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Iterated Bernstein-Type L_p Inequalities for Polynomials

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ABSTRACT: We develop several new Bernstein-type L_p inequalities for complex polynomials by iterating the first-order differential operator $A_\alpha(P) := zP'(z) - \alpha P(z)$. Our results extend, unify, and sharpen L_p inequalities of Zygmund, de Bruijn, and Jain as well as the recent L_p extensions for A_α and its second-order companion shown in [11]. In particular, for any finite sequence $\alpha_1, \dots, \alpha_m$ with $Re(\alpha_j) \le n/2$ we obtain sharp bounds for $||\prod_{j=1}^{m} A_{\alpha_j} P||_p$ in terms of $||P||_p$ for all $0 \le p \le \infty$, together with refined "Erdős–Lax"-type improvements when P has no zeros in the open unit disc. As corollaries, we derive L_p-versions of higher order Bernstein inequalities for $z^k P^{(k)}$ and scale-invariant formulations on circles $\{|z| = r\}$.

Key Words: Polynomial, Bernstein-type inequality, L_p-norm.

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1. Introduction

In mathematics and the applied sciences, polynomial estimate is one of the most useful and widely used ideas. It offers a practical approach to simplifying complex functions by utilizing polynomial forms that exhibit similar behavior. This approach reduces processing effort and simplifies numerous analytical and numerical problems. These benefits have made polynomial approximation a crucial method for numerical analysis, signal processing, computer-aided design, physics, and several engineering specialties.

Various methods have been developed over time to improve the accuracy and efficiency of this procedure. The goal of the least squares approach, which is frequently connected to linear regression, is to reduce the overall error between the approximating polynomial and the data points. In contrast, the Chebyshev technique concentrates on minimizing the maximum error across an interval. Rational approximation, which expresses functions as ratios of polynomials to improve accuracy in specific situations, offers a more adaptable concept. Depending on the data, the required degree of accuracy, and the computational resources at hand, each approach has unique benefits.

The application of Bernstein's inequality has led to the emergence of another viable avenue in recent years. This method provides a novel mathematical perspective for evaluating and refining polynomial approximations. Researchers have gained a deeper knowledge of the behavior of approximating polynomials and tighter estimations by using Bernstein-type constraints. The theoretical and practical significance of polynomial approximation in contemporary mathematical research is thus further reinforced by this line of inquiry.

For polynomials $P(z)$ of degree at most n (represented by $P \in Pn$), S. Bernstein [3] examined the problem in the complex plane on the unit circle, establishing the now-famous result:

$$max_{|z|=1}|P^{'}(z)| \le nmax_{|z|=1}|P(z)|. \tag{1.1}$$

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Equality holds in (1) if and only if $P(z)$ has all its zeros at origin.
The greatest rate of change for a polynomial on the unit circle is quantified by the aforementioned inequality, also known as Bernstein's inequality. The significance of zero locations is highlighted by the fact that equality holds if and only if $P(z)$ has all of its zeros at the origin.
The research of these inequalities has grown considerably, including a wider range of function classes, domains, and norms. One important development is to use integral L^p norms in place of the supremum norm (L^∞). Let P_n denote the set of complex polynomials of degree *exactly* n. For $0 < p < \infty$ define the L_p-norm on the unit circle by

$$||P(z)||_p := \Big(\frac{1}{2\pi}\int_0^{2\pi} \left|P(e^{i\theta})\right|^p \, d\theta\Big)^{1/p}, \qquad ||P(z)||_\infty := \max_{|z|=1} |P(z)|\,.$$

The Mahler measure is $||P(z)||_0 := \exp\big(\frac{1}{2\pi}\int_0^{2\pi} \log\left|P(e^{i\theta})\right| \, d\theta\big)$; it satisfies $\lim_{p\to 0^+} ||P(z)||_p = ||P(z)||_0$ and $\lim_{p\to\infty} ||P(z)||_p = ||P(z)||_\infty$.

