Bol. Soc. Paran. Mat.
©SPM – E-ISSN-2175-1188
SPM: www.spm.uem.br/bspm

(3s.) **v. 2026 (44) 7** : 1–16.
ISSN-0037-8712
doi:10.5269/bspm.80851

Non-Linear Diffusion Dynamics with Variable Market Capacity

Vakil Shriwastav and Umesh Kumar Gupta

ABSTRACT: The diffusion of technological innovations tends to be hindered by market saturation, competitive forces, and economic obstacles. Classical diffusion models, such as the Bass model, have been effective in modelling the adoption pattern of new technologies. These models, however, tend to assume a fixed market size and ignore the impact of policy interventions. This work proposes a modified diffusion model that takes into account the effect of government subsidies on increasing the potential market. By formulating the subsidy effect as a time-varying growth in the adopter population, the paper illustrates how subsidies hasten adoption. Through numerical simulations and example applications to markets like electric vehicles, solar power, and energy-efficient appliances, the model displays high correspondence with actual trends. The results offer policymakers important guidance in their efforts to encourage innovation adoption by means of strategic financial incentives.

Key Words: Innovation diffusion model, Bass diffusion model Heaviside step function, government subsidy, stability analysis.

Contents

1. Introduction

The diffusion of technological innovations is a fundamental driver of economic growth, industrial transformation, and social progress in modern economies. Understanding how new technologies are adopted over time is important not only from an academic perspective but also for policymakers and industry stakeholders. To this end, a variety of mathematical models have been developed to study the diffusion of innovations, among which the Bass diffusion model remains the most widely recognized and influential. By incorporating innovation and imitation effects, this model has provided a clear and tractable representation of the adoption process.

Although the Bass model has produced satisfactory results in many mature and stable markets, it-and most related traditional diffusion models—relies on the critical assumption that the size of the potential market is fixed and predetermined. In the context of modern, policy-driven, and technology-intensive markets, this assumption is often unrealistic. In sectors such as electric vehicles, renewable energy systems, digital platforms, and energy-efficient appliances, market capacity evolves continuously over time.

A defining characteristic of these contemporary markets is their structural and institutional complexity. Adoption behavior is not governed solely by consumer preferences and social influence; rather, it is strongly affected by affordability constraints, infrastructure availability, regulatory frameworks, and

2020 *Mathematics Subject Classification:* 34A34, 37N40, 65L05, 90B50.

Submitted December 17, 2025. Published April 11, 2026

government support. Consequently, the set of consumers capable of adopting a new technology expands gradually instead of remaining constant, directly challenging the fixed-market assumption used in classical diffusion models.

This study advances the innovation diffusion literature by developing a unified non-linear diffusion framework in which market potential is explicitly modeled as a time-varying, policy-dependent process rather than a fixed exogenous constant. While classical and extended Bass-type models emphasize innovation, imitation, marketing, competition, or stochastic influences in isolation, the proposed model endogenizes government subsidies and policy interventions through a dynamic market capacity function, allowing institutional decisions to directly reshape adoption trajectories.

In contrast to previous studies that either assume a fixed adopter population or introduce policy variables as external shocks, the present framework captures gradual market expansion, delayed responses, and phased policy effects using interpretable parameters linked to real policy timelines. This enables systematic analysis of subsidy introduction, intensity changes, and withdrawal within a single mathematical structure.

Furthermore, by retaining non-linear adoption dynamics and permitting extensions to stochasticity and multi-generation technologies, the proposed model bridges theoretical diffusion modeling with policy evaluation, making it especially suitable for contemporary technology-intensive markets such as electric mobility, renewable energy, and digital infrastructure.

1.1. Motivation and Background

In this context, government subsidies and policy interventions emerge as central mechanisms shaping market dynamics. By reducing the effective purchase price of emerging technologies, subsidies enable economically constrained consumers to participate in the market. Importantly, subsidies do more than accelerate adoption among existing customers; they expand the effective market capacity by incorporating previously excluded consumer segments.

In addition, infrastructure development, regulatory support, standardization policies, and public awareness programs enhance adoption feasibility while reducing perceived consumer risks. As a result, the adoption process often follows a non-linear diffusion pattern, which cannot be adequately captured by traditional diffusion models that assume fixed market potential.

Empirical evidence further indicates that the effects of policy interventions are rarely instantaneous. Information delays, technological uncertainty, learning processes, social acceptance dynamics, and infrastructural limitations cause consumer responses to unfold gradually over time. Consequently, market expansion following policy intervention is inherently time-dependent and dynamic, rather than abrupt.

To highlight the conceptual gap between classical diffusion frameworks and the realities of contemporary policy-driven markets, Table 1 presents a concise comparison of their key characteristics.

Conceptual Comparison between Classical Diffusion Models and Contemporary Policy-Driven Markets

Table 1:

Aspect	**Classical Bass-Type Models**	**Contemporary Technology Markets**
Market potential	Fixed and predetermined	Time-varying and expanding
Role of government policy	Exogenous or ignored	Central determinant of market access
Effect of subsidies	Accelerates adoption rate	Expands potential adopter base
Consumer response	Instantaneous and homogeneous	Delayed and heterogeneous
Infrastructure influence	Implicitly assumed	Explicit and evolving
Model suitability	Mature, stable markets	Emerging, policy-driven markets

1.2. Research Gap

Although classical diffusion models have been extensively applied to study technology adoption, most existing frameworks assume a fixed and predetermined market potential. Such models fail to account for policy-driven expansion of market capacity, particularly in emerging technology markets where government subsidies, infrastructure development, and regulatory support play a central role. Moreover, the impact of subsidies is often modeled as an instantaneous shift, ignoring gradual consumer response, hesitation, and learning effects. There is therefore a lack of diffusion models that explicitly incorporate time-varying market capacity resulting from policy interventions. This gap motivates the development of a non-linear diffusion framework with dynamic market potential.

