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Weighted Nonlocal Kirchhoff Variational Inequality of Fourth Order

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ABSTRACT: The theory of variational inequalities has emerged as an interesting area of applied mathematics. It is a very useful tool for studying a wide class of linear and nonlinear problems in a unified, natural, and general framework. They also have applications in physics, mechanics, engineering, optimization, and elliptic inequalities.

Keywords: Variational inequality, Kirchhoff, Critical point, multiple solutions, Szulkin.

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1. Introduction

This paper is devoted to proving the existence and multiplicity of positive solutions for a class of a nonlinear variational inequality of Kirchhoff type.
Under more general superlinear assumptions on the nonlinear term, we prove the existence of multiple positive solutions via non-smooth critical point theory for Szulkin-type functional.
Consider the following inequality of Kirchhoff type:

$$\begin{cases} -\left(a+b\int_{\Omega} p(x)|\Delta u(x)|^2 dx\right)\Delta\left(p(x)\Delta u(x)\right) \geq q(x)f(x,u(x)) & \text{in } \Omega, \\ u = \dfrac{\partial u}{\partial n} = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

Where a and b are positive numbers, and $p \in A_2$ (Muckenhoupt weight), $q \in L^1(\Omega)$, and Ω is smooth bounded domain in $\mathbb{R}^N$ (N = 1, 2, or 3), and $f \in C(\Omega \times \mathbb{R}, \mathbb{R})$.
Let $K \subset H^2_{0,p}(\Omega)$ be the closed convex set in Hilbert space where :
$H^2_{0,p}(\Omega) := \{ u \in L^2(\Omega) : D^\alpha u \in L^2_p(\Omega) \text{ for all } |\alpha| \leq 2, u_{|\partial\Omega} = 0 \}$, $L^2_p(\Omega) := \{ u \in L^1_{\text{loc}}(\Omega) : \sqrt{p}\, u \in L^2(\Omega) \}$.
with the scalar product $(u,v)_{H^2_{0,p}(\Omega)} := \int_\Omega p(x)\left[u(x)v(x) + \nabla u(x)\cdot\nabla v(x) + \Delta u(x)\Delta v(x)\right]dx$, and the norm
$\|u\|_{H^2_{0,p}(\Omega)} = \left(\int_\Omega p(x)\left(|u(x)|^2 + |\nabla u(x)|^2 + |\Delta u(x)|^2\right)dx\right)^{1/2}$, $[u]_2 := \left(\int_\Omega p(x)\,|\Delta u(x)|^2\,dx\right)^{1/2}$.
Let the space $C^0 = \{u \in C(\Omega, \mathbb{R}) : u_{|\partial\Omega} = 0\}$ endowed with the norm $\|u\|_\infty = \sup_{x\in\Omega} |u(x)|$. Concerning these spaces, we make the next assumptions
(f_1) $f \in C(\Omega \times \mathbb{R}, \mathbb{R})$ and there existsc $c > 0$ and $\theta \in (1, 2^*)$ such that:

$$|f(x,u)| \leq c(1 + |u|^{\theta-1}), \forall (x,u) \in \Omega \times \mathbb{R}$$

(f_2) $\frac{F(x,u)}{|u|^4} \to +\infty$ uniformly in $x \in \Omega$ and there exists $r_0 > 0$ such that

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$$F(x,u) \geq 0, \qquad \forall (x,u) \in \Omega \times \mathbb{R}, |u| \geq r_0$$

where $F(x,u) = \int_0^u f(x,s)ds$.
(f_3) $f(x,-u) = -f(x,u)$ for all $(x,u) \in \Omega \times \mathbb{R}$
(f_4) there exists a constant $\tau \in (0,2)$ such that

$$\liminf_{|u|\to\infty} \frac{f(x,u)u - \nu F(x,u)}{|u|^\tau} > -\beta, \text{ uniformly with respect to } x \in \Omega$$

where ν and β satisfy either
(a) $\beta > 0$ and $\nu > 4$,
or
(b) $0 < \beta < a\lambda_1$ and $\nu = 4$.
We consider the fourth-order eigenvalue problem:

$$\begin{cases} (p(x)\Delta u(x)) = \lambda q(x) u(x) & \text{in } \Omega, \\ u = \dfrac{\partial u}{\partial n} = 0 & \text{on } \partial\Omega. \end{cases}$$