Bernstein's classical inequality states that $||P'(z)||_\infty \le n||P(z)||_\infty$, which extends to L_p by Zygmund and Arestov: $||P'(z)||_p \le n||P(z)||_p$ for all $0 < p \le \infty$ [12,1]. Jain [7] proved for $|z| = 1$ and $|\alpha| \le n/2$ that

$$||zP'(z) - \alpha P(z)||_\infty \le |n - \alpha| \; ||P(z)||_\infty,$$

and recent work [11] extended this to L_p with the half-plane condition $\Re a(\alpha) \le n/2$, and further obtained second-order companions and refinements under zero restrictions.
We note that Bernstein-type phenomena continue to appear across various analytic settings. For instance, Batista, De Lima, and Gomes [2]established Moser–Bernstein type rigidity results for solitons of the mean curvature flow, while Finta [5] studied King operators preserving L_p-spaces in the context of approximation theory. These works further highlight the broad relevance and applicability of Bernstein-type estimates in modern analysis.

Motivated by these developments, we introduce new *iterated* Bernstein-type inequalities for compositions

$$A_{\alpha_m} \cdots A_{\alpha_1} P(z), \qquad A_\alpha(P) := zP'(z) - \alpha P(z),$$

which for special choices of α_j recover higher order Bernstein inequalities for $z^k P^{(k)}$.

2. Auxiliary Lemmas

We now state and prove the auxiliary lemmas used throughout.

Lemma 2.1 (Location of zeros under A_α) *Let f be an n-th degree polynomial with all zeros in $|z| \le r$. If $Re(\alpha) \le n/2$, then all zeros of $zf'(z) - \alpha f(z)$ also lie in $|z| \le r$.*

Proof: *Let w with $|w| > r$ and write $f(z) = a\prod_{v=1}^n (z - z_v)$ with $|z_v| \le r$. Then*

$$\frac{wf'(w)}{f(w)} - \alpha = \sum_{v=1}^n \frac{w}{w - z_v} - \alpha = \frac{n}{2} - Re(\alpha) + \frac{1}{2}\sum_{v=1}^n \frac{|w|^2 - |z_v|^2}{|w - z_v|^2}.$$

The last expression has strictly positive real part, since $|w| > |z_v|$ for all v and $Re(\alpha) \le n/2$. Thus $wf'(w) - \alpha f(w) \ne 0$ for all $|w| > r$, which proves the claim. □

Lemma 2.2 (Comparison principle on and outside the unit circle) *Let F be of degree n with all zeros in $|z| \le 1$ and let P be any polynomial of degree at most n satisfying $|P| \le |F|$ on $|z| = 1$. If $Re(\alpha_j) \le n/2$ for $j = 1, \dots, m$, then*

$$|A_{\alpha_m} \cdots A_{\alpha_1} P(z)| \le |A_{\alpha_m} \cdots A_{\alpha_1} F(z)| \qquad (|z| \ge 1).$$

Proof: *We first treat $m = 1$. Suppose there exists w with $|w| > 1$ such that $|A_{\alpha_1} P(w)| > |A_{\alpha_1} F(w)|$. Then $A_{\alpha_1} F(w) \ne 0$ by the maximum modulus principle and Lemma 2.1 applied to F. Define*

$$\lambda := \frac{A_{\alpha_1} P(w)}{A_{\alpha_1} F(w)}, \qquad |\lambda| > 1.$$

By the hypothesis $|P| \le |F|$ on $|z| = 1$ and the maximum modulus principle applied to $\phi(z) := P(z)/F(z)$, we have $|\phi(z)| < 1$ on $|z| < 1$. Hence $P - \lambda F$ has all its zeros in $|z| \le 1$ for $|\lambda| \ge 1$ (Rouché on $|z| = 1$). Applying Lemma 2.1 to $G := P - \lambda F$ yields that $A_{\alpha_1} G$ has all zeros in $|z| \le 1$, contradicting $A_{\alpha_1} G(w) = 0$ with $|w| > 1$. Therefore $|A_{\alpha_1} P| \le |A_{\alpha_1} F|$ for $|z| \ge 1$. By continuity, the inequality holds on $|z| = 1$ as well.