2. Literature Review

The modern study of innovation diffusion is rooted in Rogers' seminal work Diffusion of Innovations (1962) [30], which conceptualized diffusion as a social process driven by communication channels, time, and the social system. Rogers identified key attributes of innovations—relative advantage, compatibility, complexity, trialability and observability—that systematically shape adoption rates.

Building on this conceptual foundation, Bass (1969) [3] proposed the celebrated new product growth model for consumer durables, which formalized diffusion as a balance between innovation (external influence) and imitation (internal, word-of-mouth influence). The Bass model became the benchmark for aggregate adoption modeling and provided a parsimonious way to forecast first purchases.

Abernathy and Townsend (1975) [1] shifted attention to the firm level by examining technology transfer and innovation within organizations. Their work underlined that technological change inside firms affects productivity, organizational structures and adoption processes, bridging micro-level innovation management with macro-level diffusion outcomes.

In the late 1970s, several influential contributions extended diffusion modeling. Dodson and Muller (1978) [9] developed models of new product diffusion that explicitly incorporated advertising and word-of-mouth, distinguishing external communication from interpersonal influence. In a companion piece the same year, they introduced a growth-function approach to capture the influence of advertising on the diffusion curve. Mahajan and Peterson (1978) [24] introduced the idea of a dynamic potential adopter population, arguing that the number of potential adopters itself may evolve over time rather than remain fixed, an early challenge to the constant-market-potential assumption of the classical Bass framework. Mahajan (1979) [22] then formalized a new product growth model with dynamic market potential, explicitly embedding time-varying market size in the adoption equation.

Sharif and Ramanathan (1981) [32] further developed binomial innovation diffusion models with dynamic potential adopter populations. They argued that as awareness and economic conditions change, the pool of potential adopters expands, particularly for technologically advancing products. Their work provided a strong theoretical basis for later models that link diffusion to changes in market capacity.

During the 1980s, attention increasingly focused on price and replacement purchases. Kamakura (1987) [19] investigated long-term forecasting with innovation diffusion models when replacement purchases are present, showing that ignoring second and subsequent purchases can bias long-term forecasts. Kamakura and Balasubramanian (1988) extended this by incorporating the role of price and adoption influence in the diffusion of durables, empirically demonstrating that price trajectories and social influence jointly shape long-run penetration. Mahajan, Muller and Bass (1990) [23] provided a comprehensive review of new product diffusion models in marketing, systematizing a wide range of extensions and setting an agenda for further research.

Parker (1992) [28] incorporated competition into diffusion models, highlighting how rival products and brands interact within the same adopter population. This was a crucial step towards multi-brand and multi-technology diffusion analysis. Parker (1993) [29] then offered a critical review of aggregate diffusion forecasting models, emphasizing that no single specification is universally superior and underscoring the importance of model choice, parameter stability, and data requirements.

The early 2000s saw the integration of marketing-mix, network effects and nonlinear dynamics. Danaher, Hardie and Putsis (2001) [7] analyzed the diffusion of successive generations of a technological innovation while explicitly accounting for marketing-mix variables, showing that dynamic price and promotion policies are essential for understanding multi-generation diffusion. Goldenberg, Libai and Muller (2001) [13] offered a complex-systems perspective on word-of-mouth networks, revealing how network topology and social contagion mechanisms shape diffusion beyond simple mean-field assumptions. Eskinasi (2002) [11] linked diffusion of innovation theory to nonlinear dynamics, highlighting the potential of system dynamics and nonlinear models to capture feedbacks and path dependence. Saviotti and Pyka (2004) [31] connected economic development and diffusion of innovations, showing that diffusion is intertwined with structural change in the economy, while Stremersch and Tellis (2004) [33] examined international growth of new products, stressing cross-country heterogeneity in diffusion drivers. Jiang, Bass, and Bass (2006) [18] introduced the Virtual Bass Model to correct for left-hand truncation bias in diffusion studies, showing that incomplete early adoption data can significantly distort parameter estimation and forecasting in the standard Bass model. Their approach improves the accuracy of diffusion analysis when adoption data do not begin at the true market entry point. A major step towards dynamic market potential and structural interaction came with the work of Guseo and Guidolin (2009) [15], who modelled a dynamic market potential using automata networks, treating market capacity as an evolving latent process driven by awareness and social interaction. Guseo and Mortarino (2012) [16] extended this to sequential market entries and competition in multi-innovation diffusions, proposing frameworks where incumbents and entrants jointly shape the effective market potential over time. Kapur et al. (2012) [21] introduced stochastic differential equation (SDE)-based models for innovation diffusion, explicitly representing random fluctuations around deterministic trajectories and opening a path to stochastic optimal control in diffusion contexts. Decker and Hülsenberg (2014) [8] studied consumer acceptance of innovations with network externalities, stressing that user utility depends on the installed base and that network effects can create multiple equilibria. Bassamboo and Örmeci (2015) [4] explored market expansion dynamics in diffusion-driven demand models, formalizing how market size and demand co-evolve under diffusion. Melnikov (2015) [25] developed a demand-theoretic view of technological change and diffusion, embedding discrete-choice foundations into diffusion settings, while Melnikov and Zhao (2018) [26] focused on pricing and diffusion with learning and network effects in durable goods, showing how learning about quality and network growth influence optimal pricing and adoption paths. Islam and Meade (2017) [17] examined attribute-based diffusion and its implications for forecasting accuracy, highlighting that heterogeneity in product attributes can significantly affect diffusion patterns. Alberini and Bigano (2017) [2] empirically assessed the effectiveness of energy-efficiency incentives, providing direct evidence on how government incentives influence adoption of efficient technologies. Methodological advances on the diffusion side continued with Mehta, Chaudhary and Kumar (2019) [27], who incorporated a dynamic potential market into SDE-based diffusion models, explicitly connecting uncertainty, evolving market capacity, and adoption dynamics. Beck and Goldschmidt (2020) [5] examined government incentives and clean technology diffusion, bridging diffusion modeling with environmental and energy policy evaluation. The 2020s have seen rapidly growing interest in policy-driven diffusion of low-carbon and green technologies. Bitencourt, Goldemberg and Bazzo (2021) [6] adapted the Bass model to electric vehicle (EV) diffusion in Brazil, embedding public policy instruments into the diffusion parameters. Fan et al. (2022) [13] studied how government policies affect the diffusion of green innovation, providing empirical evidence that policy design and intensity significantly accelerate eco-innovation adoption. Domarchi et al. (2023) [10] reviewed EV forecasting models, emphasizing the central role of diffusion-type models with policy, infrastructure, and cost variables. Guidolin (2023) [14] synthesized concepts, models, and empirical evidence on innovation diffusion, arguing strongly for modelsi that treat market potential, policy variables, and social interactions as dynamic, interdependent processes rather than fixed exogenous constants.

