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Pseudo Generalized Quasi-Einstein Warped Products Manifolds with Respect to Affine Connections

Mohd Vasiulla, Vineet Kumar, Mohabbat Ali, Sanjay Tyagi and Ashish Tyagi

ABSTRACT: In this paper, we investigate warped products on pseudo-generalized quasi-Einstein manifolds under affine connections. We explore their fundamental properties, establish conditions for their existence, and prove that these manifolds can be nearly quasi-Einstein and pseudo quasi-Einstein. To illustrate, we provide examples in Riemannian and Lorentzian geometries, confirming their existence. Finally, we construct and analyze an explicit example of a warped product on a pseudo-generalized quasi-Einstein manifold with respect to affine connections.

Keywords: Quasi-Einstein manifold, pseudo generalized quasi-Einstein manifold, partial differential equations, warped products manifold, affine connection.

Contents

1. Introduction

A Riemannian manifold $\mathcal{M}$ is referred to as an Einstein manifold [4] if its Ricci tensor Ric, a non-zero tensor of type $(0,2)$, satisfies the equation $Ric = \frac{scal}{n} g$, where $scal$ represents the scalar curvature and g is the metric tensor.

Some generalizations of Einstein manifolds have been defined and studied. Among these, the quasi-Einstein (QE) manifold, introduced by Chaki and Maity [14], is characterized by its Ricci tensor $Ric(\neq 0)$ satisfying

$$Ric(\Omega_1, \Omega_2) = \Phi_1 g(\Omega_1, \Omega_2) + \Phi_2 \mathcal{U}(\Omega_1)\mathcal{U}(\Omega_2), \tag{1.1}$$

where $\Phi_1, \Phi_2(\neq 0) \in \mathbb{R}$ and $\mathcal{U}(\neq 0)$ is the 1-form. In addition, the 1-form $\mathcal{U}$ is called the associated 1-form. Equation 1.1 reveals that QE manifolds become equivalent to Einstein manifolds under condition $\Phi_2 = 0$.

Several authors have contributed to the development of QE manifold theory by introducing various generalizations. These expansions include, but are not limited to, semi-quasi-Einstein manifolds [13] and pseudo generalized quasi-Einstein manifolds [1,16]. In [11], the authors studied $N(\kappa)$-quasi-Einstein manifolds admitting the Schouten tensor and obtained several characterization results under suitable geometric conditions.

A Riemannian manifold $\mathcal{M}$ is defined as a generalized quasi-Einstein (GQE) manifold [17] when its Ricci tensor Ric, which is non-zero, satisfies

$$Ric(\Omega_1, \Omega_2) = \Phi_1 g(\Omega_1, \Omega_2) + \Phi_2 \mathcal{U}(\Omega_1)\mathcal{U}(\Omega_2) + \Phi_3[\mathcal{U}(\Omega_1)\mathcal{V}(\Omega_2) + \mathcal{U}(\Omega_2)\mathcal{V}(\Omega_1)], \tag{1.2}$$

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where $\Phi_1, \Phi_2 \neq 0, \Phi_3 \neq 0 \in \mathbb{R}$, and $\mathcal{U}$, $\mathcal{V}$ are distinct non-zero 1-form satisfies

$$g(\Omega_1, \nu) = \mathcal{U}(\Omega_1), \quad g(\Omega_1, \upsilon) = \mathcal{V}(\Omega_1), \quad g(\nu, \nu) = 1, \quad g(\upsilon, \upsilon) = 1,$$

with υ and ν being mutually orthogonal unit vector fields (i.e., $g(\nu, \upsilon) = 0$)). These vector fields act as generators for the GQE manifold.

A Riemannian manifold $\mathcal{M}$ is said to be a nearly quasi-Einstein (NQE) manifold [24] if the Ricci tensor $Ric(\neq 0)$ satisfies

$$Ric(\Omega_1, \Omega_2) = \Phi_1 g(\Omega_1, \Omega_2) + \Phi_2 \mathcal{E}(\Omega_1, \Omega_2), \tag{1.3}$$

where Φ_1, $\Phi_2(\neq 0)$ scalars and $\mathcal{E}$ is a symmetric tensor of type $(0, 2)$.

A Riemannian manifold $\mathcal{M}$ is said to be a pseudo quasi-Einstein (PQE) manifold [2] if the Ricci tensor $Ric(\neq 0)$ satisfies

$$Ric(\Omega_1, \Omega_2) = \Phi_1 g(\Omega_1, \Omega_2) + \Phi_2 \mathcal{U}(\Omega_1)\mathcal{U}(\Omega_2) + \Phi_3 \mathcal{E}(\Omega_1, \Omega_2), \tag{1.4}$$

where Φ_1, $\Phi_2(\neq 0)$, $\Phi_3(\neq 0) \in \mathbb{R}$ and $\mathcal{U}(\neq 0)$ is a 1-form and $\mathcal{E}$ is a symmetric tensor of type $(0, 2)$ with zero trace that satisfies

$$\mathcal{E}(\Omega_1, \nu) = 0, \quad \forall \quad \Omega_1. \tag{1.5}$$

A Riemannian manifold $\mathcal{M}$ is classified as a pseudo generalized quasi-Einstein (PGQE) manifold [1] when its Ricci tensor, Ric, which is nonzero, satisfies

$$Ric(\Omega_1, \Omega_2) = \Phi_1 g(\Omega_1, \Omega_2) + \Phi_2 \mathcal{U}(\Omega_1)\mathcal{U}(\Omega_2) + \Phi_3 \mathcal{V}(\Omega_1)\mathcal{V}(\Omega_2) + \Phi_4 \mathcal{E}(\Omega_1, \Omega_2), \tag{1.6}$$

where $\Phi_1, \Phi_2(\neq 0), \Phi_3(\neq 0), \Phi_4(\neq 0) \in \mathbb{R}$ and $\mathcal{U}(\neq 0)$, $\mathcal{V}(\neq 0)$ are 1-forms and $\mathcal{E}$ is a symmetric tensor of type $(0, 2)$ with zero trace that satisfies

$$\mathcal{E}(\Omega_1, \nu) = 0, \quad \forall \quad \Omega_1. \tag{1.7}$$

Warped product manifolds have been extensively studied in differential geometry due to their versatile applications and rich geometric structures. In 2018, Pahan, Pal, and Bhattacharyya [22] explored compact super quasi-Einstein warped products with non-positive scalar curvature, providing information on the geometric and topological properties of such manifolds. In 2021, De et al. [23] studied mixed generalized quasi-Einstein warped product manifolds and explored their geometric properties. They demonstrated that Ricci recurrent mixed generalized quasi-Einstein manifolds are product manifolds with Ricci recurrent factors. In 2023, Dipankar Debnath [8] introduced the concept of $N(K)$-quasi-Einstein warped products for dimensions $n \geq 3$, expanding the theoretical framework of warped product geometry. In 2024, Abdallah et al. [3] characterized warped product manifolds using the W_2 curvature tensor, with applications to relativity. Their study examined how the flatness and symmetry of the W_2 tensor influence both the base manifold and the fiber manifolds. In paper [6], the authors studied mixed super quasi-Einstein manifolds, investigating their geometric structure and curvature properties. Several characterizations and rigidity results were obtained, and the relevance of these manifolds to models in general relativity was demonstrated through suitable applications

In 2024, Bang-Yen Chen et al. [7] investigated the effects of quasi-conformal curvature tensors on warped product manifolds, focusing on quasi-conformally flat, quasi-conformally symmetric and divergence-free scenarios. Blaga and Özgür [5] explored 2-Killing vector fields on multiply warped product manifolds, establishing criteria for lifting vector fields from factor manifolds. Fahad et al. [9] analyzed concircular trajectories in doubly warped product manifolds, revealing geometric properties related to the tensors of the Hessian, Riemannian, Ricci and concircular curvature.

Recently, Vasiulla et al. [20] investigated generalized quasi-Einstein warped product manifolds endowed with affine connections, thereby extending the theory of pseudo-generalized quasi-Einstein warped product manifolds and highlighting their significance in geometric analysis. In a related development, Vasiulla, Ali, Khan, and Aldayel [19] studied super quasi-Einstein warped product manifolds with respect to affine connections, further enriching the structural understanding of quasi-Einstein geometry under non-Levi-Civita connections. These contributions collectively advance our understanding of warped product manifolds in diverse geometric contexts.

In 2008, Shaikh and Jana [1] introduced a new manifold of quasi-constant curvature named the manifold of pseudo generalized quasi-constant (PGQC) curvature defined as

$$
\begin{aligned}
\overline{\mathcal{K}}(\Omega_1,\Omega_2,\Omega_3,\Omega_4) &= \Phi_1[g(\Omega_2,\Omega_3)g(\Omega_1,\Omega_4) - g(\Omega_1,\Omega_3)g(\Omega_2,\Omega_4)] \\
&+ \Phi_2[g(\Omega_1,\Omega_4)\mathcal{U}(\Omega_2)\mathcal{U}(\Omega_3) - g(\Omega_2,\Omega_4)\mathcal{U}(\Omega_1)\mathcal{U}(\Omega_3) \\
&+ g(\Omega_2,\Omega_3)\mathcal{U}(\Omega_1)\mathcal{U}(\Omega_4) - g(\Omega_1,\Omega_3)\mathcal{U}(\Omega_2)\mathcal{U}(\Omega_4)] \\
&+ \Phi_3[g(\Omega_1,\Omega_4)\mathcal{V}(\Omega_2)\mathcal{V}(\Omega_3) - g(\Omega_2,\Omega_4)\mathcal{V}(\Omega_1)\mathcal{V}(\Omega_3) \\
&+ g(\Omega_2,\Omega_3)\mathcal{V}(\Omega_1)\mathcal{V}(\Omega_4) - g(\Omega_1,\Omega_3)\mathcal{V}(\Omega_2)\mathcal{V}(\Omega_4)] \\
&+ \Phi_4[\mathcal{E}(\Omega_2,\Omega_3)g(\Omega_1,\Omega_4) - \mathcal{E}(\Omega_1,\Omega_3)g(\Omega_2,\Omega_4) \\
&+ \mathcal{E}(\Omega_1,\Omega_4)g(\Omega_2,\Omega_3) - \mathcal{E}(\Omega_2,\Omega_4)g(\Omega_1,\Omega_3)],
\end{aligned} \tag{1.8}
$$

where $\Phi_1, \Phi_2(\neq 0)$, $\Phi_3(\neq 0)$, $\Phi_4(\neq 0) \in \mathbb{R}$ and $\mathcal{U}(\neq 0)$, $\mathcal{V}(\neq 0)$ are 1-forms and $\mathcal{E}$ is a symmetric tensor of type $(0,2)$.

