right], r+\sigma\right) \not\succ \mathcal{N}\left(t\right) \right\}.$$

For $\varsigma \notin U$,

$$\begin{aligned} \xi_1 &\leq \tfrac{1}{(\gamma_u+1-\lambda_u)^{\varrho}} \left|\left\{\varsigma \in P_u^{\lambda,\gamma} : \varsigma \notin U_0 \cup U_1\right\}\right| \\ &\leq \tfrac{1}{(\gamma_u+1-\lambda_u)^{\varrho}} \left|\left\{\varsigma \in P_u^{\lambda,\gamma} : \varsigma \notin B_\varsigma\right\}\right| \end{aligned}$$

or

$$\frac{1}{(\gamma_u+1-\lambda_u)^{\varrho}} \left|\left\{\varsigma \in P_u^{\lambda,\gamma} : \varsigma \in B_\varsigma\right\}\right| < 1-\xi_1 < \xi.$$

Hence,

$$U^c \subset \left\{u \in \mathbb{N} : \frac{1}{(\gamma_u+1-\lambda_u)^{\varrho}} \left|\left\{\varsigma \in P_u^{\lambda,\gamma} : \varsigma \in B_\varsigma\right\}\right| < \xi\right\}.$$

Since $U^c \in \mathcal{F}(\mathcal{I})$, then

$$\left\{u \in \mathbb{N} : \frac{1}{(\gamma_u+1-\lambda_u)^{\varrho}} \left|\left\{\varsigma \in P_u^{\lambda,\gamma} : \varsigma \in B_\varsigma\right\}\right| < \xi\right\} \in \mathcal{F}(\mathcal{I}).$$

Therefore,

$$\left\{u \in \mathbb{N} : \frac{1}{(\gamma_u+1-\lambda_u)^{\varrho}} \left|\left\{\varsigma \in P_u^{\lambda,\gamma} : \varsigma \in B_\varsigma\right\}\right| \geq \xi\right\} \in \mathcal{I}.$$

This proves that $(1-\zeta)\beta_1 + \zeta\beta_2 \in \mathcal{I}_{St_{\lambda\gamma}^{\varrho,\mathcal{L}}} - \mathrm{LIM}^r \varpi_\varsigma$, and therefore $\mathcal{I}_{St_{\lambda\gamma}^{\varrho,\mathcal{L}}} - \mathrm{LIM}^r \varpi_\varsigma$ is convex. □

Theorem 3.4 *A sequence $\varpi = (\varpi_\varsigma)$ in an $\mathcal{L}$-FNS (T, ρ, Ξ) is rough ideal statistically convergent to $\varpi_0 \in \mathrm{T}$ w.r.t. the norm ρ for some $r > 0$ if there exists a sequence $\varphi = (\varphi_\varsigma)$ in T such that $\mathcal{I}_{St_{\lambda\gamma}^{\varrho,\mathcal{L}}} - \lim_\varsigma \varphi_\varsigma = \varpi_0$ in T and for every $t \in \mathcal{L} \setminus \{0_\mathcal{L}\}$ and $\sigma \in \mathbb{R}^+$ we have $\rho(\varpi_\varsigma - \varphi_\varsigma, r+\sigma) \succ \mathcal{N}(t)$ for all $\varsigma \in \mathbb{N}$.*

Proof: Let $\mathcal{I}_{St_{\lambda\gamma}^{\varrho,\mathcal{L}}} - \lim_\varsigma \varphi_\varsigma = \varpi_0$ and $\rho(\varpi_\varsigma - \varphi_\varsigma, r+\sigma) \succ \mathcal{N}(t)$. For $t \in \mathcal{L} \setminus \{0_\mathcal{L}\}$, $\xi > 0$ and $\sigma \in \mathbb{R}^+$, the set

$$\left\{u \in \mathbb{N} : \frac{1}{(\gamma_u+1-\lambda_u)^{\varrho}} \left|\left\{\varsigma \in P_u^{\lambda,\gamma} : \rho(\varphi_\varsigma - \varpi_0, \sigma) \nsucc \mathcal{N}(t)\right\}\right| \geq \xi\right\} \in \mathcal{I}.$$

Define

$$\begin{aligned} U_1 &= \left\{\varsigma \in P_u^{\lambda,\gamma} : \rho(\varphi_\varsigma - \varpi_0, \sigma) \nsucc \mathcal{N}(t)\right\}, \\ U_2 &= \left\{\varsigma \in P_u^{\lambda,\gamma} : \rho(\varpi_\varsigma - \varphi_\varsigma, r) \nsucc \mathcal{N}(t)\right\}. \end{aligned}$$

