Bol. Soc. Paran. Mat.
©SPM – E-ISSN-2175-1188
SPM: www.spm.uem.br/bspm

(3s.) **v. 2026 (44) 7** : 1–11.
ISSN-0037-8712
doi:10.5269/bspm.80814

Solvability of Nonlinear Volterra-Hammerstein type Fractional Integral Equations in Orlicz Space

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ABSTRACT: In this paper, we are focused on analyzing new analytical properties of g-fractional type operators, such as continuity, boundedness and monotonicity in Orlicz spaces L_φ. Using these properties, along with Darbo's Fixed-Point theorem and the measure of noncompactness, we investigate the existence and uniqueness of solutions to a nonlinear fractional integral equation in L_φ. The g-fractional operators being investigated for the first time in the space L_φ. Here we generalizes various fractional operators and encompassing and unifying the results of many specific cases of classical and quadratic fractional issues explored in the previous literature. Lastly, we provide some examples to illustrate our main results.

Key Words: Nonlinear integral equation, $g-$fractional integral operator, measure of noncompactness (MNC), Fixed point theorem, Orlicz space.

Contents

1. Introduction

The study of fractional calculus is a powerful tool for designing memory and hereditary characteristics with a wide range of processes and materials.Researchers have conducted extensive investigations into fractional integral and differential equations in bounded and unbounded domains, including those by Miller and Rose [32] Benchohra et al. [8], Diethelm and Ford [29], Momani et al. [33] and the monograph of Kilbas et al. [24], etc. These findings are now applied in all the fields of electrochemistry, biology, viscoelasticity, fluid dynamics, electrical circuits, robotics, etc, (cf. [19,26])

Although fractional calculus and its applications made advancement in the 19^{th} and 20^{th} centuries, it's first mention can be found back from 16^{th} instances in works off G.W. Leibniz. Also, Abel [1], who used fractional calculus in earlier days, used the definition of differentiation and integration of arbitrary real order. Later on, it was O'Neil who first investigated the fractional calculus and convolution operators for Lorentz $L(p, q)$-space [34] and Orlicz space [10].

Relating fractional with $g-$fractional derivatives involves extension of the concept of differentiation to non-integer orders. Generalized fractional operators are extensions of the standard Riemann-Liouville and Caputo fractional derivatives and integrals. This operators unifies all of the Hadamard, Riemann-Liouville and Erdélyi-Kober fractional operators into a single form [4,8,9,17,40,41,42] In fact, it is able to present the memory properties connected to materials and processes in a better way.

The $g-$ fractional operator in quadratic integral equations is especially effective in modeling phenomena such as kinetic theory of gas [20], neutron transport [23] and radiative transfer [10] and other fields. The main goal of this work is to confirm and explain the essential properties of the $g-$ fractional type

2020 *Mathematics Subject Classification:* 45G10, 46E30, 47H08, 47H10, 47H30.
Submitted December 15, 2025. Published February 21, 2026

operator-specifically its continuity, boundedness, and monotonicity in Orlicz spaces L_ψ. These properties will be used to demonstrate and examine the monotonic solution of the equation

$$y(t) = f(t, y(t)) + \frac{c_1(t, s, y(t))}{\Gamma(\gamma)} \int_0^t \frac{c_2(\tau, y(\tau))}{(g(t) - g(\tau))^{1-\gamma}} g'(\tau)\, d\tau \tag{1.1}$$

$t \in [0, d], 0 < \gamma < 1$ in orlicz spaces L_ψ.
Our approach unifies and extends several types of fractional integral that have been studied independently by covering and generaliszing some of them. In [28] the author examined the fundamental characteristics of the Riemann-Liouville fractional integral operator

$$y(t) = f(t) + G(y)(t) \int_0^t \frac{(t-\tau)^{\gamma-1}}{\Gamma(\gamma)} f(\tau, y(\tau))\, d\tau, \quad 0 < \gamma < 1,\ t \in [0, d]$$

in the Orlicz space L_ψ.
In [29] the author illustrated and explored essential characteristics of the fractional operator of Hadamard type in L_ψ spaces and applied these to solve the given equation

$$y(t) = G_2(y)(t) + \frac{G_1(y)(t)}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{\tau}\right)^{\gamma-1} \frac{G_2(y)(\tau)}{\tau} d\tau, \quad t \in [1, e],\ 0 < \gamma < 1.$$

In [30], author examined the basic feature of the Erdély-Kober fractional operators within Lebesgue and Orlicz spaces and used them to investigate the equation

$$y(t) = f(t) + f_1(t, y(t)) + f_2\left(t, \frac{\beta c_1(t, y(t))}{\Gamma(\gamma)} \cdot \int_0^t \frac{\tau^{\beta-1} c_2(\tau, y(\tau))}{(t^\beta - \tau^\beta)^{1-\alpha}}\, d\tau\right), \quad t \in [0, d],$$

where $0 < \gamma < 1$ and $\beta > 0$ in Orlicz spaces L_ψ.
Moreover, the measure of noncompectness and Darbo's fixed point theorem are used to analyse various types of quadratic integral equations within Orlicz spaces L_ψ under different assumptions (cf. [31, 35]). Several nonlinear fractional integral equations can also be solved by applying Darbo's fixed point theorem in conjunction with the measure of noncompactness (cf. [13,14]). The concept of a measure of noncompactness (MNC) and fixed point theory are correlated to each other. This idea was propounded by Kuratowski in his paper [25]. Further, in the mid-19^{th} century, Darbo [15] established a theorem that ensures the existence of fixed points using the concept of measures of noncompactness (MNC).

