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Complex Valued Extended b-Metric Space and its Fixed Points

Monika Sihag[1 *] and Nawneet Hooda[2]

ABSTRACT: This work focuses on establishing a set of common fixed point theorems in complex valued extended b-metric spaces, formulated under rational contraction conditions. The results obtained not only extend the classical theorems of Azam et al. [1], Bhatt et al. [3], Bryant [4], and Rouzkard and Imdad [10], but also provide a broader framework for their application. Furthermore, several corollaries are derived, and illustrative examples are included to showcase the practical relevance of the theorems and the improvements they offer over earlier results.

Key Words: Complex valued metric space, contractive condition, unique common fixed point.

Contents

1. Introduction

The concept of a metric space was formally introduced by the French mathematician M. Fréchet in 1906. Since then, metric spaces have become fundamental in the development of various mathematical disciplines, particularly Functional Analysis. Inspired by their importance, numerous generalizations of metric spaces have emerged. The concept of a b-metric traces its origins to the pioneering work of Bakhtin [2]. Subsequently, Czerwik [5] introduced an axiom that weakens the traditional triangle inequality and formally defined the b-metric space, aiming to extend the Banach contraction mapping theorem. This foundational work laid the groundwork for broadening metric space theory and its applications. Kamran et al. [6] introduced the concept of extended b-metric space. In general, a b-metric is not a continuous functional and thus so is an extended b-metric. Azam et al. [1] proposed and studied complex-valued metric spaces, establishing several fixed point theorems based on rational contractive conditions. This novel framework holds promise for applications in complex-valued normed and inner product spaces, thereby opening new research directions.

While complex-valued metric spaces can be regarded as a special subclass of cone metric spaces as noted by Huang and Zhang [7], offering distinct advantages in dealing with rational expressions that may lack meaningful interpretation in the cone metric context. This distinction arises because cone metric spaces rely on Banach spaces, which are not division rings, whereas complex valued metric spaces accommodate division operations, allowing for more nuanced analysis.

In 2013, Rao et al. [9] introduced the concept of complex-valued b-metric spaces. Since then, considerable work has focused on the existence and uniqueness of common fixed points for self-mappings under various contractive conditions within this setting. In 2019, N. Ullah et al. [11] generalized this notion further by introducing complex-valued extended b-metric spaces.

The primary aim of this paper is to establish common fixed point results for two self-maps satisfying rational inequalities within the framework of complex-valued extended b-metric spaces.

To proceed, we first recall some notations and definitions that will be used throughout this paper. Let $\mathbb{C}$ denote the set of complex numbers, and let $z_1, z_2 \in \mathbb{C}$.

* Corresponding author.
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Define a partial order $\preceq$ on $\mathbb{C}$ as follows:

$$z_1 \precsim z_2 \text{ if and only if } \Re(z_1) \leq \Re(z_2), \quad \Im(z_1) \leq \Im(z_2).$$

Consequently, one can infer that $z_1 \precsim z_2$ if one of the following conditions is satisfied:

(*i*) $\Re(z_1) = \Re(z_2), \quad \Im(z_1) < \Im(z_2)$,

(*ii*) $\Re(z_1) < \Re(z_2), \quad \Im(z_1) = \Im(z_2)$,

(*iii*) $\Re(z_1) < \Re(z_2), \quad \Im(z_1) < \Im(z_2)$,

(*iv*) $\Re(z_1) = \Re(z_2), \quad \Im(z_1) = \Im(z_2)$.

In particular, we write $z_1 \lnsim z_2$ if $z_1 \neq z_2$ and one of (i), (ii), and(iii) is satisfied and we write $z_1 \prec z_2$ if only (iii) is satisfied.

Notice that $0 \precsim z_1 \lnsim z_2 \implies |z_1| < |z_2|$, and $z_1 \precsim z_2,\ z_2 \prec z_3 \implies z_1 \prec z_3$.

The following definitions are introduced by Azam et al. [1].

Definition 1.1. *[1] Let X be a non empty set and $\mathbb{C}$ be the set of complex numbers. Suppose that the mapping $d : X \times X \to \mathbb{C}$ satisfies the following conditions:*

(d_1) $0 \precsim d(\hbar, \vartheta)$, *for all* $\hbar, \vartheta \in X$ *and* $d(\hbar, \vartheta) = 0$ *if and only if* $\hbar = \vartheta$;

(d_2) $d(\hbar, \vartheta) = d(\vartheta, \hbar)$, *for all* $\hbar, \vartheta \in X$;

(d_3) $d(\hbar, \vartheta) \precsim d(\hbar, z) + d(z, \vartheta)$, *for all* $\hbar, \vartheta, z \in X$.

Then d is called a complex valued metric on X, and (X, d) is called a complex valued metric space.

Rao et al. [9] generalized the complex valued metric space and introduced complex valued b-metric space as follows:

Definition 1.2. *[9] Let X be a non empty set and let $s \geq 1$ be a real number. Suppose that the mapping $d : X \times X \to \mathbb{C}$ satisfies the following conditions:*

(d_1) $0 \precsim d(\hbar, \vartheta)$, *for all* $\hbar, \vartheta \in X$ *and* $d(\hbar, \vartheta) = 0$ *if and only if* $\hbar = \vartheta$;

(d_2) $d(\hbar, \vartheta) = d(\vartheta, \hbar)$, *for all* $\hbar, \vartheta \in X$;

(d_3) $d(\hbar, \vartheta) \precsim s\,[d(\hbar, z) + d(z, \vartheta)]$, *for all* $\hbar, \vartheta, z \in X$.

Then d is called a complex valued b-metric on X and (X, d) is called a complex valued b-metric space.

Ullah et al. [11] further generalized complex valued b-metric space to complex valued extended b-metric space as follows:

Definition 1.3. *[11] Let X be a non empty set and let $\phi : X \times X \to [1, \infty)$. Let the mapping $d : X \times X \to \mathbb{C}$ satisfy the following conditions:*

(d_1) $0 \precsim d(\hbar, \vartheta)$, *for all* $\hbar, \vartheta \in X$ *and* $d(\hbar, \vartheta) = 0$ *if and only if* $\hbar = \vartheta$;

(d_2) $d(\hbar, \vartheta) = d(\vartheta, \hbar)$, *for all* $\hbar, \vartheta \in X$;

(d_3) $d(\hbar, \vartheta) \precsim \phi(\hbar, \vartheta)\,[d(\hbar, z) + d(z, \vartheta)]$, *for all* $\hbar, \vartheta, z \in X$.

Then d is called a complex valued extended b-metric on X, and (X, d) is called a complex valued extended b-metric space.

Definition 1.4. *[11] Let (X, d) be a complex valued extended b-metric space and $\{\hbar_n\}$ be a sequence in X and $\hbar \in X$. We say that* *(i)* *the sequence* $\{\hbar_n\}$ *converges to* $\hbar$ *if for every* $c \in \mathbb{C}$, *with* $0 \prec c$ *there is* $n_0 \in N$ *such that for all* $n > n_0$, $d(\hbar_n, \hbar) \prec c$. *We denote this by* $\lim_{n\to\infty} \hbar_n = \hbar$, *or* $\hbar_n \to \hbar$, *as* $n \to \infty$,

(ii) *the sequence* $\{\hbar_n\}$ *is Cauchy sequence if for every* $c \in \mathbb{C}$ *with* $0 \prec c$ *there is* $n_0 \in \mathbb{N}$ *such that for all* $n > n_0$, $d(\hbar_n, \hbar_{n+m}) \prec c$,

(iii) *the space* (X, d) *is a complete complex valued extended b-metric space if every Cauchy sequence is convergent.*

Lemma 1.5. *[11] Let* (X, d) *be a complex valued extended b-metric space and let* $\{\hbar_n\}$ *be a sequence in* X. *Then* $\{\hbar_n\}$ *converges to* x *if and only if* $|d(\hbar_n, \hbar)| \to 0$ *as* $n \to \infty$.

Lemma 1.6. *[11] Let* (X, d) *be a complex valued extended b-metric space and let* $\{\hbar_n\}$ *be a sequence in* X. *Then* $\{\hbar_n\}$ *is a Cauchy sequence if and only if* $|d(\hbar_n, \hbar_m)| \to 0$ *as* $n, m \to \infty$.

2. Main Results

Theorem 2.1. *If* $\mathcal{Q}$ *and* $\mathcal{P}$ *are self-mappings with* $\mathcal{Q}(X) \subseteq \mathcal{P}(X)$, *defined on a complete complex valued extended b-metric space* (X, d) *and satisfying the condition*

$$\begin{aligned} d(\mathcal{P}\mathcal{Q}\hbar, \mathcal{Q}\mathcal{P}\vartheta) \precsim{}& \lambda d(\mathcal{Q}\hbar, \mathcal{P}\vartheta) + \mu \frac{d(\mathcal{Q}\hbar, \mathcal{P}\mathcal{Q}\hbar) d(\mathcal{P}\vartheta, \mathcal{Q}\mathcal{P}\vartheta)}{1 + d(\mathcal{Q}\hbar, \mathcal{P}\vartheta)} \\ &+ \gamma \frac{d(\mathcal{P}\vartheta, \mathcal{P}\mathcal{Q}\hbar) d(\mathcal{Q}\hbar, \mathcal{Q}\mathcal{P}\vartheta)}{1 + d(\mathcal{Q}\hbar, \mathcal{P}\vartheta)} \end{aligned} \tag{2.1}$$

for all $\hbar, \vartheta \in X$ *and* λ, μ, γ *are nonnegative reals with* $\lambda + \mu + \gamma < 1$, $\eta(1 - \mu) = \lambda$ *where* $\eta \in (0, 1)$ *be such that for each* $\hbar_0 \in X$, $\lim_{n,m\to\infty} \phi(\hbar_n, \hbar_m) < \frac{1}{\eta}$, *here* $\hbar_n = \mathcal{Q}^n \hbar_0, n = 1, 2, \ldots$ *Then* $\mathcal{P}, \mathcal{Q}, \mathcal{P}\mathcal{Q}$ *and* $\mathcal{Q}\mathcal{P}$ *have a unique common fixed point.*

Proof. Let $\hbar_0$ be an arbitrary point in X and define $\hbar_{2k+1} = \mathcal{Q}\hbar_{2k}$, $\hbar_{2k+2} = \mathcal{P}\hbar_{2k+1}$, $k = 0, 1, 2, \ldots$ Then

$$\begin{aligned} d(\hbar_{2k+2}, \hbar_{2k+3}) ={}& d(\mathcal{P}\mathcal{Q}\hbar_{2k}, \mathcal{Q}\mathcal{P}\hbar_{2k+1}) \\ \precsim{}& \lambda d(\mathcal{Q}\hbar_{2k}, \mathcal{P}\hbar_{2k+1}) \\ &+ \mu \frac{d(\mathcal{Q}\hbar_{2k}, \mathcal{P}\mathcal{Q}\hbar_{2k}) d(\mathcal{P}\hbar_{2k+1}, \mathcal{Q}\mathcal{P}\hbar_{2k+1})}{1 + d(\mathcal{Q}\hbar_{2k}, \mathcal{P}\hbar_{2k+1})} \\ &+ \gamma \frac{d(\mathcal{P}\hbar_{2k+1}, \mathcal{P}\mathcal{Q}\hbar_{2k}) d(\mathcal{Q}\hbar_{2k}, \mathcal{Q}\mathcal{P}\hbar_{2k+1})}{1 + d(\mathcal{Q}\hbar_{2k}, \mathcal{P}\hbar_{2k+1})} \end{aligned}$$

and so

$$\begin{aligned} |d(\hbar_{2k+2}, \hbar_{2k+3})| \leq{}& \lambda |d(\hbar_{2k+1}, \hbar_{2k+2})| \\ &+ \mu \frac{|d(\hbar_{2k+1}, \hbar_{2k+2})||d(\hbar_{2k+2}, \hbar_{2k+3})|}{|1 + d(\hbar_{2k+1}, \hbar_{2k+2})|} \\ &+ \gamma \frac{|d(\hbar_{2k+2}, \hbar_{2k+2})||d(\hbar_{2k+1}, \hbar_{2k+3})|}{|1 + d(\hbar_{2k+1}, \hbar_{2k+2})|} \\ \leq{}& \lambda |d(\hbar_{2k+1}, \hbar_{2k+2})| + \mu |d(\hbar_{2k+2}, \hbar_{2k+3})| \end{aligned}$$

which gives

$$|d(\hbar_{2k+2}, \hbar_{2k+3})| \leq \frac{\lambda}{1 - \mu} |d(\hbar_{2k+1}, \hbar_{2k+2})|. \tag{2.2}$$