2. Preliminaries

In a linear connection, if $Dg = 0$, the connection D on a Riemannian manifold $\mathcal{M}$ is referred to as a quarter symmetric metric connection. Otherwise, it is called a quarter-symmetric non-metric connection. If a linear connection is a Levi-Civita connection, it is symmetric. A linear connection D on $\mathcal{M}$ is said to be a quarter symmetric connection if its torsion tensor Tr satisfies the following relation

$$Tr(\Omega_1,\Omega_2) = \overline{D}_{\Omega_1}\Omega_2 - \overline{D}_{\Omega_2}\Omega_1 - [\Omega_1,\Omega_2] \tag{2.1}$$

and

$$Tr(\Omega_1,\Omega_2) = \mathcal{U}(\Omega_2)\phi\Omega_1 - \mathcal{U}(\Omega_1)\phi\Omega_2, \tag{2.2}$$

where $\mathcal{U}$ is the 1-form on $\mathcal{M}$ with the associated vector field ρ defined by $g(\Omega_1,\rho) = \mathcal{U}(\Omega_1)$, for all Ω_1.

The relation between the Levi-Civita connection D and a quarter-symmetric connection $\overline{D}$ on $\mathcal{M}$ is given by [15]

$$\overline{D}_{\Omega_1}\Omega_2 = D_{\Omega_1}\Omega_2 + \mu_1\mathcal{U}(\Omega_2)\Omega_1 - \mu_2 g(\Omega_1,\Omega_2)\rho, \tag{2.3}$$

where μ_1, μ_2 are non zero scalar functions.
It is easy to observe the following cases:

(a) when $\mu_1 = \mu_2 = 1$, $\overline{D}$ is a semi-symmetric metric connection,

(b) when $\mu_1 = \mu_2 \neq 1$, $\overline{D}$ is a quarter-symmetric metric connection,

(c) when $\mu_1 \neq \mu_2$, $\overline{D}$ is a quarter-symmetric non-metric connection.

Let $\mathcal{K}$ and $\overline{\mathcal{K}}$ be the curvature tensors of D and $\overline{D}$, respectively. By virtue of (3.13) in [15], we can get

$$
\begin{aligned}
\overline{\mathcal{K}}(\Omega_1,\Omega_2,\Omega_3) &= \mathcal{K}(\Omega_1,\Omega_2,\Omega_3) + \mu_1 g(\Omega_3, D_{\Omega_1}\rho)\Omega_2 - \mu_2 g(\Omega_3, D_{\Omega_2}\rho)\Omega_1 \\
&+ \mu_2[g(\Omega_1,\Omega_3)D_{\Omega_2}\rho - g(\Omega_2,\Omega_3)D_{\Omega_1}\rho] \\
&+ \mu_1\mu_2\mathcal{U}(\rho[g(\Omega_1,\Omega_3)\Omega_2 - g(\Omega_2,\Omega_3)\Omega_1] + \mu_2^2[g(\Omega_2,\Omega_3)\mathcal{U}(\Omega_1) \\
&- g(\Omega_1,\Omega_3)\mathcal{U}(\Omega_2)]\rho + \mu_1^2\mathcal{U}(\Omega_3)[\mathcal{U}(\Omega_2)\Omega_1 - \mathcal{U}(\Omega_1)\Omega_2],
\end{aligned} \tag{2.4}
$$

for all Ω_1, Ω_2, Ω_3 on $\mathcal{M}$.

3. Warped product manifolds admitting affine connection

The concept of a warped product generalizes the notion of a revolution surface. It was first introduced in [21] to study negative curvature manifolds. Let $(\mathrm{B}, g_{\mathrm{B}})$ and $(\mathcal{F}, g_{\mathcal{F}})$ be two Riemannian manifolds with dim $\mathrm{B} = p > 0$, dim $\mathcal{F} = q > 0$ and $f : \mathrm{B} \to (0,\infty)$, $f \in C^\infty(\mathrm{B})$. Consider the product manifold $\mathrm{B} \times \mathcal{F}$

with its projections $u : \mathrm{B} \times \mathcal{F} \to \mathrm{B}$ and $v : \mathrm{B} \times \mathcal{F} \to \mathcal{F}$. The warped product $\mathrm{B} \times_f \mathcal{F}$ is the manifold $\mathrm{B} \times \mathcal{F}$ with a Riemannian structure such that for any vector field Ω_1 on $\mathcal{M}$, the following relation holds

$$||\Omega_1||^2 = ||u^*(\Omega_1)||^2 + f^2(u(m))||v^*(\Omega_1)||^2.$$

Thus, we have the desired structure for the warped product

$$g_{\mathcal{M}} = g_{\mathrm{B}} + f^2 g_{\mathcal{F}}, \tag{3.1}$$

holds on $\mathcal{M}$, where B is the base of $\mathcal{M}$, $\mathcal{F}$ is the fiber, and f is the function defined on $\mathcal{M}$, known as the warping function of the warped product [12].

Since $\mathrm{B} \times_f \mathcal{F}$ is a warped product, we have the following relation between the covariant derivatives of vector fields:

$$D_{\Omega_1}\Omega_3 = D_{\Omega_3}\Omega_1 = (\Omega_1 lnf)\Omega_3,$$

for all Ω_1, Ω_3 on B and $\mathcal{F}$, respectively. Consequently, the curvature R of the manifold M is expressed as:

$$R(\Omega_1 \wedge \Omega_3) = g(D_{\Omega_3}D_{\Omega_1}\Omega_1 - D_{\Omega_1}D_{\Omega_3}\Omega_1, \Omega_3) = \frac{1}{f}\{(D_{\Omega_1}\Omega_1)f - \Omega_1^2 f\}.$$

Let $\{e_1, ..., e_n\}$ be a local orthonormal basis, where $e_1, ..., e_{n_1}$ are tangential to B and $e_{n_1+1}, ..., e_n$ are tangential to $\mathcal{F}$. In this basis, we have the following expression for the Laplacian of the warping function f:

$$\frac{\Delta f}{f} = \sum_{i=1}^{n} R(e_i \wedge e_j), \tag{3.2}$$

for each $j = n_1 + 1, ..., n$ [12].

The two lemmas outlined above provide important results for further work on the study of warped products, particularly in the context of curvature computations and the behavior of vector fields on the base and fiber spaces.

Lemma 3.1 *Let $\mathcal{M} = \mathrm{B} \times_f \mathcal{F}$ be a warped product, and let $\mathcal{K}_{\mathcal{M}}$ be the Riemannian curvature tensor of $\mathcal{M}$. Suppose Ω_1, Ω_2, Ω_3 are vector fields on* B *and P, Q, Ω_4 are vector fields on $\mathcal{F}$. Then, the following holds*

(i) $\mathcal{K}_{\mathcal{M}}(\Omega_1, \Omega_2)\Omega_3 = \mathcal{K}_{\mathrm{B}}(\Omega_1, \Omega_2)\Omega_3$,

(ii) $\mathcal{K}_{\mathcal{M}}(\Omega_1, Q)\Omega_2 = \frac{H^f(\Omega_1,\Omega_2)}{f}Q$, *where H^f is the Hessian of f,*

(iii) $\mathcal{K}_{\mathcal{M}}(\Omega_1, \Omega_2)Q = \mathcal{K}_{\mathcal{M}}(Q, \Omega_2)\Omega_1 = 0$,

(iv) $\mathcal{K}_{\mathcal{M}}(\Omega_1, Q)\Omega_4 = -(\frac{g(Q,\Omega_4)}{f})D_{\Omega_1}(gradf)$,

(v) $\mathcal{K}_{\mathcal{M}}(Q, \Omega_4)P = \mathcal{K}_{\mathcal{F}}(Q, \Omega_4)P + (\frac{|gradf|^2}{f^2})g(Q, P)\Omega_4 - g(\Omega_4, P)Q$.

Lemma 3.2 *Let $\mathcal{M} = \mathrm{B} \times_f \mathcal{F}$ be a warped product, and let $Ric_{\mathcal{M}}$ be the Ricci tensor. Suppose Ω_1, Ω_2, and Ω_3 are vector fields on* B *and Q, Ω_4 are vector fields on $\mathcal{F}$. Then, the following holds*

(i) $Ric_{\mathcal{M}}(\Omega_1, \Omega_2) = Ric_{\mathrm{B}}(\Omega_1, \Omega_2) - \frac{m}{f}H^f(\Omega_1, \Omega_2)$,

(ii) $Ric_{\mathcal{M}}(\Omega_1, Q) = 0$,

(iii) $Ric_{\mathcal{M}}(Q, \Omega_4) = Ric_{\mathcal{F}}(Q, \Omega_4) - g(Q, \Omega_4)(\frac{\Delta f}{f} + \frac{m-1}{f^2}|gradf|^2)$,
where H^f and Δf denote the Hessian of f and the Laplacian of f given by $\Delta f = -tr(H^f)$, respectively.

Furthermore, the condition is satisfied by the scalar curvature $scal_{\mathcal{M}}$ of the manifold $\mathcal{M}$.

$$scal_{\mathcal{M}} = scal_{\mathrm{B}} + \frac{scal_{\mathcal{F}}}{f^2} - 2m\frac{\Delta f}{f} - m(m-1)\frac{|gradf|^2}{f^2}, \tag{3.3}$$

where $scal_{\mathrm{B}}$ and $scal_{\mathcal{F}}$ are the scalar curvatures of B and $\mathcal{F}$, respectively.

Quan and Yong investigated warped product manifolds with quarter-symmetric connections in their paper [10], where they presented the four propositions. We refer to Propositions 3.1, 3.2, 3.3, and 3.4, denoted (3.1), (3.2), (3.3), and (3.4), respectively, which will help us to prove our results.