Then the first eigenvalue is given by:

$$\lambda_1(q) = \inf_{u \in H^2_{0,p}(\Omega)\setminus\{0\}} \frac{\int_\Omega p(x)|\Delta u(x)|^2 dx}{\int_\Omega q(x)|u(x)|^2 dx}.$$

The study of Kirchhoff-type problems originates from the classical model proposed by Kirchhoff to describe the transversal oscillations of elastic strings, where the tension depends on the entire deformation of the string [1]. In recent decades, such nonlocal problems have attracted considerable attention due to their rich mathematical structure and physical relevance [7,9,10].

Our work extends and generalizes several recent contributions. In particular, we build upon the frameworks developed in [9,10,11], which address second-order Kirchhoff-type variational inequalities. The key novelties of the present paper are twofold: first, we treat a fourth-order differential operator, which models more complex physical phenomena such as the bending of elastic plates; second, we incorporate Muckenhoupt weights $p \in A_2$ and a general potential $q \in L^1(\Omega)$, thereby working in a weighted Sobolev space $H^2_{0,p}(\Omega)$ that allows for degenerate or singular coefficients [2,5,8].

To handle the variational inequality constrained to the convex cone K, we employ the non-smooth critical point theory introduced by Szulkin [12], which is particularly well-suited for functionals that split as the sum of a C^1 functional and a convex, lower semicontinuous term. This approach has proven effective in treating variational inequalities and obstacle problems [5,6,13,14]. Under the above conditions, this infimum is achieved and $\lambda_1(q) > 0$.

Theorem 1.1 *1.1 Assume that (f_1), (f_2), (f_3), and (f_4) hold. Then problem (1.1) has an unbounded sequence of solutions.*

2. Main Result

Lemma 2.1 *2.1 $H^2_{0,p}(\Omega)$ embeds in $L^2_p(\Omega)$*

Proof: In what follows, we denote by $\|.\|_r$ as $L^r(\Omega) - norm$, for $u \in H^2_{0,p}(\Omega)$, using weighted Poincare inequalities for $C_1, C_2 > 0$ we get

$$\|u\|_{L^2_p(\Omega)} = \|\sqrt{p}u\|_2^2 \leq C_1 \|\sqrt{p}\nabla u\|_2^2 \leq C_1.C_2 \|\sqrt{p}\Delta u\|_2^2 \leq C_1.C_2 \|u\|_{H^2_{0,p}(\Omega)}$$

That is

$$\|u\|_{L^2_p(\Omega)} \leq \sqrt{C_1.C_2}\, \|u\|_{H^2_{0,p}(\Omega)}$$

□

Lemma 2.2 *2.2 On $H^2_{0,p}$, the quantity $[u]_2$ defines a norm which is equivalent to the $H^2_{0,p}-norm$.*

Proof: Given $u \in H^2_{0,p}(\Omega)$, by weighted Poincaré inequality, we have

$$\|\sqrt{p}u\|_2^2 \leq C_1.C_2\,[u]_2^2\,, \quad [u]_1^2 \leq C_2\,[u]_2^2\,.$$

Then

$$\|u\|^2_{H^2_{0,p}} = [u]_2^2 + [u]_1^2 + \|\sqrt{p}u\|_2^2 \leq (1 + C_2 + C_1.C_2)\,[u]_2^2\,.$$

That is

$$[u]_2 \leq \|u\|_{H^2_{0,p}} \leq \sqrt{1 + C_2 + C_1.C_2}\,[u]_2\,.$$

Where $[u]_1 := \left(\int_\Omega p(x)\,|\nabla u(x)|^2\,dx\right)^{1/2}$. □

Lemma 2.3 *2.3 (i) $H^2_{0,p}$ is continuously embedded in C^0, more precisely for every $u \in H^2_{0,p}$, one has $\|u\|_\infty \leq d\,[u]_2$ where $d > 0$.*
(ii) The embedding $H^2_{0,p}(\Omega) \hookrightarrow C^0(\Omega)$ is compact.