A number of integral equations can be solved using fixed point theorem and the measure of noncompectness (MNC), e.g., [3,5,6,18]. The article is certain to illustrate various aspects of the g-fractional operator such as boundedness, continuity, and monotonicity. For this purpose, we use the properties of MNC and FPT within the L_ψ-space in order to check the existence and uniqueness of the solution for an integral equation (1.1).

2. Preliminaries

Assume $\mathbb{I} = [0, d]$,$\mathbb{R} = (-\infty, \infty)$ and $\mathbb{R}^+ = (0, \infty)$. If a function $\mathbb{P} : \mathbb{R}^+ \to \mathbb{R}^+$ is a Young function, then

$$P(v) = \int_0^v u(s) ds$$

for $v \geq 0$, where u is a function that increases in the left continuous and is neither equal to zero nor exactly infinite on $\mathbb{R}^+$. The complementary Young functions are indicated by the functions P and Q if $Q(x) = \sup_{y \geq 0}(xy - P(x))$. In particular, if P has finite values, where $\lim_{v\to\infty} \frac{P(v)}{v} = 0$, $\lim_{v\to\infty} \frac{P(v)}{v} = \infty$ and $P(v) > 0$ if $v > 0$ ($P(v) = 0 \Leftrightarrow v = 0$). Denoted by $L_P = L_P(\mathcal{I})$ the Orlicz space of bounded functions $f : \mathbb{I} \to \mathbb{R}$ using the norm

$$||y||_p = \inf_{\varepsilon > 0} \left\{ \int P(y(\tau)) d\tau \leq 1 \right\} < \infty.$$

Recall that, for every Young function P, we have $p(v+\tau) \leq p(v) + p(\tau)$ and

$$P(\mu\tau) \leq \mu P(\tau),$$

where $v, \tau \in \mathbb{R}$ and $\mu \in [0,1]$.
For all bounded sets whose norms are absolutely continuous, let $D_P(\mathcal{I})$ denote the closure within $L_P(\mathcal{I})$.

Lemma 2.1 *[22] Suppose that $c(t,s,y)$ is a function defined on $\mathbb{I} \times \mathbb{I} \times \mathbb{R}$ that fulfils the Caratheódory criteria,for $y \in \mathbb{R}$ and nearly every $(t,s) \in \mathbb{I}$,the function remains continuity in y and it is measurable in (t,s). Then the superposition operator $G_c = c(t,s,y) : D_{Q_2} \to L_P = D_P$ is bounded and continuous such that*

$$|c(t,s,y)| \leq m(t,s) + n\varphi^{-1}(Q_2(y)), \text{for all} y \in \mathbb{R}, (s,t) \in \mathbb{I}$$

Lemma 2.2 *[27, Theorem 10.2] Consider Q_1, Q_2, Q_3, are arbitrary Q-function. Then the following criteria are equivalent:*

1. *For every $v \in L_{Q_2}(\mathcal{I})$ and $z \in L_{Q_3}, v, z \in L_{Q_1(\mathcal{I})}$.*
2. *$\exists$ a constants $k > 0$ s.t. every measurable v, z on $\mathcal{I}$ then*
$$||vz||_{Q_1} \leq k||v||_{Q_2}||z||_{Q_3}.$$
3. *$\exists$ numbers $C > 0, t_0 \geq 0$ s.t. any $t, \tau \geq t_0, Q_1(\frac{t\tau}{C}) \leq Q_2(t) + Q_3(\tau)$.*
4. *$\limsup\limits_{t\to\infty} \frac{Q_2^{-1}(t)Q_3^{-1}(t)}{Q_1(t)} < \infty$.*

Lemma 2.3 *[11] Let $X \subset L_P(\mathcal{I})$ be a bounded set. Consider a family $(\Omega_q)_{0\leq q\leq d} \subset \mathcal{I}$ then meas $\Omega_q = q$ for each $q \in [0,d]$ and $f \in X$,*

$$y(l_1) \geq y(l_2), (l_1 \in \Omega_q, l_2 \notin \Omega_q).$$

As a result, X has the measure of non-compactness in $L_P(\mathcal{I})$.

Definition 2.1 [7] Let $X \subset Z$ be nonempty and bounded set. Then Hausdorff measure of non-compactness $\alpha(X)$ is defined by

$$\alpha(X) = \inf\{q > o : \text{there is a finite subset } Y \; of \; Z \; such \; that \; X \subset Y + B_q\},$$

where the ball is centered at the origin with radius q and $B_q = \{y \in L_{Q_1}(\mathbb{I}) : \|y\|_{L_{Q_1}} \leq q\}$.
And the equi-intergrability of the set $X \in L_{Q_1}$ will be measured by h for any $\eta > 0$(Definition 2 in [7])

$$h(X) = \lim_{\eta\to o} \sup_{mesF\leq\eta} \sup_{f\in X} ||y.\chi_F||_{L_{Q_1}(\mathcal{I})},$$

where χ_F is a measurable subsets of the characteristic function of $F \subset \mathcal{I}$.

