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Solvability of a System of Katugampola Fractional Integral Equations Using a Fixed Point Result Obtained via New Condensing Operators

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ABSTRACT: The aim of this specific paper is to scrutinize a system of Katugampola fractional integral equations. To attain this, a new idea of condensing operators has been used to establish a new fixed point theorem. Furthermore, we present the discussion of the application of this new fixed-point theorem to the Katugampola fractional integral equations in a Banach space. An example has been illustrated to support the effectiveness of the obtained results.

Key Words: Measure of noncompactness ($\mathcal{M}\mathbb{N}\mathcal{C}$), fractional integral equation ($\mathcal{F}\mathbb{I}\mathcal{E}$), Fixed Point theorem ($\mathcal{F}\mathbb{P}\mathcal{T}$).

Contents

1. Introduction

There is an immense use of fractional integral equation ($\mathcal{F}\mathbb{I}\mathcal{E}$) in solving real-world practical problems in diverse fields. Because of the pivotal role of $\mathcal{F}\mathbb{I}\mathcal{E}$ in various branches of science and engineering, it is crucial to grasp an understanding of such equations. The idea of measure of noncompactness ($\mathcal{M}\mathbb{N}\mathcal{C}$) plays a significant role in fixed point theorems ($\mathcal{F}\mathbb{P}\mathcal{T}s$). Kuratowski [20] introduced the concept of $\mathcal{M}\mathbb{N}\mathcal{C}$ in 1930. In 1955, G. Darbo [8] utilised the notion of $\mathcal{M}\mathbb{N}\mathcal{C}$ to develop a result which demonstrated that fixed point is present for the condensing operators. $\mathcal{F}\mathbb{P}\mathcal{T}$ and $\mathcal{M}\mathbb{N}\mathcal{C}$ have several applications in the analysis of various integral equations that arise in innumerable problems imperative to physical scenarios. Numerous research work related to $\mathcal{F}\mathbb{I}\mathcal{E}$ has been discussed by the usage of $\mathcal{M}\mathbb{N}\mathcal{C}$ and $\mathcal{F}\mathbb{P}\mathcal{T}$. Some of such research works can be seen in [2,3,9,10,11,12,14,15,16,17,18].

In [13] Das.et.al demonstrated the existence of solution of a system of generalised proportional fractional integral equations with the help of a generalised Darbo Fixed Point Theorem.

Motivated by this work we are inspired to investigate the existence of a solution of a system of Katugampola fractional integral equations using a newly constructed generalisation of the Darbo fixed point theorem.

The article is organized as follows. In Section 2, we have recalled the necessary concepts and results from the literature that help us throughout the paper. In Section 3 and 4, we have presented a newly obtained Fixed point theorem in an arbitrary Banach space and the measure of noncompactness in that space respectively. The existence of solution of a system of Katugampola fractional integral equations via the newly obtained fixed point result has been given in section 5 alongwith a simulative example of our theoretical works in the previous sections, which will make it easier for the readers to understand our findings. The conclusion of our research article have been presented in Section 6.

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2. Preliminaries

Let $\mathbb{H}$ be a real Banach space with the norm $\parallel . \parallel$. Assume $\mathrm{B}(\theta, r)=\{t \in \mathbb{H}: \parallel t-\theta \parallel \leq r\}$. If $\mathfrak{B}(\neq \emptyset)$ is a subset of $\mathbb{H}$ so by $\bar{\mathfrak{B}}$ and Conv$\mathfrak{B}$ we denote the closure and convex closure of $\mathfrak{B}$. Additionally, let

- $\mathfrak{M}_{\mathbb{H}}=$ the set of all bounded and non-empty subsets of $\mathbb{H}$,
- $\mathfrak{N}_{\mathbb{H}}=$ the set of all relatively compact sets,

The definition of $\mathcal{M}\mathbb{N}\mathcal{C}$ follows from [13].

Definition 2.1 A map $\hat{\aleph}: \mathfrak{M}_{\mathbb{H}} \rightarrow \Re_{+}$ is called a $\mathcal{M}\mathbb{N}\mathcal{C}$ in $\mathbb{H}$ if it satisfies the following axioms:

(i) for all $\mathfrak{B} \in \mathfrak{M}_{\mathbb{H}}$, we get $\hat{\aleph}(\mathfrak{B})=0$ which shows $\mathfrak{B}$ has relative compactness.

(ii) $\ker \hat{\aleph}=\left\{\mathfrak{B} \in \mathfrak{M}_{\mathbb{H}}: \hat{\aleph}(\mathfrak{B})=0\right\} \neq \emptyset$ and $\ker \hat{\aleph} \subset \mathfrak{N}_{\mathbb{H}}$.

(iii) $\mathfrak{B}_1 \subseteq \mathfrak{B}_2 \implies \hat{\aleph}(\mathfrak{B}_1) \leq \hat{\aleph}(\mathfrak{B}_2)$.

(iv) $\hat{\aleph}(\bar{\mathfrak{B}})=\hat{\aleph}(\mathfrak{B})$.

(v) $\hat{\aleph}(Conv\mathfrak{B})=\hat{\aleph}(\mathfrak{B})$.

(vi) $\hat{\aleph}(\rho \mathfrak{B}+(1-\rho) \mathfrak{B}_1) \leq \rho \hat{\aleph}(\mathfrak{B})+(1-\rho) \hat{\aleph}(\mathfrak{B}_1)$ for any $\rho \in[0,1]$.

(vii) If $\mathfrak{B}_l \in \mathfrak{M}_{\mathbb{H}}, \mathfrak{B}_l=\bar{\mathfrak{B}}_l, \mathfrak{B}_{l+1} \subset \mathfrak{B}_l$ for $l=1,2,3,4, \ldots$ and $\lim\limits_{l \rightarrow \infty} \hat{\aleph}(\mathfrak{B}_l)=0$ then $\mathfrak{B}_{\infty}=\bigcap_{l=1}^{\infty} \mathfrak{B}_l \neq \emptyset$.

The set $\mathfrak{B}_{\infty}=\bigcap_{l=1}^{\infty} \mathfrak{B}_l \in ker\hat{\aleph}$. Since $\hat{\aleph}(\mathfrak{B}_{\infty}) \leq \hat{\aleph}(\mathfrak{B}_l)$ for any l, we conclude $\hat{\aleph}(\mathfrak{B}_{\infty})=0$.

Theorem 2.1 *(Schauder [1]) Let $\mathfrak{B}$ be a closed, non-empty and convex subset of a Banach space $\mathbb{H}$. Then for every continuous and compact map $\mathfrak{G}: \mathfrak{B} \rightarrow \mathfrak{B}$ $\exists$ at least one fixed point.*

Theorem 2.2 *(Darbo [8]) Let $\mathfrak{B}$ be a non-empty, bounded, closed and convex subset(NBCCS) of a Banach space $\mathbb{H}$ and let $\mathfrak{H}: \mathfrak{B} \rightarrow \mathfrak{B}$. Assume that for a constant $b \in[0,1)$*

$$\hat{\aleph}(\mathfrak{H}\mathfrak{J}) \leq b \hat{\aleph}(\mathfrak{J}),\ \mathfrak{J} \subseteq \mathfrak{B}.$$

Then there exists a fixed point in $\mathfrak{B}$ for $\mathfrak{H}$ provided that $\mathfrak{H}$ is a continuous mapping.

The following related concepts are needed in order to establish an extension of Darbo's fixed point theorem:

Definition 2.2 [12] Let $\mathfrak{S}$ be the collection of functions $\lambda: \Re_{+} \times \Re_{+} \rightarrow \Re_{+}$ satisfying:
(1) $\max \{k, k'\} \leq \lambda(k, k')$ for $k, k' \geq 0$.
(2) $\lambda(k_1+k_2, k_1'+k_2') \leq \lambda(k_1, k_1')+\lambda(k_2, k_2')$.
(3) λ is non-decreasing and continuous.

Definition 2.3 [4] A function $\mathfrak{J}: \Re_{+} \times \Re_{+} \rightarrow \Re$ is a continuous function of $\mathcal{C}$- class which is symbolized by $\mathcal{C}$ if the following axioms hold true:
(1) $\mathfrak{J}(g, k) \leq g$,
(2) $\mathfrak{J}(g, k)=g$ implies that either $g=0$ or $k=0$. Also $\mathfrak{J}(0,0)=0$.

