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Advanced Results on Independent and Hinge Domination Numbers of Some Graphs Formed by Performing Certain Graph Operations

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ABSTRACT: The concept of domination in graphs plays an eminent role in graph theory. The sources of domination can be seen in a wide range of real-world situations such as radio broadcasting, computer communication networks, school bus routing, electrical power networks, influence in social networks, surveying, resource allocation, and even the transportation of hazardous materials. A dominating set (DS) of a finite, simple, connected, and undirected graph $\Gamma(V, E)$, or simply Γ with non-empty node set $V(\Gamma)$ and line set $E(\Gamma)$, is a set $S_d \subseteq V(\Gamma)$ such that every node $\alpha \notin S_d$ is adjacent to a node $\beta \in S_d$. The domination number (DN) of Γ, represented by $\gamma(\Gamma)$, is the cardinality of a minimum DS. Similarly, a set $S_I \subseteq V(\Gamma)$ of nodes is called an independent set (IS) if no two nodes in S_I are adjacent to each other. An independent dominating set (IDS) of Γ is a set S_I that is both dominating and independent in Γ. The independent domination number (IDN) of Γ, represented by $i(\Gamma)$, is the cardinality of a minimum IDS. A dominating set $S_h \subseteq V(\Gamma)$ is called a hinge dominating set (HDS) if for every node $x \in V \setminus S_h$, there exists a node $y \in S_h$ and another node $z \in V \setminus S_h$ such that $(y, z) \notin E$. The minimum cardinality of such a set is called the hinge domination number (HDN), denoted by $\gamma_h(\Gamma)$. In this paper, the IDN and HDN of the splitting graph of a given graph, and graphs obtained by performing the graph operations duplication of a node by a node (DNN), duplication of a node by a line (DNL), duplication of a line by a node (DLN), duplication of a line by a line (DLL), and an extension of a node by a node (ENN) on certain classes of graphs are obtained. Several interesting conjectures and open problems are also proposed.

Keywords: Domination in graphs, independent domination, independent domination number, hinge domination, hinge domination number, graph operations.

Contents

1. Introduction

The study of domination in graphs has a long and rich history, with its origins dating back to 1862 in problems related to chessboards. Since then, domination theory has become one of the central topics in graph theory due to its wide range of theoretical developments and applications in computer science, communication networks, and optimization problems. Among the various domination parameters, the notion of independent domination plays a particularly important role. Although the idea was present in early combinatorial problems, it was formally introduced and studied in the seminal works of Berge [1] and Ore [2] in 1962. Later, Cockayne and Hedetniemi [3,4] provided the formal definition of the independent domination number and established its representation by $i(\Gamma)$ for a graph Γ.

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A dominating set (DS) of a graph $\Gamma = (V, E)$ is a subset $S_D \subseteq V$ such that every node $x \in V \setminus S_D$ is adjacent to some node $y \in S_D$. The domination number of Γ, denoted by $\gamma(\Gamma)$, is the minimum cardinality of a dominating set. On the other hand, a set $S_I \subseteq V$ is called *independent* if no two nodes of S_I are adjacent. A subset $S_I \subseteq V$ that is both dominating and independent is referred to as an independent dominating set (IDS). The independent domination number of Γ, denoted by $i(\Gamma)$, is the cardinality of a minimum IDS. Closely related to this is the independence number of Γ, denoted by $\alpha(\Gamma)$, which is the cardinality of a maximum independent set in Γ. These parameters are related by the inequality $\gamma(\Gamma) \leq i(\Gamma) \leq \alpha(\Gamma)$. Independent domination has been extensively studied in the literature, and a detailed survey of results and applications can be found in [5]. The rich structure and challenging nature of the problem have led to the introduction of several variations of domination parameters in recent years. In this direction, Kavitha and Kelkar [10] introduced a new domination parameter called the hinge domination number (HDN) in 2018. A dominating set $S_h \subseteq V$ of a graph Γ is called a *hinge dominating set (HDS)* if for every node $x \in V \setminus S_h$, there exists a node $y \in S_h$ adjacent to x and a node $z \in V \setminus S_h$ adjacent to x such that $(y, z) \notin E$. The minimum cardinality of such a set is called the *hinge domination number* of Γ, denoted by $\gamma_h(\Gamma)$. This newly defined parameter combines the concepts of domination and structural restrictions on the adjacency of dominators, offering new perspectives on the study of graph domination. The investigation of $\gamma_h(\Gamma)$ for different graph classes and its comparison with existing domination parameters is an active area of research.

Domination and its variants have found broad applicability in analysing the structural efficiency, controllability, and robustness of complex networks. A significant practical application appears in the optimal placement of Phasor Measurement Units (PMUs) in electric power systems [11,12]. PMUs measure voltage phase angles at electrical nodes, and because of their high installation cost, it is essential to minimise the number of units while ensuring complete system observability. This requirement is effectively modelled through domination theory by redefining adjacency as "observability": the power domination number determines the minimum number of PMUs needed, while power domination integrity identifies optimal locations that guarantee efficient and reliable monitoring. Beyond power systems, domination-based invariants are used in communication networks for identifying backbone nodes, in social networks to detect influential individuals who can efficiently spread information, and in biological and ecological networks to recognise key species whose presence ensures functional stability. They also appear in algorithm design, optimisation, and graph-based security models. Although these classical domination variants capture coverage and influence, they do not fully describe the structural sensitivity of a network to the failure of crucial nodes. The recently introduced hinge domination number addresses this gap by identifying hinge nodes whose removal substantially alters domination properties. Analysing hinge domination may provide a refined perspective on network vulnerability, resilience, and structural dependencies, offering deeper insights into control-oriented applications.

