Bol. Soc. Paran. Mat.
©SPM – E-ISSN-2175-1188
SPM: www.spm.uem.br/bspm

(3s.) **v. 2026 (44) 7** : 1–12.
ISSN-0037-8712
doi:10.5269/bspm.80745

Fixed Point Theorems for Various Contractions in Super Metric Spaces

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ABSTRACT: In this paper we prove some fixed point theorems for various contractions in setting of super metric spaces.

Key Words: Super metric spaces, Kannan contraction spaces, Reich type contraction.

Contents

1. Introduction

The basic tool of metric fixed point theory is the Banach contraction principle, which states that every contraction map on a complete metric space has a unique fixed point i.e, let T be a mapping from a complete metric space (X, d) into itself satisfying

$$d(Tx, Ty) \leq d(x, y)$$

for all $x, y \in X$,where $0 \leq \alpha < 1$.Then T has a unique fixed point."

This principle gives existence and uniqueness of fixed points and methods for obtaining approximate fixed points. This principle was generalised by several authors by using different types of minimal commutative(weak commutative and its variants,compatible and its variants and weakly compatible maps) along with continuity of one of the mappings. In 2022, Karapinar and Khojasteh [2] introduced a new generalisation of metric space known as super metric space. We begin with definition of super metric.

Definition 1.1. Let X be a non-empty set. We say that $m : X \times X \to [0, \infty)$ is super-metric or super metric if

1. if $m(x, y) = 0$,then $x = y$ *for all* $x, y \in X$
2. $m(x, y) = m(y, x)$, *for all* $x, y \in X$
3. there exists $s \geq 1$ such that for all $y \in X$ there exist distinct sequences $(x_n), (y_n) \subset X$, with $m(x_n, y_n) \longrightarrow 0$ when n tends to infinity, such that

$$\lim_{n\to\infty} \sup\, m(y_n, y) \leq s \lim_{n\to\infty} \sup\, m(x_n, y).$$

Definition 1.2. Let (X, m) be a super metic space and $a \in X$,$r \geq 0$. The set

$$B(a, r) = \{x \in X : m(a, x) < r\}$$

is called a super ball centered at a with radius r. The topology generated by the collection of all super balls acts as a basis of super metric spaces.

The notion of convergence and Cauchy sequence are defined as follows:

Definition 1.3. Let (X, m) be a super metric space. A sequence $\langle x_n \rangle$ in X is said to be

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2020 *Mathematics Subject Classification:* 47H10, 52H25.
Submitted December 12, 2025. Published April 11, 2026

1. convergent to a point x, if , $\lim_{n\to\infty} m(x_n, x) = 0$.
2. Cauchy sequence if for every $\lim_{n,m\to\infty} \sup m(x_n, x_m) = 0$, that is, for each $\epsilon > 0$, there exists n_0 such that $m(x_n, x_m) < \epsilon$ for all $n, m \geq n_0$.

Definition 1.4. A super metric space is said to be complete if every Cauchy sequence in X converges to a point in X.

2. Main Results

In this paper we prove various contraction in the setting of super metric space. We first prove Kannan contraction [4] in the setting of super metric spaces as follows:

Theorem 2.1. Let (X, m) be a complete super metric space and T be a self map on X satisfying Kannan contraction:

$$m(Tx, Ty) \leq \alpha\{m(x, Tx) + m(y, Ty)\} \tag{2.1}$$

for all $x, y \in X$ and $0 < \alpha < 1/2$. Then T has a fixed point in X.
Proof Let $x_0 \in X$ and consider the iterate of sequence $x_{n+1} = Tx_n$.
From (2.1), we have

$$\begin{aligned} m(x_1, x_2) = m(Tx_0, Tx_1) &\leq \alpha\{m(x_0, Tx_0) + m(x_1, Tx_1)\} \\ &\leq \alpha\{m(x_0, x_1) + m(x_1, x_2)\} \\ m(x_1, x_2) &\leq \frac{\alpha}{1-\alpha} m(x_0, x_1) \\ i.e, m(x_1, x_2) &\leq \beta\, m(x_0, x_1) \qquad (2.2) \\ &\mathit{where}\ \beta = \frac{\alpha}{1-\alpha} < 1 \end{aligned}$$

Continuing, this way we get

$$m(x_n, x_{n+1}) \leq \beta^n m(x_0, x_1)$$

Proceeding limit as $n \to \infty$, we have $\lim_{n\to\infty} m(x_n, x_{n+1}) = 0$ as $0 \leq \alpha < 1$.
Now by definition of super metric space, for $s \geq 1$ and for all $x_{n+2} \in X$, there exist distinct sequences $\langle x_n\rangle, \langle x_{n+1}\rangle$ with $m(x_n, x_{n+1}) \to 0$ such that

$$\lim_{n\to\infty} \sup m(x_n, x_{n+2}) \leq s \lim_{n\to\infty} \sup m(x_{n+1}, x_{n+2}).$$

Since $\lim_{n\to\infty} m(x_n, x_{n+1}) = 0$, therefore $\lim_{n\to\infty} \sup m(x_n, x_{n+2}) = 0$. Further,for $s \geq 1$ and for all $x_{n+3} \in X$ there exist distinct sequences $\langle x_n\rangle, \langle x_{n+2}\rangle$ with $m(x_n, x_{n+2}) \to 0$ such that

$$\lim_{n\to\infty} \sup m(x_n, x_{n+3}) \leq s \lim_{n\longrightarrow\infty} \sup m(x_{n+2}, x_{n+3}).$$

This implies $\lim_{n\to\infty} \sup m(x_n, x_{n+3}) = 0$.
Inductively, one can conclude that

$$\lim_{n\to\infty} \sup m(x_n, x_m) = 0\ , m > n\ \mathit{and}\ m, n \in \mathbb{N}.$$

Thus $\langle x_n\rangle$ is Cauchy sequence in X. Since (X, m) is a complete super metric space, therefore, the sequence $\langle x_n\rangle$ converges to a point, say $z \in X$.
We claim that z be a fixed point of T.