For general m, set $H_k := A_{\alpha_k} \cdots A_{\alpha_1} P$ and $K_k := A_{\alpha_k} \cdots A_{\alpha_1} F$ for $k = 1, \ldots, m$. By Lemma 2.1, each K_k has all its zeros in $|z| \le 1$, and by the $m = 1$ case (applied iteratively) $|H_k| \le |K_k|$ on $|z| \ge 1$. The claim follows for $k = m$. □

Lemma 2.3 (Arestov's inequality for admissible diagonal operators) *Let $\phi(x) = \psi(\log x)$ with ψ convex, non-decreasing. If $\Lambda_\delta : \mathcal{P}_n \to \mathcal{P}_n$ is diagonal in the monomial basis,*

$$\Lambda_\delta\Big(\sum_{j=0}^{n} a_j z^j\Big) := \sum_{j=0}^{n} \delta_j a_j z^j,$$

and is admissible *in the sense that it preserves the class of polynomials all of whose zeros lie in $|z| \le 1$ (or $|z| \ge 1$), then*

$$\int_0^{2\pi} \phi\big(|\Lambda_\delta P(e^{i\theta})|\big)\, d\theta \le \int_0^{2\pi} \phi\big(c(\delta)|P(e^{i\theta})|\big)\, d\theta, \quad c(\delta) := \max\{|\delta_0|, |\delta_n|\}.$$

In particular, for $p > 0$ and $\phi(x) = x^p$,

$$||\Lambda_\delta P||_p \le c(\delta)\, ||P||_p.$$

Proof: *See [1].* □

Lemma 2.4 (Averaging device) *For $p > 0$ and $t \ge 1$,*

$$\int_0^{2\pi} |t + e^{i\beta}|^p \, d\beta \ \ge\ \int_0^{2\pi} |1 + e^{i\beta}|^p \, d\beta.$$

Proof: *This follows from the fact that the Poisson kernel averages the subharmonic function $\beta \mapsto |t + e^{i\beta}|^p$ and the map $t \mapsto \int |t + e^{i\beta}|^p \, d\beta$ is increasing for $t \ge 0$; see [4,11].* □

We write $A_\alpha(P) := zP' - \alpha P$. Note that A_α is diagonal on monomials: $A_\alpha(z^j) = (j - \alpha) z^j$. Hence the m-fold composition $A_{\alpha_m} \cdots A_{\alpha_1}$ is diagonal with eigenvalues $\prod_{k=1}^{m}(j - \alpha_k)$, $j = 0, \ldots, n$.

3. Main results

Theorem 3.1 (Iterated L_p Bernstein-type inequality) *Let $P(z) \in \mathcal{P}_n$, $m \in \mathbb{N}$, and $\alpha_1, \ldots, \alpha_m \in \mathbb{C}$ with $Re(\alpha_k) \le n/2$ for each k. Then for all $0 \le p \le \infty$,*

$$||A_{\alpha_m} \cdots A_{\alpha_1} P(z)||_p \ \le\ \Big(\prod_{k=1}^{m} |n - \alpha_k|\Big)\, ||P(z)||_p.$$

The inequality is sharp and equality holds for $P(z) = cz^n$, $c \ne 0$.