2. Some Well-Known Results

Theorem 2.1 *[5] For a path P_β and a cycle C_β on β nodes, $i(P_\beta) = i(C_\beta) = \left\lceil \frac{\beta}{3} \right\rceil$.*

Theorem 2.2 *[5] For a complete bipartite graph K_{x_1,x_2}, $i(K_{x_1,x_2}) = \min(x_1, x_2)$.*

Theorem 2.3 *[5] If Γ_c is a bipartite graph without isolated nodes on k_r nodes, then $i(\Gamma_c) \leq \frac{k_r}{2}$.*

Theorem 2.4 *[6] For a graph Γ with β nodes and maximum degree Δ_β, $\left\lceil \frac{\beta}{1+\Delta_\beta} \right\rceil \leq i(\Gamma) \leq \beta - \Delta_\beta$.*

Theorem 2.5 *[7] If Γ_t is a tree with α_1 nodes and β_1 leaves, then $i(\Gamma_t) = \frac{\alpha_1+\beta_1}{3}$.*

Theorem 2.6 *[8] If Γ_P is a planar graph on α_1 nodes, then $i(\Gamma_P) = \frac{3\alpha_1}{4} - 2$.*

Theorem 2.7 *[8] If Γ_P is a planar graph on α_1 nodes with diameter 2, then $i(\Gamma_P) = \left\lceil \frac{\alpha_1}{3} \right\rceil$.*

Theorem 2.8 *[10] If P_x is a path on x nodes, then*

1. *$\gamma_h(P_x) = 2$ if $x = 2$,*

2. $\gamma_h(P_x) = k + 2$ *if* $x = 3k$,
3. $\gamma_h(P_x) = \left\lceil \frac{x-1}{3} \right\rceil + 1$ *if* $x \neq 3k$.

Theorem 2.9 *[10] If* C_x *is a cycle on* x *nodes, then*

1. $\gamma_h(C_x) = k$ *if* $x = 3k$,
2. $\gamma_h(C_x) = k + 1$ *if* $x = 3k + 1$,
3. $\gamma_h(C_x) = k + 2$ *if* $x = 3k + 2$.

Theorem 2.10 *[10] If* K_x *is a complete graph on* x *nodes, then* $\gamma_h(K_x) = x$.

3. Methodology

This section describes the methodological framework adopted to compute the IDN and HDN for the graphs examined in this study. Our analysis integrates structural graph-theoretic reasoning with algorithmic verification. For each graph family, we begin by deriving structural properties that restrict the possible configurations of dominating, independent, and hinge dominating sets. These structural insights are then used to obtain exact expressions for the domination-based parameters in terms of the number of nodes or defining parameters of the graph family. To ensure the correctness of the theoretically derived results and to handle cases where structural arguments become intricate, we employ computational algorithms for both the independent domination number and the hinge domination number. These algorithms provide a unified computational framework and were implemented in Python using the `NetworkX` library. The following subsections present the algorithmic procedures used in this work.

3.1. Algorithm for Computing the Independent Domination Number

For completeness, we first present the algorithm used to compute the IDN of an arbitrary graph. The Algorithm 1 enumerates all subsets of the node set in increasing order of cardinality, checks independence, and then verifies whether the set is dominating.

Algorithm 1: Computation of Independent Domination Number $i(\Gamma)$

```
Input: A graph $\Gamma = (V, E)$
Output: $i(\Gamma)$ and all minimum independent dominating sets
$i(\Gamma) \leftarrow \infty$;
$\mathcal{S} \leftarrow \emptyset$;
foreach subset $S \subseteq V$ in increasing order of size do
    if $S$ is not independent then
        continue;
    if $S$ is not dominating then
        continue;
    if $|S| < i(\Gamma)$ then
        $i(\Gamma) \leftarrow |S|$;
        $\mathcal{S} \leftarrow \{S\}$;
    else if $|S| = i(\Gamma)$ then
        $\mathcal{S} \leftarrow \mathcal{S} \cup \{S\}$;
return $i(\Gamma)$ and $\mathcal{S}$;
```

Algorithm 2: Computation of the Hinge Domination Number $\gamma_h(\Gamma)$

```
Input: A simple undirected graph $\Gamma = (V, E)$
Output: The hinge domination number $\gamma_h(\Gamma)$ and a minimum HDS $S_h$
Function IsHingeDominatingSet($S$):
    Let $C \leftarrow V \setminus S$
    // Check dominating set condition
    For each $x \in C$:
        If no $y \in S$ is adjacent to $x$, return False
    // Check hinge condition
    For each $x \in C$:
        Let $N(x)$ be the neighbors of $x$
        Let $Y \leftarrow N(x) \cap S$ // $y$ candidates
        Let $Z \leftarrow N(x) \cap C$ // $z$ candidates
        Set valid $\leftarrow$ False
        For each $y \in Y$:
            For each $z \in Z$:
                If $(y, z) \notin E$, set valid $\leftarrow$ True and break
            If valid is True, break
        If valid is False, return False
    Return True
Main Procedure:
For $r = 1$ to $|V|$:
    For each subset $S \subseteq V$ of size $r$:
        If IsHingeDominatingSet($S$) is True:
            Output $\gamma_h(\Gamma) = r$ and $S_h = S$
            Terminate the algorithm
```

3.2. Algorithm for Computing the Hinge Domination Number

To evaluate $\gamma_h(\Gamma)$, we use an exhaustive but systematic approach similar in spirit to the previous algorithm. A set $S \subseteq V$ is a hinge dominating set if it satisfies both the dominating set condition and the hinge condition involving the adjacency relations between the dominating nodes and undominated nodes. The Algorithm 2 implements this definition exactly.

These two algorithms collectively form the computational backbone of our analysis and were used to verify the closed-form expressions derived for the independent domination number and hinge domination number of the graph families considered.

3.3. Complexity Analysis

Both algorithms presented in this study follow a brute-force subset enumeration strategy. Let $n = |V|$. The algorithm 1 checks every subset of V, resulting in a worst-case time complexity of

$$O(2^n \cdot n^2),$$

where n^2 accounts for checking independence and domination conditions. The hinge condition requires an additional adjacency check between triples (x, y, z), where $x \in V \setminus S$, $y \in S$, and $z \in V \setminus S$. Thus, the time complexity of algorithm 2 becomes

$$O(2^n \cdot n^3).$$

While exponential in nature, these algorithms are practical for graphs of moderate size and sufficient for verifying theoretical results.

4. Main Results

In this section, the IDN and HDN of some special graphs are derived.