Table 2:

SI. No	**Author (Year)**	**Matematical Model**	**Potential Market Size**	**Key Contribution**	**Compared to Present Work**
1	Bass (1969)	$$\frac{dn(t)}{dt} = \left[p + q\frac{n(t)}{N}\right](N - n(t))$$, Where: • n(t)=cumulative adoption at time t, • N=Potential Market size, assumed to be constant, • p=coefficient of innovation, • q= coefficient of imitation	Fixed N	Fundamental innovation imitation diffusion model	Market size invarient; no expansion
2	Mahajan & Peterson (1978)	$$\frac{dn(t)}{dt} = \left[p + q\frac{n(t)}{N(t)}\right](N(t) - n(t))$$, Where: • n(t)=cumulative adoption at time t, • N=Potential Market size, assumed to be constant, • p=coefficient of innovation • q= coefficient of imitation	Exogenous time-varying $N(t)$	Introduced dynamic adopter population	Growth mechanism unspecified
3	Mahajan (1979)	$$\frac{dn(t)}{dt} = \left[p + q\frac{n(t)}{N(t)}\right](N(t) - n(t))$$, Where: • n(t)=cumulative adoption at time t, • N=Potential Market size, assumed to be constant, • p=coefficient of innovation • q= coefficient of imitation	Dynamic $N(t)$	Generalized Bass model	Market growth not policy driven
4	Sharif & Ramanathan (1981)	$$\frac{dn(t)}{dt} = \left[p + q\frac{n(t)}{N(t)}\right](N(t) - n(t))$$, • n(t): cumulative number of adopters at time t, • N(t): time-varying potential market size, • p = coefficient of innovation (external influence), • q= coefficient of imitation (internal or social influence). $$N(t) = N_0 + g(t), \quad \text{or} \quad \frac{dN(t)}{dt} > 0$$ Where: • N_0 is the initial market size • $g(t)$ is a monotonic increasing function representing demographic or economic expansion.	Expanding population	Binomial diffusion framework	Lacks economic/policy drivers

5	Kamakura & Balasubramanian (1988)	$$\frac{dn(t)}{dt} = \left[p + q\frac{n(t)}{N(t)}\right](N(t) - n(t))$$ $$\frac{dn(t)}{dt} = g(N(t)), \qquad g^{'}(N) > 0$$ $$N(t) = N_0 + \int_0^t g(N(s))ds$$ Where: • N_0 =initial market size • $g(N(t))$= market development function. The innovation parameter is assumed to depend on price P(t): $$p(t) = p_0 e^{-\beta P(t)}$$ Where: • p_0= Baseline innovation rate • $\beta > 0$ = Price sensitivity parameter	Gradual evolution of market	Long-run diffusion modeling	Phenomenological growth
6	Bassamboo & Örmeci (2015)	$$\frac{dn(t)}{dt} = \left[p + q\frac{n(t)}{N(t)}\right](N(t) - n(t))$$ $$\frac{dN(t)}{dt} = \rho(L - N(t))$$ where • $n(t)$ = cumulative adoption at time t, • N = Potential market size, • p = coefficient of innovation, • q = coefficient of imitation • L = long-run market capacity • ρ = represents the speed of adjustment	Endogenous $N(t)$	Co-evolution of diffusion & market size	No explicit policy term
7	Guseo & Guidolin (2009)	$$\frac{dn(t)}{dt} = \left[p + q\frac{n(t)}{N(t)}\right](N(t) - n(t))$$ Where: • $p(t)$=potentially time-varying innovation coefficient • q=imitation coefficient The market potential evolves independently as $$\frac{dn(t)}{dt} = g(N(t), n(t))$$ Where: $g(N(t), n(t))$ = market expansion function • Expansion may depend on adoption level $n(t)$ or exogenous constraints.	Dynamic market $N(t)$	Automata diffusion networks	Policy excluded
8	Guseo & Mortarino (2012)	$$\frac{dn_k(t)}{dt} = \left[p_k + q_k\frac{n_k(t)}{N_k(t)} + \sum_{j\neq k} n_{k_j}\frac{n_j(t)}{N_j(t)}\right](N_k(t) - n_k(t))$$ • $p_k > 0$: innovation (external influence) coefficient for technology k, • $q_k > 0$: imitation (within-technology social influence), • η_{kj}: cross-imitation or competitive influence of earlier eta kj technology join k. $$\frac{dN_k(t))}{dt} = g_k(N_k(t), X_1(t), ..., X_k(t))$$ where $g_k(N_k(t), X_1(t), ..., X_k(t))$; a market evolution function, representing factors such as:	Sequential dynamic markets	Competitive diffusion modeling	Market growth not policy-linked