Lemma 2.4 *[11,16] Assume that X is a compact, bounded and nonempty subset of $D_P(\mathcal{I})$. Then we get*

$$\alpha(X) = h(X).$$

Theorem 2.1 *[7] Suppose that $A \neq \emptyset$ is a bounded, close and convex subset of Z and $V : A \to A$ is a continuous mapping that contracts with the measure of noncompactness α, i.e.,*

$$\alpha(V(X)) \leq k\alpha(X), k \in [0,1)$$

for all nonempty $X \subset Z$. Then, the map V has a one fixed point in the set A.

3. Main Results

3.1. Generalized Fractional Operator

Definition 3.1 [24,37] For a well-defined function y of order γ, the generalized fractional, or g-fractional, integral of a distinct function $g(\tau)$,is defined by

$$L_g^\gamma y(t) = \frac{1}{\Gamma(\gamma)} \int_0^t \frac{y(\tau)}{(g(t)-g(\tau))^{1-\gamma}} g'(\tau) d\tau, 0<\gamma<1, \tag{3.1}$$

where $g : (0,\infty) \to \mathbb{R}$ is a positive and increasing function with a continuous derivative on $(0,\infty)$.

Remark 3.1 *(1) If $g(t) = t$, the operator L_g^γ (3.1) transforms into the Riemann-Liouville fractional operator, which has been examined in [24,28,32]*

$$L_g^\gamma y(t) = \frac{1}{\Gamma(\gamma)} \int_0^t \frac{y(\tau)}{(t-\tau)^{1-\gamma}} d\tau.$$

(2) If $g(t) = t^\beta, \beta > 0$ the operator L_g^γ (3.1) transforms into the Erd élyi Köber operator, which has been examined in [24,29,32]

$$L_{t^\beta}^\gamma y(t) = \frac{1}{\Gamma(\alpha)} \int_0^t \frac{\beta \tau^{\beta-1} y(\tau)}{(t^\beta - \tau^\beta)^{1-\gamma}} d\tau.$$

(3) If $g'(\tau) \neq 0, \gamma > 0$, the operator L_g^γ (3.1)transforms into the generalized $g-$fractional tempered integral operator, which has been examined in [21]

$$L_g^\gamma y(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (g(t)-g(\tau))^{\gamma-1} e^{-\mu(g(t)-g(\tau))} y(\tau) g'(\tau) \, d\tau.$$

(4) If $g(t) = \log(t)$, the operator L_g^γ (3.1) transforms into the Hadamard fractional operator, which has been examined in [24,30,32]

$$L_{\log(t)}^\gamma y(t) = \frac{1}{\Gamma(\gamma)} \int_1^t \left(\log \frac{t}{\tau}\right)^{\gamma-1} \frac{y(\tau)}{\tau} d\tau.$$

(5) If $g(t) = t^2$, the operator L_g^γ transforms into the fractional integral operator, which has been examined in [38]

$$L_{t^2}^\gamma y(t) = \frac{2}{\Gamma(\gamma)} \int_0^t \frac{y(t)}{(t^2-\tau^2)^{1-\gamma}} \tau \, d\tau.$$

Lemma 3.1 *The fractional integral operator $L_g^\gamma, \gamma > 0$ with $g(0) = 0$ takes the almost everywhere non-negative and almost everywhere nondecreasing functions into functions having similar properties.*

Proof: Let $0 \leq t_1 < t_2 \leq d$ Since $y(\tau) \geq 0$ for almost every $\tau \in [0,d]$, and $g'(\tau) > 0$. Therefore we know that the integrand

$$\frac{y(\tau)}{(g(t)-g(\tau))^{1-\gamma}} g'(\tau) \geq 0$$

for $0 \leq \tau < t$, Also $(g(t)-g(\tau))^{1-\gamma} > 0$ when g is strictly increasing.
Thus, the entire integral is nonnegative

$$(L_g^\gamma y)(t) \geq 0.$$

Now we show that $L_g^\gamma y$ is nondecreasing and almost everywhere.
Consider two points $0 < t_1 < t_2 \le d$. Then

$$\begin{aligned}&(L_g^\gamma y)(t_2) - (L_g^\gamma y)(t_1)\\&= \frac{1}{\Gamma(\gamma)}\left[\int_0^{v_1}\left(\frac{y(\tau)}{(g(t_2)-g(\tau))^{1-\gamma}} - \frac{y(\tau)}{(g(t_1)-g(\tau))^{1-\gamma}}\right) g'(\tau)\, d\tau + \int_{t_1}^{t_2} \frac{y(\tau)}{(g(t_2)-g(\tau))^{1-\gamma}} g'(\tau)\, d\tau\right].\end{aligned}$$

The second integral is nonnegative since the integrand is nonnegative. For the first integral, since $y(t)$ is nondecreasing and $g(t_2) > g(t_1)$, we have

$$(g(t_2)-g(\tau))^{1-\gamma} < (g(t_1)-g(\tau))^{1-\gamma},$$

which implies the difference inside the integral is nonnegative. Therefore, this expression is nonnegative and thus $(L_g^\gamma y)(t)$ is nondecreasing almost everywhere on $[0, d]$. □

Proposition 3.1 *[36] Let P is a Young function.For any $t \in \mathbb{R}^+$ and $\gamma \in (0,1)$, then*

$$P(\tau) = \left\{\eta > 0 : \frac{1}{||g'||_p}\int_0^{g(t)\zeta^{\frac{1}{1-\gamma}}} P(u^{\gamma-1}) du \le \zeta^{\frac{1}{1-\gamma}}\right\} \ne \emptyset,\ \zeta = \frac{\eta}{||g'||_p}$$

is a continuous function that increasing with $P(0) = 0$, where Definition 3.1 defines the function g.