Proof: (i) Using Hölder's inequality, for any such α

$$\|D^\alpha u\|_1^2 = \left\|\frac{\sqrt{p}}{\sqrt{p}}D^\alpha u\right\|_1^2 \leq \left\|\frac{1}{p}\right\|_1 \left\|pD^\alpha u^2\right\|_1 = M\,\|D^\alpha u\|^2_{L^2_p}$$

where $M := \left\|\frac{1}{p}\right\|_1$, therefore

$$\|u\|^2_{H^2} = \|u\|_2^2 + \|\nabla u\|_2^2 + \|\Delta u\|_2^2 \leq M(\|u\|^2_{L^2_p} + \|\nabla u\|^2_{L^2_p} + \|\Delta u\|^2_{L^2_p}) = M\,\|u\|^2_{H^2_{0,p}}\,.$$

Since $\Omega \subset \mathbb{R}^N$ and $N < 4$, the Sobolev embedding theorem implies:

$$H^2(\Omega) \hookrightarrow C^0(\Omega)$$

thus, there exists a constant $C_4 > 0$ such that for all $u \in H^2$

$$\|u\|_\infty \leq C_4\,\|u\|_{H^2} \leq C_4\sqrt{M}\,\|u\|_{H^2_{0,p}}\,.$$

which proves the continuous embedding $H^2_{0,p}(\Omega) \hookrightarrow C^0(\Omega)$
(ii) Consider a bounded sequence $\{u_n\} \subset H^2_{0,p}(\Omega)$. Then:
$\{u_n\}$ is equicontinuous and uniformly bounded in $C^0(\Omega)$. By the Arzelà–Ascoli theorem, there exists a subsequence still noted $\{u_n\}$ that converges uniformly on Ω, hence in $C^0(\Omega)$, Therefore, the embedding $H^2_{0,p}(\Omega) \hookrightarrow C^0(\Omega)$ is compact. □

Consider the Hilbert spaces

$$L^4_q(\Omega) = \{u : \Omega \to \mathbb{R} \text{ measurable such that } \sqrt[4]{q}u \in L^4(\Omega)\}$$

equipped with the norm

$$\|u\|_{L^4_q} = \left(\int_\Omega qu^4 dx\right)^{\frac{1}{4}}$$

Since $\|u\|^4_{L^4_q} \leq \|q\|_1\,\|u\|^4_\infty$, we have

Proposition 2.1 *2.1 C^0 is continuously embedded in L^4_q.*

Corollary 2.1 *2.1 $H^2_{0,p}$ is compactly embedded in L^4_q .*

We consider the problem, denoted by (P):

$$(a+b[u]_2^2)\int_\Omega p(x)\,\Delta u\,\Delta(v-u)\,dx-\int_\Omega q(x)\,f(x,u)(v-u)\,dx\geq 0\quad \forall v\in K.$$

