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An Implicit Function Implies Several Contraction Conditions in Perturbed Metric Spaces

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ABSTRACT: In this paper, we define an implicit function on perturbed metric spaces and use it to prove some common fixed point theorems for pairs of weakly compatible mappings satisfying the common property (E.A.).

Keywords: Perturbed metric space, common fixed points, weakly compatible mappings, common property (E.A.), implicit function.

Contents

1. Introduction

Fixed point and common fixed point theorems are important tools in mathematical analysis because they have many theoretical and practical applications. In a metric space, the existence of a common fixed point for two or more mappings usually depends on certain conditions such as commutativity, continuity, contraction, and a containment condition between the ranges of the mappings. Therefore, to develop new results in this area, researchers often try to relax or improve one or more of these conditions.

The first attempt to weaken the commutativity condition was made by Sessa [2], who introduced the concept of weakly commuting mappings. Following this, several other generalized forms of weak commutativity were introduced, such as compatible mappings and compatible mappings of types (A), (B), (C), and (P). A detailed comparison of these can be found in Murthy [3] . Among all these, the simplest and most natural condition was introduced by Jungck [5], called weak compatibility. In this work, we use this notion of weak compatibility to establish our main results.

To improve the continuity requirement, Kannan [6] proved a fixed point theorem for self-mappings without assuming continuity and showed that a mapping can be discontinuous at its fixed point. Later, Singh and Mishra [7] and Pant [8] established common fixed point theorems that do not require continuity. The approach used in this paper follows the idea of Singh and Mishra [7].

Another major area of improvement in fixed point theory is the relaxation of contraction conditions. Many different types of contraction conditions have been developed over the years, as summarized by Rhoades [4]. To unify several of these conditions, Popa [9] introduced the concept of implicit relations. This approach allows many existing contraction conditions to be included as special cases and also helps in forming new ones. Because of its general and flexible nature, we use the idea of implicit relations in this paper.

In real-life situations, the distance between two points cannot always be measured exactly because of small errors that may occur during measurement. These errors can be positive, negative, or zero. To handle this kind of inaccuracy, Jleli and Samet [3] introduced the concept of a perturbed metric space. A perturbed metric space is a generalization of a metric space that takes into account small errors in measurement. When the error is small, the space still keeps the main properties of a metric space. Therefore, perturbed metric spaces form a useful link between ideal mathematical models and real-world problems where exact distances are not measurable.

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Motivated by these ideas, the aim of this paper is to establish new common fixed point theorems for weakly compatible mappings in perturbed metric spaces using implicit relations. The results presented here generalize several existing theorems and demonstrate how perturbed metric spaces can be used to model practical situations more effectively.

In 2025, Mohamed Jleli and Bessem Samet [12] introduced a more general form of distance function, known as perturbed metric space as follows :

Definition 1.1. [12] Let $D, P : X \times X \to [0, \infty)$ be two given functions. The function D is called a perturbed metric on X with respect to P, if the function

$$D - P : X \times X \to \mathbb{R},$$

defined by the relation

$$(D - P)(x, y) = D(x, y) - P(x, y),$$

for all $x, y, z \in X$, is a exact metric on X, i.e., for all $x, y, z \in X$, it satisfies the following conditions

(i) $(D - P)(x, y) \geq 0$;

(ii) $(D - P)(x, y) = 0$ if and only if $x = y$;

(iii) $(D - P)(x, y) = (D - P)(y, x)$;

(iv) $(D - P)(x, y) \leq (D - P)(x, z) + (D - P)(z, y)$.

P is called a *perturbing function* and $D = d + P$ be an *perturbed metric.*
The set X endowed with D and *perturbed function* P denoted by (X, D, P) is known as *perturbed metric spaces.*

Notice that a perturbed metric on X is not necessarily a metric on X. But a metric is always perturbed metric when perturbed error is zero.

Example 1.1. [10] Let $D : \mathbb{R} \times \mathbb{R} \to [0, \infty)$ be the function defined by

$$D(x, y) = |x - y| + x^2 y^4, \ for \ all \ x, y \in \mathbb{R}.$$

Then D is a perturbed metric on $\mathbb{R}$ with respect to the perturbed function

$$P : \mathbb{R} \times \mathbb{R} \to [0, \infty)$$

given by

$$P(x, y) = x^2 y^4, \quad x, y \in \mathbb{R}.$$

In this case, the exact metric is the function $d : \mathbb{R} \times \mathbb{R} \to [0, \infty)$ defined by

$$d(x, y) = D(x, y) - P(x, y) \ , \ where$$

$$d(x, y) = |x - y|, \quad x, y \in \mathbb{R}.$$

Here we note that D is not necessarily a metric, because $D(1, 1) = 1 \neq 0$ as $x = y$, but D is perturbed metric on X with respect to perturbed function P.

