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Perturbed G-Metric Space and Fixed Point Results

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ABSTRACT: In this paper, we introduce and investigate fixed point results within the framework of perturbed G-metric space, a generalization of metric and perturbed metric spaces involving a generalized distance function defined on triples of points. We establish the fixed point theorems of Banach, Kannan, Jleli and Samet, and Nutu and Pacurar for mappings that contract with respect to a perturbed G-metric, proving existence and uniqueness of fixed points. Further, the results of Mustafa, Mustafa and Obiedat, and Gaba are obtained as corollaries.

Key Words: Perturbed G-metric space, contractive condition, unique common fixed point.

Contents

1. Introduction

In classical metric space theory, the notion of distance measurement plays a fundamental role in various mathematical analyses and fixed point results. However, real-world measurements are often subject to errors and perturbations arising from instrumental inaccuracies or environmental factors, which motivates the study of perturbed metric spaces. Recently, the concept of perturbed metric spaces [6] has been introduced to model metric structures alongside a perturbation function that captures such deviations, thereby generalizing the classical metric space framework. A perturbed metric space is characterized by a pair of functions—one representing a perturbed distance and the other representing the perturbation map—such that the difference yields a genuine metric.

Taking this into consideration, Jleli and Samet [6] introduced the notion of perturbed metric spaces and established Banach fixed point theorem in such spaces.

Definition 1.1. *[6] Let $D, P : X \times X \to [0, \infty)$ be two given mappings. We say that D is a perturbed metric on X with respect to P, if*

$$D - P : X \times X \to \mathbb{R}, \quad (x, y) \mapsto D(x, y) - P(x, y)$$

is a metric on X; that is, for all $x, y, z \in X$,

(i) $(D - P)(x, y) \geq 0$;

(ii) $(D - P)(x, y) = 0$ *if and only if* $x = y$;

(iii) $(D - P)(x, y) = (D - P)(y, x)$;

(iv) $(D - P)(x, y) \leq (D - P)(x, z) + (D - P)(z, y)$.

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We call P a perturbation mapping, *$d = (D - P)$ an* exact metric, *and (X, D, P) a* perturbed metric space.

Sihag et al. [15] extended the concept of perturbed metric space into complex valued perturbed metric spaces and extended the Banach fixed point theorem from perturbed metric spaces to complex valued perturbed metric space. Further, the results of Azam et al. [1] are derived as corollaries.

Definition 1.2. *[15] Let $D, P : X \times X \to \mathbb{C}$ be two given mappings. We say that D is a complex valued perturbed metric on X with respect to P if*

$$D - P : X \times X \to \mathbb{C},$$

$$(x, y) \mapsto D(x, y) - P(x, y)$$

is a complex valued metric on X, i.e., for all $x, y, z \in X$:

(i) $(D - P)(x, y) \succcurlyeq 0$*;*

(ii) $(D - P)(x, y) = 0$ *if and only if* $x = y$*;*

(iii) $(D - P)(x, y) = (D - P)(y, x)$*;*

(iv) $(D - P)(x, y) \preccurlyeq (D - P)(x, z) + (D - P)(z, y)$.

We call P a complex valued perturbed mapping, $d = (D - P)$ an exact complex valued metric, and (X, D, P) a complex valued perturbed metric space.

Notice that a complex valued perturbed metric on X is not necessarily a complex valued metric on X.

G-metric Space:

Mustafa and Sims [10] critically examined the foundational claims regarding the topological properties of Dhage's D-metric spaces [3] and demonstrated that many of these claims are incorrect (also see [11]). In particular, a D-metric need not be continuous with respect to its variables, and despite Dhage's efforts to define an associated topology, the notion of D-convergence for a sequence (x_n) to a point x, characterized by

$$D(x_m, x_n, x) \to 0 \quad \text{as} \quad m, n \to \infty,$$

does not necessarily correspond to convergence within any topological framework.

Addressing these fundamental flaws in Dhage's theory—flaws that undermine many of the established results— a more robust alternative was proposed: the G-metric space.

Definition 1.3. *[10] Let X be a nonempty set, and let the function $G : X \times X \times X \to [0, \infty)$ satisfy the following properties:*

(G_1) $G(x, y, z) = 0$ *if* $x = y = z$ *for all* $x, y, z \in X$*;*

(G_2) $G(x, x, y) > 0$ *whenever* $x, y \in X$ *with* $x \neq y$*;*

(G_3) $G(x, x, y) \leq G(x, y, z)$ *whenever* $x, y, z \in X$ *with* $z \neq y$*;*

(G_4) $G(x, y, z) = G(y, z, x) = G(z, x, y) = \dots$ *(symmetry in all three variables);*

(G_5) $G(x, y, z) \leq G(x, a, a) + G(a, y, z), \quad for\, any\, x, y, z, a \in X, \quad (rectangle\, inequality).$

Then the function G is called a generalized metric, or, more specifically a G-metric on X, and the pair (X, G) is a G-metric space.

Clearly these properties are satisfied when $G(x, y, z)$ is the perimeter of the triangle with vertices at x, y and z in $\mathbb{R}^2$, further taking a in the interior of the triangle shows that (G_5) is best possible.

Definition 1.4. *[10] Let (X, G) be a G-metric space, and let $\{x_n\}$ be a sequence of points in X. A point $x \in X$ is said to be the limit of the sequence $\{x_n\}$ if*

$$\lim_{n,m\to\infty} G(x, x_n, x_m) = 0,$$

and one says that the sequence $\{x_n\}$ is G-convergent to x.

Consequently, if $\{x_n\} \to x$ in the G-metric space (X, G), then for any $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that

$$G(x, x_n, x_m) < \varepsilon \quad \text{for all } n, m \geq N.$$

Definition 1.5. *[10] Let (X, G) be a G-metric space. A sequence $\{x_n\}$ is called G-Cauchy if for every $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that*

$$G(x_n, x_m, x_l) < \varepsilon, \quad \text{for all } n, m, l \geq N,$$

that is,

$$\lim_{n,m,l\to\infty} G(x_n, x_m, x_l) = 0.$$

Proposition 1.6. *[10] If (X, G) is a G-metric space, then the following are equivalent:*

(i) *The sequence $\{x_n\}$ is G-Cauchy.*

(ii) *For every $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that*

$$G(x_n, x_m, x_m) < \varepsilon, \quad \text{for all } n, m \geq N.$$

Definition 1.7. *[10] Let (X, G) and (X', G') be two G-metric spaces and $f : X \to X'$ a function. We say that f is G-continuous at a point $a \in X$ if and only if for every $\varepsilon > 0$, there exists $\delta > 0$ such that whenever $x, y \in X$ satisfy*

$$G(a, x, y) < \delta,$$

then

$$G'(f(a), f(x), f(y)) < \varepsilon.$$

The function f is G-continuous on X if it is G-continuous at every point $a \in X$.