Further,

$$\begin{aligned} d(\hbar_{2k+3},\hbar_{2k+4}) = d(\hbar_{2k+4},\hbar_{2k+3}) &= d(\mathcal{P}\mathcal{Q}\hbar_{2k+2},\mathcal{Q}\mathcal{P}\hbar_{2k+1}) \\ &\precsim \lambda d(\mathcal{Q}\hbar_{2k+2},\mathcal{P}\hbar_{2k+1}) \\ &\quad + \mu\frac{d(\mathcal{Q}\hbar_{2k+2},\mathcal{P}\mathcal{Q}\hbar_{2k+2})d(\mathcal{P}\hbar_{2k+1},\mathcal{Q}\mathcal{P}\hbar_{2k+1})}{1+d(\mathcal{Q}\hbar_{2k+2},\mathcal{P}\hbar_{2k+1})} \\ &\quad + \gamma\frac{d(\mathcal{P}\hbar_{2k+1},\mathcal{P}\mathcal{Q}\hbar_{2k+2})d(\mathcal{Q}\hbar_{2k+2},\mathcal{Q}\mathcal{P}\hbar_{2k+1})}{1+d(\mathcal{Q}\hbar_{2k+2},\mathcal{P}\hbar_{2k+1})} \end{aligned}$$

and so

$$\begin{aligned} |d(\hbar_{2k+3},\hbar_{2k+4})| &\le \lambda|d(\hbar_{2k+3},\hbar_{2k+2})| \\ &\quad + \mu\frac{|d(\hbar_{2k+3},\hbar_{2k+4})||d(\hbar_{2k+2},\hbar_{2k+3})|}{|1+d(\hbar_{2k+3},\hbar_{2k+2})|} \\ &\quad + \gamma\frac{|d(\hbar_{2k+2},\hbar_{2k+4})||d(\hbar_{2k+3},\hbar_{2k+3})|}{|1+d(\hbar_{2k+3},\hbar_{2k+2})|} \\ &\le \lambda|d(\hbar_{2k+2},\hbar_{2k+3})| + \mu|d(\hbar_{2k+3},\hbar_{2k+4})| \end{aligned}$$

which gives

$$|d(\hbar_{2k+3},\hbar_{2k+4})| \le \frac{\lambda}{1-\mu}|d(\hbar_{2k+2},\hbar_{2k+3})|.$$

Putting $\eta = \frac{\lambda}{1-\mu}$, we get (for all n)

$$|d(\hbar_n,\hbar_{n+1})| \le \eta|d(\hbar_{n-1},\hbar_n)| \le \eta^2|d(\hbar_{n-2},\hbar_{n-1}) \le \cdots \le \eta^n|d(\hbar_0,\hbar_1)|.$$

Therefore, for any $m > n$, we have

$$\begin{aligned} d(\hbar_n,\hbar_m) &\precsim \phi(\hbar_n,\hbar_m)\,[d(\hbar_n,\hbar_{n+1}) + d(\hbar_{n+1},\hbar_m)] \\ &\precsim \phi(\hbar_n,\hbar_m)\,\eta^n\,d(\hbar_0,\hbar_1) + \phi(h_n,h_m)d(\hbar_{n+1},\hbar_m) \\ &\precsim \phi(\hbar_n,\hbar_m)\,\eta^n\,d(\hbar_0,\hbar_1) \\ &\quad + \phi(\hbar_n,\hbar_m)\phi(\hbar_{n+1},\hbar_m)\Big[d(\hbar_{n+1},\hbar_{n+2}) + d(\hbar_{n+2},\hbar_m)\Big] \\ &\precsim \phi(\hbar_n,\hbar_m)\,\eta^n\,d(\hbar_0,\hbar_1) + \phi(\hbar_n,\hbar_m)\phi(\hbar_{n+1},\hbar_m)\eta^{n+1}d(\hbar_0,\hbar_1) \\ &\quad + \phi(\hbar_n,\hbar_m)\phi(\hbar_{n+1},\hbar_m)d(\hbar_{n+1},\hbar_m) \\ &\vdots \\ &\precsim \phi(\hbar_n,\hbar_m)\,\eta^n\,d(\hbar_0,\hbar_1) + \phi(\hbar_n,\hbar_m)\phi(\hbar_{n+1},\hbar_m)\eta^{n+1}d(\hbar_0,\hbar_1) \\ &\quad + \cdots + \phi(\hbar_n,\hbar_m)\phi(\hbar_{n+1},\hbar_m)\ldots\phi(\hbar_{m-1},\hbar_m)\eta^{m-1}d(\hbar_0,\hbar_1) \end{aligned}$$

which implies that

$$\begin{aligned} |d(\hbar_n,\hbar_m)| \le |d(\hbar_0,\hbar_1)|\Big[&\phi(\hbar_n,\hbar_m)\eta^n + \phi(\hbar_n,\hbar_m)\phi(\hbar_{n+1},\hbar_m)\eta^{n+1} \\ &+ \cdots + \phi(\hbar_n,\hbar_m)\phi(\hbar_{n+1},\hbar_m)\ldots\phi(\hbar_{m-1},\hbar_m)\eta^{m-1}\Big]. \end{aligned}$$

Since $\lim_{n,m\to\infty}\phi(\hbar_n,\hbar_m)\eta < 1$, so the series

$$\sum_{n=1}^{\infty}\eta^n\prod_{i=1}^{n}\phi(\hbar_i,\hbar_m)$$

converges by the ratio test for each $m \in \mathbb{N}$.

Let

$$\mathcal{S} = \sum_{n=1}^{\infty} \eta^n \prod_{i=1}^{n} \phi(\hbar_i, \hbar_m), \quad \mathcal{S}_n = \sum_{j=1}^{n} \eta^j \prod_{i=1}^{j} \phi(\hbar_i, \hbar_m).$$

Thus for $m > n$, the above inequality can be written as

$$|d(\hbar_n, \hbar_m)| \leq |d(\hbar_0, \hbar_1)||\mathcal{S}_{m-1} - \mathcal{S}_n|.$$

Now, by taking the limit as $n, m \to \infty$ we get

$$|d(\hbar_n, \hbar_m)| \to 0 \quad \text{as } n, m \to \infty.$$

In view of Lemma 1.6, the sequence $\{\hbar_n\}$ is Cauchy. Since X is complete, $\mathcal{Q}$ and $\mathcal{P}$ are continuous, so there exists some $u \in X$ such that $\hbar_n \to u$ as $n \to \infty$ and as such $\mathcal{Q}\hbar_n \to \mathcal{Q}u$, and $\mathcal{P}\mathcal{Q}\hbar_n \to \mathcal{P}\mathcal{Q}u$. This follows that $u = \mathcal{Q}u$ as

$$\begin{aligned} d(u, \mathcal{Q}u) &\precsim \phi(u, \mathcal{Q}u) \Big[d(u, \hbar_{2k+1}) + d(\hbar_{2k+1}, \mathcal{Q}u)\Big] \\ &\precsim \phi(u, \mathcal{Q}u) \Big[d(u, \hbar_{2k+1}) + d(\mathcal{Q}\hbar_{2k}, \mathcal{Q}u)\Big], \end{aligned}$$

and so

$$|d(u, \mathcal{Q}u)| \leq \phi(u, \mathcal{Q}u) \Big[|d(u, \hbar_{2k+1})| + |d(\mathcal{Q}\hbar_{2k}, \mathcal{Q}u)|\Big],$$

which on taking $k \to \infty$ yields

$$u = \mathcal{Q}u. \tag{2.3}$$

Further $u = \mathcal{P}\mathcal{Q}u$, otherwise $d(u, \mathcal{P}\mathcal{Q}u) = z > 0$ and we would then have

$$\begin{aligned} z &\precsim \phi(u, \mathcal{P}\mathcal{Q}u) \left[d(u, \hbar_{2k+3}) + d(\hbar_{2k+3}, \mathcal{P}\mathcal{Q}u)\right] \\ &\precsim \phi(u, \mathcal{P}\mathcal{Q}u) \left[d(u, \hbar_{2k+3}) + d(\mathcal{P}\mathcal{Q}u, \mathcal{Q}\mathcal{P}\hbar_{2k+1})\right] \\ &\precsim \phi(u, \mathcal{P}\mathcal{Q}u) \Bigg[d(u, \hbar_{2k+3}) + \lambda\, d(\mathcal{Q}u, \mathcal{P}\hbar_{2k+1}) \\ &\quad + \mu \frac{d(\mathcal{Q}u, \mathcal{P}\mathcal{Q}u) d(\mathcal{P}\hbar_{2k+1}, \mathcal{Q}\mathcal{P}\hbar_{2k+1})}{1 + d(\mathcal{Q}u, \mathcal{P}\hbar_{2k+1})} \\ &\quad + \gamma \frac{d(\mathcal{P}\hbar_{2k+1}, \mathcal{P}\mathcal{Q}u) d(\mathcal{Q}u, \mathcal{Q}\mathcal{P}\hbar_{2k+1})}{1 + d(\mathcal{Q}u, \mathcal{P}\hbar_{2k+1})}\Bigg] \\ &\precsim \phi(u, \mathcal{P}\mathcal{Q}u) \Bigg[d(u, \hbar_{2k+3}) + \lambda d(\mathcal{Q}u, \hbar_{2k+2}) \\ &\quad + \mu \frac{d(\mathcal{Q}u, \mathcal{P}\mathcal{Q}u) d(\hbar_{2k+2}, \hbar_{2k+3})}{1 + d(\mathcal{Q}u, \hbar_{2k+2})} \\ &\quad + \gamma \frac{d(\hbar_{2k+2}, \mathcal{P}\mathcal{Q}u) d(\mathcal{Q}u, \hbar_{2k+3})}{1 + d(\mathcal{Q}u, \hbar_{2k+2})}\Bigg], \end{aligned}$$

and so

$$\begin{aligned} |z| \leq \phi(u, \mathcal{P}\mathcal{Q}u) \Bigg[&|d(u, \hbar_{2k+3})| + \lambda |d(\mathcal{Q}u, \hbar_{2k+2})| \\ &+ \mu \frac{|d(\mathcal{Q}u, \mathcal{P}\mathcal{Q}u)||d(\hbar_{2k+2}, \hbar_{2k+3})|}{|1 + d(\mathcal{Q}u, \hbar_{2k+2})|} \\ &+ \gamma \frac{|d(\hbar_{2k+2}, \mathcal{P}\mathcal{Q}u)||d(\mathcal{Q}u, \hbar_{2k+3})|}{1 + |d(\mathcal{Q}u, \hbar_{2k+2})|}\Bigg]. \end{aligned}$$

Therefore, on taking $k \to \infty$, we have

$$\begin{aligned} |z| \leq \phi(u, \mathcal{P}\mathcal{Q}u) \Bigg[& |d(u,u)| + \lambda |d(u,u)| \\ & + \mu \frac{|d(\mathcal{Q}u, \mathcal{P}\mathcal{Q}u)||d(u,u)|}{1 + d(u,u)} \\ & + \gamma \frac{|d(u, \mathcal{P}\mathcal{Q}u)||d(u,u)|}{|1 + d(u,u)|} \Bigg]. \end{aligned}$$