Proposition 3.1 *Let $\mathcal{M} = \mathrm{B} \times_f \mathcal{F}$ be a warped product. Let Ric and $\overline{Ric}$ denote the Ricci tensors of $\mathcal{M}$ with respect to the Levi-Civita connection and a quarter-symmetric connection, respectively. Let dim B $= n_1$, dim $\mathcal{F} = n_2$, dim $\mathcal{M} = \overline{n} = n_1 + n_2$. If Ω_1, $\Omega_2 \in \mathfrak{X}(\mathrm{B})$, Q, $\Omega_4 \in \mathfrak{X}(\mathcal{F})$ and $\rho \in \mathfrak{X}(\mathrm{B})$, then the following holds*

$$\begin{aligned}
&(i)\ \overline{Ric}(\Omega_1,\Omega_2) = \overline{Ric}_{\mathrm{B}}(\Omega_1,\Omega_2) + n_2[\frac{H^f_{\mathrm{B}}(\Omega_1,\Omega_2)}{f} + \mu_2\frac{\rho f}{f}g(\Omega_1,\Omega_2) + \mu_1\mu_2\mathcal{U}(\rho)g(\Omega_1,\Omega_2)\\
&\qquad + \mu_1 g(\Omega_2, D_{\Omega_1}\rho) - \mu_1^2\mathcal{U}(\Omega_1)\mathcal{U}(\Omega_2)],\\
&(ii)\ \overline{Ric}(\Omega_1,V) = \overline{Ric}(Q,\Omega_1),\\
&(iii)\ \overline{Ric}(V,\Omega_4) = Ric_{\mathcal{F}}(Q,\Omega_4) + \{\mu_2 div_{\mathrm{B}}\rho + (n_2-1)\frac{|grad_{\mathrm{B}}f|^2_{\mathrm{B}}}{f^2}[(\overline{n}-1)\mu_1\mu_2\\
&\qquad - \mu_2^2]\mathcal{U}(\rho) + [(\overline{n}-1)\mu_1 + (n_2-1)\mu_2]\frac{\rho f}{f} + \frac{\Delta_{\mathrm{B}}f}{f}\}g(Q,\Omega_4),
\end{aligned}$$

where $div_{\mathrm{B}}\rho = \sum_{k=1}^{n_1}\epsilon_k\langle D_{W_k}\rho, W_k\rangle$ and W_k, $1 \le k \le n_1$, is an orthonormal basis of B *with $\epsilon_k = g(W_k, W_k)$.*

Proposition 3.2 *Let $\mathcal{M} = \mathrm{B}\times_f\mathcal{F}$ be a warped product, dim B $= n_1$, dim $\mathcal{F} = n_2$, dim $\mathcal{M} = \overline{n} = n_1+n_2$. If Ω_1, $\Omega_2 \in \mathfrak{X}(\mathrm{B})$, Q, $\Omega_4 \in \mathfrak{X}(\mathcal{F})$ and $\rho \in \mathfrak{X}(\mathrm{B})$, then the following holds*

$$\begin{aligned}
&(i)\ \overline{Ric}(\Omega_1,\Omega_2) = \overline{Ric}_{\mathrm{B}}(\Omega_1,\Omega_2) + [(\overline{n}-1)\mu_1\mu_2 - \mu_2^2]\mathcal{U}(\rho)g(\Omega_1,\Omega_2) + n_2\frac{H^f_{\mathrm{B}}(\Omega_1,\Omega_2)}{f}\\
&\qquad + \mu_2 g(\Omega_1,\Omega_2)div_{\mathcal{F}}\rho,\\
&(ii)\ \overline{Ric}(\Omega_1,Q) = [(\overline{n}-1)\mu_1 - \mu_2]\mathcal{U}(Q)\frac{\Omega_1 f}{f},\\
&(iii)\ \overline{Ric}(V,\Omega_1) = [\mu_2 - (\overline{n}-1)\mu_1]\mathcal{U}(Q)\frac{\Omega_1 f}{f},\\
&(iv)\ \overline{Ric}(V,\Omega_4) = \overline{Ric}_{\mathcal{F}}(Q,\Omega_4) + g(Q,\Omega_4)\{(n_2-1)\frac{|grad_{\mathrm{B}}f|^2_{\mathrm{B}}}{f^2} + \frac{\Delta_{\mathrm{B}}f}{f} + [(\overline{n}-1)\mu_1\mu_2\\
&\qquad - \mu_2^2]\mathcal{U}(\rho) + \mu_2 div_{\mathcal{F}}\rho\} + [(\overline{n}-1)\mu_1 - \mu_2]g(\Omega_4, D_Q\rho)\\
&\qquad + [\mu_2^2 + (1-\overline{n})\mu_1^2]\mathcal{U}(Q)\mathcal{U}(\Omega_4).
\end{aligned}$$

Proposition 3.3 *Let $\mathcal{M} = \mathrm{B}\times_f\mathcal{F}$ be a warped product, dim B $= n_1$, dim $\mathcal{F} = n_2$, dim $\mathcal{M} = \overline{n} = n_1+n_2$. If $\rho \in \mathfrak{X}(\mathrm{B})$, then the following holds*

$$\begin{aligned}
\overline{scal}_{\mathcal{M}} &= \overline{scal}_{\mathrm{B}} + \frac{scal_{\mathcal{F}}}{f^2} + n_2(n-1)\frac{|grad_{\mathrm{B}}f|^2_{\mathrm{B}}}{f^2} + n_2(\overline{n}-1)(\mu_1+\mu_2)\frac{\rho f}{f}\\
&+ 2n_2\frac{\Delta_{\mathrm{B}}f}{f} + [n_2(\overline{n}+n_1-1)\mu_1\mu_2 - n_2(\mu_1^2+\mu_2^2)]\mathcal{U}(\rho)\\
&+ n_2(\mu_1+\mu_2)div_{\mathrm{B}}\rho.
\end{aligned} \tag{3.4}$$

Proposition 3.4 *Let $\mathcal{M} = \mathrm{B}\times_f \mathcal{F}$ be a warped product, dim $\mathrm{B} = n_1$, dim $\mathcal{F} = n_2$, dim $\mathcal{M} = \overline{n} = n_1+n_2$. If $\rho \in \mathfrak{X}(\mathcal{F})$, then the following holds*

$$\begin{aligned}\overline{scal}_{\mathcal{M}} &= \overline{scal}_{\mathrm{B}} + \frac{scal_{\mathcal{F}}}{f^2}(\overline{n}-1)(\mu_1+\mu_2)div_{\mathcal{F}}\rho + [\overline{n}(\overline{n}-1)\mu_1\mu_2 \\ &\quad + (1-\overline{n})(\mu_1^2+\mu_2^2)]\mathcal{U}(\rho) + n_2(n-1)\frac{|grad_{\mathrm{B}} f|^2_{\mathrm{B}}}{f^2} + 2n_2\frac{\Delta_{\mathrm{B}} f}{f}.\end{aligned} \tag{3.5}$$

4. PGQE warped products

In this section, we investigate PGQE warped product manifolds and present several key results related to their properties.

Theorem 4.1 *Let $(\mathcal{M}, g)$ be a warped product manifold $\mathcal{M} = I \times_f \mathcal{F}$, where I is an open interval in $\mathbb{R}$, with dim $I = 1$ and dim $\mathcal{F} = n-1$, and $n \geq 3$. Then the following statements hold:*

(i) If $(\mathcal{M}, g)$ is a PGQE manifold with respect to a quarter-symmetric connection, then $\mathcal{F}$ is a PGQE manifold for $\rho = \frac{\partial}{\partial t}$ with respect to the Levi-Civita connection.

(ii) If $(\mathcal{M}, g)$ is a PGQE manifold with respect to a quarter-symmetric connection, then the warping function f is constant on I for $\rho \in \mathfrak{X}(\mathcal{F})$, provided $\mu_2 \neq (n-1)\mu_1$.

Proof. Let $\rho \in \mathfrak{X}(\mathrm{B})$ and let g_I be the metric on I.b Taking $f = e^{\frac{q}{2}}$ and applying Proposition 3.1, we obtain

$$\overline{Ric}_{\mathcal{M}}\Big(\frac{\partial}{\partial t}, \frac{\partial}{\partial t}\Big) = (1-n)\Big[\frac{1}{2}q'' + \frac{1}{4}q'^2 - \frac{1}{2}\mu_2 q' + \mu_1\mu_2 - \mu_1^2\Big] g_1\Big(\frac{\partial}{\partial t}, \frac{\partial}{\partial t}\Big), \tag{4.1}$$

$$\overline{Ric}\Big(\frac{\partial}{\partial t}, Q\Big) = 0, \tag{4.2}$$

$$\begin{aligned}\overline{Ric}(Q,\Omega_4) &= Ric_{\mathcal{F}}(Q,\Omega_4) + e^q\Big[\frac{n-1}{4}(q')^2 + \frac{1}{2}\Big\{(n-1)\mu_1 + (n-2)\mu_2\Big\}q' \\ &\quad + \mu_2^2 + \frac{1}{2}q'' + (1-n)\mu_1\mu_2\Big] g_{\mathcal{F}}(Q,\Omega_4),\end{aligned} \tag{4.3}$$

for all Q, Ω_4 on $\mathcal{F}$.

Since $\mathcal{M}$ is PGQE manifold with respect to the quarter-symmetric connection, form 1.6, we have

$$\begin{aligned}\overline{Ric}_{\mathcal{M}}\Big(\frac{\partial}{\partial t}, \frac{\partial}{\partial t}\Big) &= \Phi_1 g\Big(\frac{\partial}{\partial t}, \frac{\partial}{\partial t}\Big) + \Phi_2\mathcal{U}\Big(\frac{\partial}{\partial t}\Big)\mathcal{U}\Big(\frac{\partial}{\partial t}\Big) + \Phi_3\mathcal{V}\Big(\frac{\partial}{\partial t}\Big)\mathcal{V}\Big(\frac{\partial}{\partial t}\Big) \\ &\quad + \Phi_4\mathcal{E}\Big(\frac{\partial}{\partial t}, \frac{\partial}{\partial t}\Big)\end{aligned} \tag{4.4}$$

and

$$\begin{aligned}\overline{Ric}_{\mathcal{M}}(Q,\Omega_4) &= \Phi_1 g(Q,\Omega_4) + \Phi_2\mathcal{U}(Q)\mathcal{U}(\Omega_4) + \Phi_3\mathcal{V}(Q)\mathcal{V}(\Omega_4) \\ &\quad + \Phi_4\mathcal{E}(Q,\Omega_4).\end{aligned} \tag{4.5}$$

Decomposing the vector fields P and P' separately into their components P_I, $P_{\mathcal{F}}$ and P'_I, $P'_{\mathcal{F}}$ on I and $\mathcal{F}$, respectively, we obtain $P = P_I + \eta_1 P_{\mathcal{F}}$ and $P' = P'_I + \eta_2 P'_{\mathcal{F}}$, where η_1, η_2 are functions on $\mathcal{M}$. Since dim $I = 1$, we take $P_I = \frac{\partial}{\partial t}$, which implies $P = \frac{\partial}{\partial t} + \eta_1 P_{\mathcal{F}}$ and $P'_I = \frac{\partial}{\partial t}$ leading to $P' = \frac{\partial}{\partial t} + \eta_2\frac{\partial}{\partial t} + P'_{\mathcal{F}}$. Thus, we have

$$\begin{aligned}\mathcal{U}\Big(\frac{\partial}{\partial t}\Big) &= g\Big(\frac{\partial}{\partial t}, P\Big) = 1, \\ \mathcal{V}\Big(\frac{\partial}{\partial t}\Big) &= g\Big(\frac{\partial}{\partial t}, P'\Big) = 1.\end{aligned} \tag{4.6}$$