Lemma 3.2 *Suppose that, P and Q are complementary Q-functions and that Q_1 is a Q-function,where p verifies that $P(u^{\gamma-1})du < \infty, \gamma \in (0,1)$. Then the operator $K_g^\gamma : L_Q(\mathcal{I}) \to L_{Q_1}(\mathcal{I})$ is continuous, where*

$$k(t) = \frac{\zeta^{\frac{1}{1-\gamma}}}{\|g'\|_p}\int_0^{g(t)\zeta^{\frac{1}{1-\gamma}}} P(u^{\gamma-1})\, du \in D_{Q_1}(\mathbb{I}), \quad \eta > 0, \quad \zeta = \frac{\eta}{\|g'\|_p}.$$

Proof: Suppose that

$$K(t,\tau) = \begin{cases}\frac{(g(t)-g(\tau))^{\gamma-1}}{\Gamma(\gamma)} g'(\tau) & \text{if } \tau \in [0,t],\ t > 0,\\ 0 & \text{otherwise.}\end{cases}$$

Now, $y \in L_Q(\mathbb{I})$ and from Hölder inequality, we get;

$$\begin{aligned}&\left|L_g^\gamma y(t)\right| = \left|\int_0^\infty K(t,\tau) y(\tau)\, d\tau\right|\\&\le 2\|K(t,\cdot)\|_p \|y\|_Q\\&\le \frac{2}{\Gamma(\gamma)} \inf_{\eta>0}\left\{\int_{\mathbb{I}} \mathcal{P}\left(\frac{(g(t)-g(\tau))^{\gamma-1}}{\eta} g'(\tau)\right) d\tau \le 1\right\}\|y\|_Q\\&\le \frac{2}{\Gamma(\gamma)} \inf_{\eta>0}\left\{\int_{\mathbb{I}} \mathcal{P}\left(\frac{(g(t)-g(\tau))^{\gamma-1}}{\eta\cdot\|g'\|_p} g'(\tau)\|g'\|_p\right) d\tau \le 1\right\}\|y\|_Q\\&= \frac{2}{\Gamma(\gamma)} \inf_{\eta>0}\left\{\int_{\mathbb{I}} \mathcal{P}\left(\frac{(g(t)-g(\tau))^{\gamma-1}}{\eta}\cdot\|g'\|_p\right) \frac{g'(\tau)}{\|g'\|_p} d\tau \le 1\right\}\|y\|_Q.\end{aligned}$$

Putting $u = (g(t)-g(\tau))\zeta^{\frac{1}{1-\gamma}}$, where $\zeta = \frac{\eta}{\|g'\|_p}$, and by we get

$$\begin{aligned}\|L_g^\gamma y\|_{Q_1} &\le \frac{2}{\Gamma(\gamma)} \inf_{\epsilon>0}\left\{\frac{1}{\|g'\|_p}\int_0^{g(t)\zeta^{\frac{1}{1-\gamma}}} \mathcal{P}(u^{\gamma-1})\, du \le \zeta^{\frac{1}{1-\gamma}}\right\}\|y\|_Q\\&\le \frac{2}{\Gamma(\gamma)}\|K\|_{Q_1}\|y\|_Q.\end{aligned}$$

Then, by applying Proposition 3.1 and [22, Lemma 16.3], we get $I_\beta^\gamma : L_Q(\mathcal{I}) \to L_{q_2}(\mathcal{I})$ is continuous. □

3.2. New results

Now from equation (1.1) we have

$$y = A(y) = f(y) + W(y),$$

where

$$W(y) = Gc_1(y).T(y), T(y)(t,s) = L_g^{\gamma} Gc_2(y)(t,s)$$

s.t. L_β^γ is given Definition 3.1 and $G_{C_i}(y), i = 1,2,3$ are superposition operators.
Now, we shall establish an existance theorem in L_ϕ spaces, specifically in the most significant case where the associated $N-$ functions satisfy the Δ_2 condition (cf. [2,12,31]). These enable us to apply a few general conditions to the functions under study.

Theorem 3.1 *Suppose, that P and Q are complementary Q-functions and Q_1, Q_2, Q_3 are Q-functions. Assume further that Q, Q_1, Q_2 satisfy Δ_2 condition and $\int_0^{g(t)} P(u^{\gamma-1})\, du < \infty$, $\gamma \in (0,1)$ and the following conditions hold*

(F1) There exist a constant $k_1 > 0$ such that for each $v \in L_{Q_2}(\mathbb{I})$ and $z \in L_{Q_3}(\mathbb{I})$ we get $\|vz\|_{Q_1} \leq k_1 \|v\|_{Q_2} \|z\|_{Q_3}$.