		• N_k=Market potential of technology k • $X_k(t)$=Actual adoption of technology k, k=1,2,3,...			
9	Mehta et al. (2019)	$$\frac{dn(t)}{dt} = \left[p + q\frac{n(t)}{N(t)}\right](N(t) - n(t))dt + \sigma n(t)dW_t$$ Where: • $n(t)$ denotes cumulative adoption at time t, • $N(t)$ is the time-varying potential market size, assumed to evolve stochastically, • $p > 0$ is the coefficient of innovation, • $q > 0$ is the coefficient of imitation, • $\sigma > 0$ measures the intensity of random fluctuations, • W_t is a standard Wiener process.	Stochastic $N(t)$	Captures uncertainty & randomness	Policy absent
10	Bassamboo & Ormeci (2019)	$$\frac{dn(t)}{dt} = \left[p + q\frac{n(t)}{N(t)}\right](N(t) - n(t))$$ $$\frac{dN(t)}{dt} = \rho(L - N(t))$$ where L is the long-run market capacity and mu $\rho > 0$ represents the speed of adjustment.	Mixed (surveyed)	Unified literature synthesis	No governing model
11	Guidolin 2023	$$\frac{(dn(t))}{dt} = F(n(t), N(t))$$ $$\frac{(dN(t))}{dt} = G(N(t), \text{policy}(t))$$ where $n(t)$ represents effective market capacity shaped by policy actions. In most formulations, policy effects modify diffusion parameters such as por q, while the evolution of market capacity remains external to the policy environment	Time-varying $N(t)$	Introduced dynamic market potential	not offer a unified nonlinear framework

The analytical foundation of innovation diffusion modeling originates from the classical Bass diffusion model introduced by Bass (1969). This model characterizes the evolution of cumulative adopters n(t)in a market with a fixed potential size N. The governing equation is given by

$$\frac{dn(t)}{dt} = \left[p + q\frac{n(t)}{N}\right](N - n(t))$$

where $p > O$ and $q > 0$ denote the coefficients of innovation and imitation, respectively, and M represents the number of eventual adopters.

Mahajan (1979) and Mahajan and Peterson (1978) generalized the Bass framework by allowing the market potential to vary with time. The diffusion equation is then modified as

$$\frac{dn(t)}{dt} = \left[p + q\frac{n(t)}{N(t)}\right](N(t) - n(t))$$

where $N(t)$ denotes the time-varying potential market size. In many early applications, $N(t)$ is specified exogenously, for example as

$$N(t) = N_0 + \gamma t \quad \text{or} \quad N(t) = N_0 e^{\gamma t}$$

Where $N_0 > O$ representing the initial market size and $\gamma > O$ the expansion rate.

Bassamboo and Ormeci (2015) proposed a model, in which adoption and market potential co-evolve through a coupled dynamical system:

$$\frac{dn(t)}{dt} = \left[p + q\frac{n(t)}{N(t)}\right](N(t) - n(t))$$

$$\frac{dN(t)}{dt} = \rho(L - N(t))$$

where L is the long-run market capacity and $\rho > O$ represents the speed of adjustment.

Mehta et al. (2019) extended the diffusion framework by introducing stochastic market evolution. Their model is expressed as the stochastic differential equation

$$\frac{dn(t)}{dt} = \left[p + q\frac{n(t)}{N(t)}\right](N(t) - n(t))dt + \sigma n(t)dW_t$$

where $\sigma > O$ denotes the intensity of random fluctuations and W_t is a standard Wiener process. In this setting, $N(t)$ evolves stochastically, allowing the model to capture uncertainty arising from macroeconomic shocks, demand volatility, or consumer sentiment. Despite this advancement, the stochastic expansion of $N(t)$ lacks a structural economic or policy interpretation.

Guidolin (2023), provide a comprehensive synthesis of diffusion models with dynamic market potential and highlight a key unresolved issue. Although numerous formulations incorporate time-varying $N(t)$ whether deterministic, endogenous, or stochastic-they do not offer a unified nonlinear framework

Mathematically, existing models do not jointly specify a closed system of the form

$$\frac{(dn(t))}{dt} = F(n(t), N(t))$$
$$\frac{(dN(t))}{dt} = G(N(t), \text{policy}(t))$$

where $N(t)$ represents effective market capacity shaped by policy actions. In most formulations, policy effects modify diffusion parameters such as p or q, while the evolution of market capacity remains external to the policy environment.

The progression from clearly demonstrates the evolution of diffusion modelling toward increasingly flexible representations of market potential. However, the absence of a policy endogenized market capacity persists across this literature. This gap motivates the proposed model, which introduces a policy-driven, time-varying market potential and embeds it within a nonlinear diffusion framework. The resulting structure allows policy interventions to reshape not only the speed of diffusion but also the boundary of the market itself, an aspect explicitly addressed in the following section.

3. Proposed Model

The fundamental diffusion model makes the assumption that the social system's size is known, fixed, and finite, or that it can be calculated. Currently, it is realistically acceptable, particularly given the rapid changes in economic policy and technology.

The following presumptions guide the design of the suggested model:

i. There's a new technology product on the market.

ii. Only a small number of people can initially afford to acquire the product; let's say that this number is N, which stands for the initial potential.

iii. Let $n(t)$ represent the number of adopters who have embraced the product at the limit.

iv. After a certain time T, the government introduces a subsidy, which increases the affordability of the product. As a result, the number of people who can now afford the product increase to $M(M > N)$.

v. The market size Z(t) changes over time based on government intervention.

$$Z(t) = \begin{cases} N, & t \leq T \\ M, & t > T \end{cases}$$

It is possible to describe the prospective market size $Z(t)$ as a continuous function if the government subsidy impact progressively grows the market over time.

$$Z(t) = N + (M - N)\left(1 - e^{-\alpha(t-T)}\right)H(t - T) \tag{3.1}$$

Where,

- α is the market growth rate as a result of the subsidy.
- The Heaviside step function operating at $t = T$ is $H(t - T)$.