From (2.1),we have

$$m(x_{n+1},Tz)=m(Tx_n,Tz)\leq \alpha\,\{m(x_n,Tx_n)+m(z,Tz)\}$$

Proceeding as $n\to\infty$

$$\begin{aligned} m(Tz,z)&\leq \alpha\,\{m(z,Tz)+m(z,Tz)\} \qquad (2.3)\\ &=2\alpha\, m(z,Tz)\\ (1-2\alpha)\,m(z,Tz)&\leq 0 \quad \textit{since } 0\leq\alpha<1/2 \end{aligned}$$

i.e, $Tz=z$.
Therefore z is a fixed point for T.

Next we prove Reich type contraction [8] in setting of super metric spaces.

Theorem 2.2. Let (X,m) be a complete super metric space and T be a Reich type contraction map on X ,and there exits non-negative numbers a,b,c with $a+b+c<1$ such that

$$m(Tx,Ty)\leq a\,m(x,y)+b\,m(x,T(x))+c\,m(y,T(y)). \qquad (2.4)$$

Then T has a fixed point.
Proof Let $x_0\in X$ and consider the iterate of sequence $x_{n+1}=Tx_n$.
From (2.4),

$$\begin{aligned} m(x_1,x_2)=m(Tx_0,Tx_1)&\leq\ a\,m(x_0,x_1)\ +b\,m(x_0,Tx_0)\ +c\,m(x_1,Tx_1)\\ &=\ a\,m(x_0,x_1)+b\,m(x_0,x_1)\ +c\,m(x_1,x_2)\\ (1-c)\,m(x_1,x_2)&\leq\ (a+b)\,m(x_0,x_1)\\ m(x_1,x_2)&\leq\ \frac{a+b}{1-c}\,m(x_0,x_1) \qquad (2.5)\\ m(x_1,x_2)&\leq\beta\,m(x_0,x_1) \end{aligned}$$

Continuing this, way we get

$$m(x_n,x_{n+1})\leq\beta^n\,m(x_0,x_1),\ \ \textit{where } \beta<1$$

Proceeding limit as $n\to\infty$, we have $\lim\limits_{n\to\infty} m(x_n,x_{n+1})=0$ as $0\leq\alpha<1$.
Now by definition of super metric space, for $s\geq 1$ and for all $x_{n+2}\in X$ there exist distinct sequences $\langle x_n\rangle,\langle x_{n+1}\rangle$ with $m(x_n,x_{n+1})\to 0$ such that

$$\lim_{n\to\infty}\sup\, m(x_n,x_{n+2})\leq\ s\ \lim_{n\to\infty}\sup\, m(x_{n+1},x_{n+2}).$$

Since $\lim\limits_{n\to\infty} m(x_n,x_{n+1})=0$. Therefore , $\lim\limits_{n\to\infty}\sup\, m(x_n,x_{n+2})=0$. Similarly we have for $s\geq 1$ and for all $x_{n+3}\in X$ there exist distinct sequences $\langle x_n\rangle,\langle x_{n+2}\rangle$ with $m(x_n,x_{n+2})\to 0$ such that

$$\lim_{n\to\infty}\sup\, m(x_n,x_{n+3})\leq\ s\ \lim_{n\longrightarrow\infty}\sup\, m(x_{n+2},x_{n+3}).$$

i.e., $\lim\limits_{n\to\infty}\sup\, m(x_n,x_{n+3})=0$.
Inductively, one can conclude

$$\lim_{n\to\infty}\sup\, m(x_n,x_m)=0\,,m>n\,\textit{and}\,m,n\in\mathbb{N}.$$

Thus $\langle x_n\rangle$ is Cauchy sequence in X. Since (X,m) is a complete super metric space, therefore, the sequence $\langle x_n\rangle$ converges to a point say $z\in X$.
We claim that z be a fixed point of T.

From(2.4) , we have,

$$\begin{aligned} m(x_{n+1},Tz) = m(Tx_n,Tz) &\leq a\,m(x_n,z) + b\,m(x_n,Tx_n) + c\,m(z,Tz),\\ \textit{letting as } n\to\infty \text{ , } \textit{we have}&\\ m(z,Tz) &\leq b\,m(z,Tz) + c\,m(z,Tz)\\ m(z,Tz) &\leq (b+c)\,m(z,Tz)\\ (1-(b+c))m(z,Tz) &\leq 0 \end{aligned} \tag{2.6}$$

since, $b+c<1$ we get ,$Tz=z$.
So we have z is a fixed point for T.

Now we prove some fixed point theorems in the setting of super metric spaces.
Theorem 2.3. Let (X,m) be a complete super metric space and T be a self-mapping of X satisfying

$$[1+pm(x,y)]m(Tx,Ty) \leq p[m(x,Tx)m(y,Ty)] + q\,max\,\{m(x,y),m(x,Tx),m(y,Ty)\} \tag{2.7}$$

for all $x,y\in X$, where $p\geq 0$ and $0<q<1$. Then T has a fixed point in X.
Proof Let $x_0\in X$. Consider the iterate sequence $x_n = Tx_{n-1}, n = 0,1,2...$.
From (2.7)

$$\begin{aligned} [1+pm(x_1,x_0)]m(Tx_1,Tx_0) &\leq p[m(x_1,Tx_1)m(x_0,Tx_0)] + q\,max\{m(x_1,x_0),m(x_0,Tx_0),m(x_1,Tx_1)\}\\ [1+pm(x_1,x_0)]m(x_2,x_1) &\leq p[m(x_1,x_2)m(x_1,x_0)] + q\,max\,\{m(x_1,x_0),m(x_0,x_1),m(x_1,x_2)\}\\ \textit{on simplification,}&\\ m(x_1,x_2) &\leq q\,max\,\{m(x_1,x_0),m(x_1,x_2)\} \end{aligned} \tag{2.8}$$