Definition 1.8. *[10] A G-metric space (X, G) is called symmetric if*

$$G(x, y, y) = G(y, x, x) \quad \text{for all } x, y \in X.$$

Proposition 1.9. *[10] Let (X, G) be a G-metric space, then the function $G(x, y, z)$ is jointly continuous in all three variables.*

Definition 1.10. *[10] A G-metric space (X, G) is said to be G-*complete *(or complete) if every G-Cauchy sequence in (X, G) is G-convergent in (X, G).*

Proposition 1.11. *[10] Let (X, G) be a G-metric space. Then the following are equivalent:*

(i) *$\{x_n\}$ is G-convergent to x.*

(ii) $\lim_{n\to\infty} G_P(x_n, x_n, x) = 0.$

(iii) $\lim_{n\to\infty} G_P(x_n, x, x) = 0.$

(iv) $\lim_{m,n\to\infty} G_P(x_m, x_n, x) = 0.$

Perturbed G-metric space:

Now we apply the ideas of perturbation to G-metric space and introduce perturbed G-metric space. This theory of perturbed G-metric space aims to establish existence, uniqueness, and stability of fixed points for mappings under perturbations, thereby broadening the classical fixed point results like Banach contraction principle.

A *perturbed G-metric space* extends the classical notion of a G-metric space by incorporating a perturbation function that modifies the underlying G-metric structure. This generalization provides a more flexible framework suitable for analyzing contractive mappings and fixed point properties under more general and realistic metric perturbations, which frequently arise in applied and theoretical contexts.

Definition 1.12. *Let X be a nonempty set, and $G_P, P : X \times X \times X \to [0, \infty)$ be two given mappings. We say that G_P is a perturbed G-metric on X with respect to P, if*

$$G_P - P : X \times X \times X \to [0, \infty), \quad (x, y, z) \mapsto G_P(x, y, z) - P(x, y, z)$$

is a G-metric on X; that is, for all $x, y, z \in X$,

(G_{P1}) $(G_P - P)(x, y, z) = 0$ *if* $x = y = z$ *for all* $x, y, z \in X$*;*

(G_{P2}) $(G_P - P)(x, x, y) > 0$ *whenever* $x, y \in X$ *with* $x \neq y$*;*

(G_{P3}) $(G_P - P)(x, x, y) \leq (G_P - P)(x, y, z)$ *whenever* $x, y, z \in X$ *with* $z \neq y$*;*

(G_{P4}) $(G_P - P)(x, y, z) = (G_P - P)(y, z, x) = (G_P - P)(z, x, y) = \ldots$ *(symmetry in all three variables);*

(G_{P5}) $(G_P - P)(x, y, z) \leq (G_P - P)(x, a, a) + (G_P - P)(a, y, z), \quad for\, any\, x, y, z, a \in X.$

We call P a perturbation mapping*, $(G_P - P) =$* G-metric*, and (X, G_P, P) a* perturbed G-metric space.

Example 1.13. *Let $X = [0, 1]$ with the standard absolute value metric $|\cdot|$ and define:*

$$G_P(x, y, z) = \max\{|x - y|, |y - z|, |z - x|\} + P(x, y, z),$$

where

$$P(x, y, z) = \frac{1}{1 + |x| + |y| + |z|}. \quad \textit{(absolute value perturbation)}$$

Then the exact G-metric is given by

$$G(x, y, z) = G_P(x, y, z) - P(x, y, z) = \max\{|x - y|, |y - z|, |z - x|\},$$

which is known to be complete on $[0, 1]$.

Here, G_P-metric is not a G-metric since $G_P(1, 1, 1) = \frac{1}{4} \neq 0$.

Example 1.14. *Let $X = \mathbb{R}$. Define the perturbed G-metric $G_P : X^3 \to [0, \infty)$ by*

$$G_P(x, y, z) = \max\{|x - y|, |y - z|, |z - x|\} + x^2y^2z^2.$$

Here the perturbation $P : X^3 \to [0, \infty)$ is

$$P(x, y, z) = x^2y^2z^2. \quad \textit{(polynomial perturbation)}$$

Then

$$G(x, y, z) = G_P(x, y, z) - P(x, y, z) = \max\{|x - y|, |y - z|, |z - x|\},$$

is a G-metric. Here, G_P-metric is not a G-metric since $G_P(1, 1, 1) = 1 \neq 0$.

In analogy of Jleli and Samet [6], we state some elementary properties of perturbed G-metric spaces.

Definition 1.15. *Let (X, G_P, P) be a perturbed G-metric space, $\{x_n\}$ a sequence in X, and $T : X \to X$.*

(i) We say that $\{x_n\}$ is a perturbed Cauchy sequence in (X, G_P, P) if $\{x_n\}$ is a Cauchy sequence in the G-metric space (X, G).

(ii) We say that $\{x_n\}$ is a perturbed convergent sequence in (X, G_P, P) if $\{x_n\}$ is a convergent sequence in the G-metric space (X, G).

(iii) We say that (X, G_P, P) is a complete perturbed G-metric space if (X, G) is a complete G-metric space.

(iv) We say that $\mathcal{T}$ is a perturbed continuous mapping in G_P-metric if $\mathcal{T}$ is continuous with respect to the G-metric.

(v) We say that the equivalence relation in Proposition 1.11 holds in perturbed G-metric space if the relation holds in G-metric space.

2. Banach fixed point theorem in Perturbed G-metric space

Theorem 2.1. *Let (X, G_P, P) be a complete perturbed G-metric space and let $T : X \to X$ be a given perturbed continuous mapping. Assume the following condition holds:*

(i) There exists $\lambda \in (0, 1)$ such that

$$G_P(Tx, Ty, Tz) \leq \lambda\, G_P(x, y, z) \tag{2.1}$$

for all $x, y, z \in X$.

Then, T admits a unique fixed point $x^* \in X$, that is, $Tx^* = x^*$.