(B1) $f \in D_{Q_1}(\mathbb{I})$ is almost everywhere nondecreasing on $\mathbb{I}$.

(B2) $(s,t,y) \to c_i(s,t,y)$ are nondecreasing and $c_i : \mathbb{I} \times \mathbb{R} \to \mathbb{R},\ i = 1,2$ satisfy Carathéodory conditions.

(B3) There exists a costant $e_i \geq 0,\ i = 1,2$ and the functions $a_1 \in D_{Q_2}(\mathbb{I})$, and $a_2 \in D_Q(\mathbb{I})$, such that

$$|c_1(s,t,y)| \leq a_1(s,t) + e_1 Q_2^{-1}(Q(y)), \quad |c_2(s,t,y)| \leq a_2(s,t) + e_2 Q^{-1}(Q_1(y)).$$

(B4) Suppose that for almost everywhere $t \in \mathbb{I}$, there exists a constant $\eta > 0$, such that

$$k(t) = \frac{\zeta^{\frac{1}{\gamma}-1}}{\|g'\|^p} \int_0^{\zeta^{\frac{1}{\gamma}} g(t)} P(u^{\gamma-1})\, du \in D_{Q_1}(\mathbb{I}), \quad \zeta = \frac{\eta}{\|g'\|^p}.$$

(B5) Suppose that $q > 0$ on $I_0 = [0, d_0] \subset \mathbb{I}$ satisfying

$$\int_{I_0} Q_1 \left(|f(t,y(t))| + \frac{2k_1}{\Gamma(\gamma)} \|k\|_{Q_3} \left(\|a_1\|_{Q_2} + e_1 q \right) \left(\|a_2\|_Q + e_2 q \right) \right) ds \leq q$$

and

$$\left[\frac{2e_1 k_1}{\Gamma(\gamma)} \|k\|_{Q_3} \left(\|a_2\|_Q + e_2 \cdot q \right) \right] < 1.$$

Hence, on $I_0 \subset \mathbb{I}$, there is almost everywhere nondecreasing solution $y \in D_{Q_1}(I_0)$ of (1.1).

Proof 3.1 ***Step I:*** *Firstly, by Lemma 2.1 and supposition (B2), (B3), implies that $G_{c_1} : D_{Q_1}(\mathbb{I}) \to L_{Q_2}(\mathbb{I})$, $G_{c_2} : D_{Q_1}(\mathbb{I}) \to L_Q(\mathbb{I})$ are continuous and from Lemma 3.2 we get $T = L_g^\gamma G_{c_2} : D_{Q_1}(\mathbb{I}) \to D_{Q_3}(\mathbb{I})$ is continuous. Also supposition (F1) imply that $W : D_{Q_1}(\mathbb{I}) \to D_{Q_1}(\mathbb{I})$, and by supposition (B1), the operator $A : D_{Q_1}(\mathbb{I}) \to D_{Q_1}(\mathbb{I})$ is continuous.*
Step II: *Next, we aim to verify and prove that the operator A remains bounded in $D_{Q_1}(\mathbb{I})$.*
Let Ω represent the closure of the set $\{y \in D_{Q_1}(\mathbb{I}) : \int_0^{d_0} Q_1(|y(\tau)|)\, d\tau \leq q - 1\}$. Therefore, Ω is not necessarily a ball in $D_{Q_1}(\mathbb{I})$, but $\Omega \subset B_q(D_{Q_1}(\mathbb{I}))$ [22] and Ω is a closed, convex and bounded subset of $D_{Q_1}(I_0)$.
Now applying [22, Theorem 10.5], for constant $k = 1$, each $y \in \Omega$ and $(s,t) \in I_0$, we have

$$\begin{aligned} \left\| Q_2^{-1}(Q_1(|y|)) \right\|_{Q_1} &\leq \|y\|_{Q_1} = 1 + \int_{I_0} Q_1(y(\tau))\, d\tau \\ \text{and} \quad \left\| Q^{-1}(Q_1(y)) \right\|_Q &\leq \|y\|_{Q_1} = 1 + \int_{I_0} Q_1(y(\tau)\, d\tau. \end{aligned} \tag{3.2}$$

Thus, using Lemma 3.2 and our suppositions, we have

$$\begin{aligned}
&|A(y)(t,y(t))| \\
&\leq |f(t,y(t))| + |W(y)(t,y(t))| \\
&\leq |f(t,y(t))| + k_1\|G_{c_1}(y)\|_{Q_2}\|T(y)\|_{Q_3} \\
&\leq |f(t,y(t))| + k_1\left\|a_1 + e_1 Q_2^{-1}(Q_1(|y|))\right\|_{Q_2}\cdot\left\|L_g^{\gamma}Gc_2(y)\right\|_{Q_3} \\
&\leq |f(t,y(t))| + k_1\left(\|a_1\|_{Q_2} + e_1\left\|Q_2^{-1}(Q_1(|y|))\right\|_{Q_2}\right)\frac{2}{\Gamma(\gamma)}\|k\|_{Q_3}\left(\|a_2\|_Q + e_2\left\|Q^{-1}(Q_1(|y|))\right\|_Q\right) \\
&\leq |f(t,y(t))| + \frac{2k_1}{\Gamma(\gamma)}\|k\|_{Q_3}\left(\|a_1\|_{Q_2} + e_1 + e_1\int_{I_0} Q_1(y(\tau))d\tau\right)\left(\|a_2\|_Q + e_2 + e_2\int_{I_0} Q_1(y(\tau))d\tau\right) \\
&\leq |ft,y(t))| + \frac{2k_1}{\Gamma(\gamma)}\|k\|_{Q_3}\left(\|a_1\|_{Q_2} + e_1 + e_1(q-1)\right)\left(\|a_2\|_Q + e_2 + e_2(q-1)\right).
\end{aligned}$$