Case 1.
If

$$max\{m(x_0,x_1),m(x_1,x_2)\} = m(x_1,x_2),$$

then from equation (2.8)

$$m(x_1,x_2) \leq q\,m(x_1,x_2),$$

which is a contradiction as $0<q<1$.
Case 2.
If

$$max\{m(x_0,x_1),m(x_1,x_2)\} = m(x_0,x_1),$$

from (2.8), we have

$$m(x_1,x_2) \leq q\,m(x_0,x_1)$$

$$m(x_2,x_3) \leq q\,m(x_1,x_2) \leq q^2\,m(x_0,x_1)$$

Continuing this way

$$m(x_{n+1},x_n) \leq q^n\,m(x_0,x_1)$$

Proceeding limit as $n\to\infty$, we have $\lim\limits_{n\to\infty} m(x_{n-1},x_n) = 0$ as $0<q<1$. By definition of super metric space, for $s\geq 1$ and for all $x_{n+2}\in X$ there exist distinct sequences $\langle x_n\rangle, \langle x_{n+1}\rangle$ with $m(x_n,x_{n+1})\to 0$ such that

$$\lim_{n\to\infty}\sup\, m(x_n,x_{n+2}) \leq s\lim_{n\to\infty}\sup\, m(x_{n+1},x_{n+2}).$$

Since $\lim m(x_n,x_{n+1}) = 0$ as $n\to\infty$. Therefore $\lim\limits_{n\to\infty}\sup\, m(x_n,x_{n+2}) = 0$. Continuing in this way, we have for $s\geq 1$ and for all $x_{n+3}\in X$ there exist distinct sequences $\langle x_n\rangle, \langle x_{n+2}\rangle$ with $m(x_n,x_{n+2})\to 0$ such that

$$\lim_{n\to\infty}\sup\, m(x_n,x_{n+3}) \leq s\lim_{n\longrightarrow\infty}\sup\, m(x_{n+2},x_{n+3}),$$

i.e, $\lim\limits_{n\to\infty} \sup\, m(x_n, x_{n+3}) = 0$.
Inductively, one can conclude

$$\lim_{n\to\infty} \sup\, m(x_n, x_m) = 0\,, m > n \textit{ and } m, n \in \mathbb{N}.$$

Thus $\langle x_n\rangle$ is Cauchy sequence in X. Since (X, m) is a complete super metric space, therefore, the sequence $\langle x_n\rangle$ converges to a point say $z \in X$.
We claim that z be a fixed point of T.
From (2.7), we have,

$$\begin{gathered}[1 + pm(x_n, z)]m(Tx_n, Tz) \leq p[m(x_n, Tx_n)m(z, Tz)] + q\, max\,\{m(x_n, z), m(x_n, Tx_n), m(z, Tz)\}, i.e,\\ [1 + pm(x_n, z)]m(x_{n+1}, Tz) \leq p[m(x_n, x_{n+1})m(z, Tz)] + q\, max\,\{m(x_n, z), m(x_n, x_{n+1}), m(z, Tz)\}\\ \textit{Taking limit } n \to \infty \\ m(z, Tz) \leq q\, m(z, Tz), \textit{a contradiction , i.e., } Tz = z.\end{gathered} \tag{2.9}$$

Therefore z is a fixed point for T.

Theorem 2.4. Let (X, m) be a complete super metric space and f, g be self mapping on X, satisfying,

$$m(fx, gy) \leq \alpha m(x, y) + \beta[m(x, fx) + m(y, gy)] \tag{2.10}$$

with $\alpha, \beta > 0$, $\alpha + 2\beta < 1$, and for all $x, y \in X$. Then f and g have fixed point in X.
Proof Let x_0 be an arbitrary point in X. Consider sequence of iterate,
$x_{2n-1} = fx_{2n-2}$, $x_n = gx_{2n-1}$.
From (2.10) ,on putting $x = x_{2n-2}$ and $y = x_{2n-1}$, we have on simplification

$$m(x_{2n-1}, x_{2n}) = m(fx_{2n-2}, gx_{2n-1}) \leq \alpha\, m(x_{2n-2}, x_{2n-1})\ + \beta\,[m(x_{2n-2}, fx_{2n-2}) + m(x_{2n-1}, gx_{2n-1})].$$

$$\textit{or}$$

$$\begin{aligned} m(x_{2n-1}, x_{2n}) &\leq \alpha\, m(x_{2n-2}, x_{2n-1}) + \beta\,[m(x_{2n-2}, x_{2n-1}) + m(x_{2n-1}, x_{2n})]\\ (1-\beta)\, m(x_{2n-1}, x_{2n}) &\leq (\alpha + \beta)\, m(x_{2n-2}, x_{2n-1})\\ m(x_{2n-1}, x_{2n}) &\leq \frac{\alpha+\beta}{1-\beta}\, m(x_{2n-2}, x_{2n-1})\\ i.e,\ m(x_{2n-1}, m_{2n}) &\leq \gamma\, m(x_{2n-2}, x_{2n}),\ \textit{where}\ \gamma = \frac{\alpha+\beta}{1-\beta}\end{aligned} \tag{2.11}$$

$$\textit{continuing in the same way}$$

$$m(x_{2n-1}, x_{2n}) \leq \gamma^{2n}\, m(x_0, x_1)$$

proceeding $lim\, n \to \infty$, we get $\lim\limits_{n\to\infty} m(x_{2n-1}, x_{2n}) = 0$.
By definition of super metric space , for $\langle x_{2n-1}\rangle$,$\langle x_{2n}\rangle$ in X there exists $x_{2n+1} \in X$ such that

$$\lim_{n\to\infty} sup\, m(x_{2n-1}, x_{2n+1}) \leq s \lim_{n\to\infty} sup\, m(x_{2n}, x_{2n-1})$$

then using (2.11), we have $\lim\limits_{n\to\infty} sup\, m(x_{2n-1}, x_{2n+1}) = 0$.
Again for $\langle x_{2n+1}\rangle$ and $\langle x_{2n-1}\rangle$ there exists $x_{2n+2} \in X$ such that

$$\lim_{n\to\infty} sup\, m(x_{2n-1}, x_{2n+2}) \leq s \lim_{n\to\infty} sup\, m(x_{2n+2}, x_{2n+1})$$

i.e., $\lim\limits_{n\to\infty} sup\, m(x_{2n-1}, x_{2n+2}) = 0$.
Proceeding in this way, we get
$\lim\limits_{n\to\infty} sup\, m(x_n, x_m) = 0$ $m > n$ and $m, n \in \mathbf{N}$. Therefore $\{x_n\}$ is Cauchy. Since (X, m) is complete, the sequence $\langle x_n\rangle$ converges to a point say $u \in X$.