Definition 2.4 [7] Let $\mathcal{A}$ be the collection of all altering distance functions $\varepsilon: \Re_{+} \rightarrow \Re_{+}$ which are continuous and fufill the conditions:
(1) $\varepsilon(s)=0 \Leftrightarrow s=0$.
(2) ε is continuous and increasing.

3. Fixed point theory

Theorem 3.1 *Let $\mathfrak{B}$ be a NBCCS of a Banach space $\mathbb{H}$. Also $\mathcal{T}_1, \mathcal{T}_2 : \mathfrak{B} \to \mathfrak{B}$ are continuous mappings with*

$$\varepsilon[\lambda\left(\hat{\aleph}\left(\mathcal{T}_1 D\right), \varphi\left(\hat{\aleph}(\mathcal{T}_2 \hat{D})\right)\right)] \leq \mathfrak{J}\left[\varepsilon\left\{\lambda\left(\hat{\aleph}(D), \varphi\left(\hat{\aleph}(\hat{D})\right)\right)\right\}, \varpi\left\{\lambda\left(\hat{\aleph}(D), \varphi\left(\hat{\aleph}(\hat{D})\right)\right)\right\}\right] \tag{3.1}$$

where $D, \hat{D} \subseteq \mathfrak{B}$ and $\hat{\aleph}$ is an arbitrary $\mathcal{M}\mathbb{N}\mathcal{C}$ and $\varepsilon \in \mathcal{A}, \lambda \in \mathfrak{S}, \mathfrak{J} \in \mathcal{C}$. Also, $\varpi, \varphi : \Re_+ \to \Re_+$ are non-decreasing and continuous mappings such that $\varpi(0) = \varphi(0) = 0$ and $\varphi(\tau) = \tau$ for all $\tau > 0$. Then the operators $\mathcal{T}_1$ and $\mathcal{T}_2$ have a minimum of one fixed point in $\mathfrak{B}$.

Proof:

Let us consider the sequences $\mathbb{C}_p$ and $\mathbb{D}_p$ with $\mathbb{C}_1 = \mathbb{D}_1 = \mathfrak{B}$ and $\mathbb{C}_{p+1} = Conv(\mathcal{T}_1\mathbb{C}_p)$ and $\mathbb{D}_{p+1} = Conv(\mathcal{T}_2\mathbb{D}_p)$ for $p \in \mathbb{N}$. Then,

$$\mathcal{T}_1\mathbb{C}_1 = \mathcal{T}_1\mathfrak{B} \subseteq \mathfrak{B} = \mathbb{C}_1, \mathbb{C}_2 = Conv(\mathcal{T}_1\mathbb{C}_1) \subseteq \mathfrak{B} = \mathbb{C}_1$$

and

$$\mathcal{T}_2\mathbb{D}_1 = \mathcal{T}_2\mathfrak{B} \subseteq \mathfrak{B} = \mathbb{D}_1, \mathbb{D}_2 = Conv(\mathcal{T}_2\mathbb{D}_1) \subseteq \mathfrak{B} = \mathbb{D}_1$$

Proceeding similarly,

$$\mathbb{C}_1 \supseteq \mathbb{C}_2 \supseteq \mathbb{C}_3 \supseteq \ldots \supseteq \mathbb{C}_p \supseteq \mathbb{C}_{p+1} \supseteq \ldots.$$

and

$$\mathbb{D}_1 \supseteq \mathbb{D}_2 \supseteq \mathbb{D}_3 \supseteq \ldots \supseteq \mathbb{D}_p \supseteq \mathbb{D}_{p+1} \supseteq \ldots.$$

If $\hat{\aleph}(\mathbb{C}_{p_0}) = \hat{\aleph}(\mathbb{D}_{p_0}) = 0$ for some $p_0 \in \mathbb{N}$ then $\mathbb{C}_{p_0}$ and $\mathbb{D}_{p_0}$ are compact sets. In this case Schauder's Theorem implies $\mathcal{T}_1$ and $\mathcal{T}_2$ have fixed points in $\mathfrak{B}$.

Assume that $\hat{\aleph}(\mathbb{C}_p) > 0$ and $\hat{\aleph}(\mathbb{D}_p) > 0$, for p $\in \mathbb{N}$. Since, $\hat{\aleph}(\mathbb{C}_{p+1}) \leq \hat{\aleph}(\mathbb{C}_p)$ and $\hat{\aleph}(\mathbb{D}_{p+1}) \leq \hat{\aleph}(\mathbb{D}_p)$, therefore there exists $u_0, v_0 \geq 0$ satisfying

$$\lim_{p\to\infty} \hat{\aleph}(\mathbb{C}_p) = u_0 \textit{ and } \lim_{p\to\infty} \hat{\aleph}(\mathbb{D}_p) = v_0.$$

Now, by (3.1) we obtain

$$\begin{aligned}
&\varepsilon\left[\lambda\Big(\hat{\aleph}(\mathbb{C}_{p+1}), \varphi(\hat{\aleph}(\mathbb{D}_{p+1}))\Big)\right] \\
&= \varepsilon\left[\lambda\Big(\hat{\aleph}(Conv\mathcal{T}_1\mathbb{C}_p), \varphi(\hat{\aleph}(Conv\mathcal{T}_2\mathbb{D}_p))\Big)\right] \\
&= \varepsilon\left[\lambda\Big(\hat{\aleph}(\mathcal{T}_1\mathbb{C}_p), \varphi(\hat{\aleph}(\mathcal{T}_2\mathbb{D}_p))\Big)\right] \\
&\leq \mathfrak{J}\left[\varepsilon\left\{\lambda\Big(\hat{\aleph}(\mathbb{C}_p), \varphi\left(\hat{\aleph}(\mathbb{D}_p)\right)\Big)\right\}, \varpi\left\{\lambda\Big(\hat{\aleph}(\mathbb{C}_p), \varphi\left(\hat{\aleph}(\mathbb{D}_p)\right)\Big)\right\}\right].
\end{aligned}$$

As $p \to \infty$, we get

$$\varepsilon\left[\lambda\Big(u_0, \varphi(v_0)\Big)\right] \leq \mathfrak{J}\left[\varepsilon\left\{\lambda\Big(u_0, \varphi\left(v_0\right)\Big)\right\}, \varpi\left\{\lambda\Big(u_0, \varphi\left(v_0\right)\Big)\right\}\right].$$

Hence by property of $\mathfrak{J}$,

$$\mathfrak{J}\left[\varepsilon\left\{\lambda\Big(u_0, \varphi\left(v_0\right)\Big)\right\}, \varpi\left\{\lambda\Big(u_0, \varphi\left(v_0\right)\Big)\right\}\right] \leq \varepsilon\left[\lambda\Big(u_0, \varphi(v_0)\Big)\right].$$

Thus, we get

$$\varepsilon\left[\lambda\Big(u_0, \varphi(v_0)\Big)\right] = \mathfrak{J}\left[\varepsilon\left\{\lambda\Big(u_0, \varphi\left(v_0\right)\Big)\right\}, \varpi\left\{\lambda\Big(u_0, \varphi\left(v_0\right)\Big)\right\}\right].$$

Therefore by property of $\mathfrak{J}$,

$$Either\ \varepsilon\left[\lambda\Big(u_0, \varphi(v_0)\Big)\right] = 0\ or\ \varpi\left\{\lambda\Big(u_0, \varphi\left(v_0\right)\Big)\right\} = 0.$$

Thus, by property of ε

$$\lambda(u_0, \varphi(v_0)) = 0.$$

By further property of λ, we get

$$u_0 = \varphi(v_0) = 0.$$

$$i.e., u_0 = v_0 = 0.$$

Therefore,

$$\lim_{p\to\infty} \hat{\aleph}\left(\mathbb{C}_p\right) = 0 = \lim_{p\to\infty} \hat{\aleph}\left(\mathbb{D}_p\right).$$

Since $\mathbb{C}_p \supseteq \mathbb{C}_{p+1}$, and $\mathbb{D}_p \supseteq \mathbb{D}_{p+1}$, by Definition 2.1, we obtain $\mathbb{C}_\infty = \bigcap_{p=1}^{\infty} \mathbb{C}_p$ and $\mathbb{D}_\infty = \bigcap_{p=1}^{\infty} \mathbb{D}_p$ are closed, convex and non-empty subsets of $\mathfrak{B}$ and $\mathbb{C}_\infty$ is $\mathcal{T}_1$ invariant while $\mathbb{D}_\infty$ is $\mathcal{T}_2$ invariant.
So, the Theorem 3.1 concludes that $\mathcal{T}_1$ and $\mathcal{T}_2$ have fixed points in $\mathfrak{B}$. □