That is $|z| = 0$, a contradiction and hence

$$u = \mathcal{P}\mathcal{Q}u. \tag{2.4}$$

From (2.3) and (2.4), u is the common fixed point of $\mathcal{Q}$ and $\mathcal{P}\mathcal{Q}$.
In a similar way, we have $\mathcal{P}u = \mathcal{Q}\mathcal{P}u = u$ and hence

$$\mathcal{Q}u = \mathcal{P}u = \mathcal{P}\mathcal{Q}u = \mathcal{Q}\mathcal{P}u = u. \tag{2.5}$$

Now, we shall show that $\mathcal{P}, \mathcal{Q}, \mathcal{P}\mathcal{Q}$ and $\mathcal{Q}\mathcal{P}$ have a unique common fixed point. For this let us assume that u^* in X is second common fixed point of $\mathcal{P}, \mathcal{Q}, \mathcal{P}\mathcal{Q}$ and $\mathcal{Q}\mathcal{P}$ that is $\mathcal{P}u^* = \mathcal{Q}\mathcal{P}u^* = \mathcal{P}\mathcal{Q}u^* = \mathcal{Q}u^* = u^*$.
Then

$$\begin{aligned} |d(u,u^*)| = |d(\mathcal{P}\mathcal{Q}u, \mathcal{Q}\mathcal{P}u^*)| & \leq \lambda |d(\mathcal{Q}u, \mathcal{P}u^*)| \\ & + \mu \frac{|d(\mathcal{Q}u, \mathcal{P}\mathcal{Q}u)||d(\mathcal{P}u^*, \mathcal{Q}\mathcal{P}u^*)|}{|1 + d(\mathcal{Q}u, \mathcal{P}u^*)|} \\ & + \gamma \frac{|d(\mathcal{P}u^*, \mathcal{P}\mathcal{Q}u)||d(\mathcal{Q}u, \mathcal{P}\mathcal{Q}u^*)|}{|1 + d(\mathcal{Q}u, \mathcal{P}u^*)|} \\ & = \lambda |d(u,u^*)| + \mu \frac{|d(u,u)||d(u^*,u^*)|}{|1 + d(u,u^*)|} \\ & + \gamma \frac{|d(u^*,u)||d(u,u^*)|}{|1 + d(u,u^*)|} \\ & \leq \lambda |d(u,u^*)| + \gamma |d(u,u^*)| \end{aligned}$$

which is a contradiction as $\lambda + \gamma < 1$ giving $u = u^*$. □

The following example demonstrates the validity of Theorem 2.1

Example 2.2. *Let $X = [0, \frac{1}{4}]$. Define the complex valued extended b-metric*

$$d(x,y) = |x-y|^2 + i|x-y|^2, \qquad \phi(x,y) = x + y + 2,$$

for all $x, y \in X$
Define $\mathcal{Q}(x) = x^2,\ \mathcal{P}(y) = y^2$.
For arbitrary $\hbar, \vartheta \in X$:
Left Hand Side:

$$d(\mathcal{P}\mathcal{Q}\hbar, \mathcal{Q}\mathcal{P}\vartheta) = |\hbar^4 - \vartheta^4|^2 + i|\hbar^4 - \vartheta^4|^2$$

Observe:

$$\begin{aligned} |\hbar^4 - \vartheta^4| & = |(\hbar^2 - \vartheta^2)(\hbar^2 + \vartheta^2)| \\ & \leq \frac{1}{8}|\hbar^2 - \vartheta^2| \\ & \leq \frac{1}{16}|\hbar - \vartheta| \end{aligned}$$

Square both sides:

$$(|\hbar^4 - \vartheta^4|)^2 \leq \left(\frac{1}{16}\right)^2 |\hbar - \vartheta|^2 = \frac{1}{256}|\hbar - \vartheta|^2$$

giving

$$d(\mathcal{P}\mathcal{Q}\hbar, \mathcal{Q}\mathcal{P}\vartheta) \leq \frac{1}{256}d(\hbar, \vartheta)$$

Right Hand Side:
Term A:

$$d(\mathcal{Q}\hbar, \mathcal{P}\vartheta) = |\hbar^2 - \vartheta^2|^2 + i|\hbar^2 - \vartheta^2|^2$$

Recall

$$|\hbar^2 - \vartheta^2| \leq \frac{1}{2}|\hbar - \vartheta| \implies |\hbar^2 - \vartheta^2|^2 \leq \frac{1}{4}|\hbar - \vartheta|^2$$

giving

$$d(\mathcal{Q}\hbar, \mathcal{P}\vartheta) \leq \frac{1}{4}d(\hbar, \vartheta)$$

Term B:

$$d(\mathcal{Q}\hbar, \mathcal{P}\mathcal{Q}\hbar) = |\hbar^2 - \hbar^4|^2 + i|\hbar^2 - \hbar^4|^2$$

$$|\hbar^2 - \hbar^4| = \hbar^2(1 - \hbar^2) \leq \frac{1}{16},$$

giving

$$|\hbar^2 - \hbar^4|^2 \leq \left(\frac{1}{16}\right)^2 = \frac{1}{256}.$$

Thus

$$d(\mathcal{Q}\hbar, \mathcal{P}\mathcal{Q}\hbar) \leq \frac{1}{256} + i\frac{1}{256}.$$

Term C:

$$d(\mathcal{P}\vartheta, \mathcal{Q}\mathcal{P}\vartheta) = |\vartheta^2 - \vartheta^4|^2 + i|\vartheta^2 - \vartheta^4|^2$$

By the same logic as Term B,

$$|\vartheta^2 - \vartheta^4| \leq \frac{1}{16} \implies d(\mathcal{P}\vartheta, \mathcal{Q}\mathcal{P}\vartheta) \leq \frac{1}{256} + i\frac{1}{256}.$$

Term D:

$$d(\mathcal{P}\vartheta, \mathcal{P}\mathcal{Q}\hbar) = |\vartheta^2 - \hbar^4|^2 + i|\vartheta^2 - \hbar^4|^2.$$

Since $|\vartheta^2|, |\hbar^4| \leq \frac{1}{16}$,

$$|\vartheta^2 - \hbar^4| \leq \frac{1}{16} \implies d(\mathcal{P}\vartheta, \mathcal{P}\mathcal{Q}\hbar) \leq \frac{1}{256} + i\frac{1}{256}.$$

Term E:

$$d(\mathcal{Q}\hbar, \mathcal{Q}\mathcal{P}\vartheta) = |\hbar^2 - \vartheta^4|^2 + i|\hbar^2 - \vartheta^4|^2.$$

Again $|\hbar^2|, |\vartheta^4| \leq \frac{1}{16}$

$$|\hbar^2 - \vartheta^4| \leq \frac{1}{16} \implies d(\mathcal{Q}\hbar, \mathcal{Q}\mathcal{P}\vartheta) \leq \frac{1}{256} + i\frac{1}{256}.$$

The denominator

$$1 + d(\mathcal{Q}\hbar, \mathcal{P}\vartheta) \geq 1$$

since $d(\mathcal{Q}\hbar, \mathcal{P}\vartheta) \geq 0$.

Right Side Calculation: Let $\lambda = 0.06$*,* $\mu = 0.015$*,* $\gamma = 0.015$*, so* $\lambda + \mu + \gamma = 0.09 < 1$ *and* $\eta = \frac{\lambda}{1-\mu} \approx 0.06$*.*

$$\lambda d(\mathcal{Q}\hbar, \mathcal{P}\vartheta) \le 0.06 \cdot \frac{1}{4} d(\hbar, \vartheta) = 0.015 d(\hbar, \vartheta),$$

$$\mu d(\mathcal{Q}\hbar, \mathcal{P}\mathcal{Q}\hbar) d(\mathcal{P}\vartheta, \mathcal{Q}\mathcal{P}\vartheta) \le 0.015 \cdot \frac{1}{256} \cdot \frac{1}{256} \approx 2.29 \times 10^{-6},$$

$$\gamma d(\mathcal{P}\vartheta, \mathcal{P}\mathcal{Q}\hbar) d(\mathcal{Q}\hbar, \mathcal{Q}\mathcal{P}\vartheta) \le 0.015 \cdot \frac{1}{256} \cdot \frac{1}{256} \approx 2.29 \times 10^{-6}.$$

Thus the total right side is:

$$0.015 d(\hbar, \vartheta) + 4.58 \times 10^{-6}.$$

Contractive inequality:

$$d(\mathcal{P}\mathcal{Q}\hbar, \mathcal{Q}\mathcal{P}\vartheta) \le \frac{1}{256} d(\hbar, \vartheta)$$

and the right side is $0.015 d(\hbar, \vartheta) + 4.58 \times 10^{-6}$*.*

Since $\frac{1}{256} \approx 0.0039 < 0.015$*, the inequality is satisfied for all* $\hbar, \vartheta \in X$*.*

For $x_0 \in X$*,*

$$\mathcal{Q}^n x_0 = x_0^{2^n} \to 0 \text{ as } n \to \infty$$

hence

$$\lim_{n,m\to\infty} \phi(\mathcal{Q}^n x_0, \mathcal{Q}^m x_0) = 2 < \frac{1}{\eta} = \frac{1}{0.09} \approx 16.66.$$

All terms of the contractive condition are satisfied, so by Theorem 2.1, $\mathcal{Q}$*,* $\mathcal{P}$*,* $\mathcal{Q}\mathcal{P}$*, and* $\mathcal{P}\mathcal{Q}$ *have a unique common fixed point in* X*.*

The mappings $\mathcal{Q}, \mathcal{P}, \mathcal{Q}\mathcal{P}$*, and* $\mathcal{P}\mathcal{Q}$ *clearly have 0 as a unique common fixed point.*

Remark 2.3. *By setting* $\mathcal{P}\hbar = \hbar'$ *and* $\mathcal{Q}\vartheta = \vartheta'$ *in Theorem 2.1, we generalize the corresponding result of Rouzkard and Imdad [10, Theorem 2.1] in complex valued metric space.*

Remark 2.4. *By setting* $\mathcal{P}\hbar = \hbar'$*,* $\mathcal{Q}\vartheta = \vartheta'$ *and* $\gamma = 0$ *in Theorem 2.1, we generalize the corresponding result of Azam et al. [1, Theorem 4] in complex valued metric space.*

Setting $\mathcal{Q} = \mathcal{P}$ in Theorem 2.1, we have

Theorem 2.5. *Let* (X, d) *be a complete complex valued extended b-metric space and let* $\mathcal{Q} : X \to X$ *satisfy the condition*

$$d(\mathcal{Q}^2\hbar, \mathcal{Q}^2\vartheta) \precsim \lambda d(\mathcal{Q}\hbar, \mathcal{Q}\vartheta) + \mu \frac{d(\mathcal{Q}\hbar, \mathcal{Q}^2\hbar) d(\mathcal{Q}\vartheta, \mathcal{Q}^2\vartheta)}{1 + d(\mathcal{Q}\hbar, \mathcal{Q}\vartheta)} + \frac{\gamma d(\mathcal{Q}\vartheta, \mathcal{Q}^2\hbar) d(\mathcal{Q}\hbar, \mathcal{Q}^2\vartheta)}{1 + d(\mathcal{Q}\hbar, \mathcal{Q}\vartheta)} \tag{2.6}$$

for all $\hbar, \vartheta \in X$ *and* λ, μ, γ *are nonnegative reals with* $\lambda + \mu + \gamma < 1$*,* $\eta(1-\mu) = \lambda$ *where* $\eta \in (0,1)$ *be such that for each* $\hbar_0 \in X$*,* $\lim_{n,m\to\infty} \phi(\hbar_n, \hbar_m) < \frac{1}{\eta}$*, here* $\hbar_n = \mathcal{Q}^n \hbar_0$*,* $n = 1, 2, \ldots$ *Then* $\mathcal{Q}$ *and* $\mathcal{Q}^2$ *have a unique fixed point.*

If we designate $\mathcal{Q}\hbar = \hbar'$ and $\mathcal{Q}\vartheta = \vartheta'$ in Theorem 2.5, we generalize [10, Corollary 2.3] by the following corollary:

Corollary 2.6. *Let (X, d) be a complete complex valued extended b-metric space and let $\mathcal{Q} : X \to X$ satisfy the condition*

$$d(\mathcal{Q}\hbar', \mathcal{Q}\vartheta') \precsim \lambda d(\hbar', \vartheta') + \frac{\mu d(\hbar', \mathcal{Q}\hbar')d(\vartheta', \mathcal{Q}\vartheta') + \gamma d(\vartheta', \mathcal{Q}\hbar')d(\hbar', \mathcal{Q}\vartheta')}{1 + d(\hbar', \vartheta')} \tag{2.7}$$

for all $\hbar', \vartheta' \in X$ and λ, μ, γ are nonnegative reals with $\lambda + \mu + \gamma < 1$, $\eta(1 - \mu) = \lambda$ where $\eta \in (0, 1)$ be such that for each $\hbar_0 \in X$, $\lim_{n,m\to\infty} \phi(\hbar_n, \hbar_m) < \frac{1}{\eta}$, here $\hbar_n = \mathcal{Q}^n\hbar_0$, $n = 1, 2, \ldots$ Then $\mathcal{Q}$ and $\mathcal{Q}^2$ have a unique fixed point.