On putting $x = x_{2n-2}$ and $y = u$ in (2.10), we have
$m(x_{2n-1}, gu) = m(fx_{2n-2}, gu) \leq \alpha\, m(x_{2n-1}, u) + \beta\, [m(x_{2n-2}, fx_{2n-2}) + m(u, gu)]$
$m(x_{2n-1}, gu) \leq \alpha\, m(x_{2n-1}, u) + \beta\, [m(x_{2n-2}, fx_{2n-2}) + m(u, gu)]$.
Proceeding $\lim n \to \infty$ we get
$m(u, gu) \leq \beta\, m(u, gu)$, i.e., $(1-\beta)m(u, gu) \leq 0$ if $1 - \beta \leq 0$ then $\beta \geq 1$ but $\alpha + 2\beta \leq 1$ and $\alpha > 0$ hence $m(u, gu) = 0$ i.e. $gu = u$ similarly we can show that $fu = u$.
Hence f and g have a fixed point in X.

Theorem 2.5. Let (X, m) be a complete super metric space and S, T be self map on X satisfying,

$$m(Sx, Ty) \leq kmax\{m(x, y), m(x, Sx), m(y, Ty)\} \tag{2.12}$$

for all $x, y \in X$ and $0 \leq k < 1$, then S and T have a fixed point.
Proof Let x_0 be arbitrary point in X. Consider sequence $\{x_n\}$ and define

$$x_{2n-1} = Sx_{2n-2} \ \ , x_{2n} = Tx_{2n-1}.$$

From (2.12),

$$\begin{aligned} m(x_{2n-1}, x_{2n}) = m(Sx_{2n-2}, Tx_{2n-1}) &\leq k\,\{m(x_{2n-2}, x_{2n-1}), m(x_{2n-2}, Sx_{2n-2}), m(x_{2n-1}, Tx_{2n-1})\} \\ &= k\, max\{m(x_{2n-2}, x_{2n-1}), m(x_{2n-2}, x_{2n-1}), m(x_{2n-1}, x_{2n})\} \end{aligned} \tag{2.13}$$

Case 1.
Let $max\,\{m(x_{2n-1}, x_{2n-2}), m(x_{2n-1}, x_{2n})\} = m(x_{2n-1}, x_{2n})$
then, $m(x_{2n-1}, x_{2n}) \leq \, k\, m(x_{2n-1}, x_{2n})$, a contradiction as $k < 1$.
Case 2.
Let $max\,\{m(x_{2n-1}, x_{2n}), m(x_{2n-2}, x_{2n-1})\} = m(x_{2n-2}, x_{2n-1})$
then,

$$\begin{aligned} m(x_{2n-1}, x_{2n}) &\leq \, k\, m(x_{2n-2}, x_{2n-1}) \\ m(x_{2n-1}, x_{2n}) &\leq \, k^2\, m(x_{2n-3}, x_{2n-2}) \\ &\leq \, k^3\, m(x_{2n-4}, x_{2n-3}) \\ &\leq k^4\, m(x_{2n-5}, x_{2n-3}) \\ &\vdots \\ &\leq k^2 n\, m(x_0, x_1). \end{aligned} \tag{2.14}$$

Proceeding limit as $n \to \infty$, we have $\lim_{n\to\infty} m(x_{2n-1}, x_{2n}) = 0$ as $0 \leq k < 1$.
Now by definition of super metric space, for $s \geq 1$ and for all $x_{2n+1} \in X$ there exist distinct sequences $\langle x_{2n} \rangle, \langle x_{2n-1} \rangle$ with $m(x_{2n-1}, x_{2n}) \to 0$ such that

$$\lim_{n\to\infty} \sup\, m(x_{2n-1}, x_{2n+1}) \leq \, s \, \lim_{n\to\infty} \sup\, m(x_{2n-1}, x_{2n}).$$

Since $\lim m(x_{2n-1}, x_{2n+1}) = 0$ as $n \to \infty$. Therefore $\lim_{n\to\infty} \sup m(x_{2n-1}, x_{2n+1}) = 0$. Continuing in this way,we have for $s \geq 1$ and for all $x_{2n+2} \in X$ there exist distinct sequences $\langle x_{2n-1} \rangle, \langle x_{2n+1} \rangle$ with $m(x_{2n-1}, x_{2n+1}) \to 0$ such that

$$\lim_{n\to\infty} \sup\, m(x_{2n-1}, x_{2n+2}) \leq \, s \, \lim_{n\longrightarrow\infty} \sup\, m(x_{2n+2}, x_{2n+1}).$$

$i.e,$ $\lim_{n\to\infty} \sup\, m(x_{2n-1}, x_{2n+2}) = 0$.
Inductively, we can conclude

$$\lim_{n\to\infty} \sup\, m(x_n, x_m) = 0 \ \text{ for all } m > n \, and \, m, n \in \mathbf{N}.$$

Thus $\langle x_n \rangle$ is Cauchy sequence in X. Since (X, m) is complete there exists sequence $\{x_n\}$ converges to a point say z.
From (2.12)

$$m(x_{2n-1}, Tz) = m(Sx_{2n-2}, Tz) \leq k\, max\{m(x_{2n-2}, z), m(x_{2n-2}, Sx_{2n-2}), m(z, Tz)\}.$$