Initial phase $t \leq T$, only those who initially have the purchasing power N are part of the potential market. The product adoption follows the standard Bass model. The government introduce a subsidy at time T reducing, the effective prize of the product. This makes it affordable to a large group of consumers, expanding the market size from N to M, where $M > N$. After the subsidy introduced $(t > T)$, the adoption, process accelerate, as more consumers are now capable of purchasing the product. In modern industries such as electric vehicles, digital services and education technology, the market does not remain static. Instead, it expands gradually due to government initiatives, infrastructural development, technological upgrades and changing consumer awareness. We introduce a more realistic and adaptable framework by incorporating a time-dependent market potential function, denoted as $Z(t)$, into the classical model-leading to a Modified Bass Diffusion

$$\frac{dn(t)}{dt} = \left[p + \frac{q}{Z(t)}n(t)\right][Z(t) - n(t)]$$

$$\frac{dn(t)}{dt} = pZ(t) - pn(t) + qn(t) - \frac{q}{Z(t)}n^2(t)$$

$$\frac{dn(t)}{dt} = -\frac{q}{Z(t)}n^2(t) + (q - p)n(t) + pZ(t) \tag{3.2}$$

Putting the value of $Z(t)$ from equation (3.1) into equation (3.2), we get

$$\frac{dn(t)}{dt} = -q\left[N + (M - N)\left(1 - e^{-\alpha(t-T)}\right)H(t - T)\right]^{-1} n^2(t) + (q - p)n(t) + p\left[N + (M - N)\left(1 - e^{-\alpha(t-T)}\right)H(t - T)\right] \tag{3.3}$$

The equation is nonlinear in $n(t)$, specifically quadratic in $n^2(t)$ with time varying coefficients. The coefficient $Z(t)$ is not constant and is switches at $t = T$ due to the Heaviside function $H(t - T)$. The Differential Equation (3.3) exhibits a piecewise behaviour with two distinct phase, corresponding to the periods before and after the introduction of the subsidy.
i.e.

Phase(I). For $t \leq T$: $H(t - T) = 0$

$$Z(t) = N$$

Phase(II): for $t > T$: $H(t - T) = 1$

$$Z(t) = N + (M - N)(1 - e^{-\alpha(t-T)})$$

For case(1)

$$Z(t) = N$$

$$\frac{dn(t)}{dt} = -\frac{q}{N}n^2(t) + (q - p)n(t) + pN$$

$$n(t) = N \cdot \frac{1 - e^{-(p+q)t}}{1 + \frac{q}{p}e^{-(p+q)t}}$$

Case(2): for $t > T$

$$Z(t) = N + (M - N)(1 - e^{-\alpha(t-T)})$$

$$\frac{dn(t)}{dt} = -q\left[N + (M-N)\left(1 - e^{-\alpha(t-T)}\right)H(t-T)\right]^{-1} n^2(t) + (q-p)n(t)$$
$$+\, p\left[N + (M-N)\left(1 - e^{-\alpha(t-T)}\right)H(t-T)\right]$$

This is still nonlinear Riccati but non constant coefficients. So we can apply Runge Kutta Method:

$$n(t+\Delta t) = n(t) + \frac{dn(t)}{dt}\Delta t$$

$$\frac{n(t+\Delta t) - n(t)}{\Delta t} = \frac{1}{6}(K_1 + 2K_2 + 2K_3 + K_4)$$

Where,

$$K_1 = f(t, n)$$
$$K_2 = f\big(t + \tfrac{\Delta t}{2},\ n + \tfrac{\Delta t}{2}K_1\big)$$
$$K_3 = f\big(t + \tfrac{\Delta t}{2},\ n + \tfrac{\Delta t}{2}K_2\big)$$
$$K_4 = f(t + \Delta t,\ n + \Delta t K_3)$$

This Modified Bass Diffusion Model provides a comprehensive framework to understand the dynamics of product adoption in modern, evolving markets. The modified Bass model adds a time-dependent market potential function Z(t) to represent the real-world growth of the addressable market over time as a result of external interventions like government subsidies, policy changes, or infrastructure advancements, in contrast to the classical Bass model, which assumes a fixed and static market potential. The model emphasizes that the rate of product adoption is not exclusively determined by external factors like advertising and promotions (innovation impact) or the number of adopters and the internal influence they have on others (imitation effect). Instead, a crucial third dimension is added the evolution of the market potential itself. As the number of people who are aware, capable or permitted to adopt the product increases over time. This growing market potential significantly affects both the magnitude and timing of adoption. It can lead to scenarios where the adoption curve shows a delayed but sharper rise, multiple inflection points or extended periods of growth phenomena that the original model cannot capture. By accounting for the dynamic nature of modern markets, this model becomes significantly more realistic, flexible and applicable to contemporary industries where consumer accessibility, awareness and market reach are continuously expanding. Thus, in addition to increasing the adoption forecasting's prediction accuracy, the Modified Bass Model is a useful tool for decision-making for companies, marketers, and legislators working in rapidly evolving technology and economic environments.

4. Stability Analysis of Model

Equilibrium Points:

$$\frac{dn(t)}{dt} = 0$$

$$\big[Z(t) - n(t)\big]\left[p + \frac{q}{Z(t)}n(t)\right] = 0$$

$$n^*(t) = Z(t), \quad -\frac{p}{q}Z(t)$$

$$n^*(t) \neq -\frac{p}{q}Z(t) \quad (p > 0,\ q > 0)$$

Thus equation has only one feasible equilibrium

$$n(t) = n^*(t) = Z(t)$$

The number of adopters finally approaches the entire possible market at this point:
Before subsidy $n^*(t) = N$
After subsidy $n^*(t) = M$

Linearization around the equilibrium:

We linearize the system around the equilibrium $n^*(t)$' in order to investigate stability. Define a small perturbation

$$n(t) = n^*(t) + \epsilon(t)$$

where, $\epsilon(t)$ is a small deviation from the equilibrium.

$$\frac{d}{dt}\big(n^(t) + \epsilon(t)\big) = pZ(t) - n^(t) + \epsilon(t) + \frac{q}{Z(t)}\big(n^(t) + \epsilon(t)\big)\big(Z(t) - (n^(t) + \epsilon(t))\big)$$

Since $n^*(t) = Z(t)$,

$$\begin{aligned}
\frac{df}{dt} &= -p\,\epsilon(t) - q\,\epsilon(t) \\
&= -(p+q)\,\epsilon(t) \\
\frac{df}{\epsilon} &= -(p+q)\,dt \\
\epsilon(t) &= \epsilon(0)e^{-(p+q)t} \\
\lim_{\epsilon\to 0} \epsilon(t) &= 0 \\
\lim_{t\to 0} n(t) &= n^*(t) = Z(t)
\end{aligned}$$