By using supposition (B5), we have

$$\int_{I_0} Q_1(A(y)(t,y(t)))\,ds \leq \int_{I_0}\psi\left[|f(t,y(t))| + \frac{2k_1}{\Gamma(\gamma)}\|k\|_{Q_3}\left(\|a_1\|_{Q_2} + e_1 q\right)\left(\|a_2\|_Q + e_2 q\right)\right]ds \leq q,$$

then $A(\overline{\Omega}) \subset \overline{A(\Omega)} \subset \overline{\Omega} = \Omega$ *and* $A(\Omega) \subset \Omega$*, Hence, the operator* $A : \Omega \to \Omega$ *is continuous on* $\Omega \subset B_q(D_{Q_1}(I_0))$.
Step (III)*: Consider* $\Omega_q \subset \Omega$ *that consist of all functions that are almost everywhere nondecreasing on* I_0*. The set is bounded,nonempty,convex and compact in measure in* $L_{Q_1}(\mathcal{I})$*. Also it is closed set in* $L_{Q_1}(\mathcal{I})$ *(cf. [12]).*
Step (IV)*: Again, we will demonstrate that the operator* A *maintains the monotonicity of functions. Taking* $y \in \Omega_r$*, we get* y *is almost everywhere nondecreasing on* I_0*. Moreover, the operator* $G_{c_i}(y), i = 1,2$ *are almost everywhere nondecreasing on* I_0 *and from Lemma 3.1 the operator* A *is the same property [24], then the operator* $W(y) = G_{c_1}(y)T(y)$ *is almost everywhere nondecreasing on* I_0 *and by above supposition (B1) it follows that* $A : \Omega_q \to \Omega_q$ *is continuous.*
Step (V)*: Here. we verify that the operator* A *fulfills the contraction condition relative to* $\alpha(X)$
Assume that a set $F \subset I_0$ *where* $meas F \leq \eta, \eta > 0$*. Then for* $y \in X$ *and* $\phi \neq X \subset Q_q$ *we get*

$$\begin{aligned}
\|A(y)\cdot\chi_F\|_{Q_1} &\leq \|y\cdot\chi_F\|_{Q_1} + \|G_{c_1}(y)T(y)\cdot\chi_F\|_{Q_1} \\
&\leq \|y\cdot\chi_F\|_{Q_1} + k_1\|G_{c_1}(y)\cdot\chi_F\|_{Q_2}\|T(y)\cdot\chi_F\|_{Q_3} \\
&\leq \|y\cdot\chi_F\|_{Q_1} + k_1\left\|(a_1 + e_1 Q_2^{-1}(Q_1(|y|))\cdot\chi_F\right\|\cdot\left\|L_g^{\gamma}G_{c_2}(y)\right\|_{Q_3} \\
&\leq \|y\cdot\chi_F\|_{Q_1} + k_1\left(\|a_1\cdot\chi_F\|_{Q_2} + e_1\left\|Q_2^{-1}(Q_1(|y|))\cdot\chi_F\right\|_{Q_2}\right) \\
&\quad\times\frac{2}{\Gamma(\gamma)}\|k\|_{Q_3}\left(\|a_2\|_Q + e_2\left\|Q^{-1}(Q_1(|y|))\right\|_Q\right) \\
&\leq \|y\cdot\chi_F\|_{Q_1} + \frac{2k_1}{\Gamma(\gamma)}\|k\|_{Q_3}\left(\|a_1\cdot\chi_F\|_{Q_2} + e_1\|y\cdot\chi_F\|_{Q_1}\right)\left(\|a_2\|_Q + e_2\cdot q\right).
\end{aligned}$$

Then, for $y \in D_{Q_1}$ *and* $a_1 \in D_{Q_2}$ *we get*

$$\lim_{\eta\to 0}\left\{\sup_{meas\ F\leq\eta}\left[\sup_{y\in X}\|y\cdot\chi_F\|_{Q_1}\right]\right\} = 0,$$

$$\text{and}\quad \lim_{\eta\to 0}\left\{\sup_{meas\ F\leq\eta}\left[\sup_{y\in X}\|b_1\cdot\chi_F\|_{Q_2}\right]\right\} = 0.$$

Now applying formula of $h(X)$*, we obtain*

$$h(A(X)) \leq \left(\frac{2e_1 k_1}{\Gamma(\gamma)}\|k\|_{Q_3}\left(\|a_2\|_Q + e_2\cdot q\right)\right)c(X).$$

Now using Lemma 2.4 based on the previously defined properties, we have

$$\alpha(A(X)) \leq \left(\frac{2e_1k_1}{\Gamma(\gamma)}\|k\|_{Q_3}\left(\|a_2\|_Q + e_2 \cdot q\right)\right)\alpha(X).$$

Since $\left(\frac{2d_1k_1}{\Gamma(\alpha)}\|k\|_{\psi_2}\left[\|b_2\|_Q + d_2 \cdot r\right]\right) < 1$ *Then, by Theorem 2.1, the proof of the theorem is completed.*

3.3. Uniqueness of the solution

From the above suppositions, we found the existence of the solution of the equation (1.1). Now, for the uniqueness of the solution,we replace the inequalities of supposition of (B3).