Definition 4.1 *[9] An $SF(\alpha,\beta)$ is a graph consisting of a cycle C_α, $\alpha \geq 3$, and α sets of β independent nodes, where each set is joined to a distinct node of C_α.*

Theorem 4.1 *For an $SF(\alpha,\beta)$,*

1.
$$i(SF(\alpha,\beta)) = \begin{cases} \dfrac{\alpha}{2}(1+\beta), & \text{if } \alpha \text{ is even,} \\ \left\lfloor \dfrac{\alpha}{2} \right\rfloor + \beta\left(\left\lfloor \dfrac{\alpha}{2} \right\rfloor + 1\right), & \text{if } \alpha \text{ is odd,} \end{cases}$$

2.
$$\gamma_h(SF(\alpha,\beta)) = \alpha\beta.$$

Proof:

1. Let $\Gamma = SF(\alpha,\beta)$ be the given graph with $V(\Gamma) = \{v_i, v_i^j \colon 1 \leq i \leq \alpha, 1 \leq j \leq \beta\}$ and $E(\Gamma) = \{v_i v_{i+1} \colon 1 \leq i \leq \alpha - 1\} \cup \{v_i v_i^j \colon 1 \leq i \leq \alpha, 1 \leq j \leq \beta\}$. Clearly, $|V(\Gamma)| = \alpha(1+\beta)$. Now to construct the IDS, say S_α, there arise the following two cases:
   Case (i) When α is even. Without loss of generality, one has to select the nodes, say $v_i \colon 1 \leq i \leq \alpha$ (i is odd), in C_α. Then selecting in C_α because $v_i^j \colon 1 \leq j \leq \beta$ (i is even) constitutes the required IDS of Γ with $i(\Gamma) = \frac{\alpha}{2}(1+\beta)$.
   Case (ii) When α is odd. Without loss of generality, one has to select the nodes, say v_i: $1 \leq i \leq \alpha - 2$ (i is odd) , selection of v_α violates the independent domination property as it is adjacent with v_1. Then selecting $v_j^i \cup v_{\alpha-1}^j \colon 1 \leq j \leq \beta$ (i is even) constitutes the required IDS of Γ with $i(\Gamma) = \lfloor \frac{\alpha}{2} \rfloor + \beta(\lfloor \frac{\alpha}{2} \rfloor + 1)$ (see Figure 1).

2. Denote by $S_h = \{v_i^j \colon 1 \leq i \leq \alpha, 1 \leq j \leq \beta\}$, the collection of all $\alpha\beta$ pendant nodes of Γ. Since each pendant node is adjacent to one of the nodes of C_α the entire C_α is dominated by S_h. Consider a node $v_i \in V \setminus S_h$. Then v_i is hinge dominated by some pendant node $v_i^j \in S_h$. Indeed, each v_i is adjacent to $v_{i\pm1}$ that is also adjacent to a pendant node v_i^j, and moreover $v_{i\pm1}$ and v_i^j are non-adjacent. Hence, every v_i is hinge dominated, and S_h forms a hinge dominating set of Γ. Therefore, $\gamma_h(SF(\alpha,\beta)) = \alpha\beta$.

□

Definition 4.2 *[9] The splitting graph $S'(\Gamma)$ of a graph Γ is obtained by introducing, for each node $x \in V(\Gamma)$, a new node x' with the property that $N(x) = N(x')$.*

Theorem 4.2 *For a $S'(K_{1,\beta})$,*

1. $i(S'(K_{1,\beta})) = 2$,
2. $\gamma_h(S'(K_{1,\beta})) = \beta + 1$.

Proof:

1. Let $K_{1,\beta}$ be the given star graph with $V(K_{1,\beta}) = \{v_i \colon 0 \leq i \leq \beta\}$ and $E(K_{1,\beta}) = \{v_0 v_i \colon 1 \leq i \leq \beta\}$. Obtain the splitting graph of $K_{1,\beta}$ with $V(S'(K_{1,\beta})) = V(K_{1,\beta}) \cup \{v_i' \colon 0 \leq i \leq \beta\}$. Clearly, $|V(S'(K_{1,\beta}))| = 2|V(K_{1,\beta})|$ and $|E(K_{1,\beta})| = 3\beta$. Now to construct the IDS, say S_β, one has to select the central node v_0 to be in S_β as it dominates all the nodes from v_1 to v_β. From the remaining $\beta + 1$ nodes, again, it is enough to choose the node v_0' to be in S_β as it dominates all the nodes from v_1' to v_β'. Clearly, the two nodes in S_β are not adjacent and S_β is independent. Therefore, $|S_\beta| = 2$ and thus $i(S'(K_{1,\beta})) = 2$.

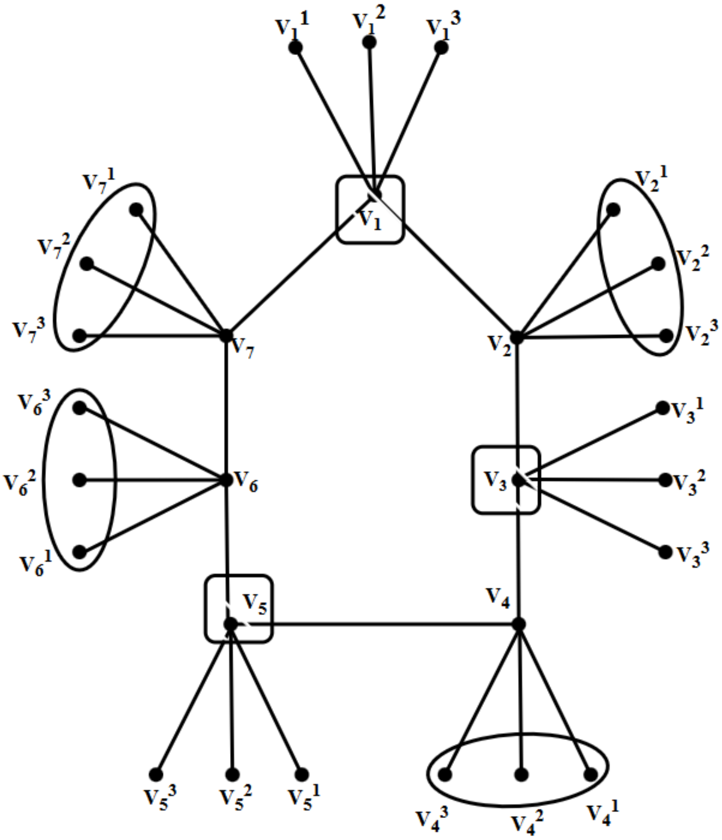


Figure 1: $i(SF(7,3)) = 15$.

2. To construct a hinge dominating set S_k of $S'(K_{1,\beta})$, one must first include all the pendant nodes in S_k, contributing β nodes. Among the remaining $\beta + 2$ nodes, namely $\{v_i' : 0 \leq i \leq \beta\} \cup \{v_0\}$, it is sufficient to include v_0' in S_k, to satisfy the hinge domination condition. This selection ensures that every node not in S_k is hinge dominated by a node in S_k, that has a non-neighbor outside S_k, while keeping the set minimal. Therefore, $\gamma_h(S'(K_{1,\beta})) = \beta + 1$.