Corollary 3.1 *Let $\mathfrak{B}$ be a NBCCS of a Banach space $\mathbb{H}$. Also $\mathcal{T}_1$ and $\mathcal{T}_2 : \mathfrak{B} \to \mathfrak{B}$ be continuous mappings with*

$$\varepsilon\left[\lambda\left(\hat{\aleph}(\mathcal{T}_1 D), \varphi(\hat{\aleph}(\mathcal{T}_2 \hat{D}))\right)\right] \leq \nabla\left[\varepsilon\left\{\lambda\left(\hat{\aleph}(D), \varphi(\hat{\aleph}(\hat{D}))\right)\right\}\right],\ \nabla \in [0, 1), \tag{3.2}$$

where $D, \hat{D} \subseteq \mathfrak{B}$ and $\hat{\aleph}$ is an arbitrary $\mathcal{M}\mathbb{N}\mathcal{C}$ and $\varepsilon \in \mathcal{A}, \lambda \in \mathfrak{S}$. Also, $\varphi : \Re_+ \to \Re_+$ is a non-decreasing and continuous mapping such that $\varphi(0) = 0$ and $\varphi(\tau) = \tau$ for all $\tau > 0$. Then the operators $\mathcal{T}_1$ and $\mathcal{T}_2$ have a minimum of one fixed point in $\mathfrak{B}$.

Proof: Taking $\mathfrak{J}(v, w) = \nabla v$ in Theorem 3.1 the above mentioned result can be obtained. □

Corollary 3.2 *Let $\mathfrak{B}$ be a NBCCS of a Banach space $\mathbb{H}$. Also $\mathcal{T}_1$ and $\mathcal{T}_2 : \mathfrak{B} \to \mathfrak{B}$ be continuous mappings with*

$$\hat{\aleph}(\mathcal{T}_1 D) + \hat{\aleph}(\mathcal{T}_2 \hat{D}) \leq \nabla[\hat{\aleph}(D) + \hat{\aleph}(\hat{D})],\ \nabla \in [0, 1), \tag{3.3}$$

where $D, \hat{D} \subseteq \mathfrak{B}$ and $\hat{\aleph}$ is an arbitrary $\mathcal{M}\mathbb{N}\mathcal{C}$. Then the operators $\mathcal{T}_1$ and $\mathcal{T}_2$ have a minimum of one fixed point in $\mathfrak{B}$.

Proof: Taking $\varepsilon(x) = x, \lambda(a, b) = a + b$ and $\varphi(x) = x$ in the Corollary 3.1, the above mentioned result can be obtained. □

Corollary 3.3 *Using $\mathcal{T} = \mathcal{T}_1 = \mathcal{T}_2$ and by taking $\varepsilon(x) = x, \lambda(a, b) = a + b, \varphi(x) = x$ in Corollary 3.1 we get*

$$\hat{\aleph}\left(\mathcal{T} D\right) \leq \nabla \hat{\aleph}(D),$$

where $\nabla \in [0, 1)$ and $D \subseteq \mathfrak{B}$. Thus, we obtain the Darbo's $\mathcal{F}\mathbb{P}\mathcal{T}$.

Definition 3.1 An element $(h_1, n_1) \in \lambda \times \lambda$ of a mapping $\mathfrak{A} : \lambda \times \lambda \to \lambda$ is called a coupled fixed point theorem if $\mathfrak{A}(h_1, n_1) = h_1$ and $\mathfrak{A}(h_1, n_1) = n_1$.

Theorem 3.2 *Suppose $\hat{\aleph}_1, \hat{\aleph}_2, ..., \hat{\aleph}_n$ are the respective $\mathcal{M}\mathbb{N}\mathcal{C}$s in $\mathbb{H}_1, \mathbb{H}_2, ..., \mathbb{H}_n$, and a convex mapping $\mathfrak{A} : \Re_n^+ \to \Re^+$ with $\mathfrak{A}(q_1, q_2, ..., q_n) = 0 \iff q_t = 0, t \in \{1, ..., n\}$. Then $\hat{\aleph}(\Omega) = \mathfrak{A}(\hat{\aleph}_1(\Omega_1), \hat{\aleph}_2(\Omega_2), ..., \hat{\aleph}_n(\Omega_n))$ defines an $\mathcal{M}\mathbb{N}\mathcal{C}$ in $\mathbb{H}_1 \times \mathbb{H}_2 \times ... \times \mathbb{H}_n$ in which Ω_q identifies a natural projection of Ω into $\mathbb{H}_q, q \in \{1, ..., n\}$.*

Example 3.1 Consider $\hat{\aleph}$ to be an $\mathcal{M}\mathbb{N}\mathcal{C}$ on $\mathbb{H}$ and $\mathfrak{A}(h_1, n_1) = h_1 + n_1, h_1, n_1 \in \Re^+$. Then $\hat{\aleph}^{cf}(\Omega) = \hat{\aleph}(\Omega_1) + \hat{\aleph}(\Omega_2)$ is an $\mathcal{M}\mathbb{N}\mathcal{C}$ in $\mathbb{H} \times \mathbb{H}$.

Theorem 3.3 *Let* $\mathfrak{B}$ *be a NBCCS of a BS* $\mathbb{H}$*. Let* $\mathcal{T}_1, \mathcal{T}_2 : \mathfrak{B} \times \mathfrak{B} \to \mathfrak{B}$ *be continuous with*

$$\begin{aligned}
&\varepsilon \Big[\lambda \Big(\hat{\aleph}\left(\mathcal{T}_1\left(M_1 \times M_2\right)\right), \varphi \Big(\hat{\aleph}(\mathcal{T}_2\left(M_1 \times M_2\right))\Big)\Big)\Big] \\
&\leq \frac{1}{2} \mathfrak{J} \Big[\varepsilon \Big\{\lambda \Big(\hat{\aleph}\left(M_1\right)+\hat{\aleph}\left(M_2\right), \varphi(\hat{\aleph}\left(M_1\right)+\hat{\aleph}\left(M_2\right))\Big)\Big\}, \varpi \Big\{\lambda \Big(\hat{\aleph}\left(M_1\right)+\hat{\aleph}\left(M_2\right), \varphi(\hat{\aleph}\left(M_1\right)+\hat{\aleph}\left(M_2\right))\Big)\Big\}\Big]
\end{aligned}$$

where $M_1, M_2 \subseteq \mathfrak{B}$ and $\hat{\aleph}$ is an arbitrary $\mathcal{M}\mathbb{N}\mathcal{C}$ and $\varepsilon \in \mathcal{A}, \lambda \in \mathfrak{S}, \mathfrak{J} \in \mathcal{C}$ and $\varpi, \varphi : \Re_+ \to \Re_+$ are non-decreasing and continuous mappings such that $\varpi(0) = \varphi(0) = 0$ and $\varphi(\tau) = \tau$ for all $\tau > 0$. Also $\varphi(s_1 + s_2) \leq \varphi(s_1) + \varphi(s_2)$ and $\varepsilon(s_1 + s_2) \leq \varepsilon(s_1) + \varepsilon(s_2),\ s_1, s_2 \geq 0$. Then the operators $\mathcal{T}_1$ and $\mathcal{T}_2$ have a coupled fixed point in $\mathfrak{B} \times \mathfrak{B}$.