For any $\xi > 0$, we have

$$\left\{u \in \mathbb{N} : \frac{1}{(\gamma_u+1-\lambda_u)^{\varrho}} \left|\left\{\varsigma \in P_u^{\lambda,\gamma} : \varsigma \in U_1 \cup U_2\right\}\right| \geq \xi\right\} \in \mathcal{I}.$$

Select $0 < \xi_1 < 1$ such that $0 < 1 - \xi_1 < \xi$. Then

$$U = \left\{u \in \mathbb{N} : \frac{1}{(\gamma_u+1-\lambda_u)^{\varrho}} \left|\left\{\varsigma \in P_u^{\lambda,\gamma} : \varsigma \in U_1 \cup U_2\right\}\right| \geq \xi_1\right\}$$

For $u \notin U$

$$\begin{aligned} &\tfrac{1}{(\gamma_u+1-\lambda_u)^{\varrho}} \left|\left\{\varsigma \in P_u^{\lambda,\gamma} : \varsigma \in U_1 \cup U_2\right\}\right| < 1 - \xi_1 \\ \Rightarrow &\tfrac{1}{(\gamma_u+1-\lambda_u)^{\varrho}} \left|\left\{\varsigma \in P_u^{\lambda,\gamma} : \varsigma \notin U_1 \cup U_2\right\}\right| \geq \xi_1. \end{aligned}$$

Then,

$$\left\{\varsigma \in P_u^{\lambda,\gamma} : \varsigma \notin U_1 \cup U_2\right\} \neq \emptyset.$$

Let $\varsigma \in (U_0 \cup U_1)^c = U_0^c \cap U_1^c$. Then,

$$\begin{aligned} \rho(\varpi_\varsigma - \varpi_0, r+\sigma) &\succeq \Xi\left(\rho(\varpi_\varsigma - \varphi_\varsigma, r), \rho(\varphi_\varsigma - \varpi_0, \sigma)\right) \\ &\succ \mathcal{N}(t). \end{aligned}$$

This implies $U_1^c \cap U_2^c \subset B^c$, where

$$B := \left\{ \varsigma \in P_u^{\lambda,\gamma} : \rho\left(\varpi_\varsigma - \varpi_0, r+\sigma\right) \not\succ \mathcal{N}\left(t\right) \right\}.$$

For $\varsigma \notin U$,

$$\begin{aligned}\xi_1 &\leq \tfrac{1}{(\gamma_u+1-\lambda_u)^\varrho} \left|\left\{\varsigma \in P_u^{\lambda,\gamma} : \varsigma \notin U_1 \cup U_2\right\}\right| \\ &\leq \tfrac{1}{(\gamma_u+1-\lambda_u)^\varrho} \left|\left\{\varsigma \in P_u^{\lambda,\gamma} : \varsigma \notin B\right\}\right|,\end{aligned}$$

or

$$\frac{1}{(\gamma_u+1-\lambda_u)^\varrho} \left|\left\{\varsigma \in P_u^{\lambda,\gamma} : \varsigma \in B\right\}\right| < 1-\xi_1 < \xi.$$

Hence,

$$U^c \subset \left\{ u \in \mathbb{N} : \frac{1}{(\gamma_u+1-\lambda_u)^\varrho} \left|\left\{\varsigma \in P_u^{\lambda,\gamma} : \varsigma \in B\right\}\right| < \xi \right\}.$$

Since $U^c \in \mathcal{F}\left(\mathcal{I}\right)$, then

$$\left\{ u \in \mathbb{N} : \frac{1}{(\gamma_u+1-\lambda_u)^\varrho} \left|\left\{\varsigma \in P_u^{\lambda,\gamma} : \varsigma \in B\right\}\right| < \xi \right\} \in \mathcal{F}\left(\mathcal{I}\right).$$

Therefore,

$$\left\{ u \in \mathbb{N} : \frac{1}{(\gamma_u+1-\lambda_u)^\varrho} \left|\left\{\varsigma \in P_u^{\lambda,\gamma} : \varsigma \in B\right\}\right| \geq \xi \right\} \in \mathcal{I}.$$

This gives that $\mathcal{I}^r_{St^{\varrho,\mathcal{L}}_{\lambda\gamma}} - \lim_\varsigma \varpi_\varsigma = \varpi_0$. □

Theorem 3.5 *Let $\varpi = (\varpi_\varsigma)$ be a sequence in $\mathcal{L}$-FNS (T, ρ, Ξ). There do not exist two distinct elements $\alpha_1, \alpha_2 \in \mathcal{I}_{St^{\varrho,\mathcal{L}}_{\lambda\gamma}} - \mathrm{LIM}^r \varpi_\varsigma$ for $r > 0$ and $t \in \mathcal{L} \setminus \{0_\mathcal{L}\}$ such that $\rho\left(\alpha_1 - \alpha_2, hr\right) \succ N(t)$ for $h > 2$.*

Proof: We proceed by contradiction to establish the result. Suppose there exist two distinct points $\alpha_1, \alpha_2 \in \mathcal{I}_{St^{\varrho,\mathcal{L}}_{\lambda\gamma}} - \mathrm{LIM}^r \varpi_\varsigma$ such that the following inequality

$$\rho\left(\alpha_1 - \alpha_2, hr\right) \not\succ \mathcal{N}(t) \text{ for } h > 2. \tag{3.5}$$

holds.