Stability Condition:

$$\begin{aligned}
\frac{dn(t)}{dt} &= [Z(t) - n(t)]\left[p + \frac{q}{Z(t)}n(t)\right] \\
f(n) &= [Z(t) - n(t)]\left[p + \frac{q}{Z(t)}n(t)\right] \\
f(n) &= [Z(t) - n(t)]p + \left(p + \frac{q}{Z(t)}n(t)\right)[Z(t) - n(t)] \\
f(n) &= [Z(t) - n(t)]p + qn(t) - \frac{q}{Z(t)}n^2(t) \\
f(n) &= Z(t)p - n(t)p + qn(t) - \frac{q}{Z(t)}n^2(t) \\
f(n) &= -\frac{q}{Z(t)}n^2(t) + (q-p)n(t) + pZ(t)
\end{aligned}$$

The Jacobian Matrix:

$$\begin{aligned}
J &= \frac{d}{dn}f(n) \\
J &= -\frac{2q}{Z(t)}n(t) + (q-p) \\
\left(\frac{d}{dn}f(n)\right)_{n(t)=Z(t)} &= -(p+q) < 0
\end{aligned}$$

Since $J < 0$, the Eigenvalue is negative and the equilibrium is locally asymptotically stable.

Theorem 4.1 (Existence and Uniqueness of Equilibrium):

For the equation (4.1), with $p > 0$, $q > 0$, and $Z(t) > 0$, there exists a unique equilibrium at $n^*(t) = Z(t)$.

Proof: We are given the ODE,

$$\frac{dn(t)}{dt} = f(n(t)) = (Z(t) - n(t)) \left(p + \frac{q}{Z(t)} n(t) \right)$$

We want to prove that there exists a unique equilibrium point n^* such that

$$\frac{dn(t)}{dt} = 0 = f(n^*) = 0$$

$$(Z(t) - n(t)) \left(p + \frac{q}{Z(t)} n(t) \right) = 0$$

This product is zero if either,

- $Z(t) - n^* = 0, \quad n^* = Z(t),$
- $p + \dfrac{q}{Z(t)} n^* = 0, \quad n^* = -\frac{p}{q} Z(t)$

But, $p > 0$, $q > 0$, $Z(t) > 0$.
Therefore, $n^* = -\dfrac{p}{q} Z(t) < 0$ is not valid (adoption cannot be negative).
Thus the only physically meaningful equilibrium is

$$n^* = Z(t).$$

We examine the derivative $\dfrac{d}{dn} f(n)$ at $n = Z(t)$:

$$f(n(t)) = (Z(t) - n(t)) \left(p + \frac{q}{Z(t)} n(t) \right)$$

$$f'(n(t)) = -\left(p + \frac{q}{Z(t)} n(t) \right) + (Z(t) - n(t)) \frac{q}{Z(t)}$$

At $n = Z(t)$,

$$f'(Z(t)) = -\left(p + \frac{q}{Z(t)} Z(t) \right) + (Z(t) - Z(t)) \frac{q}{Z(t)} = -(p + q) < 0$$

$$\text{So,} \quad f'(Z(t)) < 0$$

Since the function $f(n)$ is continuous and differentiable on $[0, Z(t)]$ and the sole equilibrium in this range is $Z(t)$, the equilibrium at $n^*(t) = Z(t)$ is thus locally asymptotically stable and unique.
Under the given conditions $p > 0$, $q > 0$, $Z(t) > 0$, there exists a unique, locally asymptotically stable equilibrium at $n^* = Z(t)$.

Theorem 4.2 (Local Asymptotic Stability of Equilibrium):

If $p > 0$, $q > 0$, then the equilibrium $n^* = Z(t)$ is locally asymptotically stable.

Proof: Linearization leads to a perturbation decay governed by

$$\frac{d\epsilon}{dt} = -(p + q)\epsilon$$

$$\epsilon(t) = \epsilon(0) e^{-(p+q)t}$$

Since $(p + q) > 0$, $\epsilon(t) \to 0$ as $t \to \infty$.
Thus, local asymptotic stability is established.

Theorem 4.3 (Stability under Dynamic Market Expansion):
The Jacobian $J = -(p+q)$ is independent of $Z(t)$ and remains negative. Thus, the equilibrium $n^* = Z(t)$ is locally asymptotically stable at each time t.

Proof: The Jacobian $J = -(p+q)$ is independent of $Z(t)$ and remains negative. Thus, the equilibrium $n^*(t) = Z(t)$ is locally asymptotically stable at each time t.

5. Numerical Simulation of Model

To examine the feasibility and validity of the analytical results obtained in the preceding sections-particularly those concerning the existence and local stability of the equilibrium point-we perform a set of numerical simulations. These simulations are designed to verify whether the qualitative behavior predicted by the theoretical analysis is supported by numerical evidence.

For this purpose, numerical calculations are carried out using the parameter values summarized in Table 3. The selected parameter is economically and mathematically meaningful. It has been chosen carefully so that the existence conditions derived for the equilibrium solution are satisfied for at least one admissible range of parameters. This allows a clear illustration of the system dynamics under theoretically consistent conditions.