Proof: We have $\hat{\aleph}^{cf}(M) = \hat{\aleph}(M_1) + \hat{\aleph}(M_2)$ is an $\mathcal{M}\mathbb{N}\mathcal{C}$ on $\mathbb{H} \times \mathbb{H}$ for any bounded subset $M \subseteq \mathbb{H} \times \mathbb{H}$, where M_1, M_2 are natural projections of M. Let $\mathbb{T}_1^{cf} : \mathfrak{B} \times \mathfrak{B} \to \mathfrak{B} \times \mathfrak{B}$ and $\mathbb{T}_2^{cf} : \mathfrak{B} \times \mathfrak{B} \to \mathfrak{B} \times \mathfrak{B}$ be defined by $\mathbb{T}_1^{cf}(u_0, v_0) = (\mathfrak{A}(u_0, v_0), \mathfrak{A}(v_0, u_0))$ and $\mathbb{T}_2^{cf}(u_0, v_0) = (\mathfrak{A}(u_0, v_0), \mathfrak{A}(v_0, u_0))$, respectively. It is trivial to see that both $\mathbb{T}_1^{cf}$ and $\mathbb{T}_2^{cf}$ are continuous. Let $M \subseteq \mathfrak{B} \times \mathfrak{B}$. Now,

$$\begin{aligned}
&\varepsilon \Big[\lambda \Big(\hat{\aleph}^{cf}(\mathcal{T}_1^{cf}(M)), \varphi(\hat{\aleph}^{cf}(\mathcal{T}_2^{cf}(M)))\Big)\Big] \\
&\leq \varepsilon \Big[\lambda \Big(\hat{\aleph}^{cf}(\mathcal{T}_1(M_1 \times M_2) \times \mathcal{T}_1(M_2 \times M_1)), \varphi(\hat{\aleph}^{cf}(\mathcal{T}_2(M_1 \times M_2) \times \mathcal{T}_2(M_2 \times M_1)))\Big)\Big] \\
&= \varepsilon \Big[\lambda \Big(\hat{\aleph}(\mathcal{T}_1(M_1 \times M_2)) + \hat{\aleph}(\mathcal{T}_1(M_2 \times M_1)), \varphi(\hat{\aleph}(\mathcal{T}_2(M_1 \times M_2)) + \hat{\aleph}(\mathcal{T}_2(M_2 \times M_1)))\Big)\Big] \\
&\leq \varepsilon \Big[\lambda \Big(\hat{\aleph}(\mathcal{T}_1(M_1 \times M_2)) + \hat{\aleph}(\mathcal{T}_1(M_2 \times M_1)), \varphi \Big(\hat{\aleph}(\mathcal{T}_2(M_1 \times M_2))\Big) + \varphi \Big(\hat{\aleph}\left(M_2 \times M_1\right)\Big)\Big)\Big] \\
&\leq \varepsilon \Big[\lambda \Big(\hat{\aleph}(\mathcal{T}_1(M_1 \times M_2)) + \varphi(\hat{\aleph}(\mathcal{T}_2(M_1 \times M_2)))\Big) + \lambda \Big(\hat{\aleph}(\mathcal{T}_1(M_2 \times M_1)) + \varphi(\hat{\aleph}(\mathcal{T}_2(M_2 \times M_1))))\Big)\Big] \\
&\leq \varepsilon \Big[\lambda \Big(\hat{\aleph}(\mathcal{T}_1(M_1 \times M_2)) + \varphi(\hat{\aleph}(\mathcal{T}_2(M_1 \times M_2)))\Big)\Big] + \varepsilon \Big[\lambda \Big(\hat{\aleph}(\mathcal{T}_1(M_2 \times M_1)) + \varphi(\hat{\aleph}(\mathcal{T}_2(M_2 \times M_1))))\Big)\Big] \\
&\leq \mathfrak{J} \Big[\varepsilon \Big\{\lambda \Big(\hat{\aleph}\left(M_1\right)+\hat{\aleph}\left(M_2\right), \varphi(\hat{\aleph}\left(M_1\right)+\hat{\aleph}\left(M_2\right))\Big)\Big\}, \varpi \Big\{\lambda \Big(\hat{\aleph}\left(M_1\right)+\hat{\aleph}\left(M_2\right), \varphi(\hat{\aleph}\left(M_1\right)+\hat{\aleph}\left(M_2\right))\Big)\Big\}\Big] \\
&= \mathfrak{J} \Big[\varepsilon \Big\{\lambda \Big(\hat{\aleph}^{cf}\left(M\right), \varphi(\hat{\aleph}^{cf}(M))\Big)\Big\}, \varpi \Big\{\lambda \Big(\hat{\aleph}^{cf}\left(M\right), \varphi(\hat{\aleph}^{cf}(M))\Big)\Big\}\Big]
\end{aligned}$$

By Theorem 3.1 we arrive at the conclusion that $\mathcal{T}_1^{cf}$ and $\mathcal{T}_2^{cf}$ have fixed points in $\mathfrak{B} \times \mathfrak{B}$, i.e., both $\mathcal{T}_1$ and $\mathcal{T}_2$ have coupled fixed points in $\mathfrak{B} \times \mathfrak{B}$. □

4. Measure of noncompactness on $C([0, T])$

Let $\mathbb{H} = C(I)$ be the space of continuous real functions on $I = [0, T]$. So, implemented with the norm

$$\parallel \Upsilon \parallel = \sup \left\{|\Upsilon(\sigma)| : \sigma \in I\right\}, \ \Upsilon \in \mathbb{H}.$$

Let $\Xi(\neq \emptyset) \subseteq \mathbb{H}$ be bounded. For $\Upsilon \in \Xi$ with $\hbar > 0$, denote by $\psi(\Upsilon, \varepsilon)$ the modulus of the continuity of Υ, i.e.,

$$\psi(\Upsilon, \hbar) = \sup \left\{|\Upsilon(\sigma_1) - \Upsilon(\sigma_2)| : \sigma_1, \sigma_2 \in I, |\sigma_2 - \sigma_1| \leq \hbar\right\}.$$

In addition, we define

$$\psi(\Xi, \hbar) = \sup \left\{\psi(\Upsilon, h) : \Upsilon \in \Xi\right\}; \ \psi_0(\Xi) = \lim_{\hbar \to 0} \psi(\Xi, \hbar).$$

The function ψ_0 is therefore generally known as an $\mathcal{M}\mathbb{N}\mathcal{C}$ in $\mathbb{H}$, with $\mathit{\Theta}(\Xi) = \frac{1}{2}\psi_0(\Xi)$ (see [5]) as the Hausdorff $\mathcal{M}\mathbb{N}\mathcal{C}$ $\mathit{\Theta}$.

5. Solvability of fractional integral equations

In this section, we will prove the existence of solutions of a system of $\mathcal{F}\mathbb{I}\mathcal{E}$s in a Banach space $\mathbb{H}$ via the application of our conclusions.
We define the left Katugampola fractional integral [19] of order $\alpha > 0$ of f by

$$ {}^{\rho}I^{\alpha}_{a+}f(\bar{s}) = \frac{\rho^{1-\alpha}}{\Gamma(\alpha)}\int_a^{\bar{s}} \frac{s^{\rho-1}}{(\bar{s}^{\rho}-s^{\rho})^{1-\alpha}}f(s)ds. \tag{5.1}$$

with $\rho > 0$ if the integral exists.
In the following section we scrutinize the given system of $\mathcal{F}\mathbb{I}\mathcal{E}$s

$$\mathfrak{a}(\bar{s}) = \mathfrak{L}_1(\bar{s},\mathfrak{a}(\bar{s})) + \frac{\mathcal{H}_1(\bar{s},\mathfrak{a}(\bar{s}))\rho^{1-\alpha}}{\Gamma(\alpha)}\int_0^{\bar{s}} \frac{s^{\rho-1}\mathfrak{P}_1(s,\mathfrak{a}(s))}{(\bar{s}^{\rho}-s^{\rho})^{1-\alpha}}ds \tag{5.2}$$

and

$$\mathfrak{b}(\bar{s}) = \mathfrak{L}_2(\bar{s},\mathfrak{b}(\bar{s})) + \frac{\mathcal{H}_2(\bar{s},\mathfrak{b}(\bar{s}))\rho^{1-\alpha}}{\Gamma(\alpha)}\int_0^{\bar{s}} \frac{s^{\rho-1}\mathfrak{P}_2(s,\mathfrak{b}(s))}{(\bar{s}^{\rho}-s^{\rho})^{1-\alpha}}ds, \tag{5.3}$$

where $\alpha > 0, \rho > 0, \bar{s} \in I = [0,T]$.
Let

$$\mathbb{Q}_{r_0} = \{\alpha \in \mathbb{H} : \parallel \alpha \parallel \leq r_0\}.$$