Since $\alpha_1, \alpha_2 \in \mathcal{I}_{St^{\varrho,\mathcal{L}}_{\lambda\gamma}} - \mathrm{LIM}^r \varpi_\varsigma$ then for any $t \in \mathcal{L} \setminus \{0_\mathcal{L}\}$ and $\sigma \in \mathbb{R}^+$. Define,

$$\begin{aligned}U_1 &= \left\{ \varsigma \in P_u^{\lambda,\gamma} : \rho\left(\varpi_\varsigma - \alpha_1; r + \tfrac{\sigma}{2}\right) \not\succ \mathcal{N}(t) \right\}, \\ U_2 &= \left\{ \varsigma \in P_u^{\lambda,\gamma} : \rho\left(\varpi_\varsigma - \alpha_2; r + \tfrac{\sigma}{2}\right) \not\succ \mathcal{N}(t) \right\}.\end{aligned}$$

Then, for $u \in \mathbb{N}$, the following inequality is satisfied due to the properties of cardinality and the union of sets:

$$\begin{aligned}&\tfrac{1}{(\gamma_u+1-\lambda_u)^\varrho} \left|\left\{\varsigma \in P_u^{\lambda,\gamma} : \varsigma \in U_1 \cup U_2\right\}\right| \\ &\leq \tfrac{1}{(\gamma_u+1-\lambda_u)^\varrho} \left|\left\{\varsigma \in P_u^{\lambda,\gamma} : \varsigma \in U_1\right\}\right| \\ &+ \tfrac{1}{(\gamma_u+1-\lambda_u)^\varrho} \left|\left\{\varsigma \in P_u^{\lambda,\gamma} : \varsigma \in U_2\right\}\right|.\end{aligned}$$

From the property of $\mathcal{I}$-convergence, we obtain

$$\begin{aligned}&\mathcal{I} - \lim_{u\to\infty} \tfrac{1}{(\gamma_u+1-\lambda_u)^\varrho} \left|\left\{\varsigma \in P_u^{\lambda,\gamma} : \varsigma \in U_1 \cup U_2\right\}\right| \\ &\leq \mathcal{I} - \lim_{u\to\infty} \tfrac{1}{(\gamma_u+1-\lambda_u)^\varrho} \left|\left\{\varsigma \in P_u^{\lambda,\gamma} : \varsigma \in U_1\right\}\right| \\ &+ \mathcal{I} - \lim_{u\to\infty} \tfrac{1}{(\gamma_u+1-\lambda_u)^\varrho} \left|\left\{\varsigma \in P_u^{\lambda,\gamma} : \varsigma \in U_2\right\}\right| = 0.\end{aligned}$$

This leads to the conclusion that, for any $\xi > 0$, the set

$$\left\{ \varsigma \in \mathbb{N} : \frac{1}{(\gamma_u+1-\lambda_u)^\varrho} \left|\left\{\varsigma \in P_u^{\lambda,\gamma} : \varsigma \in U_1 \cup U_2\right\}\right| \geq \xi \right\} \in \mathcal{I}.$$

Now, choose a real number $\xi_1 \in (0,1)$ such that $0 < 1-\xi_1 < \xi$. Let

$$Y = \left\{ \varsigma \in P_u^{\lambda,\gamma} : \frac{1}{(\gamma_u + 1 - \lambda_u)^{\varrho}} \left| \left\{ \varsigma \in P_u^{\lambda,\gamma} : \varsigma \in U_1 \cup U_2 \right\} \right| \geq \xi_1 \right\} \in \mathcal{I}.$$

For $\varsigma \notin Y$

$$\begin{aligned} & \tfrac{1}{(\gamma_u+1-\lambda_u)^{\varrho}} \left| \left\{ \varsigma \in P_u^{\lambda,\gamma} : \varsigma \in U_1 \cup U_2 \right\} \right| < 1-\xi_1. \\ \Rightarrow & \tfrac{1}{(\gamma_u+1-\lambda_u)^{\varrho}} \left| \left\{ \varsigma \in P_u^{\lambda,\gamma} : \varsigma \notin U_1 \cup U_2 \right\} \right| \geq 1-(1-\xi_1) = \xi_1. \end{aligned}$$