Theorem 3.2 *If the inequality of (B3) of the Theorem 3.1 substitute by the following inequality*

(B6) $|c_i(t,s,0)| \leq a_i(t,s),\ a_1 \in D_{Q_2}(\mathbb{I}),\ a_2 \in D_Q(\mathbb{I})$ *and*

$|c_1(t,s,y) - c_1(t,s,z)| \leq e_1 Q_2^{-1}\left(\varphi(|y-z|)\right),$

$|c_2(t,s,y) - c_2(t,s,z)| \leq e_2 Q^{-1}\left(\varphi(|y-z|)\right), \quad y,z \in \Omega_q,$

where $e_i \geq 0$, *and* Ω_q *is as in Theorem 3.1 for* $i = 1,2$.

(B7) Suppose that

$$\frac{2k_1\|k\|_{Q_3}}{\Gamma(\gamma)}\left(e_2\left(\|a_1\|_{Q_2} + e_1 \cdot q\right) + e_1\left(\|a_2\|_{Q_3} + e_2 \cdot q\right)\right) < 1, \tag{3.3}$$

where q *is given in the supposition (B5). Then (1.1) has a unique solution* $y \in D_{Q_1}$ *in* Ω_q.

Proof 3.2 *Applying supposition (B6), we get*

$$\Big||c_1(t,s,y) - c_1(t,s,0)|\Big| \leq |c_1(t,s,y) - c_1(t,s,0)| \leq e_1 Q_2^{-1}\left(\varphi(y)\right)$$

$$\Rightarrow \quad |c_1(t,s,y)| \leq |c_1(t,s,0)| + e_1 Q_2^{-1}\left(\varphi(y)\right) \leq a_1(t,s) + e_1 Q_2^{-1}\left(\varphi(y)\right).$$

Similarly,

$$|h_2(t,s,y)| \leq b_2(t,s) + d_2 Q^{-1}\left(\varphi(y)\right).$$

Therefore, according to Theorem 3.1, there is a nondecreasing solution $y \in D_{Q_1}$ *of Equation (1.1) in* Ω_q. *Now, suppose that* $y, z \in \Omega_q$ *are two different solutions of equation (1.1), then by applying the inequalities of (4) and the supposition (B6), we get*

$$\begin{aligned}
&\|y-z\|_{Q_1} \\
&\leq \|G_{c_1}(y)T(y) - G_{c_1}(z)T(z)\|_{Q_1} \\
&\leq \|G_{c_1}(y)T(y) - G_{c_1}(y)T(z)\|_{Q_1} + \|G_{c_1}(y)T(z) - G_{c_1}(z)T(z)\|_{Q_1} \\
&\leq k_1\|G_{c_1}(y)\|_{Q_2}\|T(y) - T(z)\|_{Q_3} + k_1\|G_{c_1}(y) - G_{c_1}(z)\|_{Q_2}\|T(z)\|_{Q_3} \\
&\leq k_1\left\|(a_1 + e_1Q_2^{-1}\left(Q_1(y)\right)\right\|_{Q_2}\left\|L_g^\gamma\left(G_{c_2}(y) - G_{c_2}\right)\right\|_{Q_3} + k_1Q_2^{-1}\left(Q_1(|y-z|)\right)\left\|L_g^\gamma G_{c_2}(z)\right\|_{\psi_1} \\
&\leq k_1\left(\|a_1\|_{Q_2} + e_1 \cdot r\right)\frac{2\|k\|_{Q_3}}{\Gamma(\gamma)}\|G_{c_2}(y) - G_{c_2}(z)\|_Q + e_1k_1\|y-z\|_{Q_1}\frac{2\|k\|_{Q_3}}{\Gamma(\gamma)}\|G_{c_2}(z)\|_Q \\
&\leq \frac{2k_1\|k\|_{Q_3}}{\Gamma(\gamma)}\left(\|a_1\|_{Q_2} + e_1 \cdot q\right)\|e_2Q^{-1}\left(Q_1(|y-z|)\right)\|_Q \\
&+ \frac{2e_1k_1\|k\|_{Q_3}}{\Gamma(\gamma)}\|y-z\|_{Q_1}\|a_2 + e_2Q^{-1}(Q_1(|z|))\|_Q \\
&\leq \frac{2k_1e_2\|k\|_{Q_3}}{\Gamma(\gamma)}\left(\|a_1\|_{Q_2} + e_1 \cdot q\right)\|y-z\|_{Q_1} + \frac{2e_1k_1\|k\|_{Q_3}}{\Gamma(\gamma)}\|y-z\|_{Q_1}\left(\|a_2\|_{Q_3} + e_2 \cdot q\right) \\
&= \frac{2k_1\|k\|_{Q_3}}{\Gamma(\gamma)}\left(e_2\left(\|a_1\|_{Q_2} + e_1 \cdot q\right) + e_1\left(\|a_2\|_{Q_3} + e_2 \cdot q\right)\right)\|y-z\|_{Q_1}.
\end{aligned}$$

Hence, along with the supposition (B7), this completes the proof.