Definition 2.1 *2.1 **(Szulkin-type functional):** Let X be a real Banach space and X^* its dual . Let ϕ be a functional, that is of class C^1 and let $\psi : X\to\mathbb{R}\cup\{+\infty\}$ be proper (i.e, $\psi\neq+\infty$), convex and lower semicontinuous functional, we say that $E=\phi+\psi$ is a Szulkin-type functional. An element $u\in X$ is called a critical point of $E=\phi+\psi$ if*

$$\phi'(u)(v-u)+\psi(v)-\psi(u)\geq 0\quad \textit{for all}\quad v\in X,$$

Definition 2.2 *2.2 **(Palais-Smale):** The functional $E=\phi+\psi$ satisfies the Palais-Smale condition at level $c\in\mathbb{R}$, denoted by $(PSZ)_c$, if every sequence $\{u_n\}\subset X$ such that $\lim_{n\to\infty}E(u_n)=c$ and*

$$\langle\phi'(u_n),v-u_n\rangle+\psi(v)-\psi(u_n)\geq\varepsilon_n\,\|v-u_n\| \tag{2.1}$$

where $\varepsilon_n\mapsto 0$, possesses a convergent subsequence.

Theorem 2.1 *2.1([?],Theorem 3.6.) **(Fountain theorem):** Let X be a Banach space such that $Z_k=\overline{\bigoplus_{j=k}^{\infty}X_j}$, $\quad Y_k:=\bigoplus_{j=0}^{k}X_j$, where each Y_k is a finite-dimensional subspace. And let $E=\phi+\psi$:$X\to\overline{\mathbb{R}}$ a Szulkin-type functional be even. If, for every $k\in\mathbb{N}$, there exists $\rho_k>r_k>0$ such that*
(A_1) $\max_{\substack{u\in Y_k\\ \|u\|=\rho_k}} E(u)\leq 0$
(A_2) $\inf_{\substack{u\in Z_k\\ \|u\|=r_k}} E(u)\to\infty,\quad k\to\infty,$
(A_3) *E satisfies the $(PS)_c$ condition for every $c>0$, then E has an unbounded sequence of critical values.*

We define the functional E : $H^2_{0,p}(\Omega)\to\mathbb{R}$ by

$$E(u):=\frac{a}{2}\,[u]_2^2+\frac{b}{4}\,[u]_2^4-\int_\Omega q(x)\,F(x,u)\,dx,$$

Because f is continuous, by using Lebesgue dominated convergence theorem and the compact embedding of $H^2_{0,p}(\Omega)$ in $C^0(\Omega)$ and (f_1), one can prove easily that $E\in C^1(H^2_{0,p}(\Omega),\mathbb{R})$.
We define the indicator functional of the set K by

$$\psi_K(u)=\begin{cases}0 & \text{if}\quad u\in K\\ +\infty & \text{if}\quad u\notin K.\end{cases}$$

We remark that the functional ψ_K is convex, proper, and lower semicontinuous. Then, $I=E+\psi_K$ is a Szulkin-type functional, in what follows we denote $\|u\|=\|u\|_{H^2_{0,p}}$.

Proposition 2.2 *2.3 Every critical point $u\in H^2_{0,p}(\Omega)$ of $I=E+\psi_K$ s a solution of (P).*

Proof: Since $u\in H^2_{0,p}(\Omega)$ is a critical point of $I=E+\psi_K$, we have

$$E'(u)(v-u)+\psi_K(v)-\psi_K(u)\geq 0,\qquad \forall v\in H^2_{0,p}(\Omega).$$

Note that u belongs to K. If this is not true we have $\psi_K(u)=+\infty$ and taking $v=0\in K$ in the above inequality, we obtain a contradiction. We fix $v\in K$. Since

$$0\leq E'(u)(u-v)=(a+b[u]_2^2)\int_\Omega p(x)\,\Delta u\,\Delta(v-u)\,dx-\int_\Omega q(x)\,f(x,u)(v-u)\,dx,$$

the inequality is proved. ◻

Proposition 2.3 *2.4 If the function f satisfies (f_1) and (f_4), then $I = E + \psi_K$ satisfies $(PSZ)_c$ for every $c \in \mathbb{R}$.*