Proof: Fix $0 < p < \infty$; the cases $p = 0$ and $p = \infty$ follow by limiting procedures. Let $z_1, \ldots, z_k$ be the zeros of P with $|z_j| > 1$. Define the standard "innerization"

$$T(z) := P(z) \prod_{j=1}^{k} \frac{1 - \overline{z_j}\, z}{z - z_j}.$$

Then T has all zeros in $|z| \le 1$ and $|T| = |P|$ on $|z| = 1$. By Lemma 2.2,

$$|A_{\alpha_m} \cdots A_{\alpha_1} P(e^{i\theta})| \le |A_{\alpha_m} \cdots A_{\alpha_1} T(e^{i\theta})| \quad \text{for all } \theta,$$

hence $||A_{\alpha_m} \cdots A_{\alpha_1} P||_p \le ||A_{\alpha_m} \cdots A_{\alpha_1} T||_p$.

Next, since T has zeros in $|z| \le 1$, Lemma 2.1 implies that $A_{\alpha_1}T$ also has zeros in $|z| \le 1$, and by iteration the same holds for $A_{\alpha_k} \cdots A_{\alpha_1} T$ $(k = 1, \ldots, m)$. Therefore the diagonal operator

$$\Lambda_\delta := A_{\alpha_m} \cdots A_{\alpha_1} : \sum_{j=0}^{n} a_j z^j \mapsto \sum_{j=0}^{n} \Big[\prod_{k=1}^{m} (j - \alpha_k) \Big] a_j z^j$$

is admissible on the class of polynomials whose zeros lie in $|z| \le 1$. By Lemma 2.3,

$$||A_{\alpha_m} \cdots A_{\alpha_1} T||_p \le c(\delta)\, ||T||_p, \qquad c(\delta) = \max \Big\{ \prod_{k=1}^{m} |-\alpha_k|, \prod_{k=1}^{m} |n - \alpha_k| \Big\}.$$

A direct computation shows

$$|n - \alpha_k|^2 - |\alpha_k|^2 = n(n - 2Re(\alpha_k)) \ge 0 \quad \text{if } Re(\alpha_k) \le \frac{n}{2},$$

hence $|\alpha_k| \le |n - \alpha_k|$ and $c(\delta) = \prod_{k=1}^{m} |n - \alpha_k$. Combining these bounds and using $||T||_p = ||P||_p$ yields the claim for $0 < p < \infty$; the cases $p = 0, \infty$ follow by taking $p \to 0^+$ and $p \to \infty$ respectively. For $P(z) = cz^n$ one has $A_{\alpha_k} P = (n - \alpha_k)P$, hence equality. □

Remark 3.1 *For $m = 1$ Theorem 3.1 recovers the L_p extension of Jain's inequality with the optimal half-plane condition $Re(\alpha) \le n/2$; for $m = 2$ it recovers the second-order inequality (with operator $z^2P'' + (1 - \alpha - \gamma)zP' + \alpha\gamma P$, note the* plus *sign in front of $(1 - \alpha - \gamma)zP'$) proved in [11]. The iteration is new for general m.*

Define for $m \in \mathbb{N}$ the symbol

$$W_m(z) := \prod_{k=1}^{m} \big((n - \alpha_k)z - \alpha_k \big).$$

Theorem 3.2 (Refined L_p inequality for zero-free polynomials) *Let $P(z) \in P_n$ be zero-free in the open unit disc $|z| < 1$, let $m \in \mathbb{N}$ and $\alpha_1, \ldots, \alpha_m \in C$ with $Re(\alpha_k) \le n/2$. Then for all $p > 0$,*

$$||A_{\alpha_m} \cdots A_{\alpha_1} P(z)||_p \le \left(\prod_{k=1}^{m} \frac{\Big(\int_0^{2\pi} |(n - \alpha_k)e^{i\beta} - \alpha_k|^p \, d\beta \Big)^{1/p}}{\Big(\int_0^{2\pi} |1 + e^{i\beta}|^p \, d\beta \Big)^{1/p}} \right) ||P(z)||_p.$$

For $p = \infty$, replace each L_p-norm by the L_∞-norm; the inequality then gives the sharp sup-norm refinement analogous to Erdős–Lax/Jain.