Taking lim $as\, n \to \infty$

$$m(z, Tz) \leq k\, m(z, Tz),\ \ a\ contradiction\ i.e, Tz = z\ as\ k < 1.$$

Hence T has a fixed point in X. Similarly we can show that S has a fixed point in X.
Theorem 2.6. Let S and T be self maps on a be complete super metric space (X, m) satisfying

$$m(STx, TSy) \leq k\, max\{m(x, y), m(x, STx), m(y, TSy)\} \tag{2.15}$$

for all $x, y \in X$ and $0 < k < 1$. Then S and T has a fixed point .
Proof By Theorem (2.5) z is a common fixed point of ST and TS. Then

$$ST(Sz) = S(TSz) = Sz,$$

and so $Sz = z$. Similarly , $Tz = z$. So we have proved that S and T have a common fixed point.
Theorem 2.7. Let (X, d) and (Y, e) be complete metric spaces. If T is a mapping of X into Y and S a mapping of Y into X satisfying inequalities

$$e^p(Tx, TSy) \leq c_1 max\, \{e^p(y, TSy), d^p(x, Sy)\} \tag{2.16}$$

$$d^p(Sy, STx) \leq c_2 max\, \{e^p(y, Tx), d^p(x, STx)\} \tag{2.17}$$

for all $x \in X$ and $y \in Y$, where $0 \leq c_1, c_2 < 1$, then ST has a fixed point in X and TS has a fixed point in Y. Further $Tz = w$ and $Sw = z$
Proof Let x_0 be arbitrary point in X. Define sequences $\{x_n\}$ and $\{y_n\}$ in X and Y respectively by

$$(ST)^n x_0 = x_n\ \ T(ST)^{n-1} x_0 = y_n$$

for n=1,2,3,.... Taking $x = x_n$ and $y = y_n$ in inequality (2.17) , we obtain

$$\begin{aligned} &d^p(Sy_n, STx_n) = d^p(x_n, x_{n+1}) \\ &\quad d^p(x_n, x_{n+1}) \leq c_2\, max\{e^p(y_n, Tx_n), d^p(x_n, STx_n)\} \\ &\quad d^p(x_n, x_{n+1}) \leq c_2 max\{e^p(y_n, y_{n+1}), d^p(x_n, STx_n)\} \\ &\quad d^p(x_n, x_{n+1}) \leq c_2 max\{e^p(y_n, y_{n+1}), d^p(x_n, x_{n+1})\} \end{aligned} \tag{2.18}$$

Case 1.
$max\{e^p(y_n, y_{n+1}), d^p(x_n, x_{n+1})\} = d^p(x_n, x_{n+1})$ will give contradiction as $c_2 < 1$.
Case 2.
$max\{e^p(y_n, y_{n+1}), d^p(x_n, x_{n+1})\} = e^p(y_n, y_{n+1})$

$$d^p(x_n, x_{n+1}) \leq c_2\, e^p(y_n, y_{n+1})$$

$$d(x_n, x_{n+1}) \leq t_2 e(y_n, y_{n+1})$$

where $t_2 = c_2^{1/p}$.

Taking $x = x_{n-1}$ and $y = y_n$ in inequality (2.16)

$$\begin{aligned} &e^p(Tx_{n-1}, TSy_n) = e^p(y_n, y_{n+1}) \\ &\quad e^p(y_n, y_{n+1}) \leq c_1\, max\{e^p(y_n, TSy_n), d^p(x_{n-1}, TSy_n)\} \\ &\quad e^p(y_n, y_{n+1}) \leq c_1 max\{e^p(y_n, y_{n+1}), d^p(x_{n-1}, x_n)\}. \end{aligned} \tag{2.19}$$

Case 1.
$max\{e^p(y_n, y_{n+1}), d^p(x_{n-1}, x_n)\} = e^p(y_n, y_{n+1})$ will give contradiction as $c_1 < 1$.
Case 2.
$max\{e^p(y_n, y_{n+1}), d^p(x_{n-1}, x_n)\} = d^p(x_{n-1}, x_n)$

$$e^p(y_n, y_{n+1}) \leq c_1\, d^p(x_{n-1}, x_n)$$

$$e(y_n, y_{n+1}) \leq t_1 d(x_{n-1}, x_n),$$

where $t_1 = c_1^{1/p}$.
Now $d(x_n, x_{n+1}) \leq t_1t_2d(x_{n-1}, x_n) .... \leq (t_1t_2)^n d(x_0, x_1)$ and since $0 \leq t_1t_2 < 1$ we get $\lim\limits_{n\to\infty} d(x_n, x_{n+1}) = 0$.
Now by definition of super metric space, for $s \geq 1$ and for all $x_{n+2} \in X$ there exist distinct sequences $\langle x_n\rangle, \langle x_{n+1}\rangle$ with $\lim\limits_{n\to\infty} m(x_{n+1}, x_n) \to 0$ such that

$$\lim_{n\to\infty} \sup m(x_n, x_{n+2}) \leq s \lim_{n\to\infty} \sup m(x_{n+1}, x_{n+2}).$$

Since $\lim m(x_n, x_{n+2}) = 0$ as $n \to \infty$. Therefore $\lim\limits_{n\to\infty} \sup m(x_n, x_{n+2}) = 0$. Continuing in this way, similarly we have for $s \geq 1$ and for all $x_{n+3} \in X$ there exist distinct sequences $\langle x_{n+2}\rangle, \langle x_n\rangle$ with $m(x_n, x_{n+2}) \to 0$ such that

$$\lim_{n\to\infty} \sup m(x_n, x_{n+3}) \leq s \lim_{n\longrightarrow\infty} \sup m(x_{n+2}, x_{n+3}).$$

$i.e$, $\lim\limits_{n\to\infty} \sup m(x_n, x_{n+3}) = 0$.
Inductively, we can conclude

$$\lim_{n\to\infty} \sup m(x_n, x_m) = 0 \text{ for all } a \in X, m > n\, and\, m, n \in \mathbf{N}.$$