Proof: Let $c \in \mathbb{R}$. be fixed. Let (u_n) be a sequence in $H^2_{0,p}(\Omega)$ such that

$$I(u_n) = E(u_n) + \psi_K(u_n) \to c, \tag{2.2}$$

and

$$E'(u_n)(v - u_n) + \psi_K(v) - \psi_K(u_n) \geq \varepsilon_n \|v - u_n\|, \qquad \forall v \in H^2_{0,p}(\Omega) \tag{2.3}$$

(ε_n) a sequence in Ω with $\varepsilon_n \to 0$. By (2.2), we obtain that the sequence (u_n) is in K .
Setting $v = 2u_n$ in (2.3), we obtain

$$E'(u_n)u_n \geq -\varepsilon_n \|u_n\|.$$

Therefore

$$E'(u_n)u_n = a\,[u_n]_2^2 + b\,[u_n]_2^4 - \int_\Omega q(x) f(x, u_n) u_n \,dx \geq -\varepsilon_n \|u_n\|, \tag{2.4}$$

by (2.2) for large $n \in \mathbb{N}$ we get

$$c + 1 > E(u_n) = \frac{a}{2}[u_n]_2^2 + \frac{b}{4}[u_n]_2^4 - \int_\Omega q(x)\, F(x, u_n)\, dx. \tag{2.5}$$

When $\|u_n\| \to +\infty$, then (f_4) implies that there exists $\delta > 1$ such that

$$f(x, u_n)u_n - \nu F(x, u_n) \geq -\beta\, |u_n|^\tau, \forall\, \|u_n\| > \delta.$$

For large n enough, we put $A_n = \{x \in \Omega : \|u_n(x)\| > \delta\}$.
Then, for large n, the combination of (2.4) and (2.5) implies that

$$\begin{aligned}
\nu(c+1) + \|u_n\| &\geq a\nu(\frac{1}{2} - \frac{1}{\nu})[u_n]_2^2 + b\nu(\frac{1}{4} - \frac{1}{\nu})[u_n]_2^4 \\
&+ \int_\Omega (q(x) f(x, u_n(x)) u_n(x) - \nu F(x, u_n(x))) dx \\
&= a\nu(\frac{1}{2} - \frac{1}{\nu})[u_n]_2^2 + b\nu(\frac{1}{4} - \frac{1}{\nu})[u_n]_2^4 \\
&+ \int_{A_n} (q(x) f(x, u_n(x)) u_n(x) - \nu F(x, u_n(x))) dx \\
&+ \int_{\Omega \setminus A_n} (q(x) f(x, u_n(x)) u_n(x) - \nu F(x, u_n(x))) dx \\
&\geq a\nu(\frac{1}{2} - \frac{1}{\nu})[u_n]_2^2 + b\nu(\frac{1}{4} - \frac{1}{\nu})[u_n]_2^4 - \beta \int_{A_n} q(x)\, |u_n|^\tau \,dx \\
&+ \int_{\Omega \setminus A_n} (q(x) f(x, u_n(x)) u_n(x) - \nu F(x, u_n(x))) dx
\end{aligned}$$

by (f_1), there exists$C > 0$ such that

$$q(x)(f(x, u_n))u_n - \nu F(x, u_n) \geq -Cq(x)(1 + |u_n|^{p-1} + \nu C\frac{1}{p}\,|u_n|^p) \geq -Cq(x)$$