Proof: The case $m = 1$ is the standard de Bruijn/Arestov refinement (see [4,1,11]). For general m, proceed by induction. Assume the claim holds for $m - 1$ and set $H_{m-1} := A_{\alpha_{m-1}} \cdots A_{\alpha_1} P$. By the $m = 1$ case applied to H_{m-1} (using the conjugate-polynomial construction $Q(z) = z^n \overline{H_{m-2}(1/\overline{z})}$ inside the standard proof when needed), we obtain

$$||A_{\alpha_m} H_{m-1}||_p \le \frac{\big\| (n - \alpha_m)e^{i\beta} - \alpha_m \big\|_{L_p(0,2\pi)}}{\big\| 1 + e^{i\beta} \big\|_{L_p(0,2\pi)}} \, ||H_{m-1}||_p.$$

Applying the induction hypothesis to $||H_{m-1}||_p$ yields the desired product bound for m. □

Remark 3.2 *For $m = 1$ Theorem 3.2 coincides with the L_p extension of the Erdős–Lax improvement due to de Bruijn (with $\alpha = 0$) and with its α-version in [11]. The general m-step refinement produces the clean product of L_p-ratios; we do not claim an identity $\|W_m\|_{L_p} = \prod \|(n-\alpha_k)z - \alpha_k\|_{L_p}$ for $p \neq \infty$.*

Corollary 3.1 (Higher order Bernstein-type inequalities) *Let $1 \le k \le n$ and set $\alpha_j = j - 1$ for $j = 1, \dots, k$. Then*

$$A_{\alpha_k} \cdots A_{\alpha_1} P(z) = z^k P^{(k)}.$$

Consequently, for all $0 \le p \le \infty$,

$$||z^k P^{(k)}||_p \ \le\ \big(n(n-1)\cdots(n-k+1)\big)\, ||P(z)||_p,$$

and if P has no zeros in $|z| < 1$ and $p > 0$, then

$$||z^k P^{(k)}||_p \ \le\ \left(\prod_{j=0}^{k-1} \frac{\big\|(n-j)e^{i\beta} - j\big\|_{L_p(0,2\pi)}}{\big\|1 + e^{i\beta}\big\|_{L_p(0,2\pi)}}\right) ||P(z)||_p.$$

Proof: The identity $A_{k-1} \cdots A_0 P = z^k P^{(k)}$ is checked on monomials: for $P(z) = z^m$,

$$A_{k-1} \cdots A_0(z^m) = \prod_{j=0}^{k-1} (m-j)\, z^m = \frac{m!}{(m-k)!}\, z^m = z^k \cdot \frac{m!}{(m-k)!} z^{m-k} = z^k P^{(k)}.$$

The asserted inequalities are the corresponding instances of Theorems 3.1 and 3.2. □

Theorem 3.3 (Self-inversive polynomials) *If P is self-inversive, i.e., $P(z) = uz^n \overline{P(1/\overline{z})}$ with $|u| = 1$, then for any $\alpha_1, \dots, \alpha_m$ with $Re(\alpha_k) \le n/2$ and $p > 0$ the inequality of Theorem 3.2 holds with the same right-hand side, and the constants are best possible. In particular, for $m = 1, 2$ this recovers the sharp statements in [11].*

Proof: When P is self-inversive, one checks directly that $|A_\alpha P(e^{i\theta})| = |A_\alpha Q(e^{i\theta})|$ for $Q(z) = z^n \overline{P(1/\overline{z})}$, so the comparison step used in the proof of Theorem 3.2 becomes an identity; the remainder of the argument proceeds verbatim. Sharpness follows from testing on $P(z) = az^n + \overline{a}$. □