This implies $\left\{ \varsigma \in P_u^{\lambda,\gamma} : \varsigma \notin U_1 \cup U_2 \right\} \neq \emptyset$. Then for $\varsigma \in U_1^c \cap U_2^c$ we have

$$\begin{aligned} \rho\left(\alpha_1-\alpha_2, 2r+\sigma\right) & \succeq \Xi\left(\rho\left(\varpi_\varsigma - \alpha_2; r+\tfrac{\sigma}{2}\right), \rho\left(\varpi_\varsigma - \alpha_1; r+\tfrac{\sigma}{2}\right)\right) \\ & \succ \mathcal{N}(t). \end{aligned}$$

Hence,

$$\rho\left(\alpha_1-\alpha_2, 2r+\sigma\right) \succ \mathcal{N}(t). \tag{3.6}$$

From (3.6) we have $\rho\left(\alpha_1-\alpha_2, hr\right) \succ \mathcal{N}(t)$ for $h > 2$. which results contradiction to (3.5). As a result, the condition $\rho\left(\alpha_1-\alpha_2, hr\right) \not\succ \mathcal{N}(\varepsilon)$ for $h > 2$ cannot be satisfied by any pair of elements α_1, α_2. □

Definition 3.4 *Let (T, ρ, Ξ) be an $\mathcal{L}$-FNS. An element $\varpi_0 \in \mathrm{T}$ is termed a rough $\mathcal{I}$-$\lambda\gamma$-statistical cluster point of the sequence $\varpi = (\varpi_\varsigma) \in \mathrm{T}$ w.r.t. the fuzzy norm ρ for some $r > 0$ if for any $t \in \mathcal{L} \setminus \{0_{\mathcal{L}}\}$, $\xi > 0$ and $\sigma \in \mathbb{R}^+$*

$$\left\{ u \in \mathbb{N} : \frac{1}{(\gamma_u + 1 - \lambda_u)^{\varrho}} \left| \left\{ \varsigma \in P_u^{\lambda,\gamma} : \rho\left(\varpi_\varsigma - \varpi_0; r+\sigma\right) \not\succ \mathcal{N}(t) \right\} \right| < \xi \right\} \notin \mathcal{I}.$$

We denote the set of all rough $\mathcal{I}$-$\lambda\gamma$-statistically cluster points of the sequence $\varpi = (\varpi_\varsigma)$ by $\Gamma^r_{\mathcal{I}_{St^{\varrho,\mathcal{L}}_{\lambda\gamma}}}(\varpi)$.

If $r = 0$, this reduces to the definition of an $\mathcal{I}$-$\lambda\gamma$-statistically cluster point, i.e., $\Gamma^0_{\mathcal{I}_{St^{\varrho,\mathcal{L}}_{\lambda\gamma}}}(\varpi) = \Gamma_{\mathcal{I}_{St^{\varrho,\mathcal{L}}_{\lambda\gamma}}}(\varpi)$.

Theorem 3.6 *Let (T, ρ, Ξ) be an $\mathcal{L}$-FNS. The set $\Gamma^r_{\mathcal{I}_{St^{\varrho,\mathcal{L}}_{\lambda\gamma}}}(\varpi)$ of all rough $\mathcal{I}$-$\lambda\gamma$-statistical cluster points w.r.t. ρ of a sequence $\varpi = (\varpi_\varsigma)$ is closed for $r > 0$.*

Proof: Assume that $\Gamma^r_{\mathcal{I}_{St^{\varrho,\mathcal{L}}_{\lambda\gamma}}}(\varpi)$ is empty. In this case, there is nothing to prove. So, suppose instead that $\Gamma^r_{\mathcal{I}_{St^{\varrho,\mathcal{L}}_{\lambda\gamma}}}(\varpi) \neq \emptyset$. Let $\varphi = (\varphi_\varsigma) \subseteq \Gamma^r_{\mathcal{I}_{St^{\varrho,\mathcal{L}}_{\lambda\gamma}}}(\varpi)$ be a sequence converging to some point φ_0. To establish the desired result, it is sufficient to show that $\varphi_0 \in \Gamma^r_{\mathcal{I}_{St^{\varrho,\mathcal{L}}_{\lambda\gamma}}}(\varpi)$. Since $\varphi_\varsigma \xrightarrow{\rho} \varphi_0$, then for any $t \in \mathcal{L} \setminus \{0_{\mathcal{L}}\}$ and $\sigma \in \mathbb{R}^+$, there exists $\varsigma_t \in \mathbb{N}$ such that $\rho\left(\varphi_\varsigma - \varphi_0; \frac{\sigma}{2}\right) \succ \mathcal{N}(t)$ for $\varsigma \geq \varsigma_t$. Now select any $\varsigma_0 \in \mathbb{N}$ such that $\varsigma_0 \geq \varsigma_t$. Then, we get $\rho\left(\varphi_{\varsigma_0} - \varphi_0; \frac{\sigma}{2}\right) \succ \mathcal{N}(t)$. From the assumption that $\varphi = (\varphi_\varsigma) \subseteq \Gamma^r_{\mathcal{I}_{St^{\varrho,\mathcal{L}}_{\lambda\gamma}}}(\varpi)$, it follows that $\varphi_{\varsigma_0} \in \Gamma^r_{\mathcal{I}_{St^{\varrho,\mathcal{L}}_{\lambda\gamma}}}(\varpi)$. Then