The topological structure of the perturbed metric space (X, D, P) corresponds to the balls in metric spaces and is induced by the exact metric $d = D - P$. That is, the topology $\tau_{D,P}$ on X is defined as:

$$\tau_{D,P} := \tau_d = \{U \subseteq X \mid \forall x \in U, \exists r > 0 \text{ such that } B_d(x, r) \subseteq U\},$$

where the open ball with respect to d is given by:

$$B_d(x,r) = \{y \in X \mid d(x,y) = D(x,y) - P(x,y) < r\}\,.$$

Definition 1.2. Let (X, D, P) be a perturbed metric space with perturbed function P. A sequence $\{x_n\}$ in X is said to be

(i) *perturbed convergent sequence*, if $\{x_n\}$ is convergent in the metric space (X, d), where $d = D - P$ is the exact metric.

(ii) *perturbed Cauchy sequence*, if $\{x_n\}$ is a Cauchy sequence in the metric space (X, d).

A mapping T defined on (X, D, P) is a *perturbed continuous mapping*, if T is continuous with respect to the exact metric d.

We recall some elementary properties of perturbed metric spaces [12] .

Proposition 1.1. [12] Let (X, D, P) and (X, D, Q) be two perturbed metric spaces, where P and Q are perturbed function $P, Q : X \times X \to [0, \infty)$, then following holds,

(i) If (X, D, P) and (X, D, Q) be two perturbed metric spaces, then $\left(X, D, \frac{P+Q}{2}\right)$ is a perturbed metric space.

(ii) If (X, D, P) is a perturbed metric space, then $(X, \alpha D, \alpha P)$ is a perturbed metric space, where α is any scalar, $\alpha > 0$.

Here for the convenience of readers, we provide the proof of the proposition 1.1.

Proof.

(i) Since $D - P$ and $D - Q$ are two metrics on X, then

$$\frac{1}{2}[(D - P) + (D - Q)] = D - \frac{P + Q}{2}$$

is a metric on X, which proves (i).

(ii) Since $D - P$ is a metric on X and $\alpha > 0$, then

$$\alpha(D - P) = \alpha D - \alpha P$$

is a metric on X, which proves (ii).

2. Implicit Functions

Now, we introduce an implicit function in perturbed metric space and present a collection of examples present in literature [11] that encompass many of the well-known contractions found in the existing literature, while also incorporating several new ones. It is noteworthy that some of these examples are of the nonexpansive type (e.g., Examples 2.16 and 2.19) and the Lipschitzian type (e.g., Examples 2.12, 2.14, and 2.15). It should also be mentioned that most of the following examples do not satisfy the conditions of the implicit function proposed by Popa [9]. In order to describe our implicit function, let Ψ be the family of lower semi-continuous functions

$$F : \mathbb{R}_+^6 \to \mathbb{R}$$

satisfying the following conditions:

(F_1) $F(t, 0, t, 0, 0, t) > 0$, for all $t > 0$,

(F_2) $F(t,0,0,t,t,0) > 0$, for all $t > 0$,

(F_3) $F(t,t,0,0,t,t) > 0$, for all $t > 0$.

Now for the convenience of readers we are giving examples from the paper of [11]
Example 2.1. [11]Define $F(t_1,t_2,\ldots,t_6) : \mathbb{R}_+^6 \to \mathbb{R}$ as

$$F(t_1,t_2,\ldots,t_6) = t_1 - k\max\{t_2,t_3,t_4,t_5,t_6\}, \quad \text{where } k \in [0,1).$$

Then,

(F_1) $F(t,0,t,0,0,t) = t(1-k) > 0$, for all $t > 0$,

(F_2) $F(t,0,0,t,t,0) = t(1-k) > 0$, for all $t > 0$,

(F_3) $F(t,t,0,0,t,t) = t(1-k) > 0$, for all $t > 0$.

Example 2.2. [11] Define $F(t_1,t_2,\ldots,t_6) : \mathbb{R}_+^6 \to \mathbb{R}$ as

$$F(t_1,t_2,\ldots,t_6) = t_1 - k\max\{t_2,t_3,t_3t_5,t_4t_6\}, \quad \text{where } k \in [0,1).$$

Then,

(F_1) $F(t,0,t,0,0,t) = t(1-k) > 0$, for all $t > 0$,

(F_2) $F(t,0,0,t,t,0) = t > 0$, for all $t > 0$,

(F_3) $F(t,t,0,0,t,t) = t(1-k) > 0$, for all $t > 0$.

Example 2.3. [11]Define $F(t_1,t_2,\ldots,t_6) : \mathbb{R}_+^6 \to \mathbb{R}$ as

$$F(t_1,t_2,\ldots,t_6) = t_1 - k\left[\max\{t_2^2,t_3t_4,t_5t_6,t_3t_5,t_4t_6\}\right]^{\frac{1}{2}}, \quad \text{where } k \in [0,1).$$

Example 2.4. [11]Define $F(t_1,t_2,\ldots,t_6) : \mathbb{R}_+^6 \to \mathbb{R}$ as

$$F(t_1,t_2,\ldots,t_6) = t_1 - \alpha\big[\beta\max\{t_2,t_3,t_4,t_5,t_6\} + (1-\beta)\big(\max\{t_2^2,t_3t_4,t_5t_6,t_3t_6,t_4t_5\}\big)^{\frac{1}{2}}\big],$$

where $\alpha \in [0,1)$ and $\beta \geq 0$.