Proof. Let $x_0 \in X$ be fixed. Consider the Picard sequence $\{x_n\} \subset X$ defined by

$$x_{n+1} = Tx_n, \quad n \in \mathbb{N}.$$

Applying the contractive condition (2.1) to the triple (x_0, x_1, x_1), we have

$$G_P(Tx_0, Tx_1, Tx_1) \leq \lambda G_P(x_0, x_1, x_1),$$

which implies

$$G_P(x_1, x_2, x_2) \leq \lambda G_P(x_0, x_1, x_1). \tag{2.2}$$

Similarly, for the triple (x_1, x_2, x_2), we get

$$G_P(x_2, x_3, x_3) \leq \lambda G_P(x_1, x_2, x_2),$$

which gives

$$G_P(x_2, x_3, x_3) \leq \lambda^2 G_P(x_0, x_1, x_1).$$

By induction, it follows that

$$G_P(x_n, x_{n+1}, x_{n+1}) \leq \lambda^n \tau, \quad n \in \mathbb{N}, \tag{2.3}$$

where $\tau = G_P(x_0, x_1, x_1)$.

Since $G = G_P - P$ is the G-metric, so

$$G(x_n, x_{n+1}, x_{n+1}) + P(x_n, x_{n+1}, x_{n+1}) \leq \lambda^n \tau, \quad n \in \mathbb{N}.$$

Since

$$G(x_n, x_{n+1}, x_{n+1}) \leq G(x_n, x_{n+1}, x_{n+1}) + P(x_n, x_{n+1}, x_{n+1}),$$

we have

$$G(x_n, x_{n+1}, x_{n+1}) \leq \lambda^n \tau, \quad n \in \mathbb{N}.$$

Since $\lambda \in (0,1)$, so $\{x_n\}$ is a Cauchy sequence in (X,G), that is,$\{x_n\}$ is a perturbed Cauchy sequence in the perturbed G-metric space (X,G_P,P). By the completeness of perturbed G-metric space (X,G_P,P), there exists $x^* \in X$ such that

$$\lim_{n\to\infty} G(x_n,x_n,x^*) = 0. \tag{2.4}$$

Claim: x^* is a fixed point of T.

Since T is perturbed continuous, it follows that

$$\lim_{n\to\infty} G(Tx_n,Tx_n,Tx^*) = 0,$$

which is

$$\lim_{n\to\infty} G(x_{n+1},x_{n+1},Tx^*) = 0.$$

By uniqueness of the limit,

$$x^* = Tx^*,$$

so x^* is a fixed point of T.

To show uniqueness, suppose $u,v \in X$ are distinct fixed points. Then

$$G(u,v,v) = G(Tu,Tv,Tv) \le \lambda G(u,v,v).$$

Since $u \neq v$, the left side is nonzero, giving $\lambda \ge 1$, a contradiction. Thus, x^* is the unique fixed point of T. □

Example 2.2. *Let $X = [0,1]$. Then (X,G) is G-metric space where the G-metric is given by*

$$G(x,y,z) = \max\{|x-y|,|y-z|,|z-x|\}.$$

Define

$$G_P(x,y,z) = G(x,y,z) + P(x,y,z) = \max\{|x-y|,|y-z|,|z-x|\} + xyz.$$

Then G_P is a perturbed G-metric on $[0,1]$ with respect to perturbed mapping

$$P : X \times X \times X \to [0,\infty)$$

given by

$$P(x,y,z) = xyz.$$

Here, G_P-metric is not a G-metric since $G(1,1,1) = 1 \neq 0$.

Define the mapping $T : X \to X$ by

$$Tx = \frac{x}{3}.$$

For all $x,y,z \in [0,1]$,

$$\begin{aligned} G_P(Tx,Ty,Tz) &= \max\left\{\left|\frac{x}{3}-\frac{y}{3}\right|,\left|\frac{y}{3}-\frac{z}{3}\right|,\left|\frac{z}{3}-\frac{x}{3}\right|\right\} + \frac{xyz}{27} \\ &= \frac{1}{3}\max\{|x-y|,|y-z|,|z-x|\} + \frac{xyz}{27} \\ &\le \frac{1}{3}\left(\max\{|x-y|,|y-z|,|z-x|\} + xyz\right) \\ &= \lambda\, G_P(x,y,z), \quad \textit{where}\, \lambda = \frac{1}{3} \in (0,1). \end{aligned}$$

Thus all the condition of Theorem 2.1 are satisfied and indeed $x = 0$ is the unique fixed point.

Corollary 2.3. *[9] Let (X, G) be a complete G-metric space and $T : X \to X$ a given mapping. Assume there exists $\lambda \in (0, 1)$ such that*

$$G(Tx, Ty, Tz) \leq \lambda G(x, y, z)$$

for all $x, y, z \in X$. Then T admits a unique fixed point.

Proof. Set $P \equiv 0$, i.e., $P(x, y, z) = 0$ for all $x, y, z \in X$. Then (X, G_P, P) is a perturbed G-metric space where the perturbed G_P-metric coincides with G, and the contractive condition holds directly. Consequently, Theorem 2.1 applies to guarantee the existence and uniqueness of the fixed point for T. □

Since a complete perturbed G-metric space is a generalization of perturbed metric space, so the Theorem 2.1 is a proper extension of the following result of Jleli and Samet [6].

Corollary 2.4. *Let (X, D, P) be a complete perturbed metric space and $T : X \to X$ be a given mapping. Assume that the following conditions hold:*

(i) T is a perturbed continuous mapping;

(ii) There exists $\lambda \in (0, 1)$ such that

$$D(Tx, Ty) \leq \lambda D(x, y)$$

for all $x, y \in X$.

Then T admits one and only one fixed point.

We present, without proof, the following result which can be proved by taking $y = z$ in Theorem 2.1.

Theorem 2.5. *Let (X, G_P, P) be a complete perturbed G-metric space and let $T : X \to X$ be a given perturbed continuous mapping. Assume the following condition holds:*

(i) There exists $\lambda \in (0, 1)$ such that

$$G_P(Tx, Ty, Ty) \leq \lambda\, G_P(x, y, y) \tag{2.5}$$

for all $x, y \in X$.

Then, T admits a unique fixed point $x^* \in X$, that is, $Tx^* = x^*$.

Corollary 2.6. *[9] Let (X, G) be a complete G-metric space and $T : X \to X$ a given mapping. Assume there exists $\lambda \in (0, 1)$ such that*

$$G(Tx, Ty, Ty) \leq \lambda G(x, y, y)$$

for all $x, y \in X$. Then T admits a unique fixed point.

Proof. Set $P \equiv 0$, i.e., $P(x, y, y) = 0$ for all $x, y \in X$. Then (X, G_P, P) is a perturbed G-metric space where the perturbed G_P-metric coincides with G, and the contractive condition holds directly. Consequently, Theorem 2.5 applies to guarantee the existence and uniqueness of the fixed point for T. □

3. Kannan fixed point theorem in Perturbed G-metric space

Kannan extended the Banach fixed point theorem by defining general contractions in [7] and points out that the operator is not necessarily continuous. Recent research introduces notions such as perturbed Kannan mappings in perturbed metric spaces [13], proving fixed point results that extends Kannan fixed point theorem without requiring the continuity of operators. These perturbed metric spaces and the related mapping conditions allow for unique fixed points under the influence of perturbations, which is important in generalized metric spaces.