Assume that

(i) $\mathfrak{L}_1, \mathfrak{L}_2, \mathcal{H}_1, \mathcal{H}_2, \mathfrak{P}_1, \mathfrak{P}_2 : I \times \Re^+ \to \Re$ are continuous and $\exists$ constants $\gamma_1, \hat{\gamma}_1, \gamma_2, \hat{\gamma}_2, \gamma_3, \hat{\gamma}_3 \geq 0$ satisfying

$$\begin{aligned}
|\mathfrak{L}_1(\bar{s},\mathfrak{a}(\bar{s})) - \mathfrak{L}_1(\bar{s},\bar{\mathfrak{a}}(\bar{s}))| &\leq \gamma_1|\mathfrak{a}(\bar{s}) - \bar{\mathfrak{a}}(\bar{s})|,\\
|\mathfrak{L}_2(\bar{s},\mathfrak{a}(\bar{s})) - \mathfrak{L}_2(\bar{s},\bar{\mathfrak{a}}(\bar{s}))| &\leq \hat{\gamma}_1|\mathfrak{a}(\bar{s}) - \bar{\mathfrak{a}}(\bar{s})|,\\
|\mathcal{H}_1(\bar{s},\mathfrak{a}(\bar{s})) - \mathcal{H}_1(\bar{s},\bar{\mathfrak{a}}(\bar{s}))| &\leq \gamma_2|\mathfrak{a}(\bar{s}) - \bar{\mathfrak{a}}(\bar{s})|,\\
|\mathcal{H}_2(\bar{s},\mathfrak{a}(\bar{s})) - \mathcal{H}_2(\bar{s},\bar{\mathfrak{a}}(\bar{s}))| &\leq \hat{\gamma}_2|\mathfrak{a}(\bar{s}) - \bar{\mathfrak{a}}(\bar{s})|,\\
|\mathfrak{P}_1(\bar{s},\mathfrak{a}(\bar{s})) - \mathfrak{P}_1(\bar{s},\bar{\mathfrak{a}}(\bar{s}))| &\leq \gamma_3|\mathfrak{a}(\bar{s}) - \bar{\mathfrak{a}}(\bar{s})|,
\end{aligned}$$

and

$$|\mathfrak{P}_2(\bar{s},\mathfrak{a}(\bar{s})) - \mathfrak{P}_2(\bar{s},\bar{\mathfrak{a}}(\bar{s}))| \leq \hat{\gamma}_3|\mathfrak{a}(\bar{s}) - \bar{\mathfrak{a}}(\bar{s})|.$$

Also, for all $\bar{s} \geq 0$,

$$\begin{aligned}
\mathfrak{L}_1(\bar{s},0) &= 0 = \mathfrak{L}_2(\bar{s},0),\\
\mathcal{H}_1(\bar{s},0) &= 0 = \mathcal{H}_2(\bar{s},0),
\end{aligned}$$

and

$$\mathfrak{P}_1(\bar{s},0) = 0 = \mathfrak{P}_2(\bar{s},0).$$

(ii) For the following inequalities-

$$\gamma_1 + \frac{\gamma_2\gamma_3 r_0 T^{\alpha}}{\rho^{\alpha}\Gamma(\alpha+1)} < 1$$

and

$$\hat{\gamma}_1 + \frac{\hat{\gamma}_2\hat{\gamma}_3 r_0 T^{\alpha}}{\rho^{\alpha}\Gamma(\alpha+1)} < 1.$$

$\exists$ a positive solution r_0.

Theorem 5.1 *If constraints (i)-(ii) sustain, then the equations (5.2) and (5.3) have solutions in* $\mathbb{H} = C(I)$.

Proof: Let us consider the following operators $\mathfrak{T}, \hat{\mathfrak{T}} : \mathbb{H} \to \mathbb{H}$ defined as:

$$(\mathfrak{T}\mathfrak{a})(\bar{s}) = \mathfrak{L}_1(\bar{s}, \mathfrak{a}(\bar{s})) + \frac{\mathcal{H}_1(\bar{s}, \mathfrak{a}(\bar{s}))\rho^{1-\alpha}}{\Gamma(\alpha)} \int_0^{\bar{s}} \frac{s^{\rho-1}\mathfrak{P}_1(s, \mathfrak{a}(s))}{(\bar{s}^\rho - s^\rho)^{1-\alpha}} ds$$

and

$$(\hat{\mathfrak{T}}\mathfrak{b})(\bar{s}) = \mathfrak{L}_2(\bar{s}, \mathfrak{b}(\bar{s})) + \frac{\mathcal{H}_2(\bar{s}, \mathfrak{b}(\bar{s}))\rho^{1-\alpha}}{\Gamma(\alpha)} \int_0^{\bar{s}} \frac{s^{\rho-1}\mathfrak{P}_2(s, \mathfrak{b}(s))}{(\bar{s}^\rho - s^\rho)^{1-\alpha}} ds.$$

Phase (1): In this phase we show that the operators $\mathfrak{T}$ and $\hat{\mathfrak{T}}$ map $\mathbb{Q}_{r_0}$ into $\mathbb{Q}_{r_0}$. Let $\mathfrak{a}, \mathfrak{b} \in \mathbb{Q}_{r_0}$. We now have

$$\begin{aligned}
&|(\mathfrak{T}\mathfrak{a})(\bar{s})| \\
&\leq |\mathfrak{L}_1(\bar{s}, \mathfrak{a}(\bar{s}))| + \frac{|\mathcal{H}_1(\bar{s}, \mathfrak{a}(\bar{s}))\rho^{1-\alpha}|}{\Gamma(\alpha)} \left| \int_0^{\bar{s}} \frac{s^{\rho-1}\mathfrak{P}_1(s, \mathfrak{a}(s))}{(\bar{s}^\rho - s^\rho)^{1-\alpha}} ds \right| \\
&\leq |\mathfrak{L}_1(\bar{s}, \mathfrak{a}(\bar{s})) - \mathfrak{L}_1(\bar{s}, 0)| + \\
&+ \frac{[|\mathcal{H}_1(\bar{s}, \mathfrak{a}(\bar{s})) - \mathcal{H}_1(\bar{s}, 0)|]\, \rho^{1-\alpha}}{\Gamma(\alpha)} \\
&\left| \int_0^{\bar{s}} \frac{s^{\rho-1}[\mathfrak{P}_1(s, \mathfrak{a}(s)) - \mathfrak{P}_1(s, 0)]}{(\bar{s}^\rho - s^\rho)^{1-\alpha}} ds \right| \\
&\leq \gamma_1 |\mathfrak{a}(\bar{s})| + \frac{\gamma_2 |\mathfrak{a}(\bar{s})|}{\rho^{\alpha-1}\Gamma(\alpha)} \left| \int_0^{\bar{s}} s^{\rho-1}(\bar{s}^\rho - s^\rho)^{\alpha-1} \gamma_3 |\mathfrak{a}(s)| \right| ds \\
&\leq \gamma_1 \|\mathfrak{a}\| + \frac{\gamma_2\gamma_3 \|\mathfrak{a}\|^2}{\rho^{\alpha-1}\Gamma(\alpha)} \left| \int_0^{\bar{s}} s^{\rho-1}(\bar{s}^\rho - s^\rho)^{\alpha-1} \right| ds \\
&\leq \gamma_1 \|\mathfrak{a}\| + \frac{\gamma_2\gamma_3 \|\mathfrak{a}\|^2 T^\alpha}{\rho^\alpha \Gamma(\alpha+1)}.
\end{aligned}$$

Hence $\| \mathfrak{a} \| \leq r_0$ gives

$$\| \mathfrak{T} \| \leq \gamma_1 r_0 + \frac{\gamma_2\gamma_3 r_0^2 T^\alpha}{\rho^\alpha \Gamma(\alpha+1)} < r_0.$$

Thus, $\mathfrak{T}$ maps $\mathbb{Q}_{r_0}$ to $\mathbb{Q}_{r_0}$. Similarly, we can prove that $\hat{\mathfrak{T}}$ maps $\mathbb{Q}_{r_0}$ to $\mathbb{Q}_{r_0}$.