Example 2.5. [11] Define $F(t_1,t_2,\ldots,t_6) : \mathbb{R}_+^6 \to \mathbb{R}$ as

$$F(t_1,t_2,\ldots,t_6) = t_1^2 - \alpha\max\{t_2^2,t_3^2,t_4^2\} - \beta\max\{t_3t_5,t_4t_6\} - \gamma t_5t_6,$$

where $\alpha,\beta,\gamma \geq 0$ and $\alpha+\gamma < 1$.

Example 2.6. [11] Define $F(t_1,t_2,\ldots,t_6) : \mathbb{R}_+^6 \to \mathbb{R}$ as

$$F(t_1,t_2,\ldots,t_6) = (1+\alpha t_2)t_1 - \alpha\max\{t_3t_4,t_5t_6\} - \beta\max\{t_2,t_3,t_4,t_5,t_6\},$$

where $\alpha \geq 0$ and $\beta \in [0,1)$.

Example 2.7. [11] Define $F(t_1,t_2,\ldots,t_6) : \mathbb{R}_+^6 \to \mathbb{R}$ as

$$F(t_1,t_2,\ldots,t_6) = t_1 - \alpha t_2 - \beta\max\{t_3,t_4\} - \gamma\max\{t_3+t_4,t_5+t_6\},$$

where $\alpha,\beta,\gamma \geq 0$ and $\alpha+\beta+2\gamma < 1$.

Example 2.8. [11] Define $F(t_1,t_2,\ldots,t_6) : \mathbb{R}_+^6 \to \mathbb{R}$ as

$$F(t_1,t_2,\ldots,t_6) = t_1 - \phi(\max\{t_2,t_3,t_4,t_5,t_6\}),$$

where $\phi : \mathbb{R}_+ \to \mathbb{R}$ is an upper semi-continuous function such that $\phi(0) = 0$ and $\phi(t) < t$ for all $t > 0$.

Example 2.9. [11] Define $F(t_1, t_2, \ldots, t_6) : \mathbb{R}_+^6 \to \mathbb{R}$ as

$$F(t_1, t_2, \ldots, t_6) = t_1 - \phi(t_2, t_3, t_4, t_5, t_6),$$

where $\phi : \mathbb{R}_+^5 \to \mathbb{R}$ is an upper semi-continuous function such that

$$\max\{\phi(0, t, 0, 0, t), \phi(0, 0, t, t, 0), \phi(t, 0, 0, t, t)\} < t \quad \text{for each } t > 0.$$

Example 2.10. [11] Define $F(t_1, t_2, \ldots, t_6) : \mathbb{R}_+^6 \to \mathbb{R}$ as

$$F(t_1, t_2, \ldots, t_6) = t_1^2 - \phi(t_2^2, t_3 t_4, t_5 t_6, t_3 t_6, t_4 t_5),$$

where $\phi : \mathbb{R}_+^5 \to \mathbb{R}$ is an upper semi-continuous function such that

$$\max\{\phi(0, 0, 0, t, 0),\ \phi(0, 0, 0, 0, t),\ \phi(t, 0, t, 0, 0)\} < t \quad \text{for each } t > 0.$$

Example 2.11. [11] Define $F(t_1, t_2, \ldots, t_6) : \mathbb{R}_+^6 \to \mathbb{R}$ as

$$F(t_1, t_2, \ldots, t_6) = \begin{cases} t_1 - \alpha t_2 - \beta \dfrac{t_3^2 + t_4^2}{t_3 + t_4} - \gamma(t_5 + t_6), & \text{if } t_3 + t_4 \neq 0, \\ t_1, & \text{if } t_3 + t_4 = 0, \end{cases}$$

where $\alpha, \beta, \gamma \geq 0$ and $\beta + \gamma < 1$.

Example 2.12. [11] Define $F(t_1, t_2, \ldots, t_6) : \mathbb{R}_+^6 \to \mathbb{R}$ as

$$F(t_1, t_2, \ldots, t_6) = \begin{cases} t_1^p - k t_2^p - \dfrac{t_3 t_4^p + t_5 t_6^p}{t_3 + t_4}, & \text{if } t_3 + t_4 \neq 0, \\ t_1, & \text{if } t_3 + t_4 = 0, \end{cases}$$

where $p \geq 1$ and $0 \leq k < \infty$.

Example 2.13. [11] Define $F(t_1, t_2, \ldots, t_6) : \mathbb{R}_+^6 \to \mathbb{R}$ as

$$F(t_1, t_2, \ldots, t_6) = \begin{cases} t_1 - \alpha t_2 - \beta \dfrac{t_5^2 + t_4^2}{t_5 + t_6} - \gamma(t_3 + t_4), & \text{if } t_5 + t_6 \neq 0, \\ t_1, & \text{if } t_5 + t_6 = 0, \end{cases}$$

where $\alpha, \beta, \gamma \geq 0$ and $\beta + \gamma < 1$.