Remark 2.7. *Taking $\mathcal{Q}\hbar = \hbar'$, $\mathcal{Q}\vartheta = \vartheta'$, and $\gamma = 0$ in Theorem 2.5, we generalize [1, Corollary 5]. Further, setting $\mathcal{Q}\hbar = \hbar'$, $\mathcal{Q}\vartheta = \vartheta'$ and $\gamma = \mu = 0$, Theorem 2.5 reduces to the Banach Contraction Principle in a complete complex valued extended b-metric space. Thus, Theorem 2.5 serves as a proper extension of the Banach Contraction Principle in this setting.*

Corollary 2.8. *Let (X, d) be a complete complex valued extended b-metric space and let $\mathcal{Q} : X \to X$ satisfy the condition*

$$\begin{aligned} d(\mathcal{Q}^n\hbar', \mathcal{Q}^n\vartheta') \precsim \lambda d(\hbar', \vartheta') &+ \frac{\mu d(\hbar', \mathcal{Q}^n\hbar')d(\vartheta', \mathcal{Q}^n\vartheta')}{1 + d(\hbar', \vartheta')} \\ &+ \gamma\frac{d(\vartheta', \mathcal{Q}^n\hbar')d(\hbar', \mathcal{Q}^n\vartheta')}{1 + d(\hbar', \vartheta')} \end{aligned} \tag{2.8}$$

for all $\hbar', \vartheta' \in X$ and λ, μ, γ are nonnegative reals with $\lambda + \mu + \gamma < 1$, $\eta(1 - \mu) = \lambda$ where $\eta \in (0, 1)$ be such that for each $\hbar_0 \in X$, $\lim_{n,m\to\infty} \phi(\hbar_n, \hbar_m) < \frac{1}{\eta}$, here $\hbar_n = \mathcal{Q}^n\hbar_0, n = 1, 2, \ldots$ Then $\mathcal{Q}$ and $\mathcal{Q}^2$ have a unique fixed point.

Proof. By Corollary 2.6 we obtain $v \in X$ such that

$$\mathcal{Q}^n v = v.$$

The result follows from the fact that

$$\begin{aligned} d(\mathcal{Q}v, v) = d(\mathcal{Q}\mathcal{Q}^n v, \mathcal{Q}^n v) &= d(\mathcal{Q}^n\mathcal{Q}v, \mathcal{Q}^n v) \\ &\precsim \lambda d(\mathcal{Q}v, v) + \frac{\mu d(\mathcal{Q}v, \mathcal{Q}^n\mathcal{Q}v)d(v, \mathcal{Q}^n v)}{1 + d(\mathcal{Q}^n v, v)} \\ &\quad + \gamma\frac{d(v, \mathcal{Q}^n\mathcal{Q}v)d(\mathcal{Q}v, \mathcal{Q}^n\mathcal{Q}v)}{1 + d(\mathcal{Q}^n v, v)} \\ &\precsim \lambda d(\mathcal{Q}v, v) + \frac{\mu d(\mathcal{Q}v, \mathcal{Q}v)d(v, v) + \gamma d(v, \mathcal{Q}v)d(\mathcal{Q}v, \mathcal{Q}v)}{1 + d(v, v)} \\ &= \lambda d(\mathcal{Q}v, v). \end{aligned}$$

Remark 2.9. *Our Corollary 2.8 corresponds to Rouzkard and Imdad [10, Corollary 2.7]. Moreover, setting $\mu = \gamma = 0$, it reduces to Bryant's theorem [4] in a complete complex valued extended b-metric space.*

We construct a non-trivial example of a mapping $\mathcal{Q}$ on a complete complex valued extended b-metric space (X, d) that satisfies Bryant's theorem but not the Banach contraction principle, thereby demonstrating the superiority of Bryant's theorem over the Banach contraction principle.

Example 2.10. *Let*

$$X = [0, 1], \quad d(x, y) = |x - y|(2 + i)$$

be a complete complex valued extended b-metric space. Define $\mathcal{Q}\colon X \to X$ by

$$\mathcal{Q}(x) = x^2.$$

Banach contraction does not hold: Take $x = 0.8$, $y = 0.4$,

$$|\mathcal{Q}(x) - \mathcal{Q}(y)| = |0.64 - 0.16| = 0.48,$$

$$|x - y| = 0.4, \quad \Rightarrow \quad \frac{|\mathcal{Q}(x) - \mathcal{Q}(y)|}{|x - y|} = \frac{0.48}{0.4} = 1.2 > 1.$$

No $\alpha < 1$ can satisfy

$$d(\mathcal{Q}x, \mathcal{Q}y) \preceq \alpha\, d(x, y),$$

so $\mathcal{Q}$ is not a Banach contraction.

Extended b-metric contraction holds: Consider $\mathcal{Q}^2(x) = x^4$. Then,

$$d(\mathcal{Q}^2x, \mathcal{Q}^2y) = |x^4 - y^4|(2 + i).$$

For $x, y \in [0, 1]$,

$$|x^4 - y^4| \leq |x - y|^4,$$

hence,

$$d(\mathcal{Q}^2x, \mathcal{Q}^2y) \leq |x - y|^4(2 + i) = \lambda\, d(x, y),$$

where $\lambda = |x - y|^3 < 1$ for $0 < |x - y| < 1$.
Value of ϕ in this space: The extended b-metric triangle inequality requires a function $\phi : X \times X \to [1, \infty)$ such that

$$d(x, z) \preceq \phi(x, z)\big(d(x, y) + d(y, z)\big).$$

For this example,

$$\phi(x, z) = 1$$

satisfies the inequality because d is a scaled version of the usual metric which satisfies the triangle inequality linearly.

Fixed points of $\mathcal{Q}$ satisfy $x^2 = x$ giving $x = 0$ or $x = 1$. The contraction condition on $\mathcal{Q}^2$ ensures $x = 0$ is the unique fixed point attracting iterates under the extended b-metric space conditions.

Our next result generalizes the result of Rouzkard and imdad [10, Theorem 2.11] in complete complex valued extended b-metric space.

Theorem 2.11. *Let (X, d) be a complete complex valued extended b-metric space and the mappings $\mathcal{Q}, \mathcal{P} : X \to X$ with $\mathcal{Q}(X) \subseteq \mathcal{P}(X)$, satisfy the inequality*

$$d(\mathcal{PQ}\hbar,\, \mathcal{QP}\vartheta) \precsim \begin{cases} \lambda\, d(\mathcal{Q}\hbar,\, \mathcal{P}\vartheta) \\ +\mu \left(\dfrac{d(\mathcal{Q}\hbar,\, \mathcal{PQ}\hbar)\, d(\mathcal{P}\vartheta,\, \mathcal{QP}\vartheta)}{d(\mathcal{PQ}\hbar,\, \mathcal{Q}\hbar) + d(\mathcal{QP}\vartheta,\, \mathcal{P}\vartheta)} \right. \\ \left. + \dfrac{d(\mathcal{P}\vartheta,\, \mathcal{PQ}\hbar)\, d(\mathcal{Q}\hbar,\, \mathcal{QP}\vartheta)}{d(\mathcal{PQ}\hbar,\, \mathcal{Q}\hbar) + d(\mathcal{QP}\vartheta,\, \mathcal{P}\vartheta)} \right) \\ +\gamma \left(\dfrac{d(\mathcal{Q}\hbar,\, \mathcal{PQ}\hbar)\, d(\mathcal{Q}\hbar,\, \mathcal{QP}\vartheta)}{d(\mathcal{PQ}\hbar,\, \mathcal{P}\vartheta) + d(\mathcal{QP}\vartheta,\, \mathcal{Q}\hbar)} \right. \\ \left. + \dfrac{d(\mathcal{P}\vartheta,\, \mathcal{PQ}\hbar)\, d(\mathcal{P}\vartheta,\, \mathcal{QP}\vartheta)}{d(\mathcal{PQ}\hbar,\, \mathcal{P}\vartheta) + d(\mathcal{QP}\vartheta,\, \mathcal{Q}\hbar)} \right), & \text{if } D \neq 0 \text{ and } D_1 \neq 0, \\ 0, & \text{if } D = 0 \text{ or } D_1 = 0. \end{cases} \tag{2.9}$$

For all $\hbar, \vartheta \in X$, where

$$D = d(\mathcal{PQ}\hbar,\, \mathcal{Q}\hbar) + d(\mathcal{QP}\vartheta,\, \mathcal{P}\vartheta), \quad D_1 = d(\mathcal{PQ}\hbar,\, \mathcal{P}\vartheta) + d(\mathcal{QP}\vartheta,\, \mathcal{Q}\hbar),$$

and λ, μ, γ *are nonnegative real numbers such that* $\lambda + \mu + \gamma = \eta \in (0, 1)$*, be such that for every* $\hbar_0 \in X$*,*

$$\lim_{n,m\to\infty} \phi(\hbar_n, \hbar_m) < \frac{1}{\eta},$$

here $\hbar_n = \mathcal{Q}^n \hbar_0$*,* $n = 1, 2, \ldots$ *Then* $\mathcal{Q}, \mathcal{P}, \mathcal{QP}$ *and* $\mathcal{PQ}$ *have a unique common fixed point.*