$$\Rightarrow \left\{ u \in \mathbb{N} : \frac{1}{(\gamma_u + 1 - \lambda_u)^{\varrho}} \left| \left\{ \varsigma \in P_u^{\lambda,\gamma} : \rho\left(\varpi_\varsigma - \varphi_{\varsigma_0}; r+\frac{\sigma}{2}\right) \not\succ \mathcal{N}(t) \right\} \right| < \xi \right\} \notin \mathcal{I}. \tag{3.7}$$

Let

$$G = \left\{ \varsigma \in P_u^{\lambda,\gamma} : \rho\left(\varpi_\varsigma - \varphi_{\varsigma_0}; r+\frac{\sigma}{2}\right) \not\succ \mathcal{N}(t) \right\}.$$

Choose $j \in G^c$, then we have $\rho\left(\varpi_j - \varphi_{\varsigma_0}; r+\frac{\sigma}{2}\right) \succ \mathcal{N}(t)$.
Now,

$$\begin{aligned} \rho\left(\varpi_j - \varphi_0; r+\sigma\right) & \succeq \Xi\left(\rho\left(\varpi_j - \varphi_{\varsigma_0}; r+\tfrac{\sigma}{2}\right), \mu\left(\varphi_{\varsigma_0} - \varphi_0; r+\tfrac{\sigma}{2}\right)\right) \\ & \succ \mathcal{N}(t). \end{aligned}$$

Hence

$$\left\{\varsigma \in P_u^{\lambda,\gamma} : \rho\left(\varpi_\varsigma - \varphi_{\varsigma_0}; r + \frac{\sigma}{2}\right) \succ \mathcal{N}(t)\right\}$$
$$\subseteq \left\{\varsigma \in P_u^{\lambda,\gamma} : \rho\left(\varpi_\varsigma - \varphi_0; r + \sigma\right) \succ \mathcal{N}(\varepsilon)\right\}.$$

$$\Longrightarrow \left\{\varsigma \in P_u^{\lambda,\gamma} : \rho\left(\varpi_\varsigma - \varphi_0; r + \sigma\right) \not\succ \mathcal{N}(t)\right\}$$
$$\subseteq \left\{\varsigma \in P_u^{\lambda,\gamma} : \rho\left(\varpi_\varsigma - \varphi_{\varsigma_0}; r + \frac{\sigma}{2}\right) \not\succ \mathcal{N}(t)\right\}.$$

Thus,

$$\left\{u \in \mathbb{N} : \tfrac{1}{(\gamma_u+1-\lambda_u)^\varrho}\left|\left\{\varsigma \in P_u^{\lambda,\gamma} : \rho\left(\varpi_\varsigma - \varphi_0; r + \sigma\right) \not\succ \mathcal{N}(t)\right\}\right| < \xi\right\}$$
$$\subseteq \left\{u \in \mathbb{N} : \tfrac{1}{(\gamma_u+1-\lambda_u)^\varrho}\left|\left\{\varsigma \in P_u^{\lambda,\gamma} : \rho\left(\varpi_\varsigma - \varphi_{\varsigma_0}; r + \tfrac{\sigma}{2}\right) \not\succ \mathcal{N}(t)\right\}\right| < \xi\right\}.$$

According to (3.7), we obtain

$$\left\{u \in \mathbb{N} : \frac{1}{(\gamma_u+1-\lambda_u)^\varrho}\left|\left\{\varsigma \in P_u^{\lambda,\gamma} : \rho\left(\varpi_\varsigma - \varphi_0; r + \sigma\right) \not\succ \mathcal{N}(t)\right\}\right| < \xi\right\} \notin \mathcal{I}.$$

Therefore, $\varphi_0 \in \Gamma^r_{\mathcal{I}_{St^{\varrho,\mathcal{L}}_{\lambda\gamma}}}(\varpi)$. This completes the proof. □