Theorem 3.4 (Scale-invariant formulations on $|z| = r$) *For $r > 0$ and $0 < p < \infty$ define*

$$||P||_{p,r} := \Big(\frac{1}{2\pi} \int_0^{2\pi} |P\big(re^{i\theta}\big)|^p \, d\theta\Big)^{1/p},$$

with the usual interpretations for $p = 0, \infty$: $||P||_{0,r} := \exp\big(\frac{1}{2\pi}\int_0^{2\pi} \log\big|P(re^{i\theta})\big|\, d\theta\big)$ and $||P||_{\infty,r} := \max_{|z|=r} |P(z)|$. Then for all $P \in P_n$, $Re(\alpha_k) \le n/2$, and $0 \le p \le \infty$,

$$||A_{\alpha_m} \cdots A_{\alpha_1} P||_{p,r} \ \le\ \Big(\prod_{k=1}^{m} |n - \alpha_k|\Big)\, ||P||_{p,r}. \tag{3.1}$$

If P has no zeros in $|z| < r$ and $p > 0$, then with

$$C_p := \Big(\int_0^{2\pi} |1 + e^{i\beta}|^p \, d\beta\Big)^{1/p}$$

we have

$$||A_{\alpha_m} \cdots A_{\alpha_1} P||_{p,r} \le \left(\prod_{k=1}^{m} \frac{\big(\int_0^{2\pi} |(n-\alpha_k)e^{i\beta} - \alpha_k|^p \, d\beta\big)^{1/p}}{C_p}\right) ||P||_{p,r}. \tag{3.2}$$

For $p = \infty$, each L_p-norm above is replaced by the supremum over $\beta \in [0, 2\pi]$.

Proof: Fix $r > 0$. Define the rescaled polynomial $Q(z) := P(rz)$. Then for any $0 \leq p \leq \infty$,

$$||Q_p = ||P||_{p,r}. \tag{3.3}$$

We first verify the *intertwining identity*

$$\big(A_{\alpha_m} \cdots A_{\alpha_1} Q\big)(z) = \big(A_{\alpha_m} \cdots A_{\alpha_1} P\big)(rz) \qquad \text{for all } z \in C. \tag{3.4}$$

This is proved by induction on m. For $m = 1$,

$$A_{\alpha_1} Q(z) = zQ'(z) - \alpha_1 Q(z) = z \cdot r\, P'(rz) - \alpha_1 P(rz) = (A_{\alpha_1} P)(rz).$$

Assume (3.4) holds for $m-1$, i.e., $A_{\alpha_{m-1}} \cdots A_{\alpha_1} Q = (A_{\alpha_{m-1}} \cdots A_{\alpha_1} P) \circ (\cdot\, r)$. Then

$$\begin{aligned} A_{\alpha_m}\big(A_{\alpha_{m-1}} \cdots A_{\alpha_1} Q\big)(z) &= z\frac{d}{dz}\Big[(A_{\alpha_{m-1}} \cdots A_{\alpha_1} P)(rz)\Big] - \alpha_m (A_{\alpha_{m-1}} \cdots A_{\alpha_1} P)(rz) \\ &= rz\,(A_{\alpha_{m-1}} \cdots A_{\alpha_1} P)'(rz) - \alpha_m (A_{\alpha_{m-1}} \cdots A_{\alpha_1} P)(rz) \\ &= \big(A_{\alpha_m}(A_{\alpha_{m-1}} \cdots A_{\alpha_1} P)\big)(rz), \end{aligned}$$

which is precisely (3.4) for m. This completes the induction.

We now prove (3.1). Applying Theorem 3.1 to Q yields

$$||A_{\alpha_m} \cdots A_{\alpha_1} Q||_p \leq \Big(\prod_{k=1}^{m} |n - \alpha_k|\Big)\, ||Q||_p.$$

Using (3.4) and (3.3) we obtain

$$||A_{\alpha_m} \cdots A_{\alpha_1} P||_{p,r} = ||A_{\alpha_m} \cdots A_{\alpha_1} Q||_p \leq \Big(\prod_{k=1}^{m} |n - \alpha_k|\Big)\, ||P||_{p,r},$$

which is (3.1).