Using equations (3.1) and (4.6), the equations (4.4) and (4.5) reduce to

$$\overline{Ric}_{\mathcal{M}}\Big(\frac{\partial}{\partial t}, \frac{\partial}{\partial t}\Big) = \Phi_1 + \Phi_2 + \Phi_3 + \Phi_4\mathcal{E}\Big(\frac{\partial}{\partial t}, \frac{\partial}{\partial t}\Big) \tag{4.7}$$

and

$$\begin{aligned}\overline{Ric}_{\mathcal{M}}(Q,\Omega_4) &= \Phi_1 e^q g_{\mathcal{F}}(Q,\Omega_4) + \Phi_2 \mathcal{U}(Q)\mathcal{U}(\Omega_4)\\ &+ \Phi_3 \mathcal{V}(Q)\mathcal{V}(\Omega_4) + \Phi_4 \mathcal{E}(Q,\Omega_4).\end{aligned} \tag{4.8}$$

By comparing the right hand side of the equations (4.1) and (4.7), we obtain

$$\Phi_1 + \Phi_2 + \Phi_3 + \Phi_4 \mathcal{E}\Big(\frac{\partial}{\partial t}, \frac{\partial}{\partial t}\Big) = -\frac{n-1}{4}\Big[2q'' + (q')^2\Big]. \tag{4.9}$$

Similarly, comparing the right hand side of the equations 4.3 and 4.8, we get

$$\begin{aligned}Ric_{\mathcal{F}}(Q,\Omega_4) = e^q\Big[a - \Big\{&\frac{\overline{n}-1}{4}(q')^2 + \frac{1}{2}\Big((n-1)\mu_1 + (\overline{n}-2)\mu_2\Big)q'\mu_2^2\\ &+ \frac{1}{2}q'' + (1-n)\mu_1\mu_2\Big\}\Big]g_{\mathcal{F}}(Q,\Omega_4) + \Phi_2\mathcal{U}(Q)\mathcal{U}(\Omega_4)\\ &+ \Phi_3\mathcal{V}(Q)\mathcal{V}(\Omega_4) + \Phi_4\mathcal{E}(Q,\Omega_4).\end{aligned} \tag{4.10}$$

This implies that $\mathcal{F}$ is a PGQE manifold with respect to Levi-Civita connection.
For $\rho \in \mathfrak{X}(\mathcal{F})$, applying Proposition (3.2), we get

$$\overline{Ric}\Big(\frac{\partial}{\partial t}, Q\Big) = \frac{q'}{2}\Big[(n-1)\mu_1 - \mu_2\Big]\mathcal{U}(Q), \tag{4.11}$$

$$\overline{Ric}\Big(Q, \frac{\partial}{\partial t}\Big) = \frac{q'}{2}\Big[\mu_2 - (n-1)\mu_1\Big]\mathcal{U}(Q), \tag{4.12}$$

for all $Q \in \mathfrak{X}(\mathcal{F})$.
Since $\mathcal{M}$ is a PGQE manifold, we have

$$\begin{aligned}\overline{Ric}\Big(\frac{\partial}{\partial t}, Q\Big) &= \overline{Ric}\Big(Q, \frac{\partial}{\partial t}\Big)\\ &= \Phi_1 g\Big(Q, \frac{\partial}{\partial t}\Big) + \Phi_2\mathcal{U}(Q)\mathcal{U}\Big(\frac{\partial}{\partial t}\Big) + \Phi_3\mathcal{V}(Q)\mathcal{V}\Big(\frac{\partial}{\partial t}\Big) + \Phi_4\mathcal{E}\Big(Q, \frac{\partial}{\partial t}\Big).\end{aligned} \tag{4.13}$$

Now, $g\Big(Q, \frac{\partial}{\partial t}\Big) = 0$ (as $\frac{\partial}{\partial t} \in \mathfrak{X}(B)$ and $Q \in \mathfrak{X}(\mathcal{F})$), from (4.13), we get

$$\overline{Ric}\Big(\frac{\partial}{\partial t}, Q\Big) = \overline{Ric}\Big(Q, \frac{\partial}{\partial t}\Big) = \Phi_2\mathcal{U}(Q)\mathcal{U}\Big(\frac{\partial}{\partial t}\Big) + \Phi_3\mathcal{V}(Q)\mathcal{V}\Big(\frac{\partial}{\partial t}\Big) + \Phi_4\mathcal{E}\Big(Q, \frac{\partial}{\partial t}\Big). \tag{4.14}$$

Hence, we have

$$\Phi_2\mathcal{U}(Q)\mathcal{U}\Big(\frac{\partial}{\partial t}\Big) + \Phi_3\mathcal{V}(Q)\mathcal{V}\Big(\frac{\partial}{\partial t}\Big) + \Phi_4\mathcal{E}\Big(Q, \frac{\partial}{\partial t}\Big) = \frac{q'}{2}\Big[(n-1)\mu_1 - \mu_2\Big]\mathcal{U}(Q), \tag{4.15}$$

$$\Phi_2\mathcal{U}(Q)\mathcal{U}\Big(\frac{\partial}{\partial t}\Big) = \Phi_3\mathcal{V}(Q)\mathcal{V}\Big(\frac{\partial}{\partial t}\Big) + \Phi_4\mathcal{E}\Big(Q, \frac{\partial}{\partial t}\Big) + \frac{q'}{2}\Big[\mu_2 - (\overline{n}-1)\mu_1\Big]\mathcal{U}(Q). \tag{4.16}$$

From (4.14) and (4.15), we get

$$q' = 0, \tag{4.17}$$

which implies that q is a constant on I. Therefore, f is constant on I.
Now, we consider the warped product $\mathcal{M} = \mathrm{B} \times_f I$ with dim $\mathrm{B} = n-1$, dim $I = 1$, $n \geq 3$. Under this assumption, we can now proceed to prove the following theorem.

Theorem 4.2 *Let $(\mathcal{M}, g)$ be a warped product $\mathrm{B} \times_f I$, where dim $I = 1$, dim $\mathrm{B} = n-1$, and $n \geq 3$. Then:*

(i) If $\rho \in \mathfrak{X}(\mathrm{B})$ is parallel on B with respect to the Levi-Civita connection on B, f is a constant on B, and $(\mathcal{M}, g)$ is a PGQE manifold with respect to a quarter-symmetric connection, then

$$a = [(n-1)\mu_1\mu_2 - \mu_2^2]\mathcal{U}(\rho).$$

(ii) f *is a constant on* B *if* $(\mathcal{M}, g)$ *is a PGQE manifold with respect to a quarter-symmetric connection for* $\rho \in \mathfrak{X}(I)$*, and* $\mu_2 \neq (n-1)\mu_1$*.*

(iii) $(\mathcal{M}, g)$ *is a PGQE manifold with respect to a quarter-symmetric connection if* f *is a constant on* B *and* B *is a PGQE manifold with respect to the Levi-Civita connection for* $\rho \in \mathfrak{X}(I)$*.*

Proof. Let $(\mathcal{M}, g)$ be a PGQE manifold with respect to a quarter-symmetric connection. Then we have

$$\overline{Ric}(\Omega_1, \Omega_2) = \Phi_1 g(\Omega_1, \Omega_2) + \Phi_2 \mathcal{U}(\Omega_1)\mathcal{U}(\Omega_2) + \Phi_3 \mathcal{V}(\Omega_1)\mathcal{V}(\Omega_2) + \Phi_4 \mathcal{E}(\Omega_1, \Omega_2). \tag{4.18}$$

Decomposing the vector fields P and Q into components P_{B}, P_I on B, I, respectively, we have

$$P = P_I + P_{\mathrm{B}} \quad \textit{and} \quad Q = Q_I + Q_{\mathrm{B}}. \tag{4.19}$$

Since dim $I = 1$, we can take $P_I = \eta_1 \frac{\partial}{\partial t}$ and $Q_I + \eta_2 \frac{\partial}{\partial t}$ which gives $P = P_{\mathrm{B}} + \eta_1 \frac{\partial}{\partial t}$ and $Q = Q_{\mathrm{B}} + \eta_2 \frac{\partial}{\partial t}$ where η_1, η_2 are functions on $\mathcal{M}$. From (4.18), (4.19) and Proposition 3.1, we obtain

$$\begin{aligned} \overline{Ric}^{\mathrm{B}}(\Omega_1, \Omega_2) = {} & \Phi_1 g_{\mathrm{B}}(\Omega_1, \Omega_2) + \Phi_2 g_B(\Omega_1, P_B) g_B(\Omega_2, P_B) + \Phi_3 g_B(\Omega_1, Q_B) \\ & g_B(\Omega_2, Q_B) + \Phi_4 \mathcal{E}_B(\Omega_1, \Omega_2) - \Big[\frac{H^f(\Omega_1, \Omega_2)}{f} + \mu_2 \frac{\rho f}{f} g(\Omega_1, \Omega_2) \\ & + \mu_1\mu_2 \mathcal{U}(\rho) g(\Omega_1, \Omega_2) + \mu_1 g(\Omega_2, D_{\Omega_1}\rho) - \mu_1^2 \mathcal{U}(\Omega_1)\mathcal{U}(\Omega_2)\Big]. \end{aligned} \tag{4.20}$$

By contracting (4.20) over Ω_1 and Ω_2, we derive

$$\begin{aligned} \overline{scal}^{\mathrm{B}} = {} & \Phi_1(n-1) + \Phi_2 g_{\mathrm{B}}(P_{\mathrm{B}}, P_{\mathrm{B}}) + \Phi_3 g_{\mathrm{B}}(Q_{\mathrm{B}}, Q_{\mathrm{B}}) + \Phi_4 \mathcal{E}_B(e_i, e_1) \\ & - \Big[\frac{\Delta_B}{f} + \mu_2(n-1)\frac{\rho f}{f} + [(n-1)\mu_1\mu_2 - \mu_1^2]\mathcal{U}(\rho) \\ & + \mu_1 \sum_{i=1}^{n-1} g(e_i, D_{e_i}\rho)\Big]. \end{aligned} \tag{4.21}$$

Again, contracting (4.18) over Ω_1 and Ω_2 gives

$$\overline{scal}^{\mathcal{M}} = \Phi_1 n + \Phi_2 g_{\mathrm{B}}(P_{\mathrm{B}}, P_{\mathrm{B}}) + \Phi_3 g_{\mathrm{B}}(Q_{\mathrm{B}}, Q_{\mathrm{B}}) + \Phi_4 \mathcal{E}_{\mathrm{B}}(e_i, e_i). \tag{4.22}$$