□

Theorem 4.3 *The IDN of the splitting graph of a K_β, $i(S'(K_\beta)) = 2$.*

Proof: Consider K_β with $V(K_\beta) = \{x_i \colon 1 \leq i \leq \beta\}$ and $E(K_\beta) = \{x_1x_i \colon 2 \leq i \leq \beta\}$. Obtain the splitting graph of K_β with $V(S'(K_\beta)) = V(K_\beta) \cup \{y_i \colon 1 \leq i \leq \beta\}$. Clearly, $|V(S'(K_\beta))| = 2|V(K_\beta)|$ and $|E(V(K_\beta))| = \beta \times (\beta - 1) + |E(K_\beta)|$. Now to construct the IDS, say S_β it is sufficient to choose any one of the nodes from $x_i \colon 1 \leq i \leq \beta$, say x_1 and y_1 to be in S_β as these two nodes dominate all the nodes. Clearly, x_1 and y_1 are not adjacent and S_β is independent. Therefore, $|S_\beta| = 2$ and thus $i(S'(K_\beta)) = 2$.
□

Theorem 4.4 *The IDN of the splitting graph of a wheel graph $W_{1,\beta}$, $i(S'(W_{1,\beta})) = \beta + 2$.*

Proof: Let $W_{1,\beta}$ be the given wheel graph with $V(W_{1,\beta}) = \{x_i \colon 0 \leq i \leq \beta\}$ and $E(W_{1,\beta}) = \{x_0x_i \colon 1 \leq i \leq \beta\} \cup \{x_ix_{i+1} \colon 1 \leq i \leq \beta - 1\} \cup \{x_\beta x_1\}$. Obtain the splitting graph of $W_{1,\beta}$ with

$$V(S'(W_{1,\beta})) = V(W_{1,\beta}) \cup \{y_i \colon 0 \leq i \leq \beta\}.$$

Clearly, $|V(S'(W_{1,\beta}))| = 2|V(W_{1,\beta})|$. Now to construct the IDS, say S_β, one has to choose the central node to be x_0 in S_β as it dominates all the nodes in the rim. Now from the remaining $\beta+1$ newly added nodes one must choose all of them to be in S_β as none of these nodes are adjacent to a dominated node. Therefore, $i(S'(W_{1,\beta})) = \beta + 2$. □

In this section, the IDN and HDN of some graphs that are derived by performing the well-known graph operation of DNN, DNL, DLN, and DLL.

Definition 4.3 *The operation DNN on a node β in Γ yields a new graph Γ', formed by introducing a new node β' into Γ and connecting it with lines so that $N(\beta') = N(\beta)$.*

Definition 4.4 [9] *Duplication of a node β_k of Γ by a line $e_\beta = \beta'_k\beta''_k$ is obtained by adding two new nodes $e_\beta = \beta'_k\beta''_k$ and a line $e_\beta = \beta'_k\beta''_k$ such that $N(\beta'_k) = \{\beta'_k\beta''_k\}$ and $N(\beta''_k) = \{\beta'_k\beta'_k\}$. The resultant graph is denoted by Γ'.*

Definition 4.5 [9] *The operation DLN on a line $e_1 = xy$ in Γ, yields a new graph Γ', formed by introducing a new node w_1 into Γ such that $N(w_1) = \{x, y\}$.*

Definition 4.6 [9] *The operation DLL on a line $e_2 = \alpha\beta$ in Γ yields a new graph Γ', formed by introducing a new line $e' = \alpha'\beta'$, where α', β' are new nodes added to Γ such that $N(\alpha') = (N(\alpha) \setminus \{\alpha\}) \cup \{\alpha'\}, \quad N(\beta') = (N(\beta) \setminus \{\beta\}) \cup \{\beta'\}$.*

Definition 4.7 [9] *The operation ENN on a node β in Γ yields a new graph Γ', formed by introducing a new node β' into Γ and connecting it with lines so that $N(\beta') = N[\beta]$.*

Theorem 4.5 *If $\Gamma = K_2 + \alpha K_1$ and Γ_α denotes the graph formed by applying the DNL operation to every node of Γ, then*

1. $i(\Gamma_\alpha) = |V(\Gamma)|$
2. $\gamma_h(\Gamma_\alpha) = 6 + 2^{\alpha-1}$ *for* $\alpha \geq 2$.

Proof:

1. Let $\Gamma = K_2 + \alpha K_1$ be the given graph with $V(\Gamma) = \{x, y, \beta_i : 1 \leq i \leq \alpha\}$ and $E(\Gamma) = \{xy, x\beta_i, y\beta_i \colon 1 \leq i \leq \alpha\}$. Clearly $|V(\Gamma)| = \alpha + 2$. Obtain Γ_α, by applying DNL on each node of Γ. Clearly, $|V(\Gamma_\alpha)| = 3(\alpha + 2)$. Name the newly added end nodes of the introduced line at a node, say β_k by β'_k and β''_k. Similarly, at a node x, by x', x'' and at a node y, by y', y''. Now to construct the IDS, say S_γ, without loss of generality, put x in S_γ. Clearly, x dominates x', x'', y, β'_is $: 1 \leq i \leq \alpha$. Similarly, from the remaining nodes, select y', and β'_is $: 1 \leq i \leq \alpha$ to be in S_γ. Clearly, the nodes in S_γ are not adjacent to each other and S_γ is independent. Therefore, $|S_\gamma| = \alpha + 2$ and thus $i(\Gamma_\alpha) = \alpha + 2$ (see Figure 2).

2. To construct a hinge dominating set S_k of Γ_α, one can include all nodes of degree 2 in S_k. This selection ensures that every node not in S_k is hinge dominated by a node in S_k that has a non-neighbor outside S_k, while maintaining minimality. Since Γ_α has $6 + 2^{\alpha-1}$ nodes of degree 2 for $\alpha \geq 2$, we have $\gamma_h(\Gamma_\alpha) = 6 + 2^{\alpha-1}$.

□

Theorem 4.6 *If Γ is $K_2 + \beta K_1$ and Γ_β is the graph obtained by applying the DLL operation to all lines of Γ, then*

1. $i(\Gamma_\beta) = |E(\Gamma)| + 1$,
2. $\gamma_h(\Gamma_\beta) = \beta + 2$.