Proof. Let $\hbar_0$ be an arbitrary point in X and define $\hbar_{2k+1} = \mathcal{Q}\hbar_{2k}$, $\hbar_{2k+2} = \mathcal{P}\hbar_{2k+1}$, $k = 0, 1, 2, \ldots$ Then

$$\begin{aligned}
d(\hbar_{2k+2}, \hbar_{2k+3}) =& d(\mathcal{PQ}\hbar_{2k}, \mathcal{QP}\hbar_{2k+1}) \\
&\precsim \lambda d(\mathcal{Q}\hbar_{2k}, \mathcal{P}\hbar_{2k+1}) \\
&\quad + \mu \left\{ \frac{d(\mathcal{Q}\hbar_{2k}, \mathcal{PQ}\hbar_{2k}) d(\mathcal{P}\hbar_{2k+1}, \mathcal{QP}\hbar_{2k+1})}{d(\mathcal{PQ}\hbar_{2k}, \mathcal{Q}\hbar_{2k}) + d(\mathcal{QP}\hbar_{2k+1}, \mathcal{P}\hbar_{2k+1})} \right. \\
&\qquad \left. + \frac{d(\mathcal{Q}\hbar_{2k}, \mathcal{QP}\hbar_{2k+1}) d(\mathcal{P}\hbar_{2k+1}, \mathcal{PQ}\hbar_{2k})}{d(\mathcal{PQ}\hbar_{2k}, \mathcal{Q}\hbar_{2k}) + d(\mathcal{QP}\hbar_{2k+1}, \mathcal{P}\hbar_{2k+1})} \right\} \\
&\quad + \gamma \left\{ \frac{d(\mathcal{Q}\hbar_{2k}, \mathcal{PQ}\hbar_{2k}) d(\mathcal{Q}\hbar_{2K}, \mathcal{QP}\hbar_{2k+1})}{d(\mathcal{PQ}\hbar_{2k}, \mathcal{P}\hbar_{2k+1}) + d(\mathcal{QP}\hbar_{2k+1}, \mathcal{Q}\hbar_{2k})} \right. \\
&\qquad \left. + \frac{d(\mathcal{P}\hbar_{2k+1}, \mathcal{PQ}\hbar_{2k}) d(\mathcal{P}\hbar_{2k+1}, \mathcal{QP}\hbar_{2k+1})}{d(\mathcal{PQ}\hbar_{2k}, \mathcal{P}\hbar_{2k+1}) + d(\mathcal{QP}\hbar_{2k+1}, \mathcal{Q}\hbar_{2k})} \right\}, \\
&\precsim \lambda d(\hbar_{2k+1}, \hbar_{2k+2}) \\
&\quad + \mu \left\{ \frac{d(\hbar_{2k+1}, \hbar_{2k+2}) d(\hbar_{2k+2}, \hbar_{2k+3})}{d(\hbar_{2k+2}, \hbar_{2k+1}) + d(\hbar_{2k+3}, \hbar_{2k+2})} \right. \\
&\qquad \left. + \frac{d(\hbar_{2k+1}, \hbar_{2k+3}) d(\hbar_{2k+2}, \hbar_{2k+2})}{d(\hbar_{2k+2}, \hbar_{2k+1}) + d(\hbar_{2k+3}, \hbar_{2k+2})} \right\} \\
&\quad + \gamma \left\{ \frac{d(\hbar_{2k+1}, \hbar_{2k+2}) d(\hbar_{2k+1}, \hbar_{2k+3})}{d(\hbar_{2k+2}, \hbar_{2k+2}) + d(\hbar_{2k+3}, \hbar_{2k+1})} \right. \\
&\qquad \left. + \frac{d(\hbar_{2k+2}, \hbar_{2k+2}) d(\hbar_{2k+2}, \hbar_{2k+3})}{d(\hbar_{2k+2}, \hbar_{2k+2}) + d(\hbar_{2k+3}, \hbar_{2k+1})} \right\} \\
&\precsim \lambda d(\hbar_{2k+1}, \hbar_{2k+2}) \\
&\quad + \mu \frac{d(\hbar_{2k+1}, \hbar_{2k+2}) d(\hbar_{2k+2}, \hbar_{2k+3})}{d(\hbar_{2k+2}, \hbar_{2k+1}) + d(\hbar_{2k+3}, \hbar_{2k+2})} \\
&\quad + \gamma \frac{d(\hbar_{2k+1}, \hbar_{2k+2}) d(\hbar_{2k+1}, \hbar_{2k+3})}{d(\hbar_{2k+3}, \hbar_{2k+1})}
\end{aligned}$$

and so

$$\begin{aligned}
|d(\hbar_{2k+2}, \hbar_{2k+3})| \leq& \lambda |d(\hbar_{2k+1}, \hbar_{2k+2})| \\
&+ \mu \frac{|d(\hbar_{2k+1}, \hbar_{2k+2})||d(\hbar_{2k+2}, \hbar_{2k+3})|}{|d(\hbar_{2k+2}, \hbar_{2k+1})| + |d(\hbar_{2k+3}, \hbar_{2k+2})|} \\
&+ \gamma \frac{|d(\hbar_{2k+1}, \hbar_{2k+2})||d(\hbar_{2k+1}, \hbar_{2k+3})|}{|d(\hbar_{2k+3}, \hbar_{2k+1})|} \\
&\leq \lambda |d(\hbar_{2k+1}, \hbar_{2k+2})| + \mu |d(\hbar_{2k+1}, \hbar_{2k+2})| + \gamma |d(\hbar_{2k+1}, \hbar_{2k+2})|
\end{aligned}$$

which gives

$$|d(\hbar_{2k+2}, \hbar_{2k+3})| \leq (\lambda + \mu + \gamma) \, |d(\hbar_{2k+1}, \hbar_{2k+2})|. \tag{2.10}$$

Further,

$$\begin{aligned}
d(\hbar_{2k+3},\hbar_{2k+4}) = d(\hbar_{2k+4},\hbar_{2k+3}) &= d(\mathcal{P}\mathcal{Q}\hbar_{2k+2},\mathcal{Q}\mathcal{P}\hbar_{2k+1})\\
&\precsim \lambda d(\mathcal{Q}\hbar_{2k+2},\mathcal{P}\hbar_{2k+1})\\
&\quad + \mu\Bigg\{\frac{d(\mathcal{Q}\hbar_{2k+2},\mathcal{P}\mathcal{Q}\hbar_{2k+2})d(\mathcal{P}\hbar_{2k+1},\mathcal{Q}\mathcal{P}\hbar_{2k+1})}{d(\mathcal{P}\mathcal{Q}\hbar_{2k+2},\mathcal{Q}\hbar_{2k+2})+d(\mathcal{Q}\mathcal{P}\hbar_{2k+1},\mathcal{P}\hbar_{2k+1})}\\
&\quad + \frac{d(\mathcal{P}\hbar_{2k+1},\mathcal{P}\mathcal{Q}\hbar_{2k+2})d(\mathcal{Q}\hbar_{2k+2},\mathcal{Q}\mathcal{P}\hbar_{2k+1})}{d(\mathcal{P}\mathcal{Q}\hbar_{2k+2},\mathcal{Q}\hbar_{2k+2})+d(\mathcal{Q}\mathcal{P}\hbar_{2k+1},\mathcal{P}\hbar_{2k+1})}\Bigg\}\\
&\quad + \gamma\Bigg\{\frac{d(\mathcal{Q}\hbar_{2k+2},\mathcal{P}\mathcal{Q}\hbar_{2k+2})d(\mathcal{Q}\hbar_{2k+2},\mathcal{Q}\mathcal{P}\hbar_{2k+1})}{d(\mathcal{P}\mathcal{Q}\hbar_{2k+2},\mathcal{P}\hbar_{2k+1})+d(\mathcal{Q}\mathcal{P}\hbar_{2k+1},\mathcal{Q}\hbar_{2k+2})}\\
&\quad + \frac{d(\mathcal{P}\hbar_{2k+1},\mathcal{P}\mathcal{Q}\hbar_{2k+2})d(\mathcal{P}\hbar_{2k+1},\mathcal{Q}\mathcal{P}\hbar_{2k+1})}{d(\mathcal{P}\mathcal{Q}\hbar_{2k+2},\mathcal{P}\hbar_{2k+1})+d(\mathcal{Q}\mathcal{P}\hbar_{2k+1},T\mathcal{Q}\hbar_{2k+2})}\Bigg\}\\
&\precsim \lambda d(\hbar_{2k+3},\hbar_{2k+2})\\
&\quad + \mu\Bigg\{\frac{d(\hbar_{2k+3},\hbar_{2k+4})d(\hbar_{2k+2},\hbar_{2k+3})}{d(\hbar_{2k+4},\hbar_{2k+3})+d(\hbar_{2k+3},\hbar_{2k+2})}\\
&\quad + \frac{d(\hbar_{2k+2},\hbar_{2k+4})d(\hbar_{2k+3},\hbar_{2k+3})}{d(\hbar_{2k+4},\hbar_{2k+3})+d(\hbar_{2k+3},\hbar_{2k+2})}\Bigg\}\\
&\quad + \gamma\Bigg\{\frac{d(\hbar_{2k+3},\hbar_{2k+4})d(\hbar_{2k+3},\hbar_{2k+3})}{d(\hbar_{2k+4},\hbar_{2k+2})+d(\hbar_{2k+3},\hbar_{2k+3})}\\
&\quad + \frac{d(\hbar_{2k+2},\hbar_{2k+4})d(\hbar_{2k+2},\hbar_{2k+3})}{d(\hbar_{2k+4},\hbar_{2k+2})+d(\hbar_{2k+3},\hbar_{2k+3})}\Bigg\}\\
&\precsim \lambda d(\hbar_{2k+3},\hbar_{2k+2})\\
&\quad + \mu\Bigg\{\frac{d(\hbar_{2k+3},\hbar_{2k+4})d(\hbar_{2k+2},\hbar_{2k+3})}{d(\hbar_{2k+4},\hbar_{2k+3})+d(\hbar_{2k+3},\hbar_{2k+2})}\Bigg\}\\
&\quad + \gamma\Bigg\{\frac{d(\hbar_{2k+2},\hbar_{2k+4})d(\hbar_{2k+2},\hbar_{2k+3})}{d(\hbar_{2k+4},\hbar_{2k+2})}\Bigg\}
\end{aligned}$$

and so

$$\begin{aligned}
|d(\hbar_{2k+3},\hbar_{2k+4})| &\le \lambda|d(\hbar_{2k+3},\hbar_{2k+2})|\\
&\quad + \mu\frac{|d(\hbar_{2k+3},\hbar_{2k+4})||d(\hbar_{2k+2},\hbar_{2k+3})|}{|d(\hbar_{2k+4},\hbar_{2k+3})|+|d(\hbar_{2k+3},\hbar_{2k+2})|}\\
&\quad + \gamma\frac{|d(\hbar_{2k+2},\hbar_{2k+4})||d(\hbar_{2k+2},\hbar_{2k+3})|}{|d(\hbar_{2k+4},\hbar_{2k+2})|}\\
&\le \lambda|d(\hbar_{2k+3},\hbar_{2k+2})| + \mu|d(\hbar_{2k+2},\hbar_{2k+3})| + \gamma|d(\hbar_{2k+2},\hbar_{2k+3})|
\end{aligned}$$

which gives

$$|d(\hbar_{2k+3},\hbar_{2k+4})| \le (\lambda+\mu+\gamma)\ |d(\hbar_{2k+2},\hbar_{2k+3})|.$$

Putting $\eta = \lambda+\mu+\gamma$, we get (for all n)

$$|d(\hbar_n,\hbar_{n+1})| \le \eta|d(\hbar_{n-1},\hbar_n)| \le \eta^2|d(\hbar_{n-2}|,\hbar_{n-1}) \le ... \le \eta^n|d(\hbar_0,\hbar_1)|.$$

Therefore, for any $m > n$, we have

$$\begin{aligned}
d(\hbar_n, \hbar_m) &\precsim \phi(\hbar_n, \hbar_m)\,[d(\hbar_n, \hbar_{n+1}) + d(\hbar_{n+1}, \hbar_m)] \\
&\precsim \phi(\hbar_n, \hbar_m)\,\eta^n\, d(\hbar_0, \hbar_1) + \phi(\hbar_n, \hbar_m) d(\hbar_{n+1}, \hbar_m) \\
&\precsim \phi(\hbar_n, \hbar_m)\,\eta^n\, d(\hbar_0, \hbar_1) \\
&\quad + \phi(\hbar_n, \hbar_m)\phi(\hbar_{n+1}, \hbar_m)\Big[d(\hbar_{n+1}, \hbar_{n+2}) + d(\hbar_{n+2}, \hbar_m)\Big] \\
&\precsim \phi(\hbar_n, \hbar_m)\,\eta^n\, d(\hbar_0, \hbar_1) + \phi(\hbar_n, \hbar_m)\phi(\hbar_{n+1}, \hbar_m)\eta^{n+1} d(\hbar_0, \hbar_1) \\
&\quad + \phi(\hbar_n, \hbar_m)\phi(\hbar_{n+1}, \hbar_m) d(\hbar_{n+1}, \hbar_m) \\
&\vdots \\
&\precsim \phi(\hbar_n, \hbar_m)\,\eta^n\, d(\hbar_0, \hbar_1) + \phi(\hbar_n, \hbar_m)\phi(\hbar_{n+1}, \hbar_m)\eta^{n+1} d(\hbar_0, \hbar_1) \\
&+ \cdots + \phi(\hbar_n, \hbar_m)\phi(\hbar_{n+1}, \hbar_m)\ldots\phi(\hbar_{m-1}, \hbar_m)\eta^{m-1} d(\hbar_0, \hbar_1)
\end{aligned}$$

which implies that

$$\begin{aligned}
|d(\hbar_n, \hbar_m)| \le |d(\hbar_0, \hbar_1)| \Bigg[&\phi(\hbar_n, \hbar_m)\eta^n + \phi(\hbar_n, \hbar_m)\phi(\hbar_{n+1}, \hbar_m)\eta^{n+1} \\
&+ \cdots + \phi(\hbar_n, \hbar_m)\phi(\hbar_{n+1}, \hbar_m)\ldots\phi(\hbar_{m-1}, \hbar_m)\eta^{m-1}\Bigg].
\end{aligned}$$

Since $\lim_{n,m\to\infty} \phi(\hbar_n, \hbar_m)\eta < 1$, so the series

$$\sum_{n=1}^{\infty} \eta^n \prod_{i=1}^{n} \phi(\hbar_i, \hbar_m)$$

converges by the ratio test for each $m \in \mathbb{N}$.