Theorem 3.7 *Let $\varpi = (\varpi_\varsigma)$ be a sequence in $\mathcal{L}$-FNS (T, ρ, Ξ), which is $\mathcal{I}$-$\lambda\gamma$-statistically convergent to ϖ_0. If $\overline{B(\varpi_0, t, r)} = \{s \in \mathrm{T} : \rho(s - \varpi_0; r) \not\succ \mathcal{N}(t)\}$ is a closed ball for some $r > 0$ and $t \in \mathcal{L} \setminus \{0_\mathcal{L}\}$, then $\overline{B(\varpi_0, t, r)} \subset \mathcal{I}_{St^{\varrho,\mathcal{L}}_{\lambda\gamma}} - \mathrm{LIM}^r \varpi_\varsigma$.*

Proof: Since $\varpi_\varsigma \overset{\mathcal{I}_{St^{\varrho,\mathcal{L}}_{\lambda\gamma}}}{\to} \varpi_0$, then for $t \in \mathcal{L} \setminus \{0_\mathcal{L}\}$, $\xi > 0$ and $\sigma \in \mathbb{R}^+$

$$U = \left\{u \in \mathbb{N} : \frac{1}{(\gamma_u+1-\lambda_u)^\varrho}\left|\left\{\varsigma \in P_u^{\lambda,\gamma} : \rho\left(\varpi_\varsigma - \varpi_0; \sigma\right) \not\succ \mathcal{N}(t)\right\}\right| > \xi\right\} \in \mathcal{I}.$$

As $\mathcal{I}$ is admissible so $W = \mathbb{N} \diagdown U \neq \emptyset$, then for $\varsigma \in W^c$,

$$\tfrac{1}{(\gamma_u+1-\lambda_u)^\varrho}\left|\left\{\varsigma \in P_u^{\lambda,\gamma} : \rho\left(\varpi_\varsigma - \varpi_0; \sigma\right) \not\succ \mathcal{N}(t)\right\}\right| < \xi.$$
$$\Rightarrow \tfrac{1}{(\gamma_u+1-\lambda_u)^\varrho}\left|\left\{\varsigma \in P_u^{\lambda,\gamma} : \rho\left(\varpi_\varsigma - \varpi_0; \sigma\right) \succ \mathcal{N}(t)\right\}\right| \geq 1 - \xi.$$

Put

$$Q = \left\{\varsigma \in P_u^{\lambda,\gamma} : \rho\left(\varpi_\varsigma - \varpi_0; \sigma\right) \succ \mathcal{N}(t)\right\}$$

for $m \geq \varsigma$. For $m \in Q$, $\rho\left(\varpi_m - \varpi_0; \sigma\right) \succ \mathcal{N}(t)$.

Let $\varpi^* \in \overline{B(\varpi_0, t, r)}$. We have to denote $\varpi^* \in \mathcal{I}_{St^{\varrho,\mathcal{L}}_{\lambda\gamma}} - \mathrm{LIM}^r \varpi_\varsigma$.

$$\rho\left(\varpi_m - \varpi^*; r + \sigma\right) \succeq \Xi\left(\rho\left(\varpi_m - \varpi_0, \sigma\right), \rho\left(\varpi^* - \varpi_0, r\right)\right) \succ \mathcal{N}(t).$$

Hence,

$$Q \subset \left\{\varsigma \in P_u^{\lambda,\gamma} : \rho\left(\varpi_\varsigma - \varpi^*; r + \sigma\right) \succ \mathcal{N}(t)\right\},$$

which gives that

$$1 - \xi \leq \frac{|Q|}{(\gamma_u+1-\lambda_u)^\varrho}$$
$$\leq \frac{1}{(\gamma_u+1-\lambda_u)^\varrho}\left|\left\{\varsigma \in P_u^{\lambda,\gamma} : \rho\left(\varpi_\varsigma - \varpi^*; r + \sigma\right) \succ \mathcal{N}(t)\right\}\right|$$

So,

$$\frac{1}{(\gamma_u+1-\lambda_u)^{\varrho}}\left|\left\{\varsigma\in P_u^{\lambda,\gamma}:\rho\left(\varpi_\varsigma-\varpi^*;r+\sigma\right)\not\succ\mathcal{N}(t)\right\}\right|<\xi.$$

Then,

$$\left\{u\in\mathbb{N}:\frac{1}{(\gamma_u+1-\lambda_u)^{\varrho}}\left|\left\{\varsigma\in P_u^{\lambda,\gamma}:\rho\left(\varpi_\varsigma-\varpi^*;r+\sigma\right)\not\succ\mathcal{N}(t)\right\}\right|\geq\xi\right\}\subset U\in\mathcal{I},$$

which gives that $\varpi^*\in\mathcal{I}_{St^{\varrho,\mathcal{L}}_{\lambda\gamma}}-\mathrm{LIM}^r\varpi_\varsigma$ in (T,ρ,Ξ). □