Definition 3.1. *[13] Let (X, D, P) be a perturbed metric space and $T : X \to X$ a mapping such that there exists $\lambda \in \left[0, \frac{1}{2}\right)$ with*

$$D(Tx, Ty) \leq \lambda\big[D(x, Tx) + D(y, Ty)\big], \quad \textit{for all } x, y \in X. \tag{3.1}$$

Then T is called a perturbed Kannan mapping.

Kannan mappings in the context of G-metric spaces [5,9,14] refer to a class of contractive mappings that generalize the Kannan fixed point theorem to new settings involving the G-metric framework.

Now we define Kannan contraction mapping in perturbed G-metric space.

Definition 3.2. *Let (X, G_P, P) be a perturbed G-metric space and $T : X \to X$ denote a mapping such that for all $x, y, z \in X$,*

$$G_P(Tx, Ty, Tz) \leq \alpha \left[G_P(x, Tx, Tx) + G_P(y, Ty, Ty) + G_P(z, Tz, Tz)\right] \tag{3.2}$$

where $0 \leq \alpha < \frac{1}{2}$. We say T is a perturbed Kannan mapping.

Theorem 3.3. *Suppose that $(X, G_P, P)$is a complete perturbed G-metric space and T is a perturbed Kannan mapping. Then, there exists a unique fixed point of T.*

Proof. Take an arbitrary element $x_0 \in X$ and define the Picard sequence by $x_{n+1} = Tx_n$, $n \in \mathbb{N}$.

By applying inequality (3.2) with $x = x_0$, $y = x_1$, $z = x_1$, we obtain

$$\begin{aligned} G_P(x_1, x_2, x_2) &= G_P(Tx_0, Tx_1, Tx_1) \\ &\leq \alpha \left[G_P(x_0, Tx_0, Tx_0) + G_P(x_1, Tx_1, Tx_1) + G_P(x_1, Tx_1, Tx_1)\right] \\ &= \alpha \left[G_P(x_0, x_1, x_1) + 2G_P(x_1, x_2, x_2)\right]. \end{aligned}$$

Thus,

$$G_P(x_1, x_2, x_2) \leq \frac{\alpha}{1 - 2\alpha} G_P(x_0, x_1, x_1).$$

Similarly,

$$G_P(x_n, x_{n+1}, x_{n+1}) \leq \frac{\alpha}{1 - 2\alpha} G_P(x_{n-1}, x_n, x_n).$$

Let $\beta = \dfrac{\alpha}{1 - 2\alpha}$ (with $0 \leq \beta < 1$ if $0 \leq \alpha < \frac{1}{2}$), and set $G_0 = G_P(x_0, x_1, x_1)$. By induction,

$$G_P(x_n, x_{n+1}, x_{n+1}) \leq \beta^n G_0.$$

Now, the sequence $\{x_n\}$ is Cauchy in the perturbed G-metric space because for $k \geq 1$,

$$\begin{aligned} G_P(x_n, x_{n+k}, x_{n+k}) &\leq G_0 \left(\beta^n + \cdots + \beta^{n+k-1}\right) \\ &= G_0 \beta^n \left(1 + \cdots + \beta^{k-1}\right) \\ &= G_0 \beta^n \frac{1 - \beta^k}{1 - \beta} \\ &\leq \frac{G_0}{1 - \beta} \beta^n. \end{aligned}$$

Since $0 \leq \beta < 1$, $\{x_n\}$ is a Cauchy sequence.

By completeness of (X, G_P, P), there exists $x^* \in X$ such that $\lim_{n \to \infty} G_P(x_n, x^*, x^*) = 0$.

We show that x^* is a fixed point of T. By the contractive condition,

$$\begin{aligned} G_P(x_{n+1}, Tx^*, Tx^*) &= G_P(Tx_n, Tx^*, Tx^*) \\ &\leq \alpha \left[G_P(x_n, Tx_n, Tx_n) + G_P(x^*, Tx^*, Tx^*) + G_P(x^*, Tx^*, Tx^*)\right]. \end{aligned}$$

Taking the limit as $n \to \infty$ and using the above results,

$$G_P(x^*, Tx^*, Tx^*) \leq 2\alpha G_P(x^*, Tx^*, Tx^*).$$

Hence, $G_P(x^*, Tx^*, Tx^*) = 0$, so $Tx^* = x^*$, establishing existence.

For uniqueness, suppose x and y are two fixed points of T. Then,

$$G_P(x, y, y) = G_P(Tx, Ty, Ty) \leq \alpha\left[G_P(x, Tx, Tx) + 2G_P(y, Ty, Ty)\right] = 0.$$

Thus, $G_P(x, y, y) = 0$, implying $x = y$. Therefore, the fixed point is unique. □

Example 3.4. *Let $X = \mathbb{R}$, and define the perturbed G-metric on X by*

$$G_P(x, y, z) = \max\left\{\, |x - y|,\ |y - z|,\ |z - x| \,\right\} + |xyz|$$

with perturbation

$$P(x, y, z) = |xyz|.$$

Let $T : X \to X$ be defined as

$$Tx = \begin{cases} 0, & x < 2, \\ 1, & x \geq 2. \end{cases}$$

For all $x, y, z \in X$, let us verify the contractive inequality (3.2) where $0 \leq \lambda < \frac{1}{2}$. The following Table 1 gives the values for (3.2).