Let

$$\mathcal{S} = \sum_{n=1}^{\infty} \eta^n \prod_{i=1}^{n} \phi(\hbar_i, \hbar_m), \quad \mathcal{S}_n = \sum_{j=1}^{n} \eta^j \prod_{i=1}^{j} \phi(\hbar_i, \hbar_m).$$

Thus for $m > n$, the above inequality can be written as

$$|d(\hbar_n, \hbar_m)| \le |d(\hbar_0, \hbar_1)||\mathcal{S}_{m-1} - \mathcal{S}_n|.$$

Now, by taking the limit as $n, m \to \infty$ we get

$$|d(\hbar_n, \hbar_m)| \to 0 \quad \text{as } n, m \to \infty.$$

In view of Lemma 1.6, the sequence $\{\hbar_n\}$ is Cauchy. Since X is complete, there exists some $u \in X$ such that $\hbar_n \to u$ as $n \to \infty$ and as such $\mathcal{Q}\hbar_n \to \mathcal{Q}u$, and $\mathcal{P}\mathcal{Q}\hbar_n \to \mathcal{P}\mathcal{Q}u$. This follows that $u = \mathcal{Q}u$ as

$$\begin{aligned}
d(u, \mathcal{Q}u) &\precsim \phi(u, \mathcal{Q}u)\Big[d(u, \hbar_{2k+1}) + d(\hbar_{2k+1}, \mathcal{Q}u)\Big] \\
&\precsim \phi(u, \mathcal{Q}u)\Big[d(u, \hbar_{2k+1}) + d(\mathcal{Q}\hbar_{2k}, \mathcal{Q}u)\Big],
\end{aligned}$$

and so

$$|d(u, \mathcal{Q}u)| \le \phi(u, \mathcal{Q}u)\Big[|d(u, \hbar_{2k+1})| + |d(\mathcal{Q}\hbar_{2k}, \mathcal{Q}u)|\Big],$$

which on taking $k \to \infty$ yields

$$u = \mathcal{Q}u. \tag{2.11}$$

Further $u = \mathcal{P}\mathcal{Q}u$, otherwise $d(u, \mathcal{P}\mathcal{Q}u) = z > 0$ and we would then have

$$\begin{aligned}
z = d(u, \mathcal{P}\mathcal{Q}u) &\precsim \phi(u, \mathcal{P}\mathcal{Q}u)\Big[d(u, \hbar_{2k+3}) + d(\hbar_{2k+3}, \mathcal{P}\mathcal{Q}u)\Big] \\
&\precsim \phi(u, \mathcal{P}\mathcal{Q}u)\Big[d(u, \hbar_{2k+3}) + d(\mathcal{Q}\mathcal{P}\hbar_{2k+1}, \mathcal{P}\mathcal{Q}u)\Big] \\
&\precsim \phi(u, \mathcal{P}\mathcal{Q}u)\Bigg[d(u, \hbar_{2k+3}) + \lambda d(\mathcal{Q}u, \mathcal{P}\hbar_{2k+1}) \\
&\quad + \mu\bigg(\frac{d(\mathcal{Q}u, \mathcal{P}\mathcal{Q}u)d(\mathcal{P}\hbar_{2k+1}, \mathcal{Q}\mathcal{P}\hbar_{2k+1})}{d(\mathcal{P}\mathcal{Q}u, \mathcal{Q}u) + d(\mathcal{Q}\mathcal{P}\hbar_{2k+1}, \mathcal{P}\hbar_{2k+1})} \\
&\quad + \frac{d(\mathcal{P}\hbar_{2k+1}, \mathcal{P}\mathcal{Q}u)d(\mathcal{Q}u, \mathcal{Q}\mathcal{P}\hbar_{2k+1})}{d(\mathcal{P}\mathcal{Q}u, \mathcal{Q}u) + d(\mathcal{Q}\mathcal{P}\hbar_{2k+1}, \mathcal{P}\hbar_{2k+1})}\bigg) \\
&\quad + \gamma\bigg(\frac{d(\mathcal{Q}u, \mathcal{P}\mathcal{Q}u)d(\mathcal{Q}u, \mathcal{Q}\mathcal{P}\hbar_{2k+1})}{d(\mathcal{P}\mathcal{Q}u, \mathcal{P}\hbar_{2k+1}) + d(\mathcal{Q}\mathcal{P}\hbar_{2k+1}, \mathcal{Q}u)} \\
&\quad + \frac{d(\mathcal{P}\hbar_{2k+1}, \mathcal{P}\mathcal{Q}u)d(\mathcal{P}\hbar_{2k+1}, \mathcal{Q}\mathcal{P}\hbar_{2k+1})}{d(\mathcal{P}\mathcal{Q}u, \mathcal{P}\hbar_{2k+1}) + d(\mathcal{Q}\mathcal{P}\hbar_{2k+1}, \mathcal{Q}u)}\bigg)\Bigg] \\
&= \phi(u, \mathcal{P}\mathcal{Q}u)\Bigg[d(u, \hbar_{2k+3}) + \lambda d(u, \hbar_{2k+2}) \\
&\quad + \mu\bigg(\frac{d(u, \mathcal{P}\mathcal{Q}u)d(\hbar_{2k+2}, \hbar_{2k+3})}{d(\mathcal{P}\mathcal{Q}u, u) + d(\hbar_{2k+3}, \hbar_{2k+2})} \\
&\quad + \frac{d(\hbar_{2k+2}, \mathcal{P}\mathcal{Q}u)d(u, \hbar_{2k+3})}{d(\mathcal{P}\mathcal{Q}u, u) + d(\hbar_{2k+3}, \hbar_{2k+2})}\bigg) \\
&\quad + \gamma\bigg(\frac{d(u, \mathcal{P}\mathcal{Q}u)d(u, \hbar_{2k+3})}{d(\mathcal{P}\mathcal{Q}u, \hbar_{2k+2}) + d(\hbar_{2k+3}, u)} \\
&\quad + \frac{d(\hbar_{2k+2}, \mathcal{P}\mathcal{Q}u)d(\hbar_{2k+2}, \hbar_{2k+3})}{d(\mathcal{P}\mathcal{Q}u, \hbar_{2k+2}) + d(\hbar_{2k+3}, u)}\bigg)\Bigg],
\end{aligned}$$

and so

$$\begin{aligned}
|z| \leq \phi(u, \mathcal{P}\mathcal{Q}u)\Bigg[&|d(u, \hbar_{2k+3})| + \lambda|d(u, \hbar_{2k+2})| \\
&+ \mu\bigg(\frac{|d(u, \mathcal{P}\mathcal{Q}u)||d(\hbar_{2k+2}, \hbar_{2k+3})|}{|d(\mathcal{P}\mathcal{Q}u, u)| + |d(\hbar_{2k+3}, \hbar_{2k+2})|} \\
&+ \frac{|d(\hbar_{2k+2}, \mathcal{P}\mathcal{Q}u)||d(u, \hbar_{2k+3})|}{|d(\mathcal{P}\mathcal{Q}u, u)| + |d(\hbar_{2k+3}, \hbar_{2k+2})|}\bigg)
\end{aligned}$$

$$+\gamma\Bigg(\frac{|d(u,\mathcal{P}\mathcal{Q}u)||d(u,\hbar_{2k+3})|}{|d(\mathcal{P}\mathcal{Q}u,\hbar_{2k+2})|+|d(\hbar_{2k+3},u)|}$$
$$+\frac{|d(\hbar_{2k+2},\mathcal{P}\mathcal{Q}u)||d(\hbar_{2k+2},\hbar_{2k+3})|}{|d(\mathcal{P}\mathcal{Q}u,\hbar_{2k+2})|+|d(\hbar_{2k+3},u)|}\Bigg)\Bigg]$$
$$=\phi(u,\mathcal{P}\mathcal{Q}u)\Bigg[|d(u,\hbar_{2k+3})|+\lambda|d(u,\hbar_{2k+2})|$$
$$+\mu\Bigg(\frac{|z||d(\hbar_{2k+2},\hbar_{2k+3})|}{|z|+|d(\hbar_{2k+3},\hbar_{2k+2})|}$$
$$+\frac{|d(\hbar_{2k+2},\mathcal{P}\mathcal{Q}u)||d(u,\hbar_{2k+3})|}{|z|+|d(\hbar_{2k+3},\hbar_{2k+2})|}\Bigg)$$
$$+\gamma\Bigg(\frac{|z||d(u,\hbar_{2k+3})|}{|d(\mathcal{P}\mathcal{Q}u,\hbar_{2k+2})|+|d(\hbar_{2k+3},u)|}$$
$$+\frac{|d(\hbar_{2k+2},\mathcal{P}\mathcal{Q}u)||d(\hbar_{2k+2},\hbar_{2k+3})|}{|d(\mathcal{P}\mathcal{Q}u,\hbar_{2k+2})|+|d(\hbar_{2k+3},u)|}\Bigg)\Bigg].$$

Therefore, on taking $k\to\infty$, we have

$$|z|\le\phi(u,\mathcal{P}\mathcal{Q}u)\Bigg[|d(u,u)|+\lambda|d(u,u)|+\mu\frac{|z||d(u,u)|+|z||d(u,u)|}{|z|+d(u,u)}$$
$$+\gamma\frac{|z||d(u,u)|+|z||d(u,u)|}{|z|+|d(u,u)|}\Bigg].$$

That is $|z|=0$, a contradiction and hence

$$u=\mathcal{P}\mathcal{Q}u. \tag{2.12}$$

From (2.11) and (2.12), u is the common fixed point of $\mathcal{Q}$ and $\mathcal{P}\mathcal{Q}$.
In a similar way, we have $\mathcal{P}u=\mathcal{Q}\mathcal{P}u=u$ and hence

$$\mathcal{Q}u=\mathcal{P}u=\mathcal{P}\mathcal{Q}u=\mathcal{Q}\mathcal{P}u=u. \tag{2.13}$$

Now, we shall show that $\mathcal{Q},\mathcal{P},\mathcal{Q}\mathcal{P}$ and $\mathcal{P}\mathcal{Q}$ have a unique common fixed point. For this let us assume that u^* in X is second common fixed point of $\mathcal{Q},\mathcal{P},\mathcal{Q}\mathcal{P}$ and $\mathcal{P}\mathcal{Q}$ that is $\mathcal{P}u^*=\mathcal{Q}\mathcal{P}u^*=\mathcal{P}\mathcal{Q}u^*=\mathcal{Q}u^*=u^*$.

