A Generalization of Darbo's Fixed Point Theorem and its Applications to ψ Hilfer Fractional Hybrid Differential Equation

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ABSTRACT: In the present paper, we develop a new fixed point theorem based on a newly introduced contraction operator. Our formulation is built upon essential ideas from the measure of noncompactness, which serve as a foundation for our analysis. By using this framework, we study the existence results for the solutions to the first order hybrid fractional differential equation involving the ψ-Hilfer fractional derivative. The applicability of our results, we conclude with an example.

Key Words: Fixed point theory(**FPT**), measure of noncompactness(**M$\mathbb{N}\mathcal{C}$**), fixed point($\mathbb{FP}$), Banach space($\mathbb{BS}$).

Contents

1. Introduction

Fractional calculus extends the concepts of classical calculus by allowing integration and differentiation of fractional order. In [5], the authors presented new tripled fixed point theorems based on **M$\mathbb{N}\mathcal{C}$** and suitable control functions. They also utilized this framework and investigate the existence of solution for system of integral equations. In [8], authors work on the existence of solution of generalized fractional integral equations of two variable and also established a new fixed point theorem and coupled fixed point theorem by using **M$\mathbb{N}\mathcal{C}$**. In [7], the authors discussed the existence of a solution of functional integral equations in two variables by using fixed point theorem in Banach algebra. In [10], the authors considered the existence of a solution of functional integral equations in two variables Caputo derivative of order $1 \le \alpha \le 2$. As compared to these mentioned works, we establish a new contraction type operator that generalizes Darbo's fixed point theorem and use it to obtain the existence of solutions for ψ-Hilfer fractional order equation. In [9], the authors studied the iterative algorithm and theoretical treatment of existence of solution for (k, z)-Riemann–Liouville fractional integral equations. Rabbani et al. [18] analyzed some computational convergent iterative algorithms to solve nonlinear problems. Savadkoohi et al. [20] discussed a fractional multi-wavelet basis in Banach space and solving fractional delay differential equations. In [17], author studied an iterative algorithm to find a closed form of solution for Hammerstein nonlinear integral equation constructed by the concept of cosm-rs. In [21], authors discussed Hyers-Ulam stability of generalized quartic mapping in non-Archimedean (n, β)-normed spaces. Radhakrishnan et al. [19] analyzed uniqueness of fixed points for multi-valued mappings in orthogonal ultrametric spaces. In [26], authors considered stability of quartic functional equation in non-Archimedean IFN-spaces. Karthikeyan et al. [13] studied thermal applications of stability analysis of cubic functional

equation in Banach spaces and intuitionistic Fuzzy normed spaces. Uthirasamy et al. [27] analyzed new common fixed point results in ultrametric spaces utilizing various contractions and their applications to integral equations. In [25], authors studied fixed point technique stability analysis of quadratic functional equation in various quasi-Banach spaces. In [2], authors considered some common fixed point results in modular ultrametric space using various contractions and their application to well-posedness.

Recently, Sousa and Oliveira [22] introduced a new general fractional derivative, termed the ψ- Hilfer fractional derivative, from which they derived approximately 22 distinct types of fractional derivatives and integrals. Many works have been carried out on fractional equations involving the ψ Hilfer fractional operator [14,22,23,24].

Motivated by the mentioned works, we investigate the existence of solution for hybrid fractional differential equations of ψ-Hilfer fractional derivative. The main advantage of the fractional operator introduced here lies in its flexibility in choosing the underlying classical differential operator. Depending on the formulation, the differential operator may act on the fractional integral operator, or vice versa. Consequently, the family of fractional derivatives generated by the ψ-Hilfer operator becomes significantly broader, establishing this operator as a genuine generalization of the previously defined fractional operators.

Fixed point theorems serve as a fundamental method for examining the nonlinear operators, particularly in establishing the solvability of various types of integral and differential equations. In recent years, fixed point theorems have attracted significant interest from researchers owing to their wide applicability and powerful analytical framework. In combination with study of measure of noncompactness ($\mathbf{M}\mathbb{N}\mathcal{C}$), **FPT** has become an effective method for addressing various classes of integral and differential equations. The concept of $\mathbf{M}\mathbb{N}\mathcal{C}$ was originally introduced by Kuratowski [15] in 1930. Later, Darbo applied Kuratowski's $\mathbf{M}\mathbb{N}\mathcal{C}$ to **FPT** and extended the Schauder fixed point theorem to noncompact operator.

In this article, first we recall some crucial ideas and results. We then establish a generalized fixed point theorem utilizing the study of $\mathbf{M}\mathbb{N}\mathcal{C}$. Finally, we demonstrate the existence of solution for hybrid fractional differential equations involving the ψ-Hilfer fractional derivative by applying our results.