Thus $\langle x_n\rangle$ is a Cauchy sequence in X and $\{y_n\}$ is a Cauchy sequence in Y. Since (X, d) and (Y, e) are complete super metric spaces $\{x_n\}$ converges to a point say z in X and $\{y_n\}$ converges to a point say w in Y.
Put $x = z$ and $y = y_{n-1}$ in equation (2.16)

$$e^p(Tx, TSy) = e^p(Tz, TSy_{n-1}) \leq c_1\, max\, m\{d^p(z, x_{n-1}), e^p(y_{n-1}, Tx_{n-1})\}.$$

Letting lim as $n \to \infty$

$$e^p(Tz, w) \leq c_1\, e^p(w, Tz)$$

which implies $Tz = w$ as $c_1 < 1$ similarly $Sw = z$. So $STz = S(Tz) = Sw = z$ and $TSw = T(Sw) = Tz = w$

$$STz = z, \quad TSw = w.$$

Example Let $X = [0, \infty)$ and define
$m : X \times X \longrightarrow [0, \infty)$ by
$m(x, y) = \frac{|x-y|+|x|+|y|}{2}$
Then (X, m) is super metric space.
Proof Clearly $m(x, y) \geq 0$
If $m(x, y) = 0$,then $\frac{|x-y|+|x|+|y|}{2} = 0$
which implies $x = y$.
But if $x = y$, then $m(x, x) = \frac{0+|x|+|x|}{2} = |x| \neq 0$
Clearly $m(x, y) = m(y, x)$
Suppose that $y \in X$ $\langle x_n\rangle$ and $\langle y_n\rangle$ be two distinct sequence in X such that $m(x_n, y_n) \to 0\, as\, n \to \infty$. Since the sequence are distinct, we have $m(x_n, y_n) \to 0$ i.e,

$$\lim_{n\to\infty} \frac{|x-y|+|x|+|y|}{2} \to 0$$

i.e, $\lim_{n\to\infty} x_n = \lim_{n\to\infty} y_n = 0$.
Now there exists $N > 0$ such that for $n \geq N$, we have

$$\begin{aligned}
\lim_{n\to\infty} \sup m(y_n, y) &= \lim_{n\to\infty} \sup \frac{|y_n - y| + |y_n| + |y|}{2} \\
&= |y| \\
&\leq s\,|y| \\
&= s \lim_{n\to\infty} \sup \frac{|x_n - y| + |x_n| + |y|}{2} \\
&= s \lim_{n\to\infty} \sup m(x_n, y)
\end{aligned} \tag{2.20}$$

In case $y = 0$, the proof is straight forward. Hence, (X, m) is a super metric space. It is worth-mentioning that it is not a metric space as $m(x, y) \neq 0$ when $x = y$.

Theorem 2.8. Let (X, m) be a complete super metric space and f, g be self mapping on X, satisfying,

$$m(fx, gy) \leq \alpha\, m(x, y) + \beta[m(x, fx) + m(y, gy)] \tag{2.21}$$

with $\alpha, \beta > 0$, $\alpha + 2\beta < 1$, and for all $x, y \in X$. Then f and g have fixed point in X.

Proof Let x_0 be an arbitrary point in X. Consider sequence of iterate,
$x_{2n-1} = fx_{2n-2}$, $x_n = gx_{2n-1}$.
From (2.21), on putting $x = x_{2n-2}$ and $y = x_{2n-1}$, we have on simplification

$$m(x_{2n-1}, x_{2n}) = m(fx_{2n-2}, gx_{2n-1}) \leq \alpha\, m(x_{2n-2}, x_{2n-1}) + \beta\, [m(x_{2n-2}, fx_{2n-2}) + m(x_{2n-1}, gx_{2n-1})].$$

$$\textit{or}$$

$$\begin{aligned}
m(x_{2n-1}, x_{2n}) &\leq \alpha\, m(x_{2n-2}, x_{2n-1}) + \beta\, [m(x_{2n-2}, x_{2n-1}) + m(x_{2n-1}, x_{2n})] \\
(1-\beta)\, m(x_{2n-1}, x_{2n}) &\leq (\alpha + \beta)\, m(x_{2n-2}, x_{2n-1}) \\
m(x_{2n-1}, x_{2n}) &\leq \frac{\alpha+\beta}{1-\beta}\, m(x_{2n-2}, x_{2n-1}) \\
i.e,\ m(x_{2n-1}, m_{2n}) &\leq \gamma\, m(x_{2n-2}, x_{2n}),\ \ where\ \gamma = \frac{\alpha+\beta}{1-\beta} \\
&\textit{continuing in the same way} \\
m(x_{2n-1}, x_{2n}) &\leq \gamma^{2n}\, m(x_0, x_1)
\end{aligned} \tag{2.22}$$

proceeding limit $n \to \infty$, we get $\lim_{n\to\infty} m(x_{2n-1}, x_{2n}) = 0$.
By definition of super metric space , for $\langle x_{2n-1}\rangle$,$\langle x_{2n}\rangle$ in X there exists $x_{2n+1} \in X$ such that

$$\lim_{n\to\infty} sup\, m(x_{2n-1}, x_{2n+1}) \leq \lim_{n\to\infty} sup\, m^s(x_{2n}, x_{2n-1})$$

we have $\lim_{n\to\infty} sup\, m(x_{2n-1}, x_{2n+1}) = 0$.
Again for $\langle x_{2n+1}\rangle$ and $\langle x_{2n-1}\rangle$ there exists $x_{2n+2} \in X$ such that

$$\lim_{n\to\infty} sup\, m(x_{2n-1}, x_{2n+2}) \leq \lim_{n\to\infty} sup\, m^s(x_{2n+2}, x_{2n+1})$$

i.e., $\lim_{n\to\infty} sup\, m(x_{2n-1}, x_{2n+2}) = 0$.
Proceeding in this way, we get
$\lim_{n\to\infty} sup\, m(x_n, x_m) = 0$ $m > n$ and $m, n \in \mathbf{N}$. Therefore $\{x_n\}$ is Cauchy. Since (X, m) is complete, the sequence $\langle x_n\rangle$ converges to a point say $u \in X$.
On putting $x = x_{2n-2}$ and $y = u$ in (2.21), we have
$m(x_{2n-1}, gu) = m(fx_{2n-2}, gu) \leq m(x_{2n-1}, u)^{\alpha}.[m(x_{2n-2}, fx_{2n-2}).m(u, gu)]^{\beta}$
$m(x_{2n-1}, gu) \leq m^{\alpha}(x_{2n-1}, u).[m(x_{2n-2}, fx_{2n-2}).m(u, gu)]^{\beta}$.
Proceeding $\lim n \to \infty$ we get