Substituting (4.22) into (4.21), we get

$$\overline{scal}^{\mathrm{B}} = \overline{scal}^{\mathcal{M}} - \Phi_1 - \frac{\Delta_{\mathrm{B}} f}{f} - \mu_2(n-1)\frac{\rho f}{f} - [(n-1)\mu_1\mu_2 - \mu_1^2]\mathcal{U}(\rho) - \mu_1 \sum_{i=1}^{n-1} g(e_i, D_{e_i}\rho)\Big]. \tag{4.23}$$

Form Proposition 3.3, we know

$$\begin{aligned} \overline{scal}^{\mathcal{M}} = {} & \overline{scal}^{\mathrm{B}} + (n-1)(\mu_1 + \mu_2)\frac{\rho f}{f} + 2\frac{\Delta_{\mathrm{B}} f}{f} + [2(n-1)\mu_1\mu_2 - (\mu_1^2 + \mu_2^2)]\mathcal{U}(\rho) \\ & + (\mu_1 + \mu_2)\sum_{i=1}^{n-1} g(e_i, D_{e_i}\rho)\Big]. \end{aligned} \tag{4.24}$$

From (4.23) and (4.24), we obtain

$$\begin{aligned} & \Phi_1 + \frac{\Delta_{\mathrm{B}} f}{f} + \mu_2(\overline{n} - 1)\frac{\rho f}{f} + [(n-1)\mu_1\mu_2 - \mu_1^2]\mathcal{U}(\rho) + \mu_1 \sum_{i=1}^{\overline{n}-1} g(e_i, D_{e_i}\rho)\Big] \\ & = (n-1)(\mu_1 + \mu_2)\frac{\rho f}{f} + 2\frac{\Delta_{\mathrm{B}} f}{f} + [2(\overline{n} - 1)\mu_1\mu_2 - (\mu_1^2 + \mu_2^2)]\mathcal{U}(\rho) \\ & + (\mu_1 + \mu_2)\sum_{i=1}^{\overline{n}-1} g(e_i, D_{e_i}\rho)\Big]. \end{aligned} \tag{4.25}$$

Since f is a constant on B and $\rho \in \mathfrak{X}(\mathrm{B})$ is parallel, we get

$$\Phi_1 = [(\overline{n}-1)\mu_1\mu_2 - \mu_2^2]\mathcal{U}(\rho).$$

(ii) Let $\rho \in \mathfrak{X}(I)$. By Proposition 3.2, we have

$$\overline{Ric}(\Omega_1, \rho) = [(n-1)\mu_1\mu_2 - \mu_2^2]\mathcal{U}(\rho)\frac{\Omega_1 f}{f} \tag{4.26}$$

and

$$\overline{Ric}(\rho, \Omega_1) = [\mu_2 - (n-1)\mu_1]\mathcal{U}(\rho)\frac{\Omega_1 f}{f}. \tag{4.27}$$

Since $\mathcal{M}$ is a PGQE manifold, the Ricci curvature satisfies

$$\overline{Ric}(\Omega_1, \rho) = \overline{Ric}(\rho, \Omega_1) = \Phi_1 g(\Omega_1, \rho) + \Phi_2\mathcal{U}(\Omega_1)\mathcal{U}(\rho) + \Phi_3\mathcal{V}(\Omega_1)\mathcal{V}(\rho) + \Phi_4\mathcal{E}(\Omega_1, \rho).$$

Again, we have $g(\Omega_1, \rho) = 0$ for $\Omega_1 \in \mathfrak{X}(\mathrm{B})$ and $\rho \in \mathfrak{X}(I)$. Thus, we obtain

$$\Omega_1 f = 0,$$

where $\mu_2 \neq (n-1)\mu_1$. This implies that f is constant on B.

(iii) Suppose that B is a PGQE manifold with respect to the Levi-Civita connection, we have

$$\overline{Ric}^{\mathrm{B}}(\Omega_1, \Omega_2) = \Phi_1 g(\Omega_1, \Omega_2) + \Phi_2\mathcal{U}(\Omega_1)\mathcal{U}(\Omega_2) + \Phi_3\mathcal{V}(\Omega_1)\mathcal{V}(\Omega_2) + \Phi_4\mathcal{E}(\Omega_1, \Omega_2), \tag{4.28}$$

for all Ω_1, Ω_2 tangent to B.
From Proposition 3.2, we know that the Ricci curvature of $\mathcal{M}$ is related to that of B by the following equation

$$\overline{Ric}^{\mathcal{M}}(\Omega_1, \Omega_2) = \overline{Ric}^{\mathrm{B}}(\Omega_1, \Omega_2) + [(n-1)\mu_1\mu_2 - \mu_2^2]\mathcal{U}(\rho)g(\Omega_1, \Omega_2) + \frac{H^f(\Omega_1, \Omega_2)}{f},$$

for all $\rho \in \mathfrak{X}(I)$.
Since f is a constant, $H^f(\Omega_1, \Omega_2) = 0 \ \forall\ \Omega_1, \Omega_2 \in \mathfrak{X}(\mathrm{B})$. Thus, the equation simplifies to

$$\overline{Ric}^{\mathcal{M}}(\Omega_1, \Omega_2) = \overline{Ric}^{\mathrm{B}}(\Omega_1, \Omega_2) + [(n-1)\mu_1\mu_2 - \mu_2^2]\mathcal{U}(\rho)g(\Omega_1, \Omega_2). \tag{4.29}$$

Now, substituting equation (4.28) into (4.29), we obtain

$$\begin{aligned}\overline{Ric}^{\mathcal{M}}(\Omega_1, \Omega_2) = (\Phi_1 + [(n-1)\mu_1\mu_2 - \mu_2^2]\mathcal{U}(\rho))g(\Omega_1, \Omega_2) + \Phi_2\mathcal{U}(\Omega_1)\mathcal{U}(\Omega_2) + \Phi_3\mathcal{V}(\Omega_1)\mathcal{V}(\Omega_2) \\ + \Phi_4\mathcal{E}(\Omega_1, \Omega_2),\end{aligned} \tag{4.30}$$

which shows that $\mathcal{M}$ is a PGQE manifold with respect to a quarter-symmetric connection.

Theorem 4.3 *Let $(\mathcal{M}, g)$ be a warped product manifold of the form $I \times_f \mathrm{B}$. If the two generators P and Q of a PGQE manifold are parallel to I with respect to a quarter-symmetric connection, then $\mathcal{M}$ is a PQE manifold with respect to a quarter-symmetric connection.*

Proof. Let P be a parallel vector field, then $\overline{K}(\Omega_1, \Omega_2)P = 0$. Thus,

$$\overline{Ric}(\Omega_1, P) = 0. \tag{4.31}$$

Consider the following expressions for P and Q

$$P = P_{\mathrm{B}} + f^2 P_I \quad \textit{and} \quad Q = Q_{\mathrm{B}} + f^2 Q_I. \tag{4.32}$$

Substituting $\Omega_2 = P$ and using (4.32) in (1.6), we get

$$\begin{aligned}\overline{Ric}(\Omega_1, P) &= \Phi_1 g(\Omega_1, P) + \Phi_2 \mathcal{U}(\Omega_1)\mathcal{U}(P) + \Phi_3 \mathcal{V}(\Omega_1)\mathcal{V}(P) + \Phi_4 \mathcal{E}(\Omega_1, P) \\ &= \{\Phi_1 + \Phi_2(f^4+1)\} g_I(\Omega_1, P_I) f^2 + c(f^4+1) g_I(\Omega_1, Q_I) f^2.\end{aligned} \tag{4.33}$$

From (4.3), we have

$$\begin{aligned}\overline{Ric}_{\mathcal{M}}(\Omega_1, \Omega_2) &= Ric_I(\Omega_1, \Omega_2) + e^q \Big[\frac{n-1}{4}(q')^2 + \frac{1}{2}\Big\{(n-1)\mu_1 \\ &\quad + (n-2)\mu_2\Big\} q' + \mu_2^2 + \frac{1}{2} q'' + (1-n)\mu_1\mu_2\Big] g_I(\Omega_1, \Omega_2),\end{aligned} \tag{4.34}$$

for all Ω_1, Ω_2 on I.

Since P is parallel to I, we can take from the above relation

$$\begin{aligned}\overline{Ric}_{\mathcal{M}}(\Omega_1, P) &= e^q \Big[\frac{n-1}{4}(q')^2 + \frac{1}{2}\Big\{(n-1)\mu_1 + (n-2)\mu_2\Big\} q' + \mu_2^2 + \frac{1}{2} q'' \\ &\quad + (1-n)\mu_1\mu_2\Big] g_I(\Omega_1, P_B + f^2 P_I) \\ &= f^2 e^q \Big[\frac{n-1}{4}(q')^2 + \frac{1}{2}\Big\{(n-1)\mu_1 + (n-2)\mu_2\Big\} q' \\ &\quad + \mu_2^2 + \frac{1}{2} q'' + (1-n)\mu_1\mu_2\Big] g_I(\Omega_1, P).\end{aligned} \tag{4.35}$$

Comparing (4.33) and (4.35), we obtain

$$\Phi_3 = 0. \tag{4.36}$$

Using (4.36) in (1.6), we get

$$Ric(\Omega_1, \Omega_2) = \Phi_1 g(\Omega_1, \Omega_2) + \Phi_2 \mathcal{U}(\Omega_1)\mathcal{U}(\Omega_2) + \Phi_4 \mathcal{E}(\Omega_1, \Omega_2),$$

i.e., PQE manifold with respect to quarter symmetric connection. Similarly, if Q is parallel to I, we also obtain

$$\Phi_3 = 0.$$

So, the manifold also becomes a PQE manifold with respect to quarter symmetric connection.

Theorem 4.4 *Let $(\mathcal{M}, g)$ be a warped product $\mathrm{B} \times_f \mathcal{F}$ of a complete connected k-dimensional $(1 < k < n)$ Riemannian manifold B and $(n-k)$-dimensional Riemannian manifold $\mathcal{F}$. Then*

(i) If $(\mathcal{M}, g)$ is a manifold of PGQC curvature, the Hessian of f is proportional to the metric tensor g_{B}, and the associated vector fields W and W' are a general vector field on $\mathcal{M}$ or satisfy W, $W' \in \mathfrak{X}(\mathrm{B})$, then B is a two-dimensional NQE manifold.

(ii) If $(\mathcal{M}, g)$ is a manifold of PGQC curvature with associated vector fields W, $W' \in \mathfrak{X}(\mathcal{F})$, then B is a NQE manifold.