Then

$$\begin{aligned}\nu(c+1)+\|u_n\|\,\varepsilon_n &\geq a\nu(\frac{1}{2}-\frac{1}{\nu})[u_n]_2^2+b\nu(\frac{1}{4}-\frac{1}{\nu})[u_n]_2^4\\ &-\beta\int_{A_n} q(x)\,|u_n|^{\tau}\,dx - C\int_{\Omega\setminus A_n} q(x)dx\\ &\geq a\nu(\frac{1}{2}-\frac{1}{\nu})[u_n]_2^2+b\nu(\frac{1}{4}-\frac{1}{\nu})[u_n]_2^4\\ &-\beta\int_{\Omega} q(x)\,|u_n|^2\,dx - C\,\|q\|_{L^1}\\ &\geq \frac{a\nu}{1+C_2+C_1.C_2}(\frac{1}{2}-\frac{1}{\nu})\,\|u_n\|^2+\frac{b\nu}{(1+C_2+C_1.C_2)^2}(\frac{1}{4}-\frac{1}{\nu})\,\|u_n\|^4\\ &-\beta\,\|u_n\|_{L_q^2}^2 - C\,\|q\|_{L^1}\\ &\geq \frac{a\nu}{1+C_2+C_1.C_2}(\frac{1}{2}-\frac{1}{\nu})\,\|u_n\|^2+\frac{b\nu}{(1+C_2+C_1.C_2)^2}(\frac{1}{4}-\frac{1}{\nu})\,\|u_n\|^4\\ &-\beta\frac{\|u_n\|^2}{\lambda_1} - C\,\|q\|_{L^1}\\ &=\frac{1}{1+C_2+C_1.C_2}\left(a\nu(\frac{1}{2}-\frac{1}{\nu})-\frac{\beta}{\lambda_1}\right)\|u_n\|_2^2+\frac{b\nu}{(1+C_2+C_1.C_2)^2}(\frac{1}{4}-\frac{1}{\nu})\,\|u_n\|_2^4\\ &-C\,\|q\|_{L^1}\,.\end{aligned}$$

Since ν and β satisfy (a) or (b), this is a contradiction. Then, (u_n) is bounded in K. Hence there exists a subsequence still denoted by (u_n) which converges weakly in $H^2_{0,p}(\Omega)$. So there exists $u \in H^2_{0,p}(\Omega)$ such that

$$u_n \rightharpoonup u \quad \text{in} \quad H^2_{0,p}(\Omega) \tag{2.6}$$

$$u_n \to u \quad \text{in} \quad L^4_q(\Omega) \tag{2.7}$$

$$u_n \to u \quad \text{in} \quad C^0(\Omega) \tag{2.8}$$

As K is weakly closed, $u \in K$. Setting $v = u$ in (2.3), we obtain that

$$\begin{aligned}&(a+b[u_n]_2^2)\int_{\Omega} p(x)\,\Delta u_n\,\Delta(u-u_n)\,dx\\ &-\int_{\Omega} q\,f(x,u_n)(u-u_n)\,dx \geq -\varepsilon_n\,\|u-u_n\|\,.\end{aligned}$$

Therefore, for large $n \in \mathbb{N}$, by using Hölder's inequality and (f_1), there exists$R > 0$ such that

$$\begin{aligned}(a+b[u_n]_2^2)\,\|u-u_n\|^2 &\leq (a+b[u_n]_2^2)\int_{\Omega} p(x)\,\Delta u\,\Delta(u-u_n)\,dx+\int_{\Omega} q(x)\,f(x,u_n)(u_n-u)\,dx+\varepsilon_n\,\|u-u_n\|\\ &\leq (a+b[u_n]_2^2)(u,u-u_n))_{H^2_{0,p}(\Omega)}+c\left(\int_{\Omega} q(x)\,\left|1+|u_n|^{\theta-1}\right|^{\frac{4}{3}}dx\right)^{\frac{3}{4}}\|u-u_n\|_{L_q^4}\\ &+\varepsilon_n\,\|u-u_n\|\\ &\leq (a+b[u_n]_2^2)(u,u-u_n))_{H^2_{0,p}(\Omega)}+c\,\|q\|_1^{\frac{3}{4}}\,(R^{\theta}+1)\,\|u-u_n\|_{L_q^4}\,dx\\ &+\varepsilon_n\,\|u-u_n\|\,.\end{aligned}$$

As $n \to \infty$, $\|u-u_n\| \to 0$ therefore $u_n \to u \quad \text{in} \quad H^2_{0,p}(\Omega)$. □