Phase (2): In this phase we show that $\mathfrak{T}$ is a continuous operator on $\mathbb{Q}_{r_0}$. Let $\varepsilon > 0$ and $\mathfrak{a}, \mathfrak{a}_1 \in \mathbb{Q}_{r_0}$ such that $\| \mathfrak{a} - \mathfrak{a}_1 \| < \varepsilon$. We now have

$$\begin{aligned}
&|(\mathfrak{T}\mathfrak{a})\,(\bar{s}) - (\mathfrak{T}\mathfrak{a}_1)\,(\bar{s})| \\
&\leq |\mathfrak{L}_1(\bar{s}, \mathfrak{a}(\bar{s})) - \mathfrak{L}_1(\bar{s}, \mathfrak{a}_1(\bar{s}))| \\
&+ \frac{1}{\rho^{\alpha-1}\Gamma(\alpha)} \Big| \mathcal{H}_1(\bar{s}, \mathfrak{a}(\bar{s})) \int_0^{\bar{s}} \frac{s^{\rho-1}}{(\bar{s}^\rho - s^\rho)^{1-\alpha}} \mathfrak{P}_1(s, \mathfrak{a}(s)) ds \\
&- \mathcal{H}_1(\bar{s}, \mathfrak{a}_1(\bar{s}) \int_0^{\bar{s}} \frac{s^{\rho-1}}{(\bar{s}^\rho - s^\rho)^{1-\alpha}} \mathfrak{P}_1(s, \mathfrak{a}_1(s)) ds \Big|
\end{aligned}$$

$$
\begin{aligned}
&\leq \gamma_1|\mathfrak{a}(\bar{s})-\mathfrak{a}_1(\bar{s})|+\frac{1}{\rho^{\alpha-1}\Gamma(\alpha)}\Big|\mathcal{H}_1(\bar{s},\mathfrak{a}(\bar{s}))\\
&\int_0^{\bar{s}}\frac{s^{\rho-1}}{(\bar{s}^{\rho}-s^{\rho})^{1-\alpha}}\left\{\mathfrak{P}_1(s,\mathfrak{a}(s))-\mathfrak{P}_1(s,\mathfrak{a}_1(s))\right\}ds\Big|\\
&+\frac{1}{\rho^{\alpha-1}\Gamma(\alpha)}\Big|\left\{\mathcal{H}_1(\bar{s},\mathfrak{a}(\bar{s}))-\mathcal{H}_1(\bar{s},\mathfrak{a}_1(\bar{s})\right\}\\
&\int_0^{\bar{s}}\frac{s^{\rho-1}}{(\bar{s}^{\rho}-s^{\rho})^{1-\alpha}}\mathfrak{P}_1(s,\mathfrak{a}_1(s))\Big|ds\\
&\leq \gamma_1\|\mathfrak{a}-\mathfrak{a}_1\|+\frac{|\gamma_2(\mathfrak{a}(\bar{s}))|}{\rho^{\alpha-1}\Gamma(\alpha)}\Big|\int_0^{\bar{s}}\frac{s^{\rho-1}}{(\bar{s}^{\rho}-s^{\rho})^{1-\alpha}}\Big|\\
&\gamma_3|\mathfrak{a}(s)-\mathfrak{a}_1(s)|ds+\frac{\gamma_2|\mathfrak{a}(\bar{s})-\mathfrak{a}_1(\bar{s})|}{\rho^{\alpha-1}\Gamma(\alpha)}\\
&\Big|\int_0^{\bar{s}}\frac{s^{\rho-1}}{(\bar{s}^{\rho}-s^{\rho})^{1-\alpha}}\Big|\gamma_3|\mathfrak{a}_1(s)|ds\\
&\leq \gamma_1\|\mathfrak{a}-\mathfrak{a}_1\|+\frac{\gamma_3\gamma_2\|\mathfrak{a}\|\|\mathfrak{a}-\mathfrak{a}_1\|T^{\alpha}}{\rho^{\alpha}\Gamma(\alpha+1)}\\
&+\frac{\gamma_3\gamma_2\|\mathfrak{a}_1\|\|\mathfrak{a}-\mathfrak{a}_1\|T^{\alpha}}{\rho^{\alpha}\Gamma(\alpha+1)}\\
&\leq \gamma_1\varepsilon+\frac{2\gamma_3\gamma_2 r_0\varepsilon T^{\alpha}}{\rho^{\alpha}\Gamma(\alpha+1)}.
\end{aligned}
$$

As $\varepsilon\to 0$ we get $|(\mathfrak{T}\mathfrak{a})(\bar{s})-(\mathfrak{T}\mathfrak{a}_1)(\bar{s})|\to 0$.
Thus the operator $\mathfrak{T}$ is continuous on $\mathbb{Q}_{r_0}$. Similarly, we can prove that $\hat{\mathfrak{T}}$ is a continuous operator on $\mathbb{Q}_{r_0}$.
Phase (3): An estimation of $\mathfrak{T}$ and $\hat{\mathfrak{T}}$ with respect to the $\mathcal{M}\mathbb{N}\mathcal{C}$ ψ_0.
Assuming $Q_{\mathfrak{a}},Q_{\mathfrak{b}}\subseteq\mathbb{Q}_{r_0}$. Let $\beta>0$ be an arbitrary constant and let $\mathfrak{a}\in Q_{\mathfrak{a}}$, $\mathfrak{b}\in Q_{\mathfrak{b}}$ and $\bar{s}_1,\bar{s}_2\in I$ such as $|\bar{s}_2-\bar{s}_1|\leq\beta$ with $\bar{s}_2\geq\bar{s}_1$.
We have,

$$
\begin{aligned}
&|(\mathfrak{T}\mathfrak{a})(\bar{s}_2)-(\mathfrak{T}\mathfrak{a})(\bar{s}_1)|\\
&\leq|\mathfrak{L}_1(\bar{s}_2,\mathfrak{a}(\bar{s}_2))-\mathfrak{L}_1(\bar{s}_1,\mathfrak{a}(\bar{s}_1))|\\
&+\frac{1}{\rho^{\alpha-1}\Gamma(\alpha)}\Big|\mathcal{H}_1(\bar{s}_2,\mathfrak{a}(\bar{s}_2))\int_0^{\bar{s}_2}\frac{s^{\rho-1}}{(\bar{s}_2^{\rho}-s^{\rho})^{1-\alpha}}\mathfrak{P}_1(s,\mathfrak{a}(s))ds\\
&-\mathcal{H}_1(\bar{s}_1,\mathfrak{a}(\bar{s}_1)\int_0^{\bar{s}_1}\frac{s^{\rho-1}}{(\bar{s}_1^{\rho}-s^{\rho})^{1-\alpha}}\mathfrak{P}_1(s,\mathfrak{a}(s))ds\Big|\\
&\leq|\mathfrak{L}_1(\bar{s}_2,\mathfrak{a}(\bar{s}_2))-\mathfrak{L}_1(\bar{s}_1,\mathfrak{a}(\bar{s}_1))|\\
&+\frac{1}{\rho^{\alpha-1}\Gamma(\alpha)}\Big|\mathcal{H}_1(\bar{s}_2,\mathfrak{a}(\bar{s}_2))\int_0^{\bar{s}_2}\frac{s^{\rho-1}}{(\bar{s}_2^{\rho}-s^{\rho})^{1-\alpha}}\mathfrak{P}_1(s,\mathfrak{a}(s))ds\\
&-\mathcal{H}_1(\bar{s}_2,\mathfrak{a}(\bar{s}_2))\int_0^{\bar{s}_1}\frac{s^{\rho-1}}{(\bar{s}_1^{\rho}-s^{\rho})^{1-\alpha}}\mathfrak{P}_1(s,\mathfrak{a}(s))ds\Big|
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{1}{\rho^{\alpha-1}\Gamma(\alpha)}\Big|\left\{\mathcal{H}_1(\bar{s}_2,\mathfrak{a}(\bar{s}_2))-\mathcal{H}_1(\bar{s}_1,\mathfrak{a}(\bar{s}_1))\right\}\\
&\int_0^{\bar{s}_1}\frac{s^{\rho-1}}{(\bar{s}_1^{\rho}-s^{\rho})^{1-\alpha}}\mathfrak{P}_1(s,\mathfrak{a}(s))ds\Big|.
\end{aligned}
$$