Proof. Let $\mathcal{M}$ be a manifold of PGQC curvature. Using equation (1.8), the curvature tensor can be expressed as

$$\begin{aligned}\overline{\mathcal{K}}(\Omega_1, \Omega_2, \Omega_3, \Omega_4) &= f_1[g(\Omega_2, \Omega_3) g(\Omega_1, \Omega_4) - g(\Omega_1, \Omega_3) g(\Omega_2, \Omega_4)] \\ &\quad + f_2[g(\Omega_1, \Omega_4)\mathcal{U}(\Omega_2)\mathcal{U}(\Omega_3) - g(\Omega_2, \Omega_4)\mathcal{U}(\Omega_1)\mathcal{U}(\Omega_3) \\ &\quad + g(\Omega_2, \Omega_3)\mathcal{U}(\Omega_1)\mathcal{U}(\Omega_4) - g(\Omega_1, \Omega_3)\mathcal{U}(\Omega_2)\mathcal{U}(\Omega_4)] \\ &\quad + f_3[g(\Omega_1, \Omega_4)\mathcal{V}(\Omega_2)\mathcal{V}(\Omega_3) - g(\Omega_2, \Omega_4)\mathcal{V}(\Omega_1)\mathcal{V}(\Omega_3) \\ &\quad + g(\Omega_2, \Omega_3)\mathcal{V}(\Omega_1)\mathcal{V}(\Omega_4) - g(\Omega_1, \Omega_3)\mathcal{V}(\Omega_2)\mathcal{V}(\Omega_4)] \\ &\quad + f_4[\mathcal{E}(\Omega_2, \Omega_3) g(\Omega_1, \Omega_4) - \mathcal{E}(\Omega_1, \Omega_3) g(\Omega_2, \Omega_4) \\ &\quad + \mathcal{E}(\Omega_1, \Omega_4) g(\Omega_2, \Omega_3) - \mathcal{E}(\Omega_2, \Omega_4) g(\Omega_1, \Omega_3)],\end{aligned} \tag{4.37}$$

for all Ω_1, Ω_2, Ω_3, Ω_4 on B.

Decomposing the vector fields W and W' uniquely into components

$$W = W_{\rm B} + W_{\mathcal{F}} \quad and \quad W' = W'_{\rm B} + W'_{\mathcal{F}},$$

where $W_{\rm B}$, $W'_{\rm B}$ and $W_{\mathcal{F}}$, $W'_{\mathcal{F}}$ in B and $\mathcal{F}$, respectively. Then

$$\begin{aligned} g(\Omega_1, W) &= g(\Omega_1, W_{\rm B}) = g_{\rm B}(\Omega_1, W_{\rm B}) = \mathcal{U}(\Omega_1), \\ g(\Omega_1, W') &= g(\Omega_1, W'_{\rm B}) = g_{\rm B}(\Omega_1, W'_{\rm B}) = \mathcal{V}(\Omega_1). \end{aligned} \tag{4.38}$$

Making use of (3.1) and (4.38) in (4.37) and applying Lemma 3.1 with $\Omega_1 = \Omega_4 = e_i$, where e_i is an orthonormal basis, we obtain

$$\begin{aligned} Ric_{\rm B}(\Omega_2, \Omega_3) = {} & [\Phi_1(k-1) + \Phi_2 g_{\rm B}(W_{\rm B}, W_{\rm B}) + \Phi_3 g_{\rm B}(W'_{\rm B}, W'_{\rm B}) + \Phi_4 \mathcal{E}_{\rm B}(e_i, e_i)] g_{\rm B}(\Omega_2, \Omega_3) \\ & + \Phi_3(k-2)\mathcal{V}(\Omega_2)\mathcal{V}(\Omega_3) + k\Phi_4 \mathcal{E}_B(\Omega_2, \Omega_3). \end{aligned} \tag{4.39}$$

This shows that B is a PQE manifold.

Again, putting $\Omega_2 = \Omega = e_i$, we obtain the scalar curvature

$$scal_{\rm B} = (k-1)[\Phi_1 k + 2\Phi_2 g_{\rm B}(W_{\rm B}, W_{\rm B}) + 2\Phi_3 g_{\rm B}(W'_{\rm B}, W'_{\rm B})] + 2k\Phi_4 \mathcal{E}_{\rm B}(e_i, e_i). \tag{4.40}$$

Using (3.2) in (4.40), we infer

$$\frac{\Delta f}{f} = \frac{\Phi_1 k + \Phi_2 g_{\rm B}(W_{\rm B}, W_{\rm B}) + \Phi_3 g_{\rm B}(W'_{\rm B}, W'_{\rm B}) + k\Phi_4 \mathcal{E}_{\rm B}(e_i, e_i)}{2}. \tag{4.41}$$

Since the metric tensor $g_{\rm B}$ is proportional to the Hesssian of f, we have

$$H^f(\Omega_1, \Omega_2) = \frac{\Delta f}{k} g_{\rm B}(\Omega_1, \Omega_2). \tag{4.42}$$

Using (4.40) and (4.41) in (4.42), we get

$$H^f(\Omega_1, \Omega_2) + Rf g_{\rm B}(\Omega_1, \Omega_2) = 0,$$

where $R = \dfrac{1}{2k(k-1)}\Big(k(3-k)d\mathcal{E}_{\rm B}(e_i, e_i) + (k-1)[\Phi_2 g_{\rm B}(W_{\rm B} W_{\rm B}) + \Phi_3 g_{\rm B}(W'_{\rm B} W'_{\rm B})] - sacl_{\rm B}\Big)$. By OBATA's theorem [18], in the $(k+1)$-dimensional Euclidean space, B is isometric to the sphere of radius $\dfrac{1}{\sqrt{R}}$. This implies B is an Einstein manifold. Since $\Phi_2 \neq 0$, $\Phi_3 \neq 0$, we conclude $k = 2$. Thus, B is a two-dimensional nearly quasi-Einstein manifold.

Suppose the associated vector fields W, $W' \in \mathfrak{X}(\rm B)$. Using equations (3.1) and (4.37) and substituting $\Omega_1 = \Omega_4 = e_i$, we derive the following expression

$$\begin{aligned} scal_{\rm B}(\Omega_2, \Omega_3) = {} & [\Phi_1(k-1) + \Phi_2 + \Phi_3 + \Phi_4 \mathcal{E}_{\rm B}(e_i, e_i)] g_{\rm B}(\Omega_2, \Omega_3) \\ & + \Phi_2(k-2) g_{\rm B}(\Omega_2, W) g_{\rm B}(\Omega_3, W) \\ & + \Phi_3(k-2) g_{\rm B}(\Omega_2, W') g_{\rm B}(\Omega_3, W') + k\Phi_4 \mathcal{E}_{\rm B}(\Omega_2, \Omega_3), \end{aligned} \tag{4.43}$$

which shows that B is a PGQE manifold.
Now, substituting $\Omega_2 = \Omega_3 = e_i$ in (4.43), we obtain

$$scal_{\rm B} = (k-1)[\Phi_1 k + 2\Phi_2 + 2\Phi_3] + 2k\Phi_4 \mathcal{E}_{\rm B}(e_i, e_i). \tag{4.44}$$

In view of (3.1) and (4.37) (for W, $W' \in \mathfrak{X}(\rm B)$), we derive

$$\frac{\Delta f}{f} = \frac{\Phi_1 k + \Phi_2 + \Phi_3 + k\Phi_4 \mathcal{E}_{\rm B}(e_i, e_i)}{2}. \tag{4.45}$$

Since the metric tensor $g_{\rm B}$ is proportional to the Hesssian of f, it can be expressed as

$$H^f(\Omega_1,\Omega_2)=\frac{\Delta f}{k}g_{\rm B}(\Omega_1,\Omega_2). \tag{4.46}$$

Using (4.44) and (4.45) in (4.46), we get

$$H^f(\Omega_1,\Omega_2)+Rfg_{\rm B}(\Omega_1,\Omega_2)=0,$$

where $R=\frac{1}{2k(k-1)}\Big(k(3-k)\Phi_4\mathcal{E}_{\rm B}(e_i,e_i)+(k-1)\Phi_2-sacl_{\rm B}\Big)$. BY OBATA's theorem [18], in a $(k+1)$-dimensional Euclidean space, B is isometric to the sphere of radius $\frac{1}{\sqrt{R}}$. Therefore, B is an Einstein manifold.

Since $\Phi_2\neq 0$ and $\Phi_3\neq 0$, it follows that $k=2$. As a result, B is a two-dimensional nearly quasi-Einstein manifold.

Suppose that the associated vector fields W, $W'\in\mathfrak{X}(\mathcal{F})$, the relation (4.37) reduces to

$$\begin{aligned}\overline{\mathcal{K}}(\Omega_1,\Omega_2,\Omega_3,\Omega_4)&=\Phi_1[g(\Omega_2,\Omega_3)g(\Omega_1,\Omega_4)-g(\Omega_1,\Omega_3)g(\Omega_2,\Omega_4)]\\&+\Phi_4[g(\Omega_2,\Omega_3)\mathcal{E}(\Omega_1,\Omega_4)-g(\Omega_1,\Omega_3)\mathcal{E}(\Omega_2,\Omega_4)\\&+g(\Omega_1,\Omega_4)\mathcal{E}(\Omega_2,\Omega_3)-g(\Omega_2,\Omega_4)\mathcal{E}(\Omega_1,\Omega_3)].\end{aligned} \tag{4.47}$$

Making use of (3.1) in (4.47), we get

$$\begin{aligned}\overline{\mathcal{K}}(\Omega_1,\Omega_2,\Omega_3,\Omega_4)&=\Phi_1[g_{\rm B}(\Omega_2,\Omega_3)g_{\rm B}(\Omega_1,\Omega_4)-g_{\rm B}(\Omega_1,\Omega_3)g_{\rm B}(\Omega_2,\Omega_4)]\\&+\Phi_4[g_{\rm B}(\Omega_2,\Omega_3)\mathcal{E}_{\rm B}(\Omega_1,\Omega_4)]-g_{\rm B}(\Omega_1,\Omega_3)\mathcal{E}_{\rm B}(\Omega_2,\Omega_4)\\&+g_{\rm B}(\Omega_1,\Omega_4)\mathcal{E}_{\rm B}(\Omega_2,\Omega_3)-g_{\rm B}(\Omega_2,\Omega_4)\mathcal{E}_{\rm B}(\Omega_1,\Omega_3).\end{aligned} \tag{4.48}$$

Contracting of (4.48) over Ω_1 and Ω_4, we obtain

$$Ric_{\rm B}(\Omega_2,\Omega_3)=[\Phi_1(k-1)+d\mathcal{E}_{\rm B}(e_i,e_i)]g_{\rm B}(\Omega_2,\Omega_3)+\Phi_4\mathcal{E}_{\rm B}(\Omega_2,\Omega_3), \tag{4.49}$$

which shows that B is a NQE manifold with $scal_{\rm B}=\Phi_1k(k-1)+2k\Phi_4\mathcal{E}_{\rm B}(e_i,e_i)$.