Example 2.14. [11] Define $F(t_1, t_2, \ldots, t_6) : \mathbb{R}_+^6 \to \mathbb{R}$ as

$$F(t_1, t_2, \ldots, t_6) = \begin{cases} t_1 - k t_2 - \dfrac{t_3 t_4 + t_5 t_6}{t_5 + t_6}, & \text{if } t_5 + t_6 \neq 0, \\ t_1, & \text{if } t_5 + t_6 = 0, \end{cases}$$

where $0 \leq k < \infty$.

Example 2.15. [11] Define $F(t_1, t_2, \ldots, t_6) : \mathbb{R}_+^6 \to \mathbb{R}$ as

$$F(t_1, t_2, \ldots, t_6) = \begin{cases} t_1 - k t_2 - \dfrac{t_3 t_4 + t_5 t_6}{t_3 + t_4} - \dfrac{t_3 t_5 + t_4 t_6}{t_5 + t_6}, & \text{if } t_3 + t_4 \neq 0 \text{ and } t_5 + t_6 \neq 0, \\ t_1, & \text{if } t_3 + t_4 = 0 \text{ or } t_5 + t_6 = 0, \end{cases}$$

where $0 \leq k < \infty$.

Example 2.16. [11] Define $F(t_1, t_2, \ldots, t_6) : \mathbb{R}_+^6 \to \mathbb{R}$ as

$$F(t_1, t_2, \ldots, t_6) = t_1 - \frac{t_3 t_4 + t_5 t_6}{1 + t_2}.$$

Example 2.17. [11] Define $F(t_1, t_2, \ldots, t_6) : \mathbb{R}_+^6 \to \mathbb{R}$ as

$$F(t_1, t_2, \ldots, t_6) = t_1 - \alpha t_2 - \beta \frac{t_3 + t_4}{1 + t_5 t_6},$$

where $\alpha, \beta \in [0, 1)$.

Example 2.18. [11] Define $F(t_1, t_2, \ldots, t_6) : \mathbb{R}_+^6 \to \mathbb{R}$ as

$$F(t_1, t_2, \ldots, t_6) = t_1^2 - \alpha t_2^2 - \beta \frac{t_5 t_6}{1 + t_3^2 + t_4^2},$$

where $\alpha, \beta \geq 0$ and $\alpha + \beta < 1$.

Example 2.19. [11] Define $F(t_1, t_2, \ldots, t_6) : \mathbb{R}_+^6 \to \mathbb{R}$ as

$$F(t_1, t_2, \ldots, t_6) = t_1^3 - \frac{t_3^2 t_4^2 + t_5^2 t_6^2}{1 + t_2}.$$

Example 2.20. [11] Define $F(t_1, t_2, \ldots, t_6) : \mathbb{R}_+^6 \to \mathbb{R}$ as

$$F(t_1, t_2, \ldots, t_6) = t_1^3 - \alpha t_1^2 t_2 - \beta t_1 t_3 t_4 - \gamma t_5^2 t_6 - \eta t_5 t_6^2,$$

where $\alpha, \beta, \gamma, \eta \geq 0$ and $\alpha + \gamma + \eta < 1$.

Since verification of requirements (F_1, F_2, F_3) for Examples 2.3–2.20 are easy, details are not included.

3. Main

Javed Ali and Imdad [11] proved the following fixed point results in metric space,using lemma

Lemma 3.1. Let A, B, S and T be self-mappings of a metric space (X, d) such that

(3.1) the pair (A, S) (or (B, T)) satisfies the property $(E.A)$,

(3.2) $A(X) \subset T(X)$ (or $B(X) \subset S(X)$) and

(3.3) for all $x, y \in X$ and $F \in \Psi$,

$$F(d(Ax, By), d(Sx, Ty), d(Ax, Sx), d(By, Ty), d(Sx, By), d(Ty, Ax)) \leq 0. \tag{3.4}$$

Then the pairs (A, S) and (B, T) satisfy the common property $(E.A)$.

Theorem 3.1 [11] Let A, B, S and T be self mappings of a metric space (X, d) which satisfy inequality (3.4). Suppose that

- pairs (A, S) and (B, T) enjoy the common property (E.A.),
- $S(X)$ and $T(X)$ are closed subsets of X.

Then the pairs (A, S) as well as (B, T) have a coincidence point. Moreover, A, B, S and T have a unique common fixed point provided both the pairs (A, S) and (B, T) are weakly compatible.

Now we prove analogue of this result in perturbed metric space. Before proving our main result we prove the following lemma in the setting of perturbed metric spaces.

Lemma 3.2 Let A, B, S and T be self-mappings of a perturbed metric space (X, D, P) such that

(3.5) the pair (A,S) (or (B,T)) satisfies the property $(E.A)$,

(3.6) $A(X)\subset T(X)$ (or $B(X)\subset S(X)$), and

(3.7) for all $x,y\in X$ and $F\in\Psi$,

$$F(D(Ax,By),D(Sx,Ty),D(Ax,Sx),D(By,Ty),D(Sx,By),D(Ty,Ax))\leq 0. \tag{3.8}$$

Then the pairs (A,S) and (B,T) satisfy the common property $(E.A)$.