Table 1: The values of (3.2) without α

x	Tx	$G_P(x,Tx,Tx)$	y	Ty	$G_P(y,Ty,Ty)$	z	Tz	$G_P(z,Tz,Tz)$	L.H.S. (3.2), R.H.S. (3.2), min R.H.S. (3.2)
< 2	0	$\|x\|$	< 2	0	$\|y\|$	< 2	0	$\|z\|$	0, $\|x\| + \|y\| + \|z\|$, 0
< 2	0	$\|x\|$	< 2	0	$\|y\|$	≥ 2	1	$\|z-1\| + \|z\|$	1, $\|x\| + \|y\| + \|z\| + \|z-1\|$, 3
< 2	0	$\|x\|$	≥ 2	1	$\|y-1\| + \|y\|$	< 2	0	$\|z\|$	1, $\|x\| + \|y\| + \|z\| + \|y-1\|$, 3
< 2	0	$\|x\|$	≥ 2	1	$\|y-1\| + \|y\|$	≥ 2	1	$\|z-1\| + \|z\|$	1, $\|x\| + \|y\| + \|z\| + \|y-1\| + \|z-1\|$, 6
≥ 2	1	$\|x-1\| + \|x\|$	< 2	0	$\|y\|$	< 2	0	$\|z\|$	1, $\|x\| + \|y\| + \|z\| + \|x-1\|$, 3
≥ 2	1	$\|x-1\| + \|x\|$	< 2	0	$\|y\|$	≥ 2	1	$\|z-1\| + \|z\|$	1, $\|x\| + \|y\| + \|z\| + \|x-1\| + \|z-1\|$, 6
≥ 2	1	$\|x-1\| + \|x\|$	≥ 2	1	$\|y-1\| + \|y\|$	< 2	0	$\|z\|$	1, $\|x\| + \|y\| + \|z\| + \|x-1\| + \|y-1\|$, 6
≥ 2	1	$\|x-1\| + \|x\|$	≥ 2	1	$\|y-1\| + \|y\|$	≥ 2	1	$\|z-1\| + \|z\|$	1, $\|x\| + \|y\| + \|z\| + \|x-1\| + \|y-1\| + \|z-1\|$, 9

Thus all the conditions of Theorem 3.3 are satisfied for $\alpha = \frac{1}{3}$ and $x = 0$ is the unique fixed point.

Corollary 3.5. *[12] Let (X, G) be a complete G-metric space and $T : X \to X$ be a mapping satisfying*

$$G(Tx, Ty, Tz) \leq k\big[G(x, Tx, Tx) + G(y, Ty, Ty) + G(z, Tz, Tz)\big], \quad \forall\, x, y, z \in X,$$

where $k \in [0, \frac{1}{3})$. Then T has a unique fixed point

Proof. Set $P \equiv 0$, i.e., $P(x, y, z) = 0$ for all $x, y, z \in X$. Then (X, G_P, P) is a perturbed G-metric space where the perturbed G_P-metric coincides with G, and the contractive condition holds directly. Consequently, Theorem 3.3 applies to guarantee the existence and uniqueness of the fixed point for T. □

Since a complete perturbed G-metric space is a generalization of perturbed metric space, so the Theorem 3.3 is a proper extension of the following theorem of Nutu and Pacurar [13].

Corollary 3.6. *[13] Let (X, D, P) be a complete perturbed metric space and $T : X \to X$ a perturbed Kannan mapping. Then T admits a unique fixed point.*

4. Existence of atleast one fixed point in Perturbed G-metric space

In this section we investigate the case where uniqueness of fixed point is not guaranteed. Such operators belong to the category of so called weakly Picard operators.

Theorem 4.1. *Let (X, G_P, P) be a symmetric perturbed G-complete G-metric space. Let $T : X \to X$ be a mapping. Suppose T satisfies the following condition for all $x, y, z \in X$:*

$$G_P(Tx, Ty, Tz) \leq \left(\frac{G_P(Tx, y, z) + G_P(x, Ty, z) + G_P(x, y, Tz)}{2G_P(x, Tx, Tx) + G_P(y, Ty, Ty) + G_P(z, Tz, Tz) + 1} \right) G_P(x, y, z). \tag{4.1}$$

Then:

(i) T has at least one fixed point $\xi \in X$;

(ii) For any $x \in X$, the iterative sequence $\{T^n x\} = \{Tx_n\}$ G_P-converges to a fixed point;

(iii) If $\xi, \kappa \in X$ are two distinct fixed points, then

$$\frac{1}{3} \leq G_P(\xi, \kappa, \kappa) = G_P(\xi, \xi, \kappa).$$

Proof. Let T satisfy the condition (4.1) and let x_0 be an arbitrary point in X. Since $T(X) \subset X$, there is $x_1 \in X$ such that $x_1 = Tx_0$. Continuing the same process, we can construct a sequence $\{x_n\}$ such that $x_{n+1} = Tx_n$ for all $n \in \mathbb{N}$. Let $x_n \neq x_{n+1}$ for all $n \in \mathbb{N}$. So for each $n \in \mathbb{N}$, by using (4.1), we obtain that

$$G_P(x_n, x_{n+1}, x_{n+1}) = G_P(Tx_{n-1}, Tx_n, Tx_n)$$

$$\begin{aligned}
&\leq \left(\frac{G_P(Tx_{n-1}, x_n, x_n) + G_P(x_{n-1}, Tx_n, x_n) + G_P(x_{n-1}, x_n, Tx_n)}{2G_P(x_{n-1}, Tx_{n-1}, Tx_{n-1}) + G_P(x_n, Tx_n, Tx_n) + G_P(x_n, Tx_n, Tx_n) + 1} \right) G_P(x_{n-1}, x_n, x_n) \\
&= \left(\frac{G_P(x_n, x_n, x_n) + G_P(x_{n-1}, x_{n+1}, x_n) + G_P(x_{n-1}, x_n, x_{n+1})}{2G_P(x_{n-1}, x_n, x_n) + G_P(x_n, x_{n+1}, x_{n+1}) + G_P(x_n, x_{n+1}, x_{n+1}) + 1} \right) G_P(x_{n-1}, x_n, x_n) \\
&= \left(\frac{2G_P(x_{n-1}, x_n, x_{n+1})}{2G_P(x_{n-1}, x_n, x_n) + 2G_P(x_n, x_{n+1}, x_{n+1}) + 1} \right) G_P(x_{n-1}, x_n, x_n) \\
&\leq \left(\frac{2G_P(x_{n-1}, x_n, x_n) + 2G_P(x_n, x_{n+1}, x_{n+1})}{2G_P(x_{n-1}, x_n, x_n) + 2G(x_n, x_{n+1}, x_{n+1}) + 1} \right) G_P(x_{n-1}, x_n, x_n).
\end{aligned}$$

Put $\dfrac{2G_P(x_{n-1}, x_n, x_n) + 2G_P(x_n, x_{n+1}, x_{n+1})}{2G_P(x_{n-1}, x_n, x_n) + 2G_P(x_n, x_{n+1}, x_{n+1}) + 1} = \rho$, then $o \leq \ \rho < 1$ and

$$G_P(x_n, x_{n+1}, x_{n+1}) \leq \ \rho \ G_P(x_{n-1}, x_n, x_n).$$

That is for each $n \in \mathbb{N}$, we have

$$\begin{aligned}
G_P(x_n, x_{n+1}, x_{n+1}) &\leq \rho \ G_P(x_{n-1}, x_n, x_n) \\
&\leq \rho^2 G_P(x_{n-2}, x_{n-1}, x_{n-1}) \\
&\vdots \\
&\leq \rho^n G_P(x_0, x_1, x_1).
\end{aligned}$$

Moreover, for all $n, m \in \mathbb{N}$; $n < m$, we have by rectangle inequality that

$$\begin{aligned} G_P(x_n, x_m, x_m) &\le G_P(x_n, x_{n+1}, x_{n+1}) + G_P(x_{n+1}, x_{n+2}, x_{n+2}) + \cdots + G_P(x_{m-1}, x_m, x_m) \\ &\le \left(\rho^n + \rho^{n+1} + \cdots + \rho^{m-1}\right) G_P(x_0, x_1, x_1) \\ &\le \frac{\rho^n}{1-\rho} G_P(x_0, x_1, x_1), \end{aligned}$$

and so $\lim G_P(x_n, x_m, x_m) = 0$, as $n, m \to \infty$.