3. Proof of Main Results

Let $\{e_j\}$ is an orthonormal basis of $H^2_{0,p}(\Omega)$ and define $x_j = \mathbb{R}e_j$

$$Y_l = \bigoplus_{j=1}^{l} X_j, \qquad Z_k = \bigoplus_{j=k}^{\infty} X_j, \qquad l,k \in \mathbb{Z}. \tag{3.1}$$

Therefore, we have the following lemma

Lemma 3.1 *3.1 Suppose that (f_2) is satisfied. Then, for any finite dimensional subspace $Y_K \subset K$, there is a $\rho_k = \rho(Y_K) > 0$ such that:*

$$E \leq 0 \quad \textit{for} \quad \|u\| = \rho_k. \tag{3.2}$$

Proof: Since Y_k is finite-dimensional, there exists $c_k > 0$ such that for all $u \in Y_k$:

$$\|u\|_{L^r} \geq c_r \|u\|. \tag{3.3}$$

By (f_2) and there exists $\rho_k > 0$ for $T > 0$ such that:

$$F(x,u) \geq T|u|^4 \quad \text{for} \quad |u| \geq \rho_k \tag{3.4}$$

It follows from (3.3) and (3.4) and the continuous of F that

$$\begin{aligned}
E(u) &\leq \frac{a}{2}\|u\|^2 + \frac{b}{4}\|u\|^4 - \int_\Omega q(x)F(x,u)dx \\
&\leq \frac{a}{2}\|u\|^2 + \frac{b}{4}\|u\|^4 - \int_{u\geq\rho_k} q(x)F(x,u)dx - \int_{u<\rho_k} q(x)F(x,u)dx \\
&\leq \frac{a}{2}\|u\|^2 + \frac{b}{4}\|u\|^4 - T\int_{u\geq\rho_k} q(x)u^4dx - C_R\|q\|_1 \\
&\leq \frac{a}{2}\|u\|^2 + \frac{b}{4}\|u\|^4 - T\int_\Omega q(x)u^4dx + T\int_{u<\rho_k} q(x)u^4dx - C_R\|q\|_1 \\
&\leq \frac{a}{2}\|u\|^2 + \frac{b}{4}\|u\|^4 - T\|u\|^4_{L^4_q} + (T\rho_k^4 - C_R)\|q\|_1
\end{aligned}$$

by corollary 2.1 there $C_q > 0$ such that $\|u\|_{L^4_q} \leq C_q\|u\|$

$$E(u) \leq \frac{a}{2}\|u\|^2 + \frac{b}{4}\|u\|^4 - TC_q^4\|u\|^4 + (T\rho_k^4 - C_R)\|q\|_1 \tag{3.5}$$

for $T > \frac{b}{4C_q^4}$, and for large ρ_k, the quartic term dominates, since Y_k is finite-dimensional, the sphere $\{u \in Y_k : \|u\| = \rho_k\}$ is compact. Therefore for large ρ_k

$$E(u) \leq 0 \quad \text{for} \quad \|u\| = \rho_k.$$

□

Lemma 3.2 *3.2 We define*

$$\beta_k := \sup_{\substack{u\in Z_k \\ \|u\|=1}} \|u\|_\infty.$$

Where $\beta_k \to 0$ as $k \to \infty$.

Proof: We have $0 < \beta_{k+1} \leq \beta_k$ hence $\beta := \lim_{k\to\infty} \beta_k$. Assume, for contradiction, that $\beta > 0$. Then there exists$\varepsilon > 0$ and $u_k \in Z_k$ such that $\|u_k\|_H = 1$ and $\|u_k\|_\infty \geq \varepsilon$. Because of the orthogonal decomposition, $u_k \rightharpoonup 0$ weakly in $H^2_{0,p}(\Omega)$. By the compact embedding $H^2_{0,p}(\Omega) \hookrightarrow C^0(\overline{\Omega})$, a subsequence (still denoted u_k) satisfies $u_k \to u$ in $C^0(\overline{\Omega})$. Since $\int_\Omega p < \infty$, this implies $u_k \to u$ in $L^2_p(\Omega)$, while the weak convergence in $H^2_{0,p}(\Omega)$ gives $u_k \rightharpoonup 0$ in $L^2_p(\Omega)$. Hence $u \equiv 0$, and therefore $\|u_k\|_\infty \to 0$, contradicting $\|u_k\|_\infty \geq \varepsilon$. Thus $\beta = 0$. □