Proof: If the pair (A,S) enjoys property $(E.A)$, then there exists a sequence $\{x_n\}$ in X such that

$$\lim_{n\to\infty} Ax_n=\lim_{n\to\infty} Sx_n=t,\quad \text{for some } t\in X.$$

Since $A(X)\subset T(X)$, hence for each $\{x_n\}$ there exists $\{y_n\}$ in X such that $Ax_n=Ty_n$. Therefore, $\lim_{n\to\infty} Ty_n=\lim_{n\to\infty} Ax_n=t$. Thus, in all we have $Ax_n\to t$, $Sx_n\to t$ and $Ty_n\to t$. Now, we assert that $By_n\to t$.

If not, then using (3.8), we have

$$F(D(Ax_n,By_n),D(Sx_n,Ty_n),D(Ax_n,Sx_n),D(By_n,Ty_n),D(Sx_n,By_n),D(Ty_n,Ax_n))\leq 0,$$

which on making $n\to\infty$ reduces to

$$F(D(t,By_n),0,0,D(By_n,t),D(t,By_n),0)\leq 0,$$

a contradiction to (F_2). Hence $By_n\to t$ which shows that the pairs (A,S) and (B,T) satisfy the common property $(E.A)$.

Now, we state and prove our main result for two pairs of weakly compatible mappings satisfying an implicit function in perturbed metric space.

Theorem 3.2. Let A,B,S and T be self-mappings of a perturbed metric space (X,D,P) which satisfy inequality (3.8). Suppose that

(3.9) pairs (A,S) and (B,T) enjoy the common property $(E.A)$,

(3.10) $S(X)$ and $T(X)$ are closed subsets of X.

Then the pair (A,S) as well as (B,T) have a coincidence point. Moreover, A,B,S and T have a unique common fixed point provided both the pairs (A,S) and (B,T) are weakly compatible.

Proof: Since the pairs (A,S) and (B,T) enjoy the common property $(E.A)$, there exist two sequences $\{x_n\}$ and $\{y_n\}$ in X such that

$$\lim_{n\to\infty} Ax_n=\lim_{n\to\infty} Sx_n=\lim_{n\to\infty} By_n=\lim_{n\to\infty} Ty_n=t,\quad \text{for some } t\in X.$$

If $S(X)$ is a closed subset of X, then $\lim_{n\to\infty} Sx_n=t\in S(X)$. Therefore, there exists a point $u\in X$ such that $Su=t$. Now we assert that $Au=Su$.

If not, then using (3.8), we have

$$F(D(Au,By_n),D(Su,Ty_n),D(Au,Su),D(By_n,Ty_n),D(Su,By_n),D(Ty_n,Au))\leq 0,$$

which on making $n\to\infty$ reduces to

$$F(D(Au,t),D(Su,t),D(Au,Su),D(t,t),D(Su,t),D(t,Au))\leq 0,$$

or

$$F(D(Au,Su),0,D(Au,Su),0,0,D(Su,Au))\leq 0.$$

a contradiction to (F_1). Hence $Au = Su$. Therefore, u is a coincidence point of the pair (A, S).

If $T(X)$ is a closed subset of X, then $\lim_{n\to\infty} Ty_n = t \in T(X)$. Therefore, there exists a point $w \in X$ such that $Tw = t$. Now we assert that $Bw = Tw$.

If not, then again using (3.8), we have

$$F(D(Ax_n, Bw), D(Sx_n, Tw), D(Ax_n, Sx_n), D(Bw, Tw), D(Sx_n, Bw), D(Tw, Ax_n)) \leq 0,$$

which on making $n \to \infty$ reduces to

$$F(D(t, Bw), D(t, Tw), D(t, t), D(Bw, Tw), D(t, Bw), D(Tw, t)) \leq 0,$$

or

$$F(D(Tw, Bw), 0, 0, D(Bw, Tw), D(Tw, Bw), 0) \leq 0,$$

a contradiction to (F_2). Hence $Bw = Tw$, which shows that w is a coincidence point of the pair (B, T). Since the pair (A, S) is weakly compatible and $Au = Su$, hence $At = ASu = SAu = St$. Now we assert that t is a common fixed point of the pair (A, S). Suppose that $At \neq t$, then using (3.8), we have

$$F(D(At, Bw), D(St, Tw), D(At, St), D(Bw, Tw), D(St, Bw), D(Tw, At)) \leq 0$$

or

$$F(D(At, t), D(At, t), 0, 0, D(At, t), D(t, At)) \leq 0,$$

a contradiction to (F_3).