Thus $\{x_n\}$ is a G-Cauchy sequence. Since (X, G_P, P) is complete, there exists $p \in X$ such that $\{x_n\}$ is G-Convergent to $p \in X$. We will show that $Tp = p$. By (4.1), we have

$$\begin{aligned} G_P(x_n, Tp, Tp) &= G_P(Tx_{n-1}, Tp, Tp) \\ &\le \left(\frac{G_P(Tx_{n-1}, p, p) + G_P(x_{n-1}, Tp, p) + G_P(x_{n-1}, p, Tp)}{2G_P(x_{n-1}, Tx_{n-1}, Tx_{n-1}) + G_P(p, Tp, Tp) + G_P(p, Tp, Tp) + 1} \right) G_P(x_{n-1}, p, p). \end{aligned}$$

Taking limit as $n \to \infty$, we have $G_P(p, Tp, Tp) = 0$ and $p = Tp$.
If p' is another fixed point of T, then

$$\begin{aligned} G_P(p, p, p') &= G(Tp, Tp, Tp') \\ &\le \left(\frac{G_P(Tp, p, p') + G_P(p, Tp, p') + G_P(p, p, Tp')}{2G_P(p, Tp, Tp) + G_P(p, Tp, Tp) + G_P(p', Tp', Tp') + 1} \right) G_P(p, p, p') \\ &= \Big[G_P(p, p, p') + G_P(p, p, p') + G_P(p, p, p') \Big] G_P(p, p, p') \\ &= 3G_P(p, p, p')^2 \end{aligned}$$

giving

$$G_P(p, p, p') \ge \frac{1}{3}.$$

□

In the following example, let $G(x, y, z)$ be the perimeter of the triangle with vertices at $0, \frac{1}{3}$ and 1 in $\mathbb{R}^2$ and a is taken as an interior point of the triangle (Figure 1). Furthermore $G_P(x, y, z)$ is the corresponding perturbed perimeter.

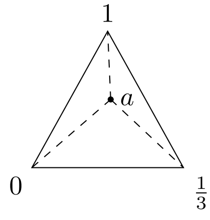

Figure 1

Example 4.2. *Let $X = [0,1]$ and $X' = [0,1] - \{0, \frac{1}{3}\}$. Take point $a\,(\neq 1) \in X'$. Let the values of $G, P, G_P \colon X^3 \to [0, \infty)$ be given as in Table 2.*

Here (X, G) is a symmetric G-metric space and (X, G_P, P) is a symmetric perturbed G-metric space. Note that (X, G_P, P) is not a G-metric space since

$$11 = G_P(0, \tfrac{1}{3}, 1) < G_P(0, a, a) = 13 \ (\textit{contradicting}\ (G_3)).$$

Table 2: Values of G, P and G_P

G-metric	Perturbation P	G_P-metric
$G(0,\frac{1}{3},\frac{1}{3}) = G(\frac{1}{3},0,0) = 5$	2	$G_P(0,\frac{1}{3},\frac{1}{3}) = G_P(\frac{1}{3},0,0) = 7$
$G(0,1,1) = G(1,0,0) = 6$	4	$G_P(0,1,1) = G_P(1,0,0) = 10$
$G(\frac{1}{3},1,1) = G(1,\frac{1}{3},\frac{1}{3}) = 8$	3	$G_P(\frac{1}{3},1,1) = G_P(1,\frac{1}{3},\frac{1}{3}) = 11$
$G(0,\frac{1}{3},1) = 10$	1	$G_P(0,\frac{1}{3},1) = 11$
$G(0,a,a) = G(a,0,0) = G(\frac{1}{3},a,a) = G(a,\frac{1}{3},\frac{1}{3}) = G(1,a,a) = G(a,1,1) = 7$	6	$G_P(0,a,a) = G_P(a,0,0) = G_P(\frac{1}{3},a,a) = G_P(a,\frac{1}{3},\frac{1}{3}) = G_P(1,a,a) = G_P(a,1,1) = 13$
$G(0,1,a) = G(\frac{1}{3},1,a) = G(0,\frac{1}{3},a) = 9$	1	$G_P(0,1,a) = G_P(\frac{1}{3},1,a) = G_P(0,\frac{1}{3},a) = 10$
$G(x,x,x) = 0$ for all $x \in X$	0	$G_P(x,x,x) = 0$ for all $x \in X$

Define the mapping $T\colon X \to X$ by

$$Tx = \begin{cases} 0, & x = 0, x = 1, \\ \frac{1}{3}, & x = \frac{1}{3}, \\ 1 & x \in X'. \end{cases}$$

We have,

$$\begin{aligned} 7 = G_P(0,\tfrac{1}{3},\tfrac{1}{3}) &= G_P(T0, T\tfrac{1}{3}, T\tfrac{1}{3}) \\ &\leq \left(\frac{G_P(T0,\frac{1}{3},\frac{1}{3}) + G_P(0,T\frac{1}{3},\frac{1}{3}) + G_P(0,\frac{1}{3},T\frac{1}{3})}{2G_P(0,T0,T0) + 2G_P(\frac{1}{3},T\frac{1}{3},T\frac{1}{3}) + 1} \right) G_P(0,\tfrac{1}{3},\tfrac{1}{3}) \\ &\leq 3G_P(0,\tfrac{1}{3},\tfrac{1}{3})\, G_P(0,\tfrac{1}{3},\tfrac{1}{3}) = 147; \end{aligned}$$