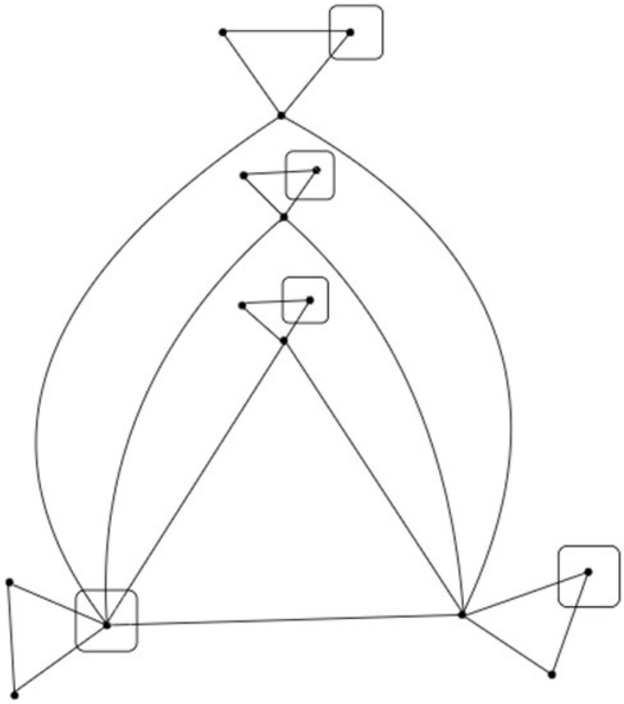


Figure 2: $i(\Gamma_3) = 5$.

Proof: **(1)** Let $\Gamma = K_2 + \beta K_1$ be the given graph on $\beta + 2$ nodes, and let Γ_β be the graph obtained by performing the DLL operation on all lines of Γ. Clearly, $|V(\Gamma_\beta)| = (\beta + 2) + 2|E(\Gamma)|$ and $|E(\Gamma_\beta)| = 2|E(\Gamma)|$. To construct an independent dominating set S_β, we first select any one of the two nodes in K_2 to ensure minimum cardinality. Additionally, from each pair of end-nodes introduced through the DLL operation, we choose exactly one node that is not already dominated. Each selected node is non-adjacent to the others, ensuring that S_β is an independent set. Therefore, $i(\Gamma_\beta) = |E(\Gamma)| + 1$ (see Figure 3).

(2) To construct a hinge dominating set S_k of Γ_β, we include all nodes of degree greater than 2. This guarantees that every node outside S_k is hinge dominated by a node in S_k that possesses at least one non-neighbour outside S_k, ensuring minimality of the set. Since Γ_β contains $\beta+2$ nodes of degree greater than 2, it follows that $\gamma_h(\Gamma_\beta) = \beta + 2$. □

Theorem 4.7 *If Γ is $K_2 + \beta K_1$ and Γ_β is constructed by applying the DNN operation to all nodes of Γ, then*

1. $i(\Gamma_\beta) = 2$,
2. $\gamma_h(\Gamma_\beta) = \beta + 2$.

Proof:

1. Let $\Gamma = K_2 + \beta K_1$ be the given graph with $|V(\Gamma)| = \beta + 2$. The graph Γ_β is obtained by applying the DNN operation to every node of Γ, giving $|V(\Gamma_\beta)| = 2|V(\Gamma)| = 2(\beta + 2)$. Let $V(\Gamma_\beta) = \{v_i : 1 \leq i \leq |V(\Gamma)|\} \cup \{v'_i : 1 \leq i \leq |V(\Gamma)|\}$, where each v'_i is the newly added node corresponding to v_i. To construct an independent dominating set S_β, we choose any one of the two nodes of K_2, say v_1, to keep the cardinality minimum. The node v_1 dominates all nodes of Γ_β except v'_1, which must therefore be included in S_β. The two nodes v_1 and v'_1 are non-adjacent, so S_β is an independent set. Thus, $i(\Gamma_\beta) = 2$ (see Figure 4(A)).

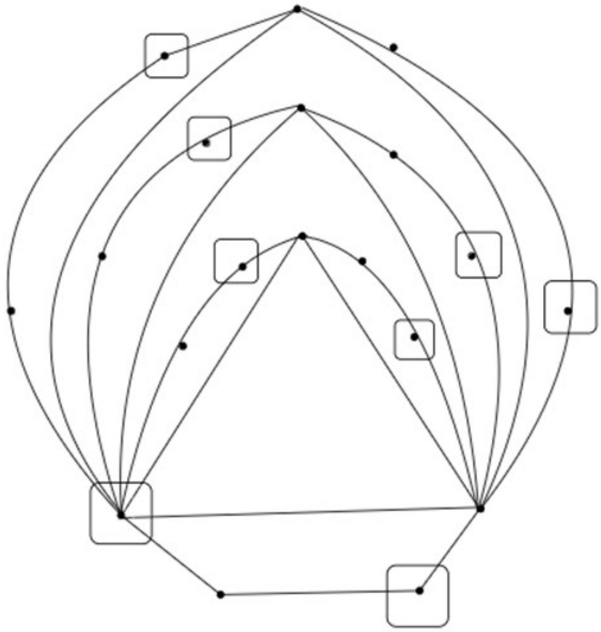


Figure 3: $i(\Gamma_3) = 8$.

2. To construct a hinge dominating set S_k of Γ_β, consider $S_k = \{v_i' : 1 \leq i \leq |V(\Gamma)|\}$, the set of all added nodes. Every $v_j \notin S_k$ is adjacent to some $v_i' \in S_k$, and by the duplication rule, each v_i' has at least one non-neighbour outside S_k. Hence, every node outside S_k is hinge dominated, so S_k is a hinge dominating set. To show minimality, note that removing any $v_i' \in S_k$ leaves the node v_i' itself without hinge domination, since no other node in S_k hinge dominates it. Hence, S_k is minimal. Therefore, $\gamma_h(\Gamma_\beta) = \beta + 2$ (see Figure 4(B)).