Also

$$\begin{aligned}
&\Big|\{\mathcal{H}_1(\bar{s}_2,\mathfrak{a}(\bar{s}_2))-\mathcal{H}_1(\bar{s}_1,\mathfrak{a}(\bar{s}_1))\}\int_0^{\bar{s}_1}\frac{s^{\rho-1}}{(\bar{s}_1^{\rho}-s^{\rho})^{1-\alpha}}\mathfrak{P}_1(s,\mathfrak{a}(s))ds\Big|\\
&\leq|\{\mathcal{H}_1(\bar{s}_2,\mathfrak{a}(\bar{s}_2))-\mathcal{H}_1(\bar{s}_1,\mathfrak{a}(\bar{s}_1))\}|\Big|\int_0^{\bar{s}_1}\frac{s^{\rho-1}}{(\bar{s}_1^{\rho}-s^{\rho})^{1-\alpha}}\mathfrak{P}_1(s,\mathfrak{a}(s))ds\Big|\\
&\leq|\mathcal{H}_1(\bar{s}_2,\mathfrak{a}(\bar{s}_2))-\mathcal{H}_1(\bar{s}_2,\mathfrak{a}(\bar{s}_1))|+|\mathcal{H}_1(\bar{s}_2,\mathfrak{a}(\bar{s}_1))-\mathcal{H}_1(\bar{s}_1,\mathfrak{a}(\bar{s}_1))|\frac{r_0\gamma_3T^{\alpha}}{\rho\alpha}\\
&\leq\{\gamma_2|\mathfrak{a}(\bar{s}_2)-\mathfrak{a}(\bar{s}_1)|+\acute{\varkappa}(\beta)\}\,\frac{r_0\gamma_3T^{\alpha}}{\rho\alpha},
\end{aligned}$$

where

$$\acute{\varkappa}(\beta)=sup\,\{|\mathcal{H}_1(\bar{s}_2,\mathfrak{a}(\bar{s}_1))-\mathcal{H}_1(\bar{s}_1,\mathfrak{a}(\bar{s}_1))|:|\bar{s}_2-\bar{s}_1|\leq\beta,\bar{s}_1,\bar{s}_2\in I,\|\mathfrak{a}\|\leq r_0\}$$

and

$$\begin{aligned}
&\Big|\mathcal{H}_1(\bar{s}_2,\mathfrak{a}(\bar{s}_2))\Big\{\int_0^{\bar{s}_2}\frac{s^{\rho-1}}{(\bar{s}_2^{\rho}-s^{\rho})^{1-\alpha}}\mathfrak{P}_1(s,\mathfrak{a}(s))ds-\int_0^{\bar{s}_1}\frac{s^{\rho-1}}{(\bar{s}_1^{\rho}-s^{\rho})^{1-\alpha}}\mathfrak{P}_1(s,\mathfrak{a}(s))ds\Big\}\Big|\\
&\leq|\mathcal{H}_1(\bar{s}_2,\mathfrak{a}(\bar{s}_2))|\Big|\Big\{\int_0^{\bar{s}_2}\frac{s^{\rho-1}}{(\bar{s}_2^{\rho}-s^{\rho})^{1-\alpha}}-\int_0^{\bar{s}_1}\frac{s^{\rho-1}}{(\bar{s}_1^{\rho}-s^{\rho})^{1-\alpha}}\Big\}ds\Big||\mathfrak{P}_1(s,\mathfrak{a}(s))|\\
&\leq\gamma_2\gamma_3r_0^2\Big|\int_0^{\bar{s}_2}\frac{s^{\rho-1}}{(\bar{s}_2^{\rho}-s^{\rho})^{1-\alpha}}ds-\int_0^{\bar{s}_1}\frac{s^{\rho-1}}{(\bar{s}_2^{\rho}-s^{\rho})^{1-\alpha}}ds\Big|+\gamma_2\gamma_3r_0^2\Big|\int_0^{\bar{s}_1}\frac{s^{\rho-1}}{(\bar{s}_2^{\rho}-s^{\rho})^{1-\alpha}}ds\\
&-\int_0^{\bar{s}_2}\frac{s^{\rho-1}}{(\bar{s}_1^{\rho}-s^{\rho})^{1-\alpha}}ds\Big|+\gamma_2\gamma_3r_0^2\Big|\int_0^{\bar{s}_2}\frac{s^{\rho-1}}{(\bar{s}_1^{\rho}-s^{\rho})^{1-\alpha}}ds-\int_0^{\bar{s}_1}\frac{s^{\rho-1}}{(\bar{s}_1^{\rho}-s^{\rho})^{1-\alpha}}ds\Big|\\
&\leq\frac{\gamma_3\gamma_2r_0^2}{\rho\alpha}\Big(2(\bar{s}_2^{\rho}-\bar{s}_1^{\rho})^{\alpha}+2|\,(\bar{s}_2^{\rho}-\bar{s}_1^{\rho})^{\alpha}\,|+\bar{s}_2^{\alpha\rho}-\bar{s}_1^{\alpha\rho}\Big)
\end{aligned}$$

and

$$\begin{aligned}
&|\mathcal{H}_1(\bar{s}_2,\mathfrak{a}(\bar{s}_2))-\mathcal{H}_1(\bar{s}_1,\mathfrak{a}(\bar{s}_1))|\\
&\leq|\mathcal{H}_1(\bar{s}_2,\mathfrak{a}(\bar{s}_2))-\mathcal{H}_1(\bar{s}_1,\mathfrak{a}(\bar{s}_2))|+|\mathcal{H}_1(\bar{s}_1,\mathfrak{a}(\bar{s}_2))-\mathcal{H}_1(\bar{s}_1,\mathfrak{a}(\bar{s}_1))|\\
&\leq\gamma_{r_0}(\mathcal{H}_1,\beta)+\gamma_1|\mathfrak{a}(\bar{s}_2)-\mathfrak{a}(\bar{s}_1)|\\
&\leq\gamma_{r_0}(\mathcal{H}_1,\beta)+\gamma_1\psi(\mathfrak{a},\beta),
\end{aligned}$$

where

$$\gamma_{r_0}(\mathcal{H}_1,\beta)=\sup\{|\mathcal{H}_1(\bar{s}_2,\mathfrak{a})-\mathcal{H}_1(\bar{s}_1,\mathfrak{a})|:|\bar{s}_2-\bar{s}_1|\leq\beta,\ \bar{s}_1,\bar{s}_2\in I,\ \|\,\mathfrak{a}\,\|\leq r_0\}\,.$$

Therefore

$$\begin{aligned}
&|(\mathfrak{T}\mathfrak{a})(\bar{s}_2)-(\mathfrak{T}\mathfrak{a})(\bar{s}_1)|\\
&\leq\gamma_{r_0}(\mathcal{H}_1,\beta)+\gamma_1\psi(\mathfrak{a},\beta)+\frac{\gamma_3\gamma_2r_0^2}{\rho^{\alpha}\Gamma(\alpha+1)}\Big(2(\bar{s}_2^{\rho}-\bar{s}_1^{\rho})^{\alpha}+2|\,(\bar{s}_2^{\rho}-\bar{s}_1^{\rho})^{\alpha}\,|+\bar{s}_2^{\alpha\rho}-\bar{s}_1^{\alpha\rho}\Big)\\
&\{\gamma_2\psi(\mathfrak{a},\beta)+\acute{\varkappa}(\beta)\}\,\frac{r_0\gamma_3T^{\alpha}}{\rho^{\alpha}\Gamma(\alpha+1)}.
\end{aligned}$$

i.e.,

$$\psi(\mathfrak{T}\mathfrak{a},\beta)\leq\gamma_{r_0}(\mathcal{H}_1,\beta)+\gamma_1\psi(\mathfrak{a},\beta)+\frac{\gamma_3\gamma_2r_0^2}{\rho^{\alpha}\Gamma(\alpha+1)}\Big(2(\bar{s}_2^{\rho}-\bar{s}_1^{\rho})^{\alpha}+2|\,(\bar{s}_2^{\rho}-\bar{s}_1^{\rho})^{\alpha}\,|+\bar{s}_2^{\alpha\rho}-\bar{s}_1^{\alpha\rho}\Big)$$

$$\{\gamma_2\psi(\mathfrak{a},\beta)+\acute{\varkappa}(\beta)\}\,\frac{r_0\gamma_3T^{\alpha}}{\rho^{\alpha}\Gamma(\alpha+1)}$$

which gives

$$\psi(\mathfrak{T}Q_{\mathfrak{a}},\beta)\leq\gamma_{r_0}(\mathcal{H}_1,\beta)+\gamma_1\psi(Q_{\mathfrak{a}},\beta)+\frac{\gamma_3\gamma_2r_0^2}{\rho^{\alpha}\Gamma(\alpha+1)}\Big(2(t_2^{\rho}-t_1^{\rho})^{\alpha}+2|\,(t_2^{\rho}-t_1^{\rho})^{\alpha}\,|+t_2^{\alpha\rho}-t_1^{\alpha\rho}\Big)$$

$$\{\gamma_2\psi(Q_{\mathfrak{a}},\beta)+\acute{\varkappa}(\beta)\}\,\frac{r_0\gamma_3T^{\alpha}}{\rho^{\alpha}\Gamma(\alpha+1)}.$$