$$\begin{aligned} 0 = G_P(0,0,0) &= G_P(T0,T1,T1) \\ &\leq \left(\frac{G_P(T0,1,1) + G_P(0,T1,1) + G_P(0,1,T1)}{2G_P(0,T0,T0) + G_P(1,T1,T1) + G_P(1,T1,T1) + 1} \right) G_P(0,1,1) \\ &= \left(\frac{G_P(0,1,1) + G_P(0,0,1) + G_P(0,1,0)}{2G_P(0,0,0) + G_P(1,0,0) + G_P(1,0,0) + 1} \right) G_P(0,1,1) \\ &= \frac{10+10+10}{2 \cdot 10 + 1} \times 10 \\ &= \frac{30}{21} \times 10 = 14.28; \end{aligned}$$

$$\begin{aligned} 7 = G_P(\tfrac{1}{3},0,0) &= G_P(T\tfrac{1}{3},T1,T1) \\ &\leq \left(\frac{G_P(T\frac{1}{3},1,1) + G_P(\frac{1}{3},T1,1) + G_P(\frac{1}{3},1,T1)}{2G_P(\frac{1}{3},T\frac{1}{3},T\frac{1}{3}) + G_P(1,T1,T1) + G_P(1,T1,T1) + 1} \right) G_P(\tfrac{1}{3},1,1) \\ &= \left(\frac{G_P(\frac{1}{3},1,1) + G_P(\frac{1}{3},0,1) + G_P(\frac{1}{3},1,0)}{2G_P(\frac{1}{3},\frac{1}{3},\frac{1}{3}) + G_P(1,0,0) + G_P(1,0,0) + 1} \right) G_P(\tfrac{1}{3},1,1) \\ &= \left(\frac{11+11+11}{10+10+1} \right) \times 11 = \frac{33}{21} \times 11 = 17.28; \end{aligned}$$

$$
\begin{aligned}
7 &= G_P(0, \tfrac{1}{3}, 0) = G_P(T0, T\tfrac{1}{3}, T1) \\
&\le \left(\frac{G_P(T0, \frac{1}{3}, 1) + G_P(0, T\frac{1}{3}, 1) + G_P(0, \frac{1}{3}, T1)}{2G_P(0, T0, T0) + G_P(\frac{1}{3}, T\frac{1}{3}, T\frac{1}{3}) + G_P(1, T1, T1) + 1} \right) G_P(0, \tfrac{1}{3}, 1) \\
&= \left(\frac{G_P(0, \frac{1}{3}, 1) + G_P(0, \frac{1}{3}, 1) + G_P(0, \frac{1}{3}, 0)}{G_P(1, 0, 0) + 1} \right) G_P(0, \tfrac{1}{3}, 1) \\
&= \left(\frac{11 + 11 + 7}{11} \right) \times 11 = 29;
\end{aligned}
$$

$$
\begin{aligned}
10 &= G_P(0, 1, 1) = G_P(T0, Ta, Ta) \\
&\le \left(\frac{G_P(T0, a, a) + G_P(0, Ta, a) + G_P(0, a, Ta)}{2\,G_P(0, T0, T0) + 2\,G_P(a, Ta, Ta) + 1} \right) G_P(0, a, a) \\
&= \left(\frac{G_P(0, a, a) + G_P(0, 1, a) + G_P(0, a, 1)}{2\,G_P(0, 0, 0) + 2\,G_P(a, 1, 1) + 1} \right) G_P(0, a, a) \\
&= \frac{13 + 20}{27} \times 13 = 15.88;
\end{aligned}
$$

$$
\begin{aligned}
11 &= G_P\left(\tfrac{1}{3}, 1, 1\right) = G_P\left(T\tfrac{1}{3}, Ta, Ta\right) \\
&\le \left(\frac{G_P\left(T\frac{1}{3}, a, a\right) + G_P\left(\frac{1}{3}, Ta, a\right) + G_P\left(\frac{1}{3}, a, Ta\right)}{2G_P\left(\frac{1}{3}, T\frac{1}{3}, T\frac{1}{3}\right) + 2G_P(a, Ta, Ta) + 1} \right) G_P\left(\tfrac{1}{3}, a, a\right) \\
&= \left(\frac{G_P\left(\frac{1}{3}, a, a\right) + G_P\left(\frac{1}{3}, 1, a\right) + G_P\left(\frac{1}{3}, a, 1\right)}{2G_P\left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right) + 2G_P(a, 1, 1) + 1} \right) G_P\left(\tfrac{1}{3}, a, a\right) \\
&= \frac{13 + 10 + 10}{27} \times 13 = 15.88;
\end{aligned}
$$

$$
\begin{aligned}
10 &= G_P(0, 1, 1) = G_P(T1, Ta, Ta) \\
&\le \left(\frac{G_P(T1, a, a) + G_P(1, Ta, a) + G_P(1, a, Ta)}{2G_P(1, T1, T1) + 2G_P(a, Ta, Ta) + 1} \right) G_P(1, a, a) \\
&= \left(\frac{G_P(0, a, a) + G_P(1, 1, a) + G_P(1, a, 1)}{2G_P(1, 0, 0) + 2G_P(a, 1, 1) + 1} \right) G_P(1, a, a) \\
&= \frac{13 + 13 + 13}{20 + 2 \cdot 13 + 1} \times 13 \\
&= \frac{39}{47} \times 13 = 10.78;
\end{aligned}
$$

$$
\begin{aligned}
10 &= G_P(0, 0, 1) = G_P(T0, T1, Ta) \\
&\le \left(\frac{G_P(T0, 1, a) + G_P(0, T1, a) + G_P(0, 1, Ta)}{2G_P(0, T0, T0) + G_P(1, T1, T1) + G_P(a, Ta, Ta) + 1} \right) G_P(0, 1, a) \\
&= \left(\frac{G_P(0, 1, a) + G_P(0, 0, a) + G_P(0, 1, 1)}{2G_P(0, 0, 0) + G_P(1, 0, 0) + G_P(a, 1, 1) + 1} \right) G_P(0, 1, a) \\
&= \frac{10 + 13 + 10}{10 + 13 + 1} \times 10 \\
&= \frac{33}{24} \times 10 = 13.75;
\end{aligned}
$$

$$\begin{aligned}
11 = G_P(\tfrac{1}{3}, 0, 1) &= G_P(T\tfrac{1}{3}, T1, Ta) \\
&\leq \left(\frac{G_P(T\frac{1}{3}, 1, a) + G_P(\frac{1}{3}, T1, a) + G_P(\frac{1}{3}, 1, Ta)}{2G_P(\frac{1}{3}, T\frac{1}{3}, T\frac{1}{3}) + G_P(1, T1, T1) + G_P(a, Ta, Ta) + 1} \right) G_P(\tfrac{1}{3}, 1, a) \\
&= \left(\frac{G_P(\frac{1}{3}, 1, 1) + G_P(\frac{1}{3}, 0, a) + G_P(\frac{1}{3}, 1, 1)}{2G_P(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}) + G_P(1, 0, 0) + G_P(a, 1, 1) + 1} \right) G_P(\tfrac{1}{3}, 1, a) \\
&= \frac{11 + 10 + 11}{10 + 13 + 1} \times 10 \\
&= \frac{32}{24} \times 10 = 13.33;
\end{aligned}$$