□

Theorem 4.8 *If Γ is $K_2 + \beta K_1$ and Γ_β is formed by applying the DLN operation to each line of Γ, then*

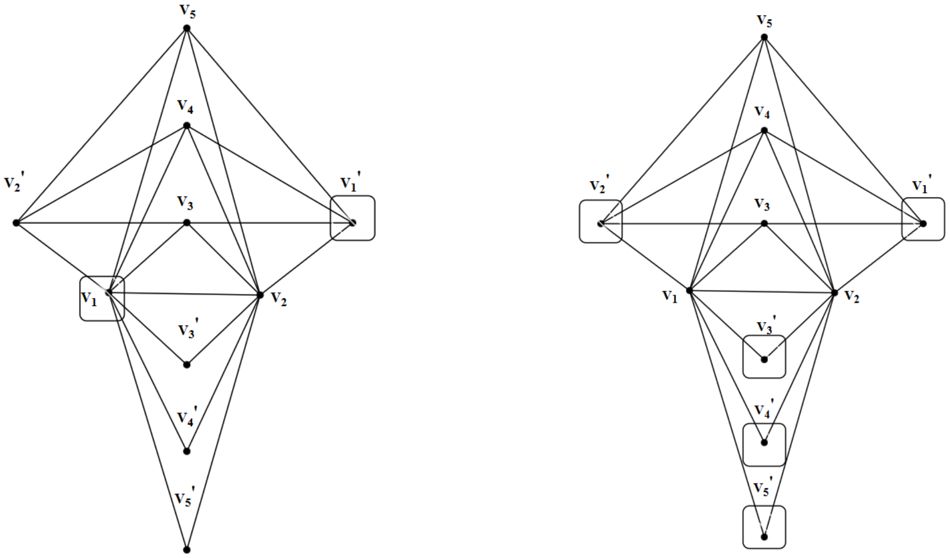
1. $i(\Gamma_\beta) = \beta + 1$,
2. $\gamma_h(\Gamma_\beta) = \beta + 2$.

Proof:

1. Let $\Gamma = K_2 + \beta K_1$ be the given graph with $|V(\Gamma)| = \beta + 2$ and $|E(\Gamma)| = 2\beta + 1$. Obtain Γ_β by applying DLN on each lines of Γ with $|V(\Gamma_\beta)| = |V(\Gamma)| + |E(\Gamma)| = (\beta + 2) + (2\beta + 1) = 3\beta + 3$. Let $V(\Gamma_\beta) = \{v_i : 1 \leq i \leq |V(\Gamma)|\} \cup \{x_i' : 1 \leq i \leq \beta\} \cup \{y_i' : 1 \leq i \leq \beta\} \cup \{z'\}$, where x_i's, y_i's and z' are the newly added nodes corresponding to the the lines x_i's, y_i's and z of Γ_β. Now to construct the IDS, say S_β, one has to choose any one of the nodes, say v_1, of K_2, as to obtain the minimum cardinality. Clearly, v_1 dominates all the nodes of Γ, the newly added nodes $x_i' : 1 \leq i \leq \beta$, and z. Moreover, one has to choose all the remaining unobserved β nodes to be in S_β to satisfy independent domination property. It is evident that the nodes in S_β are not adjacent and S_β is independent. Thus, $i(\Gamma_\beta) = \beta + 1$ (see Figure 5(A)).

2. To construct a HDS, S_k, consider the set of all newly added nodes:

$$S_k = \{x_i' : 1 \leq i \leq \beta\} \cup \{y_i' : 1 \leq i \leq \beta\} \cup \{z'\}.$$



(a) (b)

Figure 4: $(A)\ i(\Gamma_3) = 2\ (B)\ \gamma_h(\Gamma_3) = 5$.

By construction, each node $v_i \notin S_k$ is adjacent to at least one node say v_j in S_k and one node v_k in $V \setminus S_k$, where $(v_j, v_k) \notin E$. Also this set is minimum. Since $|S_k| = |E(\Gamma)| = 2\beta + 1$, it follows that $\gamma_h(\Gamma_\beta) = \beta + 2$ (see Figure 5(B)).

□

Theorem 4.9 *If Γ_β is formed by applying the DNL operation to each node of a complete graph K_β, then $i(\Gamma_\beta) = \beta$.*

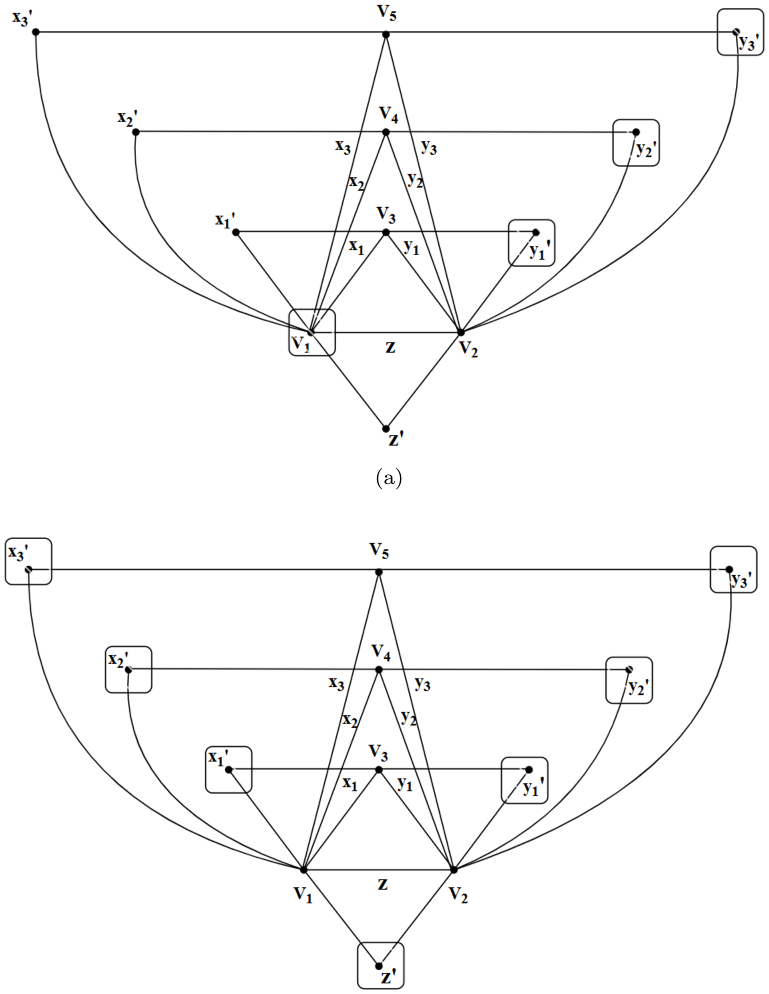
Proof: Consider K_β on β nodes. Obtain the graph Γ_β by applying DNL on each node of K_β. Let $\alpha_1, \alpha_2, \ldots, \alpha_\beta$ be the nodes of K_β and $\alpha_i^j;\ 1 \leq i \leq \beta,\ 1 \leq j \leq 2$ be the newly added nodes in Γ_β. Now, to construct the IDS, say S_β, it is enough to choose any one of the nodes of K_β, say α_1, to be in S_β. See that, α_1 clearly dominates all the nodes of K_β and α_1^1, α_1^2. Now one has to choose either of $\alpha_i^j : 2 \leq i \leq \beta, 1 \leq j \leq 2$ to be in S_β. Clearly, α_1 and $\alpha_i^j : 2 \leq i \leq \beta, 1 \leq j \leq 2$ are not adjacent to each other and S_β is independent. Therefore, $i(\Gamma_\beta) = \beta$ (see Figure 6). Now as a generalization of the above thm, we propose the following conjecture as a future work.