Also the pair (B, T) is weakly compatible and $Bw = Tw$, then $Bt = BTw = TBw = Tt$. Suppose that $Bt \neq t$, then using (3.8), we get

$$F(D(Au, Bt), D(Su, Tt), D(Au, Su), D(Bt, Tt), D(Su, Bt), D(Tt, Au)) \leq 0$$

or

$$F(D(Bt, t), D(Bt, t), 0, 0, D(Bt, t), D(t, Bt)) \leq 0,$$

a contradiction to (F_3). Therefore, $Bt = t$ which shows that t is a common fixed point of the pair (B, T). Hence t is a common fixed point of both the pairs (A, S) and (B, T). Uniqueness of the common fixed point is an easy consequence of inequality (3.8) in view of condition (F_3). This completes the proof.

Theorem 3.3. The conclusions of Theorem 3.2 remain true if the condition (3.10) of Theorem 3.2 is replaced by the following:

(3.11) $\overline{A(X)} \subset T(X)$ and $\overline{B(X)} \subset S(X)$.

As a corollary of Theorem 3.3, we can have the following result which is also a variant of Theorem 3.2.

Corollary 3.1 The conclusions of Theorems 3.2 and 3.3 remain true if the conditions (3.10) and (3.11) are replaced by the following:

(3.12) $A(X)$ and $B(X)$ are closed subsets of X provided $A(X) \subset T(X)$ and $B(X) \subset S(X)$.

Theorem 3.4. Let A, B, S and T be self-mappings of a perturbed metric space (X, D, P) satisfying inequality (3.8). Suppose that

(3.13) the pair (A, S) (or (B, T)) has property $(E.A)$,

(3.14) $A(X) \subset T(X)$ (or $B(X) \subset S(X)$), and

(3.15) $S(X)$ (or $T(X)$) is a closed subset of X.

Then the pairs (A,S) and (B,T) have coincidence points. If the pairs (A,S) and (B,T) are weakly compatible, then A,B,S and T have a unique common fixed point.

Proof: In view of Lemma 3.2, the pairs (A,S) and (B,T) satisfy the common property $(E.A)$, i.e., there exist two sequences $\{x_n\}$ and $\{y_n\}$ in X such that

$$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = \lim_{n\to\infty} By_n = \lim_{n\to\infty} Ty_n = t \in X.$$

If $S(X)$ is a closed subset of X, then on the lines of Theorem 3.1, the pair (A,S) has a coincidence point, say u, i.e. $Au = Su$. Since $A(X) \subset T(X)$ and $Au \in A(X)$, there exists $w \in X$ such that $Au = Tw$. Now we assert that $Bw = Tw$.

If not, then using (3.8), we have

$$F(D(Ax_n, Bw), D(Sx_n, Tw), D(Ax_n, Sx_n), D(Bw, Tw), D(Sx_n, Bw), D(Tw, Ax_n)) \leq 0,$$

which on making $n \to \infty$ reduces to

$$F(D(t, Bw), D(t, Tw), D(t, t), D(Bw, Tw), D(t, Bw), D(Tw, t)) \leq 0,$$

or

$$F(D(Tw, Bw), 0, 0, D(Bw, Tw), D(Tw, Bw), 0) \leq 0,$$

a contradiction to (F_2). Hence $Bw = Tw$, which shows that w is a coincidence point of the pair (B,T). The rest of the proof can be completed on the lines of Theorem 3.1.

By choosing A, B, S, and T appropriately, one can obtain corollaries for both a pair and a triad of mappings. The corollaries corresponding to a triad of mappings are not presented here. As an illustration, we present the following natural result for a pair of self-mappings.

Corollary 3.2. Let A and S be self-mappings of a perturbed metric space (X, D, P). Suppose that

(3.16) A and S have property $(E.A)$,

(3.17) for all $x, y \in X$ and $F \in \Psi$,

$$F(D(Ax, Ay), D(Sx, Sy), D(Ax, Sx), D(Ay, Sy), D(Sx, Ay), D(Sy, Ax)) \leq 0, \tag{3.18}$$

(3.19) $S(X)$ is a closed subset of X.

Then A and S have a coincidence point. Moreover, if the pair (A,S) is weakly compatible, then A and S have a unique common fixed point.

Corollary 3.3. The conclusions of Theorem 3.1 remain true if inequality (3.8) is replaced by one of the following contraction conditions. For all $x, y \in X$,

(a_1)

$$D(Ax, By) \leq k \max\{D(Sx, Ty), D(Ax, Sx), D(By, Ty), D(Sx, By), D(Ty, Ax)\},$$

where $k \in [0, 1)$.

(a_2)

$$D(Ax, By) \leq k \max\{D(Sx, Ty), D(Ax, Sx), D(Ax, Sx)D(Sx, By), D(By, Ty)D(Ty, Ax)\},$$

where $k \in [0, 1)$.

(a$_3$)

$$D(Ax,By) \le k\Big[\max\Big\{D^2(Sx,Ty),\ D(Ax,Sx)D(By,Ty),\ D(Sx,By)D(Ty,Ax),\ D(Ax,Sx)D(Sx,By),\ D(By,Ty)D(Ty,Ax)\Big\}\Big]^{\frac{1}{2}}.$$

where $k \in [0,1)$.