4. Conclusions

This study has introduced a comprehensive and highly generalized framework for analyzing sequence convergence within $\mathcal{L}$-fuzzy normed spaces: rough $\mathcal{I}$-$\lambda\gamma$-statistical convergence of order ϱ. This novel concept successfully synthesizes several powerful generalizations–ideal convergence ($\mathcal{I}$), generalized sequence transformations ($\lambda\gamma$), arbitrary order of convergence (ϱ), and the notion of roughness (r)–within the flexible environment of lattice-valued norms. The primary contribution of this work lies in the rigorous analysis of the resulting rough limit set, $\mathcal{I}_{St^{\varrho,\mathcal{L}}_{\lambda\gamma}}-\mathrm{LIM}^r(\varpi)$. We established that the existence of a non-empty rough limit set is equivalent to the sequence being $\mathcal{I}_{St^{\varrho,\mathcal{L}}_{\lambda\gamma}}$-bounded. Crucially, we proved that this limit set possesses fundamental structural properties, namely closedness and convexity, with respect to the topology induced by the $\mathcal{L}$-fuzzy norm. Furthermore, we investigated the constraints on the diameter of the limit set relative to the roughness degree r. Additionally, we defined and characterized the associated set of cluster points, $\Gamma^r_{\mathcal{I}_{St^{\varrho,\mathcal{L}}_{\lambda\gamma}}}(\varpi)$, proving its closedness and elucidating its relationship with the rough limit set. These findings provide a robust theoretical foundation for handling convergence in complex systems where uncertainty is better modeled by lattice structures rather than the standard unit interval. The established structural properties of the limit set are particularly significant for future applications in optimization theory and approximation theory within $\mathcal{L}$-fuzzy environments.

References

1. Aktuğlu, H., *Korovkin type approximation theorems proved via $\alpha\beta$-statistical convergence*, J. Comput. Appl. Math., (259)(14), 174-181, (2014).
2. Antal, R., Kaur, M., Chawla, M., *Deferred rough $\mathcal{I}$-statistical convergence in $\mathcal{L}$-fuzzy normed spaces*, Bol. Soc. Paran. Mat., 43(2), 1–13, (2025).
3. Atanassov, K.T., *Intuitionistic fuzzy sets*, Fuzzy Sets Syst., 20, 87–96, (1986).
4. Aytar, S., *Rough statistical convergence*, Numer. Funct. Anal. Optim., 29(3-4), 291–303, (2008).
5. Belen, C., Mohiuddine, S.A., *Generalized weighted statistical convergence and application*, Appl. Math. Comput., 219, 9821–9826, (2013).
6. Braha, N.L., Loku, V., *Korovkin type theorems and its applications via $\alpha\beta$-statistically convergence*, J. Math. Inequal., 14(4), 951–966, (2020).
7. Das, P., Savaş, E., Ghosal, S.Kr., *On generalizations of certain summability methods using ideals*, Appl. Math. Lett., 24(9), 1509–1514, (2011).
8. Dündar, E., Çakan, C., *Rough $\mathcal{I}$-convergence*, Demonstr. Math., 47(3), 638–651, (2014).
9. Fast, H., *Sur la convergence statistique*, Colloq. Math., 2(3-4), 241–244, (1951).
10. George, A., Veeramani, P., *On some results in fuzzy metric spaces*, Fuzzy Sets Syst., 64(3), 395–399, (1994).
11. Ghosal, S., Mandal, S., *Rough weighted $\mathcal{I}$-$\alpha\beta$-statistical convergence in locally solid Riesz spaces*, J. Math. Anal. Appl., 506(2), 125681, (2022).
12. Giles, R., *A computer program for fuzzy reasoning*, Fuzzy Sets Syst., 4(3), 221–234, (1980).
13. Goguen, J.A., *L-fuzzy sets*, J. Math. Anal. Appl., 18(1), 145–174, (1967).
14. Gürdal, M., *Description of extended eigenvalues and extended eigenvectors of integration operators on the Wiener algebra*, Expo. Math., 27(2), 153–160, (2009).
15. Gürdal, M., Yamanci, U., *Statistical convergence of operator theory*, Dyn. Syst. Appl., 24(3), 305–311, (2015).
16. Hazarika, B., Alotaibi, A., Mohiuddine, S.A., *Statistical convergence in measure for double sequences of fuzzy-valued functions*, Soft Comput., 24, 6613–6622, (2020).

17. Kadak, U., Mohiuddine, S.A., *Generalized statistically almost convergence based on the difference operator which includes the (p, q)-Gamma function and related approximation theorems*, Results Math., 73(9), 1–31, (2018).

18. Karaev, M.T., Gürdal, M., Huban, M.B., *Reproducing kernels, Englis algebras and some applications*, Studia Math., 232(2), 113–141, (2016).

19. Karthick, P., Mohiuddine, S.A., Tamilvanan, K., Narayanamoorthy, S., Maheswari, S., *Investigations of color image segmentation based on connectivity measure, shape priority and normalized fuzzy graph cut*, Appl. Soft Comput., 139, 110239, (2023).