Table 3:

Parameter	**Symbol**	**Value**	**Role in the Model**
Innovation coefficient	(p)	0.03	Measures the rate of adoption driven by independent decision-making and external information sources. The chosen value lies within the empirically established range (p $\in$ [0.01, 0.03]) reported by Jiang, Bass, and Bass (2006) [18], representing a realistic proportion of early adopters while satisfying equilibrium existence conditions.
Imitation coefficient	(q)	0.38	Represents social influence and word-of-mouth effects. The value is selected such that ($q > p$), consistent with diffusion theory and empirical findings indicating dominant imitation behaviour (q $\in$ [0.30, 0.70]) in real markets (Jiang et al., 2006) [18].
Initial market potential	(N)	100,000	Represents the number of consumers who are able to afford or access the product prior to government intervention. This ensures a positive and finite equilibrium in the pre-policy phase.
Expanded market potential	(M)	200,000	Denotes the long-run market size after subsidy-induced expansion. The condition ($M > N$) is satisfied, which is necessary for policy-driven market growth.
Market expansion rate	α	0.5	Controls the speed at which the effective market potential expands following policy intervention. The chosen value yields gradual yet significant expansion, consistent with model stability requirements.
Policy intervention time	(T)	2 years	Represents the time at which government subsidy or policy support is introduced, separating the system dynamics into pre- and post-intervention regimes.
Time step for simulation	(Δt)	1 year	Numerical step size used for discretization and implementation of the Runge-Kutta (or Euler) integration method.
Simulation horizon	(t)	1-12 years	Time interval over which adoption dynamics, convergence behaviour, and equilibrium stability are examined.

t (years)	**Market size** $Z(t)$	**Previous cumulative** $n(t-1)$	**New Adopters** $S(t)$	**Cumulative adoption** $n(t)$
1	100000.0	0.0	3000.00	0.00
2	100000.0	0.00	3575.35	3575.35
3	139346.93	3575.35	5072.51	9247.85
4	163212.06	9247.85	8791.30	18030.15
5	177686.98	18039.15	12193.47	30232.62
6	186466.47	30232.02	15091.50	45921.12
7	191791.50	45921.12	18822.76	64746.88
8	195021.29	64746.88	20053.05	85700.53
9	196980.26	85700.53	21541.86	107242.39
10	198168.44	107242.39	20431.61	127674.00
11	198889.10	127674.00	17954.93	145628.03
12	199326.21	145628.03	14750.69	100379.02

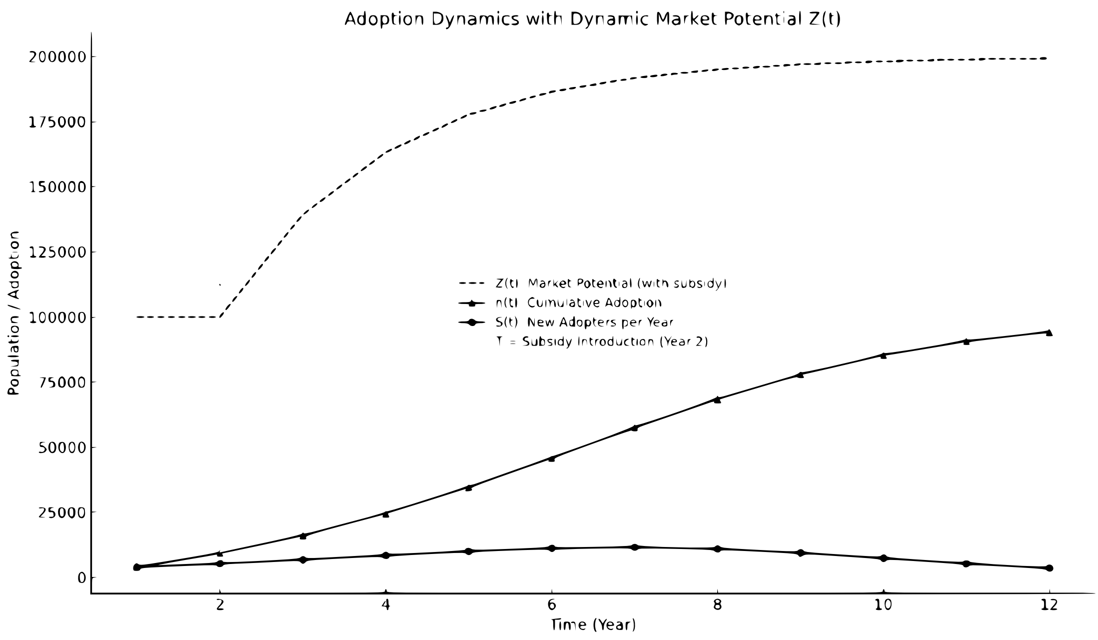

Figure 1: Adoption Dynamics Market Potential $Z(t)$

In the above figure (1) indicate that dynamic market potential $(Z(t))$ driven by policy intervention is more realistic than assuming a static market. The model helps policymakers understand how adjusting subsidy intensity can directly influence technology adoption. From the graph we can conclude that-

i. **When** $t \leq 2$ no government intervention has occurred yet,
$Z(t) = N = 100000$. Therefore **Blue line** is initially flat

ii. When $t > 2$ the government introduces a subsidy at time $T = 2$, which increases market affordability.

As a result, $Z(t)$ starts to increase gradually toward $M = 200000$, following an exponential growth pattern given by

$$Z(t) = N + (M - N)(1 - e^{-\alpha(t-T)})$$

Adoption Behaviour

- The **Green Line** Shows how Cumulative Adoption $n(t)$ increases over time. After the government subsidy, the adoption rate accelerates.
- The **Red Line** Indicates how new Adopters$S(t)$ many new users adopt the product each year. There's a noticeable increase in new adopters after $t = 3$, due to the expanded market.

$$Z(t) = N + (M - N)(1 - e^{-\alpha(t-T)})$$

- The heaviside step models the policy oneset time not instantaneous adoption actual consumer responses are gradual and are reflected through the parameters p_1, q_1, α and time varying market potential $Z(t)$. Hesitation, learry and information delays can bev interpreted as a slower rate α in $Z(t)$ & lower initial post subsidy p , q values. In empirical application estimated parameters after T typically absorb these behavioural frictions, So the jump in potential does not mean a jump in realized sales.

6. Conclusion

This paper introduces an improved diffusion model that effectively incorporates the impact of government subsidies on new technology adoption. Through dynamic modelling of the growth in market size after the introduction of subsidies, the model better reflects actual diffusion patterns than static models. Simulation outcomes confirm the theoretical model, demonstrating that subsidies have a significant impact on decreasing adoption lag and increasing market penetration. Its practical applicability is evident from the fact that it has been implemented in numerous sectors such as electric mobility, solar power and consumer electronics, thereby reflecting the model's generality. In addition to the above, a "basic influence number" serves to further develop the model's analytical capabilities to differentiate the relative influence of imitation and innovation. Finally, the study highlights the importance of policy interventions in facilitating innovation diffusion and presents a powerful data-based framework for projecting and directing such actions.