$$\begin{aligned}
11 = G_P\Big(0, \tfrac{1}{3}, 1\Big) &= G_P\big(T0, T\tfrac{1}{3}, Ta\big) \\
&\leq \left(\frac{G_P\big(T0, \frac{1}{3}, a\big) + G_P\big(0, T\frac{1}{3}, a\big) + G_P\big(0, \frac{1}{3}, Ta\big)}{2G_P(0, T0, T0) + G_P\big(\frac{1}{3}, T\frac{1}{3}, T\frac{1}{3}\big) + G_P(a, Ta, Ta) + 1} \right) G_P\Big(0, \tfrac{1}{3}, a\Big) \\
&= \left(\frac{G_P\big(0, \frac{1}{3}, a\big) + G_P\big(0, \frac{1}{3}, a\big) + G_P\big(0, \frac{1}{3}, 1\big)}{G_P(a, 1, 1) + 1} \right) G_P\Big(0, \tfrac{1}{3}, a\Big) \\
&= \left(\frac{10 + 10 + 11}{13 + 1} \right) \times 10 = \frac{310}{14} = 22.14.
\end{aligned}$$

Hence all the conditions of Theorem 4.1 are satisfied and T has two distinct fixed points 0 and $\frac{1}{3}$.

Also,

$$G_P(0, \tfrac{1}{3}, \tfrac{1}{3}) = G_P(\tfrac{1}{3}, 0, 0) = 7 \geq \tfrac{1}{3}.$$

Corollary 4.3. *[4] Let (X, G) be a symmetric complete G-metric space. Let $T : X \to X$ be a mapping and satisfy the following condition for all $x, y, z \in X$:*

$$G(Tx, Ty, Tz) \leq \left(\frac{G(Tx, y, z) + G(x, Ty, z) + G(x, y, Tz)}{2G(x, Tx, Tx) + G(y, Ty, Ty) + G(z, Tz, Tz) + 1} \right) G(x, y, z).$$

Then:

(i) T has at least one fixed point $\xi \in X$;

(ii) For any $x \in X$, the iterative sequence $\{T^n x\} = \{Tx_n\}$ G-converges to a fixed point;

(iii) If $\xi, \kappa \in X$ are two distinct fixed points, then

$$\frac{1}{3} \leq G(\xi, \kappa, \kappa) = G(\xi, \xi, \kappa).$$

Proof. Set $P \equiv 0$, i.e., $P(x, y, z) = 0$ for all $x, y, z \in X$. Then (X, G_P, P) is a perturbed G-metric space where the perturbed G_P-metric coincides with G, and the contractive condition holds directly. Consequently, Theorem 4.1 applies to guarantee the existence and uniqueness of the fixed point for T. □

5. Conclusion

In this paper, we have presented a unified fixed point framework within the setting of perturbed G-metric spaces, thereby extending and integrating both the classical metric approach and the perturbed metric structure. By establishing the results of Banach [2], Jleli and Samet [6], Kannan [7], and Nutu and Pacurar [13] tailored to these perturbed G-metric spaces, we ensured the existence and uniqueness of fixed points for a broad class of contractive and perturbed-continuous mappings on complete spaces. Furthermore, several well-known results—including those of Gaba [4], Mustafa [9], and Mustafa and Obiedat [12] emerge naturally as corollaries of our main theorems. This demonstrates that our framework not only encompasses but also significantly generalizes many foundational fixed point principles. Overall, the theory developed here provides a more comprehensive and flexible platform for fixed point analysis, with the potential to unify and streamline further advancements in generalized metric structures.

References

1. A. Azam, B. Fisher, and M. Khan, *Common Fixed Point Theorems in Complex Valued Metric Spaces,* Numerical Functional Analysis and Optimization, 32 (3): 243–253, (2011).
2. S. Banach, *Sur les opérations dans les ensembles abstraits et leur application aux équations intégrales*, Fund. Math., 3, 133-181, (1922).
3. B. C. Dhage, *Generalized metric space and mappings with fixed point*, Bull. Cal. Math. Soc., 84, 329-336, (1992).
4. Y. U. Gaba, *Fixed point theorems in G-metric spaces*, J. Math. Anal. Appl., 455, 528–537, (2017).
5. M. Jleli, C. M. Păcurar, and B. Samet, *New directions in fixed point theory in G-metric spaces and applications to mappings contracting perimeters of triangles*, arXiv preprint (2024), arXiv:2405.11648.
6. M. Jleli and B. Samet, *On Banach's Fixed Point Theorem in Perturbed Metric Spaces*, Journal of Applied Analysis and Computation, 15 (2), 993–1001, (2025).
7. R. Kannan, *Some results on fixed points*, Bull. Calc. Math. Soc. 60, 71–76 (1968).
8. N. S. Mohsenia, S. H. Parvaneh Masiha, and H. Mamghaderi, *On single valued contractions of Banach and Kannan type in G-metric spaces endowed with a graph*, International Journal of Nonlinear Analysis and Applications, 12 (2), 297–304, (2021).
9. Z. Mustafa, *A new structure for generalized metric spaces with applications to fixed point theory*, Ph.D. thesis, The University of Newcastle, Australia, 2005.
10. Z. Mustafa and B. Sims, *A new approach to generalized metric spaces*, Journal of Nonlinear and Convex Analysis, 7 (2), 289-297, (2006).
11. Z. Mustafa and B. Sims, *Some remarks concerning D-metric spaces*, Proceedings of the International Conference on Fixed Point Theory and Applications, Yokohama, Japan, 189-198, (2004).
12. Z. Mustafa and H. Obiedat, *A fixed point theorem of Reich in G-metric spaces*, *CUBO* 12 (1), 83-93, (2010).
13. M. Nuțu and C. M. Păcurar, *More general contractions in perturbed metric spaces*, arXiv preprint (2025), arXiv:2502.12936.
14. K. Roy, S. Panja, M. Saha, and R. K. Bisht, *On fixed points of generalized Kannan and Reich type contractive mappings*, Filomat, 37 (26), 9079–9087, (2023).
15. M. Sihag, P. Kumar and N. Hooda, *Complex Valued Perturbed Metric Space and its Fixed Points*, Bol. Soc. Paran. Mat., (3s.) v. 2026 (44) 2 :1-12, doi:10.5269/bspm.80021.

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