Lemma 3.3 *3.3 Suppose that* (f_1) *and* (f_2) *are satisfied, let* $h>0$ *and* $r_k := \frac{h}{\beta_k}$ *which implies* $r_k \to +\infty$ *as* $k \to +\infty$ *then:*

$$\inf_{\substack{u\in Z_K \\ \|u\|=r_k}} E(u) \to +\infty \quad as \quad k \to +\infty.$$

Proof: By the choose $r_k := \frac{h}{\beta_k}$ thus for every $u \in Z_k$ with $\|u\| = r_k$ we get

$$\|u\|_\infty \leq \beta_k \|u\| = \beta_k r_k = h$$

and by the continuity of F for $|u| < h$, there exists a constant $A > 0$ such that $|F(x,u)| \leq A$ thus

$$-\int_\Omega q(x)F(x,u)dx \geq -A\|q\|_1 . \tag{3.6}$$

Therefore,

$$\begin{aligned} E(u_n) &= \frac{a}{2}[u_n]_2^2 + \frac{b}{4}[u_n]_2^4 - \int_\Omega q(x)\,F(x,u_n)\,dx. \\ &\geq \frac{a}{2(1+C_2+C_1.C_2)}\|u\|^2 + \frac{b}{4(1+C_2+C_1.C_2)^2}\|u\|^4 - A\|q\|_1 \\ &\geq \frac{a}{2(1+C_2+C_1C_2M)} r_k^2 + \frac{b}{4(1+C_2+C_1.C_2)^2} r_k^4 - A\|q\|_1 . \end{aligned}$$

Thus $E(u) \to \infty$ as $k \to \infty$ □

Proof of Theorem 1.1 Let $X = H^2_{0,p}, X_1 = Z_m$, and $X_2 = Y_n$. By (f_3) implies that E is even. By Lemmas 3.1, 3.3 and proposition 2.4 all conditions of Theorem 2.1 are satisfied. Thus, problem (P) possesses unbounded sequence of solutions.

4. Conclusion

In our work, we have studied variational inequalities of Kircchoff type which is defined on a bounded set of $\mathbb{R}^N$. Unlike the classic variational inequalities of Kircchoff type our case has a mass function that's why we've defined a new space $H^2_{0,p}$. and we study the possible embbeds between $H^2_{0,p}$. And the L^r.
In order to establish multiple solutions for the problem we used the Palais-Smale condition under some assumptions. Through it and through the critical point theory, we were able to find solutions to the problem.
Exemple: Consider

$$f(x,u) = (1+|x|^2)\,|u|^{p-2}u, \qquad (x,u) \in \Omega \times \mathbb{R},$$

with an exponent

$$p \in (\max\{4,\nu\},\, 2^*).$$

It is straightforward to verify that f meets all the assumptions:

(f_1) $f \in C(\Omega \times \mathbb{R})$ and $|f(x,u)| \leq c(1+|u|^{p-1})$ with $\theta = p \in (1, 2^*)$.

(f_2) $\dfrac{F(x,u)}{|u|^4} \to +\infty$ uniformly in x since $p > 4$, and $F(x,u) \geq 0$ for all $|u| \geq 0$.

(f_3) $f(x,-u) = -f(x,u)$.

(f_4) $f(x,u)u - \nu F(x,u) = (1+|x|^2)\Big(1 - \dfrac{\nu}{p}\Big)|u|^p > 0$ for $p > \nu$, hence the condition holds for any $\tau \in (0,2)$.

Declarations

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