References

1. Abernathy, W.J., Townsend, P.L. *Technology transference for innovation*, Administrative Science Quarterly 20 (4) (1975) 603–622.
2. Albernini, A., Bigano, M., *How effective are energy-efficiency incentives?*, Energy Economics 68 (2017) 413–428.
3. Bass, F.M., *A new product growth model for consumer durables,* Management Science 15 (5) (1969) 215–227.
4. Bassamboo, A., Örmeci, S., *Market expansion dynamics in diffusion-driven demand models,* Omega 52 (2015) 21–32.
5. Beck, T., Goldschmidt, M., *Government incentives and clean technology diffusion,* Energy Policy 141 (2020) 111442.
6. Bitencourt, L., Goldemberg, J., Bazzo, W.A., *Bass diffusion model adaptation considering public policies applied to electric vehicle diffusion,* Energies 14 (17) (2021) 5435.
7. Danaher, P.J., Hardie, B.G.S., Putsis, W.P., *Marketing-mix variables and the diffusion of successive generations of a technological innovation,* Journal of Marketing Research 38 (4) (2001) 501–514.
8. Decker, M., Hülsenberg, A., *Consumer acceptance of innovations with network externalities,* Technovation 34 (4) (2014) 220–231.
9. Dodson, J.A., Muller, E., *Models of new product diffusion through advertising and word-of-mouth,* Management Science 24 (15) (1978) 1568–1578.
10. Domarchi, C., Fiorello, D., Fontes, T., Electric vehicle forecasts: A review of models and methods, Transport Reviews 43 (5) (2023) 623–650.
11. Eskinasi, L., Diffusion of innovation theory and nonlinear dynamics, Technological Forecasting and Social Change 69 (2) (2002) 123–135.

12. Fan, R., Liu, G., Wang, Y., Wang, J., How do government policies affect the diffusion of green innovation?, Journal of Cleaner Production 373 (2022) 133803.

13. Goldenberg, J., Libai, B., Muller, E., Talk of the network: A complex systems approach to word-of-mouth, Marketing Letters 12 (3) (2001) 211–223.

14. Guidolin, M., Innovation diffusion processes: Concepts, models, and empirical evidence, Annual Review of Statistics and Its Application 10 (2023) 397–423.

15. Guseo, R., Guidolin, M., Modelling a dynamic market potential: A class of automata networks for diffusion of innovations, Technological Forecasting and Social Change 76 (6) (2009) 806–820.

16. Guseo, R., Mortarino, C., Sequential market entries and competition modelling in multi-innovation diffusions, European Journal of Operational Research 216 (3) (2012) 658–667.

17. Islam, T., Meade, N., The impact of attribute-based diffusion on forecasting accuracy, International Journal of Forecasting 33 (3) (2017) 736–750.

18. Jiang, Z., Bass, F. M., & Bass, P. I. (2006). *The Virtual Bass Model and the left-hand data-truncation bias in diffusion of innovation studies.* International Journal of Research in Marketing, 23(1), 93–106.

19. Kamakura, W.A., Long-term forecasting with innovation diffusion models: Impact of replacement purchases, Journal of Forecasting 6 (1) (1987) 1–19.

20. Kamakura, W.A., Balasubramanian, S.K., Long-term view of the diffusion of durables: Role of price and adoption influence, International Journal of Research in Marketing 5 (1) (1988) 1–13.

21. Kapur, P.K., Pham, H., Chanda, U., Kumar, S., Innovation diffusion model using stochastic differential equations, International Journal of Operational Research 13 (2) (2012) 164–178.

22. Mahajan, V., A new product growth model with a dynamic market potential, Management Science 25 (11) (1979) 1027–1037.

23. Mahajan, V., Muller, E., Bass, F.M., New product diffusion models in marketing: A review and directions for research, Journal of Marketing 54 (1) (1990) 1–26.

24. Mahajan, V., Peterson, R.A., Innovation diffusion in a dynamic potential adopter population, Management Science 24 (15) (1978) 1589–1597.

25. Melnikov, O., Demand theory of technological change and diffusion, Journal of Economics & Management Strategy 24 (1) (2015) 214–236.

26. Melnikov, O., Zhao, X., Pricing and diffusion of durable goods with learning and network effects, Journal of Economic Theory 176 (2018) 1–38.

27. Mehta, S., Chaudhary, K., Kumar, P., Incorporating dynamic potential market in innovation diffusion using stochastic differential equations, IEOM Conference (2019).

28. Parker, P.M., Incorporating competition into diffusion models, Management Science 38 (12) (1992) 1504–1516.

29. Parker, P.M., Aggregate diffusion forecasting models in marketing: A critical review, International Journal of Forecasting 9 (4) (1993) 447–466.

30. Rogers, E.M., Diffusion of Innovations, Free Press, New York, 1962.

31. Saviotti, P.P., Pyka, A., Economic development and diffusion of innovations, Structural Change and Economic Dynamics 15 (4) (2004) 519–540.

32. Sharif, M.N., Ramanathan, K., Binomial innovation diffusion models with dynamic potential adopter populations, Technological Forecasting and Social Change 20 (1) (1981) 63–87.

33. Stremersch, S., Tellis, G.J., Understanding and managing international growth of new products, International Journal of Research in Marketing 21 (4) (2004) 421–438.

Vakil Shriwastav,

Department of Mathematics and Statistics,

D. D. U. Gorakhpur University, Gorakhpur, (U. P.),

India.

E-mail address: vakilshriwastav@gmail.com

and

Umesh Kumar Gupta,

Department of Mathematics,

Mahatma Gandhi P. G. College, Gorakhpur, (U. P.),

India.

E-mail address: ukgmgpg@gmail.com