20. Khan, V.A., Et, M., Khan, I.A., *Ideal convergence in modified IFNS and $\mathcal{L}$-fuzzy normed space*, Math. Found. Comput., (2023), 1–15.

21. Khan, V.A., Rahaman, S.A., Hazarika, B., *On deferred $\mathcal{I}$-statistical rough convergence of difference sequences in intuitionistic fuzzy normed spaces*, Filomat, 38(18), 6333–6354, (2024).

22. Kostyrko, P., Šalát, T., Wilczyński, W., *$\mathcal{I}$-convergence*, Real Anal. Exchange, 26(2), 669–686, (2000/2001).

23. Kramosil, I., Michalek, J., *Fuzzy metrics and statistical metric spaces*, Kybernetika, 11(5), 336–344, (1975).

24. Mohiuddine, S.A., Asiri, A., Hazarika, B., *Weighted statistical convergence through difference operator of sequences of fuzzy numbers with application to fuzzy approximation theorems*, Int. J. Gen. Syst., 48(5), 492–506, (2019).

25. Mohiuddine, S.A., Hazarika, B., Alotaibi, A., *On statistical convergence of double sequences of fuzzy valued functions*, J. Intell. Fuzzy Syst., 32, 4331–4342, (2017).

26. Or, A., Özcan, A.C., Karabacak, G., *Rough statistical convergence in L-fuzzy normed spaces*, Int. J. Adv. Nat. Sci. Eng. Res., 7, 307–314, (2023).

27. Pal, S.K., Debraj, C.H., Dutta, S., *Rough ideal convergence*, Hacet. J. Math. Stat., 42(6), 633–640, (2013).

28. Park, J.H., *Intuitionistic fuzzy metric spaces*, Chaos Solitons Fractals, 22(5), 1039–1046, (2004).

29. Phu, H.X., *Rough convergence in normed linear spaces*, Numer. Funct. Anal. Optim., 22(1-2), 199–222, (2001).

30. Rahaman, S.K.A., Mursaleen, M., *On rough deferred statistical convergence of difference sequences in $\mathcal{L}$-fuzzy normed spaces*, J. Math. Anal. Appl., 530(2), 127684, (2024).

31. Saadati, R., Vaezpour, S.M., *Some results on fuzzy Banach spaces*, J. Appl. Math. Comput., 17(1), 475–484, (2005).

32. Şahiner, A., Gürdal, M., Yigit, T., *Ideal convergence characterization of the completion of linear n-normed spaces*, Comput. Math. Appl., 61(3), 683–689, (2011).

33. Savaş, E., Kisi, Ö., Gürdal, M., *On statistical convergence in credibility space*, Numer. Funct. Anal. Optim., 43(8), 987–1008, (2022).

34. Shakeri, S., Saadati, R., Park, C., *Stability of the quadratic functional equation in non-archimedean $\mathcal{L}$-fuzzy normed spaces*, Int. J. Nonlinear Anal. Appl., 1(2), 72–83, (2010).

35. Sözbir, B., Altundağ, S., *$\alpha\beta$-statistical convergence on time scales*, Facta Univ. Ser. Math. Inform., 35(1), 141–150, (2020).

36. Tok, M.A., Kara, E.E., Altundağ, S., *On the $\alpha\beta$-statistical convergence of the modified discrete operators*, Adv. Differ. Equ., 2018(1), 1–6, (2018).

37. Yamanci, U., Gürdal, M., *On lacunary ideal convergence in random n-normed space*, J. Math., 868457, (2013).

38. Yamanci, U., Gürdal, M., *Statistical convergence and operators on Fock space*, New York J. Math., 22, 199–207, (2016).

39. Yapali, R., Çoşkun, H., Gürdal, U., *Statistical convergence on $\mathcal{L}$-fuzzy normed space*, Filomat, 37(7), 2077–2085, (2023).

40. Yapali, R., Korkmaz, E., Çinar, M., Çoskun, H., *Lacunary statistical convergence on $\mathcal{L}$-fuzzy normed space*, J. Intell. Fuzzy Syst., 46(1), 1985-1993, (2024).

41. Zadeh, L.A., *Fuzzy sets*, Inf. Control, 8(3), 338-353, (1965).

Ömer Kişi,
Department of Mathematics,
Bartın University,
Turkey.
E-mail address: `okisi@bartin.edu.tr`

and

Mehmet Gürdal,

Department of Mathematics,
Suleyman Demirel University,
Turkey.
E-mail address: gurdalmehmet@sdu.edu.tr

and

Selim Çetin,
Department of Mathematics,
Mehmet Akif Ersoy University,
Turkey.
E-mail address: selimcetin@mehmetakif.edu.tr