(a$_4$)

$$D(Ax,By) \le \alpha\Big[\beta\max\{D(Sx,Ty), D(Ax,Sx), D(By,Ty), D(Sx,By), D(Ty,Ax)\} + (1-\beta)\Big(\max\{D^2(Sx,Ty), D(Ax,Sx)D(By,Ty), D(Sx,By)D(Ty,Ax), D(Ax,Sx)D(Ty,Ax), D(By,Ty)D(Sx,By)\}\Big)^{\frac{1}{2}}\Big],$$

where $\alpha \in [0,1)$ and $\beta \ge 0$.

(a$_5$)

$$D^2(Ax,By) \le \alpha\max\{D^2(Sx,Ty), D^2(Ax,Sx), D^2(By,Ty)\} + \beta\max\{D(Ax,Sx)D(Sx,By), D(By,Ty)D(Ty,Ax)\} + \gamma D(Sx,By)D(Ty,Ax),$$

where $\alpha,\beta,\gamma \ge 0$ and $\alpha+\gamma < 1$.

(a$_6$)

$$(1+\alpha D(Sx,Ty))D(Ax,By) \le \alpha\max\{D(Ax,Sx)D(By,Ty), D(Sx,By), D(Ty,Ax)\} + \beta\max\{D(Sx,Ty), D(Ax,Sx), D(By,Ty), D(Sx,By), D(Ty,Ax)\},$$

where $\alpha \ge 0$ and $\beta \in [0,1)$.

(a$_7$)

$$D(Ax,By) \le \alpha D(Sx,Ty) + \beta\max\{D(Ax,Sx), D(By,Ty)\} + \gamma\max\{D(Ax,Sx)+D(By,Ty), D(Sx,By)+D(Ty,Ax)\},$$

where $\alpha,\beta,\gamma \ge 0$ and $\alpha+\beta+2\gamma < 1$.

(a$_8$)

$$D(Ax,By) \le \phi(\max\{D(Sx,Ty), D(Ax,Sx), D(By,Ty), D(Sx,By), D(Ty,Ax)\}),$$

where $\phi : \mathbb{R}^+ \to \mathbb{R}$ is an upper semi-continuous function such that $\phi(0) = 0$ and $\phi(t) < t$ for all $t > 0$.

(a$_9$)

$$D(Ax,By) \le \phi(D(Sx,Ty), D(Ax,Sx), D(By,Ty), D(Sx,By), D(Ty,Ax)),$$

where $\phi : \mathbb{R}^5_+ \to \mathbb{R}$ is an upper semi-continuous function such that

$$\max\{\phi(0,t,0,0,t),\ \phi(0,0,t,t,0),\ \phi(t,0,0,t,t)\} < t \quad \text{for each } t > 0.$$

(a$_{10}$)

$$D^2(Ax,By) \le \phi(D^2(Sx,Ty), D(Ax,Sx)D(By,Ty), D(Sx,By)D(Ty,Ax), D(Ax,Sx)D(Ty,Ax), D(By,Ty)D(Sx,By)),$$

where $\phi : \mathbb{R}^5_+ \to \mathbb{R}$ is an upper semi-continuous function such that

$$\max\{\phi(0,0,0,t,0),\ \phi(0,0,t,0,0),\ \phi(t,0,t,0,0)\} < t \quad \text{for each } t > 0.$$

In the following contraction conditions, we denote

$$D_1 = D(Ax,Sx) + D(By,Ty) \quad \text{and} \quad D_2 = D(Sx,By) + D(Ty,Ax).$$

(a$_{11}$)

$$D(Ax,By)\leq\begin{cases}\alpha D(Sx,Ty)+\beta\dfrac{D^2(Ax,Sx)+D^2(By,Ty)}{D(Ax,Sx)+D(By,Ty)}+\gamma(D(Sx,By)+D(Ty,Ax)), & \text{if } D_1\neq 0,\\ 0, & \text{if } D_1=0,\end{cases}$$

where $\alpha,\beta,\gamma\geq 0$ and $\beta+\gamma<1$.

(a$_{12}$)

$$D^p(Ax,By)\leq\begin{cases}kD^p(Sx,Ty)+\dfrac{D(Ax,Sx)D^p(By,Ty)+D(Sx,By)D^p(Ty,Ax)}{D(Ax,Sx)+D(By,Ty)}, & \text{if } D_1\neq 0,\\ 0, & \text{if } D_1=0,\end{cases}$$

where $p\geq 1$ and $0\leq k<\infty$.

(a$_{13}$)

$$D(Ax,By)\leq\begin{cases}\alpha D(Sx,Ty)+\beta\dfrac{D^2(Sx,By)+D^2(Ty,Ax)}{D(Sx,By)+D(Ty,Ax)}+\gamma(D(Ax,Sx)+D(By,Ty)), & \text{if } D_2\neq 0,\\ 0, & \text{if } D_2=0,\end{cases}$$

where $\alpha,\beta,\gamma\geq 0$ and $\beta+\gamma<1$.