We give the following examples to demonstrate our results.

Example 3.1 *Let Q-functions* $P(u) = Q(u) = \frac{u^2}{2}$ *and* $Q_1(u) = u^2 + 5u + 3$. *Now, we can established that the operator* $L_g^{\gamma} : L_Q(\mathcal{I}) \to L_{Q_1}(\mathcal{I})$ *is continuous. Also Lemma 3.2 is satisfied.*
For any $\gamma \in (0,1)$ *and* $t \in [1,e]$ *we get*

$$k(t) = \int_0^t P(u^{\gamma-1})du = \int_0^t (\frac{u^2}{2})^{\gamma-1}du = 2^{1-\gamma}\left(\frac{t^{2\gamma-1}}{2\gamma-1}\right)$$

which suggest that Proposition 3.1 has accomplished. Moreover,

$$\int_0^1 Q_1(k(t))du = \int_0^1 \left[\left(2^{1-\gamma}\frac{t^{2\gamma-1}}{2\gamma-1}\right)^2 + 5\left(2^{1-\gamma}\frac{t^{2\gamma-1}}{2\gamma-1}\right) + 3\right] dt < \infty.$$

Then for $y \in L_Q(\mathcal{I})$, *we have* $L_g^{\gamma} : L_Q(\mathcal{I}) \to L_{Q_1}(\mathcal{I})$ *is continuous, which satisfies the Lemma 3.2.*

Example 3.2 *Let* $f(t, y(t)) = f(t), c_1(t, s, y(t)) = 1, g(t) = 1$ *and* $c_2(\tau, y(\tau)) = H(\tau, y(\tau))$ *Then we get*

$$y(t) = f(t) + \frac{1}{\Gamma(\gamma)}\int_0^t (t-\tau)^{\gamma-1} H(\tau, y(\tau))\, d\tau$$

which illustrates a specific case of equation (1.1) and it is a classical form of Volterra–Hammerstein fractional integral equation.

Example 3.3 *Consider the nonlinear integral equation of Volterra-Hammesrstein type*

$$y(t) = \frac{1}{5}\sin t\, y(t) + \frac{\ln(1+t^2)}{\Gamma\big(\frac{1}{2}\big)}\int_0^t \frac{(1+\cos\tau)\,[y(\tau)]^2}{(t^2-\tau^2)^{1/2}}\, d\tau. \tag{3.4}$$

Comparing equation (1.1) with equation (3.4), we have

$$f(t, y(t)) = \frac{1}{5}\sin t\, y(t)$$

$$c_1(t, s, y(t)) = \ln(1+t^2)$$

Here independent of s *and of the inner variable* τ; *it depends only on* t *and* $y(t)$ *through* t

$$\gamma = \tfrac{1}{2}, \quad \textit{so } \Gamma(\gamma) = \Gamma\big(\tfrac{1}{2}\big) = \sqrt{\pi}$$

$$g(t) = t$$

$$c_2(\tau, y(\tau)) = (1+\cos\tau)\,[y(\tau)]^2$$

and

$$\|k\| = \sup_{t\in[0,T]} |\ln(1+t^2)| = \ln(1+T^2)$$

$$k_1 = \sup_{\tau\in[0,T]} 1 + \cos\tau \le 2 \quad \Rightarrow \quad k_1 \le 2$$

also

$$Q_1 = Q_2 = 1, \|a_1\| = 0.5, \|a_2\| = 10$$

$$e_1 = 0.1, e_2 = 0.2, q = 5$$

Since,

$$\frac{2k_1\|k\|_{Q_3}}{\Gamma(\gamma)} \le \frac{2\cdot 2\cdot \ln(1+T^2)}{\sqrt{\pi}} = \frac{4\ln(1+T^2)}{1.772}$$

and

$$e_2\big(\|a_1\|_{Q_2}+e_1q\big)+e_1\big(\|a_2\|_{Q_3}+e_2q\big)=0.2(0.5+0.1\times 5)+0.1(10+0.2\times 5)=1.3$$

So the inequality (3.3) is equivalence

$$\frac{4\ln(1+T^2)}{1.772}\left(0.2(0.5+0.1\times 5)+0.1(10+0.2\times 5)\right)<1$$

Clearly, the above inequality is true for the assumption (B7).Moreover, the remaining conditions of Theorem 3.2 can be easily checked to hold.Thus,by using Theorem 3.2 ensures that equation (3.4) has a unique solution of $y\in D_{Q_1}(I_0)$ and $I_0\in[0,d]$.

Acknowledgment: The research of the first author (Pallav Bora) is supported by SERB, DST, New Delhi, India, under the application reference number DST/INSPIRE/03/2019/002532 .

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