□

Conjecture: If Γ is a graph on β nodes and Γ_β is formed by applying the DNL operation to each node of Γ, then $i(\Gamma_\beta) = \beta$ (see Figure 7).

Theorem 4.10 *If Γ_β is formed by applying the DLN operation to all the lines of a path P_α, then $i(\Gamma_\beta) = \left\lfloor \frac{\alpha}{2} \right\rfloor$.*

Proof: Consider P_α on α nodes, say $x_j : 1 \leq j \leq \beta$. Obtain Γ_β by performing DLN at all the lines of P_α. Obviously, $|V(\Gamma_\beta)| = 2\alpha - 1$ and $y_j : 1 \leq j \leq \beta$ be the newly added nodes in Γ_β for each line, respectively. Now to construct the IDS, say S_β, two cases arise.
Case 1: P_α, α is odd.
One can choose the node $x_{(\alpha-1)/2+1}$ to be in S_β. Then choose alternative nodes on both sides until all the nodes are dominated and no two nodes in S_β are adjacent. Hence, $i(\Gamma_\beta) = \lfloor \alpha/2 \rfloor$.



(a)

(b)

Figure 5: $(A)\ i(\Gamma_3) = 3\ (B)\ \gamma_h(\Gamma_3) = 7$.

Case 2: P_α, α is even.
One can choose the node $x_{\alpha/2}$ to be in S_β. Then choose alternative nodes on both sides until all the nodes are dominated and no two nodes in S_β are adjacent. Hence $i(\Gamma_\beta) = \lfloor \alpha/2 \rfloor$ (see Figure 8).

□

Theorem 4.11 *If Γ_{ex} is formed by applying an ENN at all the nodes in a path P_β for $\beta \geq 3$, then*

$$i(\Gamma_{ex}) = \begin{cases} \lfloor \beta/3 \rfloor, & \text{if } \lfloor \beta/3 \rfloor \text{ is an integer,} \\ \lfloor \beta/3 \rfloor + 1, & \text{otherwise.} \end{cases}$$

Proof: Consider P_β on β nodes, say $y_j : 1 \leq j \leq \beta$. Form Γ_{ex} by applying an ENN at all the nodes of P_β. Clearly, $|V(\Gamma_{\text{ex}})| = 2\beta$ and $x_j : 1 \leq j \leq \beta$ be the newly added nodes in Γ_{ex} for each node, respectively. Now to construct the IDS, say S_{ex}, one can choose the node y_2 to be in S_{ex} and can skip the next two nodes, namely, y_3, y_4 and choose the node y_5 to be in S_{ex} then again skipping the next two nodes, namely, y_6, y_7 and including y_8 in S_{ex}. This process is to be continued till all the nodes are exhausted. Clearly,

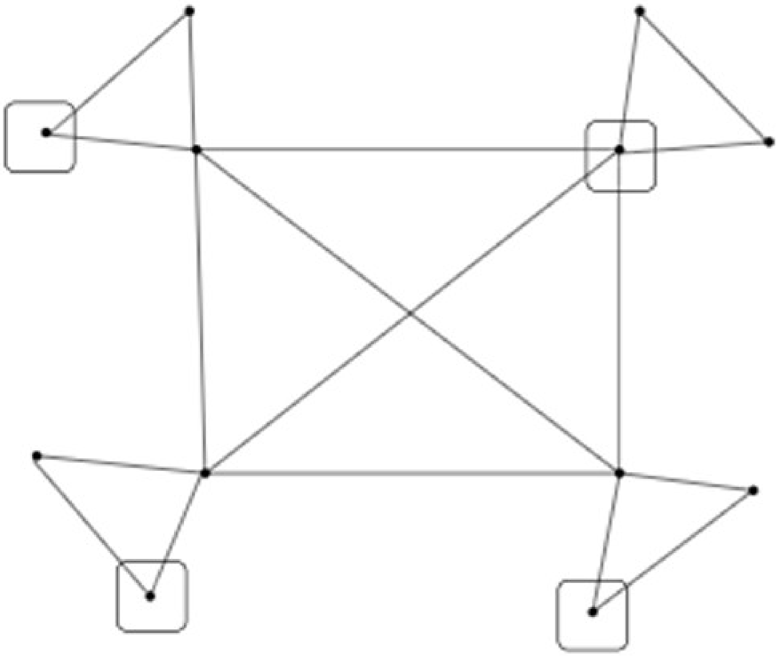


Figure 6: Γ_4 and $i(\Gamma_4) = 4$.

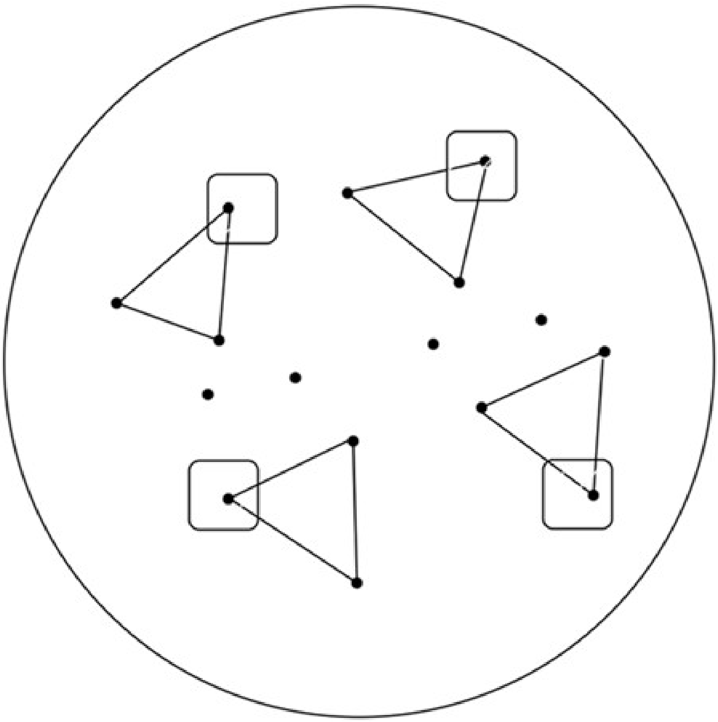


Figure 7: The graph formed by applying DNL at all the nodes in a given graph.