(a$_{14}$)

$$D(Ax,By)\leq\begin{cases}kD(Sx,Ty)+\dfrac{D(Ax,Sx)D(By,Ty)+D(Sx,By)D(Ty,Ax)}{D(Sx,By)+D(Ty,Ax)}, & \text{if } D_2\neq 0,\\ 0, & \text{if } D_2=0,\end{cases}$$

where $0\leq k<\infty$.

(a$_{15}$)

$$D(Ax,By)\leq\begin{cases}kD(Sx,Ty)+\dfrac{D(Ax,Sx)D(By,Ty)+D(Sx,By)D(Ty,Ax)}{D(Ax,Sx)+D(By,Ty)}\\ \quad+\dfrac{D(Ax,Sx)D(Sx,By)+D(By,Ty)D(Ty,Ax)}{D(Sx,By)+D(Ty,Ax)}, & \text{if } D_1\neq 0,\ D_2\neq 0,\\ 0, & \text{if } D_1=0 \text{ or } D_2=0,\end{cases}$$

where $0\leq k<\infty$.

(a$_{16}$)

$$D(Ax,By)\leq\frac{D(Ax,Sx)D(By,Ty)+D(Sx,By)D(Ty,Ax)}{1+D(Sx,Ty)}.$$

(a$_{17}$)

$$D(Ax,By)\leq\alpha D(Sx,Ty)+\beta\frac{D(Ax,Sx)+D(By,Ty)}{1+D(Sx,By)D(Ty,Ax)},$$

where $\alpha,\beta\in[0,1)$.

(a$_{18}$)

$$D^2(Ax,By)\leq\alpha D^2(Sx,Ty)+\beta\frac{D(Sx,By)D(Ty,Ax)}{1+D^2(Ax,Sx)+D^2(By,Ty)},$$

where $\alpha,\beta\geq 0$ and $\alpha+\beta<1$.

(a$_{19}$)
$$D^3(Ax,By) \leq \frac{D^2(Ax,Sx)D^2(By,Ty) + D^2(Sx,By)D^2(Ty,Ax)}{1 + D(Sx,Ty)}.$$

(a$_{20}$)
$$D^3(Ax,By) \leq \alpha D^2(Ax,By)D(Sx,Ty) + \beta D(Ax,By)D(Ax,Sx)D(By,Ty) + \gamma D^2(Sx,By)D(Ty,Ax) + \eta D(Sx,By)D^2(Ty,Ax),$$

where $\alpha, \beta, \gamma, \eta \geq 0$ and $\alpha + \gamma + \eta < 1$.

The proof follows from Theorem 3.1.

As an application of Theorem 3.2, we have the following result for four finite families of self-mappings.

Theorem 3.5. Let $\{A_1, A_2, \ldots, A_m\}$, $\{B_1, B_2, \ldots, B_p\}$, $\{S_1, S_2, \ldots, S_n\}$ and $\{T_1, T_2, \ldots, T_q\}$ be four finite families of self-mappings of a perturbed metric space (X, D, P), with $A = A_1A_2 \cdots A_m$, $B = B_1B_2 \cdots B_p$, $S = S_1S_2 \cdots S_n$ and $T = T_1T_2 \cdots T_q$ satisfying condition (3.8), and the pairs (A, S) and (B, T) share the common property $(E.A)$. If $S(X)$ and $T(X)$ are closed subsets of X, then

(3.19) the pair (A, S) has a coincidence point,

(3.20) the pair (B, T) has a coincidence point.

Moreover, if
$$A_iA_j = A_jA_i, \quad B_kB_l = B_lB_k, \quad S_rS_s = S_sS_r, \quad T_tT_u = T_uT_t,$$

$A_iB_k = B_kA_i$, and $S_rT_t = T_tS_r$ for all
$$i, j \in I_1 = \{1, 2, \ldots, m\},\ k, l \in I_2 = \{1, 2, \ldots, p\},\ r, s \in I_3 = \{1, 2, \ldots, n\},\ t, u \in I_4 = \{1, 2, \ldots, q\},$$

then (for all $i \in I_1$, $k \in I_2$, $r \in I_3$ and $t \in I_4$) A_i, B_k, S_r and T_t have a common fixed point.

By setting $A_1 = A_2 = \cdots = A_m = G$, $B_1 = B_2 = \cdots = B_p = H$, $S_1 = S_2 = \cdots = S_n = I$, and $T_1 = T_2 = \cdots = T_q = J$ in Theorem 3.4, we deduce the following:

Corollary 3.4. Let G, H, I and J be self-mappings of a perturbed metric space (X, D, P), and suppose the pairs (G^m, I^n) and (H^p, J^q) have the common property $(E.A)$ and satisfying the condition
$$F(D(G^mx, H^py), D(I^nx, J^qy), D(G^mx, I^nx), D(H^py, J^qy), D(I^nx, H^py), D(J^qy, G^mx)) \leq 0,$$

for all $x, y \in X$ and $F \in \Psi$ where m, n, p, and q are fixed positive integers. If $I^n(X)$ and $J^q(X)$ are closed subsets of X, then G, H, I, and J have a unique common fixed point provided $GI = IG$ and $HJ = JH$.

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