$m(u,gu) \leq \beta\, m(u,gu)$, i.e., $(1-\beta)m(u,gu) \leq 0$ if $1-\beta \leq 0$ then $\beta \geq 1$ but $\alpha + 2\beta \leq 1$ and $\alpha > 0$ hence $m(u,gu) = 0$ i.e. $gu = u$ similarly we can show that $fu = u$.
Hence f and g have a fixed point in X.

Theorem 2.9. Let (X,m) be a complete super metric space and S,T be self map on X satisfying,

$$m(Sx,Ty) \leq max\{m(x,y), m(x,Sx), m(y,Ty)\}^k \tag{2.23}$$

for all $x,y \in X$ and $0 \leq k < 1$, then S and T have a fixed point.

Proof Let x_0 be arbitrary point in X. Consider sequence $\{x_n\}$ and define

$$x_{2n-1} = Sx_{2n-2} \ , x_{2n} = Tx_{2n-1}.$$

From (2.23),

$$\begin{aligned} m(x_{2n-1},x_{2n}) = m(Sx_{2n-2},Tx_{2n-1}) &\leq max\{m(x_{2n-2},x_{2n-1}), m(x_{2n-2},Sx_{2n-2}), m(x_{2n-1},Tx_{2n-1})\}^k \\ &= max\{m(x_{2n-2},x_{2n-1}), m(x_{2n-2},x_{2n-1}), m(x_{2n-1},x_{2n})\}^k \end{aligned} \tag{2.24}$$

Case 1.
Let $max\,\{m(x_{2n-1},x_{2n-2}), m(x_{2n-1},x_{2n})\} = m(x_{2n-1},x_{2n})$
then, $m(x_{2n-1},x_{2n}) \leq m(x_{2n-1},x_{2n})^k$, a contradiction as $k < 1$.

Case 2.
Let $max\,\{m(x_{2n-1},x_{2n}), m(x_{2n-2},x_{2n-1})\} = m(x_{2n-2},x_{2n-1})$
then,

$$\begin{aligned} m(x_{2n-1},x_{2n}) &\leq m(x_{2n-2},x_{2n-1})^k \\ m(x_{2n-1},x_{2n}) &\leq m(x_{2n-3},x_{2n-2})^{k^2} \\ &\leq m(x_{2n-4},x_{2n-3})^{k^3} \\ &\leq m(x_{2n-5},x_{2n-3})^{k^4} \\ &\vdots \\ &\leq m(x_0,x_1)^{k^{2n}}. \end{aligned} \tag{2.25}$$

Proceeding limit as $n \to \infty$, we have $\lim\limits_{n\to\infty} m(x_{2n-1},x_{2n}) = 1$ as $0 \leq k < 1$.
Now by definition of super metric space, for $s \geq 1$ and for all $x_{2n+1} \in X$ there exist distinct sequences $\langle x_{2n}\rangle, \langle x_{2n-1}\rangle$ with $m(x_{2n-1},x_{2n}) \to 1$ such that

$$\lim_{n\to\infty} \sup\, m(x_{2n-1},x_{2n+1}) \leq \lim_{n\to\infty} \sup\, m^s(x_{2n-1},x_{2n}).$$

Since $\lim m(x_{2n-1},x_{2n+1}) = 1$ as $n \to \infty$. Therefore $\lim\limits_{n\to\infty} \sup\, m(x_{2n-1},x_{2n+1}) = 1$. Continuing in this way,we have for $s \geq 1$ and for all $x_{2n+2} \in X$ there exist distinct sequences $\langle x_{2n-1}\rangle, \langle x_{2n+1}\rangle$ with $m(x_{2n-1},x_{2n+1}) \to 1$ such that

$$\lim_{n\to\infty} \sup\, m(x_{2n-1},x_{2n+2}) \leq \lim_{n\longrightarrow\infty} \sup\, m^s(x_{2n+2},x_{2n+1}).$$

i.e, $\lim\limits_{n\to\infty} \sup\, m(x_{2n-1},x_{2n+2}) = 1$.
Inductively, we can conclude

$$\lim_{n\to\infty} \sup\, m(x_n,x_m) = 1 \text{ for all } m > n \ and \ m,n \in \mathbf{N}.$$

Thus $\langle x_n\rangle$ is Cauchy sequence in X. Since (X,m) is complete there exists sequence $\{x_n\}$ converges to a point say z.
From (2.23)

$$m(x_{2n-1},Tz) = m(Sx_{2n-2},Tz) \leq max\{m(x_{2n-2},z), m(x_{2n-2},Sx_{2n-2}), m(z,Tz)\}^k.$$

Taking lim $as\, n \to \infty$

$$m(z,Tz) \leq m^k(z,Tz),\ \ a\ contradiction\ i.e, Tz = z\ as\ k < 1.$$

Hence T has a fixed point in X. Similarly we can show that S has a fixed point in X.

Theorem 2.10. Let S and T be self maps on a be complete super metric space (X, m) satisfying

$$m(STx, TSy) \leq max\{m(x,y), m(x, STx), m(y, TSy)\}^k \tag{2.26}$$

for all $x, y \in X$ and $0 < k < 1$. Then S and T has a fixed point .

Proof By Theorem(2.9), z is a common fixed point of ST and TS. Then

$$ST(Sz) = S(TSz) = Sz,$$

and so $Sz = z$. Similarly , $Tz = z$. So we have proved that S and T have a common fixed point.