Theorem 4.5 *Let $(\mathcal{M},g)$ be a warped product $\mathrm{B}\times_f I$ of a complete connected $(n-1)$-dimensional Riemannian manifold* B *and a one-dimensional Riemannian manifold I. If $(\mathcal{M},g)$ is a PGQE manifold with constant associated scalars Φ_1, Φ_2, Φ_3 and the Hessian of f is proportional to the metric tensor $g_{\rm B}$, then $(\mathrm{B},g_{\rm B})$ is a $(n-1)$-dimensional sphere with radius $rd=\frac{n-1}{\sqrt{scal_{\rm B}+\Phi_1}}$.*

Proof. Suppose that $\mathcal{M}$ is a warped product manifold. Then by use of Lemma 3.2 we can write

$$Ric_{\rm B}(\Omega_1,\Omega_2)=Ric_{\mathcal{M}}(\Omega_1,\Omega_2)+\frac{1}{f}H^f(\Omega_1,\Omega_2), \tag{4.50}$$

for all Ω_1, Ω_2 on B.

Since $\mathcal{M}$ is a PGQE manifold, we have

$$Ric_{\mathcal{M}}(\Omega_1,\Omega_2)=ag(\Omega_1,\Omega_2)=\Phi_2\mathcal{U}(\Omega_1)\mathcal{U}(\Omega_2)+\Phi_3\mathcal{V}(\Omega_1)\mathcal{V}(\Omega_2)+\Phi_4\mathcal{E}(\Omega_1,\Omega_2). \tag{4.51}$$

Decomposing the vector fields P and P' uniquely into its components P_I, P_F and P'_I, P'_F on B and I, respectively, we can write

$$P=P_{\rm B}+P_I\quad P'=P'_{\rm B}+P'_I. \tag{4.52}$$

In view of (3.1),(4.51) and (4.52) the relation (4.50) can be write as

$$\begin{aligned}Ric_{\rm B}(\Omega_1,\Omega_2)&=\Phi_1g_{\rm B}(\Omega_1,\Omega_2)+\Phi_2g_{\rm B}(\Omega_1,P_{\rm B})g_{\rm B}(\Omega_2,P_{\rm B})\\&+\Phi_3g_{\rm B}(\Omega_1,P'_{\rm B})g_{\rm B}(\Omega_2,P'_{\rm B})+\Phi_4\mathcal{E}_{\rm B}(\Omega_1,\Omega_2)+\frac{1}{f}H^f(\Omega_1,\Omega_2).\end{aligned} \tag{4.53}$$

Contracting the above relation over Ω_1 and Ω_2, we get

$$scal_{\mathrm{B}} = \Phi_1(n-1) + \Phi_2 g_{\mathrm{B}}(P_{\mathrm{B}}, P_{\mathrm{B}}) + \Phi_3 g_{\mathrm{B}}(P'_{\mathrm{B}}, P'_{\mathrm{B}}) + \Phi_4 \mathcal{E}_{\mathrm{B}}(e_i, e_i) + \frac{\Delta f}{f}. \tag{4.54}$$

Similarly, contracting (4.51) over Ω_1 and Ω_2 yields

$$scal_{\mathcal{M}} = \Phi_1 n + \Phi_2 g_{\mathrm{B}}(P_{\mathrm{B}}, P_{\mathrm{B}}) + \Phi_3 g_{\mathrm{B}}(P'_{\mathrm{B}}, P'_{\mathrm{B}}) + \Phi_4 \mathcal{E}_{\mathrm{B}}(e_i, e_i). \tag{4.55}$$

Using (4.55) in (4.54), we get

$$scal_{\mathrm{B}} = scal_{\mathcal{M}} - \Phi_1 + \frac{\Delta f}{f}. \tag{4.56}$$

In view of Lemma 3.2, we know that

$$-\frac{scal_{\mathcal{M}}}{n} = \frac{\Delta f}{f}. \tag{4.57}$$

Substituting equation (4.57) into (4.56), we obtain

$$scal_{\mathrm{B}} = \frac{n-1}{n} scal_{\mathcal{M}} - \Phi_1. \tag{4.58}$$

Since the metric tensor g_{B} is proportional to the Hesssian of f, we can write as

$$H^f(\Omega_1, \Omega_2) = \frac{\Delta f}{n-1} g_{\mathrm{B}}(\Omega_1, \Omega_2). \tag{4.59}$$

From (4.57), we have

$$\frac{\Delta f}{n-1} = -\frac{1}{n(n-1)} scal_{\mathcal{M}} f. \tag{4.60}$$

Using equations (4.59) and (4.60), we arrive at the relation

$$H^f(\Omega_1, \Omega_2) + \frac{scal_{\mathrm{B}} + \Phi_1}{(n-1)^2} f g_{\mathrm{B}}(\Omega_1, \Omega_2) = 0.$$

Thus, B is isometric to the sphere of radius $(n-1)$ dimensional $rd = \dfrac{n-1}{\sqrt{scal_{\mathrm{B}} + \Phi_1}}$.

Corollary 4.1 *Let*

$$\mathcal{M} = \mathrm{B}_1 \times_{f_1} \mathrm{B}_2 \times_{f_2} \cdots \times_{f_n} \mathcal{F}$$

be a multiply warped product manifold, where $\mathrm{B}_1, \mathrm{B}_2, \ldots, \mathrm{B}_n$ *are Riemannian manifolds,* $\mathcal{F}$ *is a Riemannian or Lorentzian manifold, and* $f_1, f_2, \ldots, f_n$ *are smooth warping functions defined on* $\mathrm{B}_1, \mathrm{B}_2, \ldots, \mathrm{B}_n$*, respectively. Assume the following conditions:*

1. ***Hessian Condition:***
   For each warping function f_i*, the Hessian satisfies*

   $$H_{f_i}(1,2) = \Delta_{\mathrm{B}_i} f_i \frac{g_{\mathrm{B}_i}(1,2)}{\dim(\mathrm{B}_i)},$$

   where H_{f_i} *is the Hessian of* f_i*,* Δ_{B_i} *is the Laplacian on* B_i*, and* g_{B_i} *is the metric on* B_i*.*

2. ***Relation to Scalar Curvature:***
   The warping functions satisfy

   $$\Delta_{\mathrm{B}_i} f_i = -\frac{\mathrm{scal}_{\mathrm{B}_i}}{\dim(\mathrm{B}_i) + 1} f_i,$$

   where $\mathrm{scal}_{\mathrm{B}_i}$ *is the scalar curvature of* B_i*.*

3. ***Derivatives of Warping Functions:***
For each i,

$$\frac{|\nabla f_i|^2}{f_i^2} = \text{constant on each } \mathrm{B}_i.$$

4. ***Associated 1-Forms:***
The associated 1-form satisfies

$$A_{\mathrm{B}_i}(X) = \frac{\nabla_{\mathrm{B}_i} f_i}{f_i}.$$

5. ***Semi-Symmetric Affine Connection:***
The affine connection on $\mathcal{M}$ is semi-symmetric with torsion

$$T(1,2) = \mathcal{U}(2)\,\phi(1) - \mathcal{U}(1)\,\phi(2),$$

where $\mathcal{U}$ is an associated 1-form and ϕ is a vector field.

Under these conditions, the manifold $\mathcal{M}$ inherits a PGQE structure from its components, and its Ricci tensor can be written in terms of the Ricci tensors of the base manifolds B_i, the fiber $\mathcal{F}$, and the contributions of the warping functions.

Proof. Consider the Ricci tensor of the multiply warped product

$$\mathcal{M} = \mathrm{B}_1 \times_{f_1} \mathrm{B}_2 \times_{f_2} \cdots \times_{f_n} \mathcal{F}.$$

For a singly warped product, we have

$$\mathrm{Ric}_{\mathcal{M}}(1,2) = \mathrm{Ric}_{\mathrm{B}}(1,2) - m\,\frac{H_f(1,2)}{f},$$

where $m = \dim(\mathcal{F})$.

In the multiply warped case, this generalizes to

$$\mathrm{Ric}_{\mathcal{M}}(1,2) = \sum_{i=1}^{n} \left(\mathrm{Ric}_{\mathrm{B}_i}(1,2) - \dim(\mathrm{B}_{i+1})\,\frac{H_{f_i}(1,2)}{f_i} \right) + \mathrm{Ric}_{\mathcal{F}}(1,2),$$

where H_{f_i} is the Hessian of f_i and $\mathrm{Ric}_{\mathrm{B}_i}$, $\mathrm{Ric}_{\mathcal{F}}$ are the Ricci tensors of B_i and $\mathcal{F}$, respectively.

The imposed conditions on the warping functions—their Laplacians, gradients, and associated 1-forms—enter directly into these correction terms, modifying the Ricci tensor. The semi-symmetric affine connection introduces additional torsion-dependent components, producing a PGQE-type decomposition of the Ricci tensor:

$$\mathrm{Ric}_{\mathcal{M}}(1,2) = \Phi_1\, g_{\mathcal{M}}(1,2) + \Phi_2\,\mathcal{U}(1)\mathcal{U}(2) + \Phi_3\,\mathcal{V}(1)\mathcal{V}(2) + \Phi_4\,\mathcal{E}(1,2),$$

where Φ_i are scalar functions determined by the geometry of $\mathcal{M}$, and $\mathcal{E}$ is a symmetric deformation tensor.