Since $d(\mathcal{P}\mathcal{Q}\hbar,\mathcal{Q}\hbar)+d(\mathcal{Q}\mathcal{P}\vartheta,\mathcal{P}\vartheta)=d(\mathcal{P}\mathcal{Q}u,\mathcal{Q}u)+d(\mathcal{Q}\mathcal{P}u^*,\mathcal{P}u^*)=0$ implies $d(\mathcal{P}\mathcal{Q}u,\mathcal{Q}u)=d(\mathcal{Q}\mathcal{P}u^*,\mathcal{P}u^*)=0$, therefore by definition of contraction condition $d(u,u^*)=d(\mathcal{P}\mathcal{Q}u,\mathcal{P}\mathcal{Q}u^*)=0$, which proves the uniqueness of common fixed point. □

Remark 2.12. *Taking $\mathcal{Q}\hbar=\hbar'$ and $\mathcal{Q}\vartheta=\vartheta'$ in Theorem 2.11, we generalize the result of Rouzkard and Imdad [10, Theorem 2.11].*

Setting $\mathcal{Q}=\mathcal{P}$ in Theorem 2.11, we have

Corollary 2.13. *Let (X,d) be a complete complex valued extended b-metric space and the mapping*

$\mathcal{Q} : X \to X$ satisfies the inequality

$$d(\mathcal{Q}^2\hbar, \mathcal{Q}^2\vartheta) \preccurlyeq \begin{cases} \lambda d(\mathcal{Q}\hbar, \mathcal{Q}\vartheta) \\ +\mu \frac{d(\mathcal{Q}\hbar,\mathcal{Q}^2\hbar)d(\mathcal{Q}\vartheta,\mathcal{Q}^2\vartheta)+d(\mathcal{Q}\vartheta,\mathcal{Q}^2\hbar)d(\mathcal{Q}\hbar,\mathcal{Q}^2\vartheta)}{d(\mathcal{Q}^2\hbar,\mathcal{Q}\hbar)+d(\mathcal{Q}^2\vartheta,\mathcal{Q}\vartheta)} \\ +\gamma \frac{d(\mathcal{Q}\hbar,\mathcal{Q}^2\hbar)d(\mathcal{Q}\hbar,\mathcal{Q}^2\vartheta)+d(\mathcal{Q}\vartheta,\mathcal{Q}^2\hbar)d(\mathcal{Q}\vartheta,\mathcal{Q}^2\vartheta)}{d(\mathcal{Q}^2\hbar,\mathcal{Q}^2\vartheta)+d(\mathcal{Q}^2\vartheta,\mathcal{Q}\hbar)}, & if D \neq 0, \quad D_1 \neq 0 \\ 0, & if\ D = 0 \ \ or \ \ D_1 = 0 \end{cases} \tag{2.14}$$

for all $\hbar, \vartheta \in X$ where $D = d(\mathcal{Q}^2\hbar, \mathcal{Q}\hbar) + d(\mathcal{Q}^2\vartheta, \mathcal{Q}\vartheta)$ and $D_1 = d(\mathcal{Q}^2\hbar, \mathcal{Q}\vartheta) + d(\mathcal{Q}^2\vartheta, T\hbar)$ and λ, μ, γ are nonnegative reals with $\lambda + \mu + \gamma = \eta \in (0, 1)$ be such that for each $\hbar_0 \in X$, $\lim_{n,m\to\infty} \phi(\hbar_n, \hbar_m) < \frac{1}{\eta}$, here $\hbar_n = \mathcal{Q}^n\hbar_0, n = 1, 2, \ldots$. Then $\mathcal{Q}$ and $\mathcal{Q}^2$ have a unique common fixed point.

Theorem 2.14. *Let (X, d) be a complete complex valued extended b-metric space. Let the mappings $\mathcal{P}, \mathcal{Q} : X \to X$ with $\mathcal{Q}(X) \subseteq \mathcal{P}(X)$ and satisfy*

$$d(\mathcal{P}\mathcal{Q}\hbar, \mathcal{Q}\mathcal{P}\vartheta) \preccurlyeq \lambda \left\{ \frac{d(\mathcal{Q}\hbar, \mathcal{P}\mathcal{Q}\hbar)\, d(\mathcal{Q}\hbar, \mathcal{Q}\mathcal{P}\vartheta) + d(\mathcal{P}\vartheta, \mathcal{Q}\mathcal{P}\vartheta)\, d(\mathcal{P}\vartheta, \mathcal{P}\mathcal{Q}\hbar)}{d(\mathcal{Q}\hbar, \mathcal{Q}\mathcal{P}\vartheta) + d(\mathcal{P}\vartheta, \mathcal{P}\mathcal{Q}\hbar)} \right\} \tag{2.15}$$

for all $\hbar, \vartheta \in X$, where $0 < \lambda < 1$ and such that $\lim_{n,m\to\infty} \phi(\hbar_n, \hbar_m) < \frac{1}{\lambda}$, here $\hbar_n = \mathcal{Q}^n\hbar_0, n = 1, 2, \ldots$. Then $\mathcal{P}, \mathcal{Q}, \mathcal{P}\mathcal{Q}$ and $\mathcal{Q}\mathcal{P}$ have a unique common fixed point.

Proof. For any arbitrary point $\hbar_0 \in X$, construct a sequence $\{\hbar_n\}$ in X such that

$$\hbar_{2k+1} = \mathcal{Q}\hbar_{2k}, \quad \hbar_{2k+2} = \mathcal{P}\hbar_{2k+1}, \quad k = 0, 1, 2, \ldots$$

From (2.15), we have

$$\begin{aligned} d(\hbar_{2k+2}, \hbar_{2k+3}) &= d(\mathcal{P}\mathcal{Q}\hbar_{2k}, \mathcal{Q}\mathcal{P}\hbar_{2k+1}) \\ &\preccurlyeq \lambda \left\{ \frac{d(\mathcal{Q}\hbar_{2k}, \mathcal{P}\mathcal{Q}\hbar_{2k})\, d(\mathcal{Q}\hbar_{2k}, \mathcal{Q}\mathcal{P}\hbar_{2k+1})}{d(\mathcal{Q}\hbar_{2k}, \mathcal{Q}\mathcal{P}\hbar_{2k+1}) + d(\mathcal{P}\hbar_{2k+1}, \mathcal{P}\mathcal{Q}\hbar_{2k})} \right. \\ &\quad \left. + \frac{d(\mathcal{P}\hbar_{2k+1}, \mathcal{Q}\mathcal{P}\hbar_{2k+1})\, d(\mathcal{P}\hbar_{2k+1}, \mathcal{P}\mathcal{Q}\hbar_{2k})}{d(\mathcal{Q}\hbar_{2k}, \mathcal{Q}\mathcal{P}\hbar_{2k+1}) + d(\mathcal{P}\hbar_{2k+1}, \mathcal{P}\mathcal{Q}\hbar_{2k})} \right\} \\ &= \lambda \left\{ \frac{d(\hbar_{2k+1}, \hbar_{2k+2})\, d(\hbar_{2k+1}, \hbar_{2k+3}) + d(\hbar_{2k+2}, \hbar_{2k+3})\, d(\hbar_{2k+2}, \hbar_{2k+2})}{d(\hbar_{2k+1}, \hbar_{2k+3}) + d(\hbar_{2k+2}, \hbar_{2k+2})} \right\} \\ &\preccurlyeq \lambda \left\{ \frac{d(\hbar_{2k+1}, \hbar_{2k+2})\, d(\hbar_{2k+1}, \hbar_{2k+3})}{d(\hbar_{2k+1}, \hbar_{2k+3})} \right\} \\ &\preccurlyeq \lambda d(\hbar_{2k+1}, \hbar_{2k+2}), \end{aligned}$$

and so,

$$|d(\hbar_{2k+2}, \hbar_{2k+3})| \leq \lambda\, |d(\hbar_{2k+1}, \hbar_{2k+2})|. \tag{2.16}$$

Again

$$\begin{aligned} d(\hbar_{2k+3},\hbar_{2k+4}) &= d(\hbar_{2k+4},\hbar_{2k+3}) = d(\mathcal{P}\mathcal{Q}\hbar_{2k+2},\mathcal{Q}\mathcal{P}\hbar_{2k+1}) \\ &\precsim \lambda\Bigg\{\frac{d(\mathcal{Q}\hbar_{2k+2},\mathcal{P}\mathcal{Q}\hbar_{2k+2})d(\mathcal{Q}\hbar_{2k+2},\mathcal{Q}\mathcal{P}\hbar_{2k+1})}{d(\mathcal{Q}\hbar_{2k+2},\mathcal{Q}\mathcal{P}\hbar_{2k+1})+d(\mathcal{P}\hbar_{2k+1},\mathcal{P}\mathcal{Q}\hbar_{2k+2})} \\ &\quad + \frac{d(\mathcal{P}\hbar_{2k+1},\mathcal{Q}\mathcal{P}\hbar_{2k+1})d(\mathcal{P}\hbar_{2k+1},\mathcal{P}\mathcal{Q}\hbar_{2k+2})}{d(\mathcal{Q}\hbar_{2k+2},\mathcal{Q}\mathcal{P}\hbar_{2k+1})+d(\mathcal{P}\hbar_{2k+1},\mathcal{P}\mathcal{Q}\hbar_{2k+2})}\Bigg\} \\ &= \lambda\Bigg\{\frac{d(\hbar_{2k+3},\hbar_{2k+4})d(\hbar_{2k+3},\hbar_{2k+3})}{d(\hbar_{2k+3},\hbar_{2k+3})+d(\hbar_{2k+2},\hbar_{2k+4})} \\ &\quad + \frac{d(\hbar_{2k+2},\hbar_{2k+3})d(\hbar_{2k+2},\hbar_{2k+4})}{d(\hbar_{2k+3},\hbar_{2k+3})+d(\hbar_{2k+2},\hbar_{2k+4})}\Bigg\} \\ &\precsim \lambda\Bigg\{\frac{d(\hbar_{2k+2},\hbar_{2k+3})d(\hbar_{2k+2},\hbar_{2k+4})}{d(\hbar_{2k+2},\hbar_{2k+4})}\Bigg\} \\ &\precsim \lambda d(\hbar_{2k+2},\hbar_{2k+3}), \end{aligned}$$

and so

$$|d(\hbar_{2k+3},\hbar_{2k+4})| \leq \lambda|d(\hbar_{2k+2},\hbar_{2k+3})|. \tag{2.17}$$

Thus, we have for all n

$$|d(\hbar_n,\hbar_{n+1})| \leq \lambda|d(\hbar_{n-1},\hbar_n)| \leq \lambda^2|d(\hbar_{n-2},\hbar_{n-1})| \leq ... \leq \lambda^n|d(\hbar_0,\hbar_1)|.$$

Therefore, for any $m > n$, we have

$$\begin{aligned} d(\hbar_n,\hbar_m) &\precsim \phi(\hbar_n,\hbar_m)\,[d(\hbar_n,\hbar_{n+1}) + \, d(\hbar_{n+1},\hbar_m)] \\ &\precsim \phi(\hbar_n,\hbar_m)\,\lambda^n\,d(\hbar_0,\hbar_1) + \phi(\hbar_n,\hbar_m)d(\hbar_{n+1},\hbar_m) \\ &\precsim \phi(\hbar_n,\hbar_m)\,\lambda^n\,d(\hbar_0,\hbar_1) \\ &\quad + \phi(\hbar_n,\hbar_m)\phi(\hbar_{n+1},\hbar_m)\Big[d(\hbar_{n+1},\hbar_{n+2}) + \, d(\hbar_{n+2},\hbar_m)\Big] \\ &\precsim \phi(\hbar_n,\hbar_m)\,\lambda^n\,d(\hbar_0,\hbar_1) + \phi(\hbar_n,\hbar_m)\phi(\hbar_{n+1},\hbar_m)\lambda^{n+1}d(\hbar_0,\hbar_1) \\ &\quad + \phi(\hbar_n,\hbar_m)\phi(\hbar_{n+1},\hbar_m)d(\hbar_{n+1},\hbar_m) \\ &\vdots \\ &\precsim \phi(\hbar_n,\hbar_m)\,\lambda^n\,d(\hbar_0,\hbar_1) + \phi(\hbar_n,\hbar_m)\phi(\hbar_{n+1},\hbar_m)\lambda^{n+1}d(\hbar_0,\hbar_1) \\ &\quad + \cdots + \phi(\hbar_n,\hbar_m)\phi(\hbar_{n+1},\hbar_m)\ldots\phi(\hbar_{m-1},\hbar_m)\lambda^{m-1}d(\hbar_0,\hbar_1) \end{aligned}$$

which implies that

$$\begin{aligned} |d(\hbar_n,\hbar_m)| \leq |d(\hbar_0,\hbar_1)|\Big[&\phi(\hbar_n,\hbar_m)\lambda^n + \phi(\hbar_n,\hbar_m)\phi(\hbar_{n+1},\hbar_m)\lambda^{n+1} \\ &+ \cdots + \phi(\hbar_n,\hbar_m)\phi(\hbar_{n+1},\hbar_m)\ldots\phi(\hbar_{m-1},\hbar_m)\lambda^{m-1}\Big]. \end{aligned}$$

Since $\lim_{n,m\to\infty}\phi(\hbar_n,\hbar_m)\lambda < 1$, so the series

$$\sum_{n=1}^{\infty}\lambda^n\prod_{i=1}^{n}\phi(\hbar_i,\hbar_m)$$

converges by the ratio test for each $m \in \mathbb{N}$.