As $\beta\to 0$, then by the property of uniform continuity of $\mathfrak{L}_1$ and $\mathcal{H}_1$ we have $\psi(Q_{\mathfrak{a}},\beta)\to 0$ and $\acute{\varkappa}(\beta)\to 0$. Hence as $\beta\to 0$ we obtain

$$\psi_0(\mathfrak{T}Q_{\mathfrak{a}})\leq\Big[\gamma_1+\frac{\gamma_2\gamma_3r_0T^{\alpha}}{\rho^{\alpha}\Gamma(\alpha+1)}\Big]\psi_0(Q_{\mathfrak{a}}).$$

Similarly we obtain

$$\psi_0(\hat{\mathfrak{T}}Q_{\mathfrak{b}})\leq\Big[\hat{\gamma}_1+\frac{\hat{\gamma}_2\hat{\gamma}_3r_0T^{\alpha}}{\rho^{\alpha}\Gamma(\alpha+1)}\Big]\psi_0(Q_{\mathfrak{b}}).$$

Let

$$\Xi=max\left\{\gamma_1+\frac{\gamma_2\gamma_3r_0T^{\alpha}}{\rho^{\alpha}\Gamma(\alpha+1)},\hat{\gamma}_1+\frac{\hat{\gamma}_2\hat{\gamma}_3r_0T^{\alpha}}{\rho^{\alpha}\Gamma(\alpha+1)}\right\}<1$$

as

$$\gamma_1+\frac{\gamma_2\gamma_3r_0T^{\alpha}}{\rho^{\alpha}\Gamma(\alpha+1)}<1$$

and

$$\hat{\gamma}_1+\frac{\hat{\gamma}_2\hat{\gamma}_3r_0T^{\alpha}}{\rho^{\alpha}\Gamma(\alpha+1)}<1$$

Hence,

$$\psi_0(\mathfrak{T}Q_{\mathfrak{a}})+\psi_0(\hat{\mathfrak{T}}Q_{\mathfrak{b}})\leq\beta[\psi_0(Q_{\mathfrak{a}})+\psi_0(Q_{\mathfrak{b}})].$$

Hence with reference to Corollary 3.2, we can conclude that $\mathfrak{T},\hat{\mathfrak{T}}$ possess fixed points in $\mathbb{Q}_{r_0}$. That is the equations (5.2) and (5.3) have solutions in $\mathbb{H}$. □

Example 5.1 Let us suppose the fractional integral equations as follows:

$$\mathfrak{a}(\bar{s})=\frac{|\mathfrak{a}(\bar{s})|}{2}+\frac{|\mathfrak{a}(\bar{s})|\times 1^{-1}}{\Gamma(2)}\int_0^{\bar{s}}\frac{\mathfrak{a}(\bar{s})}{(e^{\bar{s}}+1)(\bar{s}-s)^{-1}}ds \tag{5.4}$$

and

$$\mathfrak{b}(\bar{s})=\frac{|\mathfrak{b}(\bar{s})|}{2}+\frac{|\mathfrak{b}(\bar{s})|\times 1^{-1}}{\Gamma(2)}\int_0^{\bar{s}}\frac{\mathfrak{b}(\bar{s})}{(\bar{s}+2)(\bar{s}-s)^{-1}}ds \tag{5.5}$$

for $\bar{s}\in[0,1]=I$.

Here, $\rho=1,\alpha=2,T=1$. Also,

$$\begin{aligned}
&\mathfrak{L}_1(\bar{s},\mathfrak{a}(\bar{s}))=\mathfrak{L}_2(\bar{s},y(\bar{s}))=\frac{|\mathfrak{a}(\bar{s})|}{2},\\
&\mathcal{H}_1(\bar{s},\mathfrak{a}(\bar{s}))=\mathcal{H}_2(\bar{s},y(\bar{s}))=|\mathfrak{a}(\bar{s})|,\\
&\mathfrak{P}_1(s,\mathfrak{a}(s))=\frac{\mathfrak{a}(\bar{s})}{e^{\bar{s}}+1},\\
&\mathfrak{P}_2(s,\mathfrak{a}(s))=\frac{\mathfrak{a}(\bar{s})}{\bar{s}+2}.
\end{aligned}$$

Therefore,

$$\begin{aligned}
|\mathfrak{L}_1(\bar{s},\mathfrak{a}(\bar{s}))-\mathfrak{L}_1(\bar{s},\bar{\mathfrak{a}}(\bar{s}))| &\le \frac{|\mathfrak{a}(\bar{s})-\bar{\mathfrak{a}}(\bar{s})|}{2},\\
|\mathfrak{L}_2(\bar{s},\mathfrak{a}(\bar{s}))-\mathfrak{L}_2(\bar{s},\bar{\mathfrak{a}}(\bar{s}))| &\le \frac{|\mathfrak{a}(\bar{s})-\bar{\mathfrak{a}}(\bar{s})|}{2},\\
|\mathcal{H}_1(\bar{s},\mathfrak{a}(\bar{s}))-\mathcal{H}_1(\bar{s},\bar{\mathfrak{a}}(\bar{s}))| &\le \ |\mathfrak{a}(\bar{s})-\bar{\mathfrak{a}}(\bar{s})|,\\
|\mathcal{H}_2(\bar{s},\mathfrak{a}(\bar{s}))-\mathcal{H}_2(\bar{s},\bar{\mathfrak{a}}(\bar{s}))| &\le \ |\mathfrak{a}(\bar{s})-\bar{\mathfrak{a}}(\bar{s})|,\\
|\mathfrak{P}_1(\bar{s},\mathfrak{a}(\bar{s}))-\mathfrak{P}_1(\bar{s},\bar{\mathfrak{a}}(\bar{s}))| &\le \frac{|\mathfrak{a}(\bar{s})-\bar{\mathfrak{a}}(\bar{s})|}{2},\\
|\mathfrak{P}_2(\bar{s},\mathfrak{a}(\bar{s}))-\mathfrak{P}_2(\bar{s},\bar{\mathfrak{a}}(\bar{s}))| &\le \frac{|\mathfrak{a}(\bar{s})-\bar{\mathfrak{a}}(\bar{s})|}{2},
\end{aligned}$$

and

$$\mathfrak{L}_1(\bar{s},0)=\mathfrak{L}_2(\bar{s},0)=\mathcal{H}_1(\bar{s},0)=0=\mathcal{H}_2(\bar{s},0)=\mathfrak{P}_1(\bar{s},0)=\mathfrak{P}_2(\bar{s},0)=0,$$

$$\gamma_1=\hat{\gamma}_1=\frac{1}{2},\gamma_2=\hat{\gamma}_2=1,\gamma_3=\hat{\gamma}_3=\frac{1}{2}.$$

Now, let us substitute the above values in the inequality illustrated in constraint (ii),

$$\frac{1}{2}+\frac{\frac{1}{2}.1.r_0.1^2}{1^2.\Gamma(3)}<1,$$

which implies that $r_0<2$.
In the similar manner by substituting the same values in the inequality provided in assumption (ii) we get

$$r_0<2.$$

Thus the constraint (ii) is also satisfied for $r_0=1$.
Hence, we observe that the constraints from (i)-(ii) in Theorem 5.1 are fulfilled. From Theorem 5.1 it can be verified that equations (5.4) and (5.5) possess solutions in $\mathbb{H}=C(I)$.

6. Conclusion

With the help of a newly obtained condensing operator, we have established a new fixed point theorem involving two distinct operators and applied it to the Katugampola fractional integral system to investigate the existence of its solutions. We have provided an example as well to give a better understanding of our obtained theories. The concept of condensing operators involving two distinct operators can be further extended to n-tuple operators and establish fixed point theorem for future research. This fixed point result can also be applied to various types of system of equations.

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