Thus, the multiply warped product manifold $\mathcal{M}$ inherits the PGQE structure, with its Ricci tensor reflecting contributions from the base manifolds, the fiber, the warping functions, and the associated 1-forms. □

5. Examples of PGQE manifolds

Example 5.1 We define a Riemannian metric g in 4-dimensional space $\mathbb{R}^4$ by the relation

$$ds^2 = g_{ij}dx^i dx^j = (1+2p)[(dx^1)^2 + (dx^2)^2 + (dx^3)^2 + (dx^4)^2] \tag{5.1}$$

where x^1, x^2, x^3, x^4 are non-zero finite and $p = e^{x^1} k^{-2}$. Then the covariant and contravariant components of the metric tensor are

$$g_{11} = g_{22} = g_{33} = g_{44} = (1 + 2p), \quad g_{ij} = 0 \quad \forall \quad i \neq j \tag{5.2}$$

and

$$g^{11} = g^{22} = g^{33} = g^{44} = \frac{1}{1+2p}, \quad g^{ij} = 0 \quad \forall \quad i \neq j. \tag{5.3}$$

The only non-vanishing components of the Christoffel symbols are

$$\begin{aligned}
&\begin{Bmatrix} 1 \\ 11 \end{Bmatrix} = \begin{Bmatrix} 2 \\ 12 \end{Bmatrix} = \begin{Bmatrix} 3 \\ 13 \end{Bmatrix} = \begin{Bmatrix} 4 \\ 14 \end{Bmatrix} = \frac{\partial}{\partial x^1} \begin{Bmatrix} 4 \\ 14 \end{Bmatrix} = \frac{p}{1+2p}, \\
&\begin{Bmatrix} 1 \\ 22 \end{Bmatrix} = \begin{Bmatrix} 1 \\ 33 \end{Bmatrix} = \begin{Bmatrix} 1 \\ 44 \end{Bmatrix} = \frac{-p}{1+2p}.
\end{aligned} \tag{5.4}$$

The non-zero derivatives of (5.4), we have partial differential equations

$$\begin{aligned}
&\frac{\partial}{\partial x^1} \begin{Bmatrix} 1 \\ 11 \end{Bmatrix} = \frac{\partial}{\partial x^1} \begin{Bmatrix} 2 \\ 12 \end{Bmatrix} = \frac{\partial}{\partial x^1} \begin{Bmatrix} 3 \\ 13 \end{Bmatrix} = \frac{p}{(1+2p)^2}, \\
&\frac{\partial}{\partial x^1} \begin{Bmatrix} 1 \\ 22 \end{Bmatrix} = \frac{\partial}{\partial x^1} \begin{Bmatrix} 1 \\ 33 \end{Bmatrix} = \frac{\partial}{\partial x^1} \begin{Bmatrix} 1 \\ 44 \end{Bmatrix} = \frac{-p}{(1+2p)^2}.
\end{aligned} \tag{5.5}$$

Thus, the non-zero components of curvature tensor, up to symmetry are as follows

$$\overline{\mathcal{K}}_{1221} = \overline{\mathcal{K}}_{1331} = \overline{\mathcal{K}}_{1441} = \frac{p}{1+2p},$$

$$\overline{\mathcal{K}}_{2332} = \overline{\mathcal{K}}_{2442} = \overline{\mathcal{K}}_{3443} = \frac{p^2}{1+2p}$$

and the Ricci tensor given by

$$Ric_{11} = g^{jh}\overline{\mathcal{K}}_{1j1h} = g^{22}\overline{\mathcal{K}}_{1212} + g^{33}\overline{\mathcal{K}}_{1313} + g^{44}\overline{\mathcal{K}}_{1414} = \frac{3p}{(1+2p)^2},$$

$$Ric_{22} = g^{jh}\overline{\mathcal{K}}_{2j2h} = g^{11}\overline{\mathcal{K}}_{2121} + g^{33}\overline{\mathcal{K}}_{2323} + g^{44}\overline{\mathcal{K}}_{2424} = \frac{p}{(1+2p)},$$

$$Ric_{33} = g^{jh}\overline{\mathcal{K}}_{3j3h} = g^{11}\overline{\mathcal{K}}_{3131} + g^{22}\overline{\mathcal{K}}_{3232} + g^{44}\overline{\mathcal{K}}_{3434} = \frac{p}{(1+2p)},$$

$$Ric_{44} = g^{jh}\overline{\mathcal{K}}_{4j4h} = g^{11}\overline{\mathcal{K}}_{4141} + g^{22}\overline{\mathcal{K}}_{4242} + g^{33}\overline{\mathcal{K}}_{4343} = \frac{p}{(1+2p)}.$$

Let us consider the associated scalars $\Phi_1, \Phi_2, \Phi_3, \Phi_4$ and the associated tensor $\mathcal{E}$ be defined by

$$\Phi_1 = \frac{3p}{(1+2p)^3}, \quad \Phi_2 = 2p, \quad \Phi_3 = \frac{-p}{(1+2p)^2}, \quad \Phi_4 = \frac{-2\sqrt{p}}{(1+2p)^2}$$

and

$$\mathcal{E}_{ij} = \begin{cases} \sqrt{p}, & \text{if i=j=1} \\ -\sqrt{p}, & \text{if i=j=3} \\ 0, & \text{otherwise} \end{cases}$$

the 1-forms

$$\mathcal{U}_i(x) = \begin{cases} \frac{1}{1+2p}, & \text{if i=1} \\ 0, & \text{otherwise} \end{cases} \quad and \quad \mathcal{V}_i(x) = \begin{cases} \sqrt{p}, & \text{if i=2} \\ -\sqrt{p}, & \text{if i=2} \\ 0, & \text{otherwise} \end{cases},$$

where generators are unit vector fields, then from (1.6), we have

$$Ric_{11} = \Phi_1 g_{11} + \Phi_2 \mathcal{U}_1\mathcal{U}_1 + \Phi_3 \mathcal{V}_1\mathcal{V}_1 + \Phi_4 \mathcal{E}_{11}, \tag{5.6}$$

$$Ric_{22} = \Phi_1 g_{22} + \Phi_2 \mathcal{U}_2\mathcal{U}_2 + \Phi_3 \mathcal{V}_2\mathcal{V}_2 + \Phi_4 \mathcal{E}_{22}, \tag{5.7}$$

$$Ric_{33} = \Phi_1 g_{33} + \Phi_2 \mathcal{U}_3\mathcal{U}_3 + \Phi_3 \mathcal{V}_3\mathcal{V}_3 + \Phi_4 \mathcal{E}_{33}, \tag{5.8}$$

$$Ric_{44} = \Phi_1 g_{44} + \Phi_2 \mathcal{U}_4\mathcal{U}_4 + \Phi_3 \mathcal{V}_4\mathcal{V}_4 + \Phi_4 \mathcal{E}_{44}, \tag{5.9}$$

$$\begin{aligned} R.H.S. \quad of \quad (5.6) &= \Phi_1 g_{11} + \Phi_2 \mathcal{U}_1\mathcal{U}_1 + \Phi_3 \mathcal{V}_1\mathcal{V}_1 + \Phi_4 \mathcal{E}_{11} \\ &= \frac{3p}{(1+2p)^2} + \frac{2p}{(1+2p)^2} - \frac{2p}{(1+2p)^2} \\ &= \frac{3p}{(1+2p)^2} \\ &= L.H.S. \quad of \quad (5.6). \end{aligned}$$

By similar argument it can be shown that (5.7) and (5.9) are also true.
Hence $(\mathbb{R}^4, g)$ is a PGQE manifold.

Example 5.2 The Lorentzian manifold $(\mathbb{R}^4, g)$ endowed with the metric given by

$$ds^2 = g_{ij}dx^i dx^j = -(1+2p)(dx^1)^2 + (1+2p)[(dx^2)^2 + (dx^3)^2 + (dx^4)^2],$$

where x^1, x^2, x^3 are non-zero finite, then $(\mathbb{R}^4, g)$ is a PGQE manifold.

6. Example of PGQE warped product manifold

In this section, we present a four-dimensional example of a PGQE warped product manifold.

Example 6.1 Let $(\mathbb{R}^4, g)$ be a Riemannian manifold equipped with the metric

$$ds^2 = g_{ij}dx^i dx^j = (1+2p)[(dx^1)^2 + (dx^2)^2 + (dx^3)^2 + (dx^4)^2],$$

where x^1, x^2, x^3, and x^4 are non-zero and finite. To construct the 4-dim PGQE warped product manifold, we define a warping function $f : \mathbb{R}^3 \to (0, \infty)$ by $f(x^1, x^2, x^3) = \sqrt{1+2p}$, where $f > 0$ is a smooth function. This allows us to define the warped product. The setup allows us to define a warped product manifold $\mathbb{R}^3 \times \mathbb{R}$ and has the form $\mathrm{B} \times_f \mathcal{F}$, where $\mathrm{B} = \mathbb{R}^3$ is the base and $\mathcal{F} = \mathbb{R}$ is the fiber.

The metric on the warped product manifold is given by

$$ds^2_{\mathcal{M}} = ds^2_{\mathrm{B}} + f^2 ds^2_{\mathcal{F}}.$$

Substituting, we can write

$$ds^2 = g_{ij}dx^i dx^j = (1+2p)[(dx^1)^2 + (dx^2)^2 + (dx^3)^2] + \sqrt{1+2p}(dx^4)^2.$$

This metric represents a four-dimensional PGQE warped product manifold, demonstrating its structure and properties.

7. Discussion

This study develops the theory of pseudo-generalized quasi-Einstein (PGQE) manifolds by exploring their behavior on warped product geometries equipped with affine connections. The results provide clear insights into how PGQE structures relate to both the global warped product and the geometry of its components. Theorems 3.1 and 3.2 show that when a warped product $I \times F$ admits a PGQE structure with respect to a quarter-symmetric connection, the fiber F automatically inherits a PGQE structure under the Levi-Civita connection. This demonstrates a strong structural link between the warped product and its factors.

A key outcome of this study is that under specific conditions on the associated vector field ρ and the parameters μ_1, μ_2, the warping function f must be constant. Moreover, Theorem 3.3 indicates that PGQE manifolds reduce to pseudo quasi-Einstein manifolds when the generating vector fields are parallel, establishing a natural hierarchy within generalized Einstein structures. Theorem 3.4 introduces dimensional rigidity by proving that certain proportionality conditions on the Hessian force the base manifold to be two-dimensional and nearly quasi-Einstein, echoing classical rigidity results. Further, Theorem 3.5 demonstrates that the base manifold becomes isometric to a sphere under suitable assumptions, linking PGQE geometry to constant curvature models with potential applications in symmetric cosmological frameworks. Collectively, these results broaden existing research on generalized quasi-Einstein manifolds and deepen the geometric framework through the inclusion of affine connections.

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Mohd Vasiulla,
Department of Mathematics, School of CS & AI,
SR University, Warangal, India.
E-mail address: vsmlk45@gmail.com

and

Vineet Kumar,
Department of Applied Sciences & Humanities,
Meerut Institute of Engineering & Technology,
India.
E-mail address: vineetaggarwal32@gmail.com

and

Mohabbat Ali,
School of Basic and Applied Sciences,
K. R. Mangalam University,
India.
E-mail address: ali.math509@gmail.com

and

Sanjay Tyagi,
Department of Physics, Sri Aurobindo College,
University of Delhi, Delhi-110017,
India
E-mail address: sanjaytyagi43@gmail.com

and

Ashish Tyagi,
Department of Physics, Swami Shraddhanand College,
University of Delhi, Delhi-110036,
India
E-mail address: ashish.tyagi@ss.du.ac.in