Let

$$\mathcal{S} = \sum_{n=1}^{\infty} \lambda^n \prod_{i=1}^{n} \phi(\hbar_i, \hbar_m), \quad \mathcal{S}_n = \sum_{j=1}^{n} \lambda^j \prod_{i=1}^{j} \phi(\hbar_i, \hbar_m),$$

Thus for $m > n$, the above inequality can be written as

$$|d(\hbar_n, \hbar_m)| \leq |d(\hbar_0, \hbar_1)||\mathcal{S}_{m-1} - \mathcal{S}_n|.$$

Now, by taking the limit as $n, m \to \infty$ we get

$$|d(\hbar_n, \hbar_m)| \to 0 \quad \text{as } n, m \to \infty$$

In view of Lemma 1.6, the sequence $\{\hbar_n\}$ is Cauchy. Since X is complete, there exists some $u \in X$ such that $\hbar_n \to u$ as $n \to \infty$ and as such $\mathcal{Q}\hbar_n \to \mathcal{Q}u$, and $\mathcal{P}\mathcal{Q}\hbar_n \to \mathcal{P}\mathcal{Q}u$. This follows that $u = \mathcal{Q}u$, since

$$\begin{aligned} d(u, \mathcal{Q}u) &\preccurlyeq \phi(u, \mathcal{Q}u)\Big[d(u, \hbar_{2k+1}) + d(\hbar_{2k+1}, \mathcal{Q}u)\Big] \\ &\preccurlyeq \phi(u, \mathcal{Q}u)\Big[d(u, \hbar_{2k+1}) + d(\mathcal{Q}\hbar_{2k}, \mathcal{Q}u)\Big], \end{aligned}$$

giving

$$|d(u, \mathcal{Q}u)| \leq \phi(u, \mathcal{Q}u)\Big[|d(u, \hbar_{2k+1})| + |d(\mathcal{Q}\hbar_{2k}, \mathcal{Q}u)|\Big],$$

which on taking $k \to \infty$ yields

$$u = \mathcal{Q}u. \tag{2.18}$$

Further $u = \mathcal{P}\mathcal{Q}u$, otherwise $d(u, \mathcal{P}\mathcal{Q}u) = z > 0$ and we would then have

$$\begin{aligned} z &\preccurlyeq \phi(u, \mathcal{P}\mathcal{Q}u)\,[d(u, \hbar_{2k+3}) + d(\hbar_{2k+3}, \mathcal{P}\mathcal{Q}u)] \\ &\preccurlyeq \phi(u, \mathcal{P}\mathcal{Q}u)\,[d(u, \hbar_{2k+3}) + d(\mathcal{P}\mathcal{Q}u, \mathcal{Q}\mathcal{P}\hbar_{2k+1})] \\ &\preccurlyeq \phi(u, \mathcal{P}\mathcal{Q}u)\,\Bigg[d(u, \hbar_{2k+3}) + \lambda\, d(\mathcal{Q}u, \mathcal{P}\hbar_{2k+1}) \\ &\quad + \mu\frac{d(\mathcal{Q}u, \mathcal{P}\mathcal{Q}u)d(\mathcal{P}\hbar_{2k+1}, \mathcal{Q}\mathcal{P}\hbar_{2k+1})}{1 + d(\mathcal{Q}u, \mathcal{P}\hbar_{2k+1})} \\ &\quad + \gamma\frac{d(\mathcal{P}\hbar_{2k+1}, \mathcal{P}\mathcal{Q}u)d(\mathcal{Q}u, \mathcal{Q}\mathcal{P}\hbar_{2k+1})}{1 + d(\mathcal{Q}u, \mathcal{P}\hbar_{2k+1})}\Bigg] \\ &\preccurlyeq \phi(u, \mathcal{P}\mathcal{Q}u)\,\Bigg[d(u, \hbar_{2k+3}) + \lambda d(\mathcal{Q}u, \hbar_{2k+2}) \\ &\quad + \mu\frac{d(\mathcal{Q}u, \mathcal{P}\mathcal{Q}u)d(\hbar_{2k+2}, \hbar_{2k+3})}{1 + d(\mathcal{Q}u, \hbar_{2k+2})} \\ &\quad + \gamma\frac{d(\hbar_{2k+2}, \mathcal{P}\mathcal{Q}u)d(\mathcal{Q}u, \hbar_{2k+3})}{1 + d(\mathcal{Q}u, \hbar_{2k+2})}\Bigg], \end{aligned}$$

and so

$$\begin{aligned} |z| \leq \phi(u, \mathcal{P}\mathcal{Q}u)\Bigg[&|d(u, \hbar_{2k+3})| + \lambda|d(\mathcal{Q}u, \hbar_{2k+2})| \\ &+ \mu\frac{|d(\mathcal{Q}u, \mathcal{P}\mathcal{Q}u)||d(\hbar_{2k+2}, \hbar_{2k+3})|}{|1 + d(\mathcal{Q}u, \hbar_{2k+2})|} \\ &+ \gamma\frac{|d(\hbar_{2k+2}, \mathcal{P}\mathcal{Q}u)||d(\mathcal{Q}u, \hbar_{2k+3})|}{1 + |d(\mathcal{Q}u, \hbar_{2k+2})|}\Bigg]. \end{aligned}$$

Therefore, on taking $k \to \infty$, we have

$$|z| \leq \phi(u, \mathcal{P}\mathcal{Q}u) \Bigg[|d(u,u)| + \lambda |d(u,u)| + \mu \frac{|d(\mathcal{Q}u, \mathcal{P}\mathcal{Q}u)||d(u,u)|}{1 + d(u,u)} + \gamma \frac{|d(u, \mathcal{P}\mathcal{Q}u)||d(u,u)|}{|1 + d(u,u)|} \Bigg].$$

That is $|z| = 0$, a contradiction and hence

$$u = \mathcal{P}\mathcal{Q}u. \tag{2.19}$$

From (2.18) and (2.19), u is the common fixed point of $\mathcal{Q}$ and $\mathcal{P}\mathcal{Q}$.

In a similar way, we have $\mathcal{P}u = \mathcal{Q}\mathcal{P}u = u$ and hence

$$\mathcal{Q}u = \mathcal{P}u = \mathcal{P}\mathcal{Q}u = \mathcal{Q}\mathcal{P}u = u. \tag{2.20}$$

Now, we shall show that $\mathcal{P}, \mathcal{Q}, \mathcal{P}\mathcal{Q}$ and $\mathcal{Q}\mathcal{P}$ have a unique common fixed point. For this let us assume that u^* in X is second common fixed point of $\mathcal{P}, \mathcal{Q}, \mathcal{P}\mathcal{Q}$ and $\mathcal{Q}\mathcal{P}$ that is $\mathcal{P}u^* = \mathcal{Q}\mathcal{P}u^* = \mathcal{P}\mathcal{Q}u^* = \mathcal{Q}u^* = u^*$.

Then

$$\begin{aligned} |d(u,u^*)| &= |d(\mathcal{P}\mathcal{Q}u, \mathcal{Q}\mathcal{P}u^*)| \\ &\leq \lambda \left\{ \frac{|d(\mathcal{Q}u, \mathcal{P}\mathcal{Q}u)|\,|d(\mathcal{Q}u, \mathcal{Q}\mathcal{P}u^*)|}{|d(\mathcal{Q}u, \mathcal{Q}\mathcal{P}u^*)| + |d(\mathcal{P}u^*, \mathcal{P}\mathcal{Q}u)|} \right. \\ &\quad \left. + \frac{|d(\mathcal{P}u^*, \mathcal{Q}\mathcal{P}u^*)|\,|d(\mathcal{P}u^*, \mathcal{P}\mathcal{Q}u)|}{|d(\mathcal{Q}u, \mathcal{Q}\mathcal{P}u^*)| + |d(\mathcal{P}u^*, \mathcal{P}\mathcal{Q}u)|} \right\} \\ &= \lambda \left\{ \frac{|d(u,u)|\,|d(u,u^*)| + |d(u^*,u^*)|\,|d(u^*,u)|}{|d(u,u^*)| + |d(u^*,u)|} \right\} \leq 0, \end{aligned}$$

giving $u = u^*$. □

Remark 2.15. *Taking $\mathcal{Q}\hbar = \hbar'$, $\mathcal{Q}\vartheta = \vartheta'$, and $\gamma = 0$ in Theorem 2.14, we have the result of Bhatt et al. [3, Theorem 2.1].*

Setting $\mathcal{P} = \mathcal{Q}$ in Theorem 2.14, we have

Corollary 2.16. *Let (X,d) be a complete complex valued extended b-metric space. Let the mapping $\mathcal{P} : X \to X$ satisfy*

$$d(\mathcal{P}^2\hbar, \mathcal{P}^2\vartheta) \preccurlyeq \lambda \left\{ \frac{d(\mathcal{P}\hbar, \mathcal{P}^2\hbar)\, d(\mathcal{P}\hbar, \mathcal{P}^2\vartheta) + d(\mathcal{P}\vartheta, \mathcal{P}^2\vartheta)\, d(\mathcal{P}\vartheta, \mathcal{P}^2\hbar)}{d(\mathcal{P}\hbar, \mathcal{P}^2\vartheta) + d(\mathcal{P}\vartheta, \mathcal{P}^2\hbar)} \right\} \tag{2.21}$$

for all $\hbar, \vartheta \in X$, where $0 < \lambda < 1$ and such that $\lim_{n,m\to\infty} \phi(\hbar_n, \hbar_m) < \frac{1}{\lambda}$, here $\hbar_n = \mathcal{Q}^n\hbar_0$, $n = 1, 2, \ldots$ Then $\mathcal{P}$ and $\mathcal{P}^2$ and have a unique common fixed point.

3. Conclusion

In this work, we have developed a comprehensive family of common fixed point theorems in complex-valued extended b-metric spaces governed by rational-type contraction conditions. The obtained results significantly generalize and refine several well-known fixed point theorems due to Azam *et al.*, Bhatt *et al.*, Bryant, and Rouzkard and Imdad by weakening the contractive assumptions and enlarging the underlying space. The inclusion of meaningful corollaries and carefully constructed examples demonstrates the effectiveness and genuine improvement of our results over the existing literature.

The proposed framework contributes to the ongoing development of fixed point theory in generalized metric structures and provides a unified setting that accommodates a wide class of nonlinear mappings. Consequently, these results may serve as useful tools for addressing problems arising in nonlinear analysis, integral and functional equations, and related areas.

Future investigations may focus on extending the present approach to hybrid and multivalued contractions, probabilistic and fuzzy extensions, as well as algorithmic and computational aspects of fixed point iterations in complex-valued extended b-metric spaces.

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[1,2] *Department of Mathematics, Deenbandhu Chhotu Ram University of Science and Technology, Murthal 131039, India.*
E-mail address: monikasihag6@gmail.com, nawneethooda@gmail.com