Theorem 2.11. Let (X, d) and (Y, e) be complete metric spaces. If T is a mapping of X into Y and S a mapping of Y into X satisfying inequalities

$$e^p(Tx, TSy) \leq c_1 max\,\{e^p(y, TSy), d^p(x, Sy)\} \tag{2.27}$$

$$d^p(Sy, STx) \leq c_2 max\,\{e^p(y, Tx), d^p(x, STx)\} \tag{2.28}$$

for all $x \in X$ and $y \in Y$, where $0 \leq c_1, c_2 < 1$, then ST has a fixed point in X and TS has a fixed point in Y.

Proof Let x_0 be arbitrary point in X. Define sequences $\{x_n\}$ and $\{y_n\}$ in X and Y respectively by

$$(ST)^n x_0 = x_n \ \ T(ST)^{n-1}x_0 = y_n$$

for n=1,2,3,.... Taking $x = x_n$ and $y = y_n$ in inequality (2.28),we obtain

$$\begin{aligned} d^p(Sy_n, STx_n) &= d^p(x_n, x_{n+1}) \\ d^p(x_n, x_{n+1}) &\leq c_2\, max\{e^p(y_n, Tx_n), d^p(x_n, STx_n)\} \\ d^p(x_n, x_{n+1}) &\leq c_2 max\{e^p(y_n, y_{n+1}), d^p(x_n, STx_n)\} \\ d^p(x_n, x_{n+1}) &\leq c_2 max\{e^p(y_n, y_{n+1}), d^p(x_n, x_{n+1})\} \end{aligned} \tag{2.29}$$

Case 1.
$max\{e^p(y_n, y_{n+1}), d^p(x_n, x_{n+1})\} = d^p(x_n, x_{n+1})$ will give contradiction as $c_2 < 1$.
Case 2.
$max\{e^p(y_n, y_{n+1}), d^p(x_n, x_{n+1})\} = e^p(y_n, y_{n+1})$

$$d^p(x_n, x_{n+1}) \leq c_2\, e^p(y_n, y_{n+1})$$

$$d(x_n, x_{n+1}) \leq t_2 e(y_n, y_{n+1})$$

where $t_2 = c_2^{1/p}$.

Taking $x = x_{n-1}$ and $y = y_n$ in inequality (2.27)

$$\begin{aligned} e^p(Tx_{n-1}, TSy_n) &= e^p(y_n, y_{n+1}) \\ e^p(y_n, y_{n+1}) &\leq c_1\, max\{e^p(y_n, TSy_n), d^p(x_{n-1}, TSy_n)\} \\ e^p(y_n, y_{n+1}) &\leq c_1 max\{e^p(y_n, y_{n+1}), d^p(x_{n-1}, x_n)\}. \end{aligned} \tag{2.30}$$

Case 1.
$max\{e^p(y_n, y_{n+1}), d^p(x_{n-1}, x_n)\} = e^p(y_n, y_{n+1})$ will give contradiction as $c_1 < 1$.
Case 2.
$max\{e^p(y_n, y_{n+1}), d^p(x_{n-1}, x_n)\} = d^p(x_{n-1}, x_n)$

$$e^p(y_n, y_{n+1}) \leq c_1\, d^p(x_{n-1}, x_n)$$

$$e(y_n, y_{n+1}) \leq t_1 d(x_{n-1}, x_n),$$

where $t_1 = c_1^{1/p}$.
Now $d(x_n, x_{n+1}) \leq t_1 t_2 d(x_{n-1}, x_n) .... \leq (t_1 t_2)^n d(x_0, x_1)$ and since $0 \leq t_1 t_2 < 1$ we get $\lim_{n\to\infty} d(x_n, x_{n+1}) = 1$.
Now by definition of super metric space, for $s \geq 1$ and for all $x_{n+2} \in X$ there exist distinct sequences $\langle x_n \rangle, \langle x_{n+1} \rangle$ with $\lim_{n\to\infty} m(x_{n+1}, x_n) \to 1$ such that

$$\lim_{n\to\infty} \sup m(x_n, x_{n+2}) \leq \lim_{n\to\infty} \sup m^s(x_{n+1}, x_{n+2}).$$

Since $\lim m(x_n, x_{n+2}) = 1$ as $n \to \infty$. Therefore $\lim_{n\to\infty} \sup m(x_n, x_{n+2}) = 1$. Continuing in this way, similarly we have for $s \geq 1$ and for all $x_{n+3} \in X$ there exist distinct sequences $\langle x_{n+2} \rangle, \langle x_n \rangle$ with $m(x_n, x_{n+2}) \to 1$ such that

$$\lim_{n\to\infty} \sup m(x_n, x_{n+3}) \leq \lim_{n\longrightarrow\infty} \sup m^s(x_{n+2}, x_{n+3}).$$

$i.e$, $\lim_{n\to\infty} \sup m(x_n, x_{n+3}) = 1$.
Inductively, we can conclude

$$\lim_{n\to\infty} \sup m(x_n, x_m) = 1,\ m > n \ and \ m, n \in \mathbf{N}.$$

Thus $\langle x_n \rangle$ is a Cauchy sequence in X and $\{y_n\}$ is a Cauchy sequence in Y. Since (X, d) and (Y, e) are complete super metric spaces $\{x_n\}$ converges to a point say z in X and $\{y_n\}$ converges to a point say w in Y.
Put $x = z$ and $y = y_{n-1}$ in equation (2.28)

$$e^p(Tx, TSy) = e^p(Tz, TSy_{n-1}) \leq c_1 \, max \, m\{d^p(z, x_{n-1}), e^p(y_{n-1}, Tx_{n-1})\}.$$

Letting limit as $n \to \infty$

$$e^p(Tz, w) \leq c_1 \, e^p(w, Tz)$$

which implies $Tz = w$ as $c_1 < 1$ similarly $Sw = z$. So $STz = S(Tz) = Sw = z$ and $TSw = T(Sw) = Tz = w$

$$STz = z, \quad TSw = w$$

3. Conflict of Interest

The authors declare there is no conflict of interests.

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