all the nodes are dominated, and no two nodes in S_{ex} are adjacent. Hence,

$$i(\Gamma_{\text{ex}}) = \lfloor \beta/3 \rfloor, \text{ if } \lfloor \beta/3 \rfloor \text{ is an integer}, \lfloor \beta/3 \rfloor + 1 \text{ otherwise.}$$

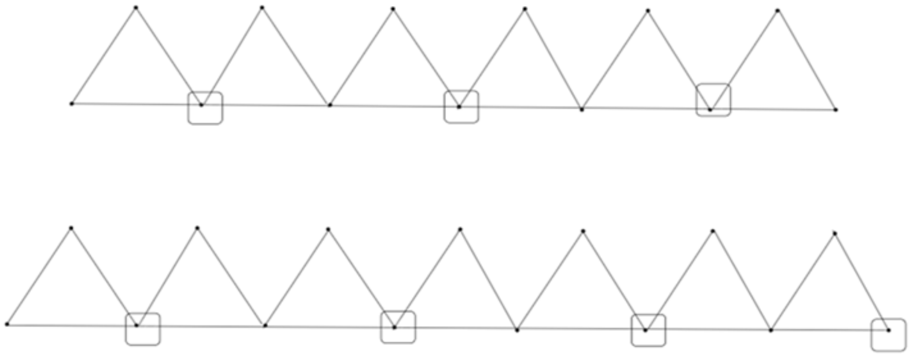


Figure 8: The graphs formed by applying DLN at all the lines of P_7 and P_8, respectively, and their IDS.

□

Theorem 4.12 *If K_{ex} is formed by applying the ENN operation to each node of the complete graph K_β, then $i(K_{\text{ex}}) = 1$.*

Proof: Consider K_β, the complete graph on β nodes, say $y_j : 1 \leq j \leq \beta$. Form K_{ex} by applying an ENN at all the nodes of K_β. Clearly, $|V(K_{\text{ex}})| = 2\beta$ and $x_i : 1 \leq i \leq \beta$ be the newly added nodes in K_{ex} for each node, respectively. Now to construct the IDS, say S_{ex}, it is sufficient to choose only one node to be in S_{ex} as all the nodes are adjacent to all the nodes in K_β. Clearly, all the nodes in K_{ex} are dominated, and no two nodes in S_{ex} are adjacent. Hence, $i(K_{\text{ex}}) = 1$.

□

Theorem 4.13 *If C_{ex} is formed by applying the ENN operation to each node of a cycle C_β for $\beta \geq 3$, then*

$$i(C_{\text{ex}}) = \begin{cases} \left\lfloor \dfrac{\beta}{3} \right\rfloor, & \text{if } \dfrac{\beta}{3} \text{ is an integer,} \\ \left\lfloor \dfrac{\beta}{3} \right\rfloor + 1, & \text{otherwise.} \end{cases}$$

Proof: Consider C_β, the cycle on β nodes, say $y_j : 1 \leq j \leq \beta$. Form C_{ex} by applying an ENN at all the nodes of C_β. Clearly, $|V(C_{\text{ex}})| = 2\beta$ and $x_i : 1 \leq i \leq \beta$ be the newly added nodes in C_{ex} for each node, respectively. Now to construct the IDS, say S_{ex}, one can choose the node y_1 to be in S_{ex} and can skip the next two nodes, namely, y_2, y_3 and choose the node y_4 to be in S_{ex}. Next, can again skip the next two nodes y_5, y_6 and selecting y_7 to S_{ex}. This process is continued till all the nodes are exhausted. Clearly, all the nodes are dominated, and no two nodes in S_{ex} are adjacent. Hence, $i(C_{\text{ex}}) = \lfloor \beta/3 \rfloor$, if $\lfloor \beta/3 \rfloor$ is an integer, $\lfloor \beta/3 \rfloor + 1$ otherwise.

□

5. Open Problems

In this section, we raise several open problems for future research:

- What is the IDN of $SF(\alpha, \beta)$ for $\alpha, \beta \geq 2$; that is, what is $i(SF(\alpha, \beta))$?
- Is there a relation between $i(\Gamma)$ and $i(S'(\Gamma))$? If so, describe the relation.
- Is determining the IDN of a graph Γ formed by applying DNN to Γ NP-complete?
- Is determining the IDN of a graph Γ formed by applying DNL to Γ NP-complete?

- Is determining the IDN of a graph Γ formed by applying DLN to Γ NP-complete?
- Is determining the IDN of a graph Γ formed by applying DLL to Γ NP-complete?
- Is determining the IDN of a graph Γ formed by applying ENN to each node of Γ NP-complete?
- For a graph Γ and Γ_{ex} formed by ENN on each node, is it always true that $i(\Gamma) = i(\Gamma_{\text{ex}})$? If not, determine the conditions or graph classes for which $i(\Gamma) < i(\Gamma_{\text{ex}})$.
- Is determining the HDN of a graph Γ formed by applying DNN NP-complete?
- Is determining the HDN of a graph Γ formed by applying DNL NP-complete?
- Is determining the HDN of a graph Γ formed by applying DLN NP-complete?
- Is determining the HDN of a graph Γ formed by applying DLL NP-complete?
- Is determining the HDN of a graph Γ formed by applying ENN NP-complete?

The study of hinge domination opens several promising directions for future research, particularly in the analysis of network vulnerability and resilience. Since hinge nodes represent structurally critical nodes whose removal significantly affects domination properties, hinge domination can be applied to assess robustness in power grids, transportation networks, communication infrastructures, social networks, and ecological systems. Its ability to capture subtle structural dependencies makes it a valuable tool for designing fault-tolerant and efficiently controllable networked systems.

6. Conclusions

This study determined the independent domination number and the hinge domination number for graphs obtained through the operations DNN, DLN, DLL, and DNL. The results reveal that further investigation into additional graph families and their duplication graphs may deepen our understanding of independent and hinge domination. Several open problems and conjectures have been proposed to guide future research in this area.

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