For the refined inequality (3.2), assume P has no zeros in $|z| < r$. Then Q has no zeros in $|z| < 1$. Applying Theorem 3.2 to Q gives, for $p > 0$,

$$||A_{\alpha_m} \cdots A_{\alpha_1} Q||_p \leq \left(\prod_{k=1}^{m} \frac{\big(\int_0^{2\pi} |(n - \alpha_k)e^{i\beta} - \alpha_k|^p\, d\beta\big)^{1/p}}{\big(\int_0^{2\pi} |1 + e^{i\beta}|^p\, d\beta\big)^{1/p}}\right) ||Q||_p.$$

Invoking (3.4) and (3.3) again yields

$$||A_{\alpha_m} \cdots A_{\alpha_1} P||_{p,r} \leq \left(\prod_{k=1}^{m} \frac{\big(\int_0^{2\pi} |(n - \alpha_k)e^{i\beta} - \alpha_k|^p\, d\beta\big)^{1/p}}{C_p}\right) ||P||_{p,r},$$

which is (3.2). The case $p = \infty$ follows by replacing each L_p-norm in β by the essential supremum over β. □

4. Further Remarks

Theorems 3.1–3.4 unify several classical and recent inequalities:

- Taking $m = 1$ and $\alpha = 0$ in Theorem 3.1 yields Zygmund–Arestov: $||P'||_p \leq n||P||_p$ [12,1].
- Taking $m = 1$ and $\alpha \in C$ with $Re(\alpha) \leq n/2$ yields the L_p-Jain inequality; letting $p \to \infty$ recovers Jain's sup-norm bound [7].
- The case $m = 2$ of Theorem 3.1 reproduces the second-order inequality in [11] with the correct sign $z^2P'' + (1 - \alpha - \gamma)zP' + \alpha\gamma P$.
- Corollary 3.1 gives higher order analogues of Bernstein-type inequalities for $z^kP^{(k)}$ in L_p.

5. Example

Let $P(z) = z^2 + 1$, so $n = 2$. Let $\alpha_1 = 0$, $\alpha_2 = 1$.
Compute the iterated operator:

$$A_0(P) = zP'(z) = z(2z) = 2z^2,$$
$$A_1(A_0(P)) = z\frac{d}{dz}(2z^2) - 1(2z^2) = z(4z) - 2z^2 = 2z^2.$$

Now compute the L^2–norms on the unit circle $|z| = 1$. Write $z = e^{i\theta}$.

$$\|P\|_2 = \left(\frac{1}{2\pi}\int_0^{2\pi} |e^{2i\theta} + 1|^2 \, d\theta\right)^{1/2}.$$

Expand the integrand:

$$|e^{2i\theta} + 1|^2 = (e^{2i\theta} + 1)(e^{-2i\theta} + 1) = 2 + e^{2i\theta} + e^{-2i\theta} = 2 + 2\cos(2\theta).$$

Since $\int_0^{2\pi} \cos(2\theta)\, d\theta = 0$, we have

$$\frac{1}{2\pi}\int_0^{2\pi} |e^{2i\theta} + 1|^2 \, d\theta = 2, \quad \text{so } \|P\|_2 = \sqrt{2}.$$

Next,

$$\|A_1 A_0(P)\|_2 = \|2z^2\|_2 = 2\|z^2\|_2 = 2,$$

since $|z^2| = 1$ on $|z| = 1$.

Theorem 2.1 predicts

$$\|A_{\alpha_2} A_{\alpha_1} P\|_2 \le |2 - \alpha_1||2 - \alpha_2|\|P\|_2 = 2 \cdot 1 \cdot \sqrt{2} = 2\sqrt{2} \approx 2.828.$$

Indeed, $2 \le 2\sqrt{2}$, so the inequality holds. Equality occurs only for monomials cz^n.

This concrete calculation confirms the bound in Theorem 2.1 for the quadratic case and demonstrates how the iterated operator A_α behaves in a simple example.

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