2. Preliminaries

Suppose $(\mathbb{S}, \|\cdot\|_{\mathbb{S}})$ be a real Banach space($\mathbb{BS}$) and ς be a non-empty subset of $\mathbb{S}$. Assume $\mathcal{B}(\mathfrak{t}, \mathfrak{r}) = \{\iota \in \mathbb{S} : \|\iota - \mathfrak{t}\|_{\mathbb{S}} \leq \mathfrak{r}\}$.

- We denote $\mathbb{M}(\mathbb{S})$ by the collection of all bounded, non-empty subsets of $\mathbb{S}$.

- $\bar{\varsigma}$ and $\mathrm{Conv}(\varsigma)$ represents the closure and the convex closure of ς respectively.

- $R^+ = [0, +\infty); R = (-\infty, \infty)$.

Definition 2.1 [11] A map $\mu : \mathbb{M}(\mathbb{S}) \to R^+$ is known as $\mathbf{M}\mathbb{N}\mathcal{C}$ in $\mathbb{S}$, if:

(i) for all $\varsigma \in \mathbb{M}(\mathbb{S})$, $\mu(\varsigma) = 0$ gives ς is relatively compact.

(ii) $\ker \mu = \{\varsigma \in \mathbb{M}(\mathbb{S}) : \mu(\varsigma) = 0\} \neq \emptyset$ and $\ker \mu \subset \mathbb{N}(\mathbb{S})\}$.

(iii) $\varsigma \subseteq \varsigma_1 \implies \mu(\varsigma) \leq \mu(\varsigma_1)$.

(iv) $\mu(\bar{\varsigma}) = \mu(\varsigma)$.

(v) $\mu(\mathrm{Conv}\varsigma) = \mu(\varsigma)$.

(vi) $\mu(\sigma\varsigma + (1 - \sigma)\varsigma_1) \leq \sigma\mu(\varsigma) + (1 - \sigma)\mu(\varsigma_1)$ for $\sigma \in [0, 1]$.

(vii) if $\varsigma_l \in \mathbb{M}(\mathbb{S})$, $\varsigma_l = \bar{\varsigma}_l$, $\varsigma_{l+1} \subset \varsigma_l$ for $l = 1, 2, ...$ and $\lim_{l \to \infty} \mu(\varsigma_l) = 0$ then $\bigcap_{l=1}^{\infty} \varsigma_l \neq \emptyset$.

We denote by $\mathcal{C}(L)$ the set of all real continuous functions on $L = [0, 1]$. This space is a $\mathbb{BS}$ with the norm

$$\|J\| = \sup\{|J(v)| : v \in L\}, \quad J \in \mathcal{C}(L).$$

For $\epsilon > 0$ and $J \in \mathcal{C}(L)$ denote by $\varpi(J, \epsilon)$ the modulus of continuity of J, i.e.,

$$\varpi(J, \epsilon) = \sup\{|J(j) - J(p)| : j, p \in [0, L], |j - p| \leq \epsilon\}.$$

Moreover, the uniform continuity of J on L gives that $\varpi(J, \epsilon) \to 0$ as $\epsilon \to 0$.

Now, we consider $\varpi(\varsigma, \varepsilon) = \sup\{\varpi(J, \epsilon) : J \in \varsigma\}$. The Hausdorff $\mathbb{MNC}$ [4] for all bounded set ς of $\mathcal{C}(L)$ is given by:

$$\varpi(\varsigma) = \lim_{\epsilon \to 0} \big\{ \sup_{J \in \varsigma} \varpi(J, \epsilon) \big\}.$$

Definition 2.2 [22] Assume Γ be the Euler gamma function. The left-sided fractional integral of a map φ with respect to a map ψ on $[a, b]$ is

$$I_{a^+}^{\vartheta; \psi} \varphi(\beta) = \frac{1}{\Gamma(\vartheta)} \int_a^\beta \psi'(\zeta)(\psi(\beta) - \psi(\nu))^{\vartheta - 1} \varphi(\zeta) d\zeta, \vartheta > 0.$$

Definition 2.3 [22] Consider $n - 1 < \vartheta < n$ with $n \in \mathbb{N}$, suppose $L = [a, b]$ with $\infty \leq a \leq \infty$ and consider $\varphi, \psi \in \mathcal{C}^n[a, b]$ be two functions so that ψ is increasing and $\psi'(\beta) \neq 0$, for all $\beta \in L$. The left ψ- Hilfer fractional derivative $^{\mathbb{H}}\mathbf{D}_{a^+}^{\vartheta, \upsilon; \psi} \varphi(\cdot)$ of a map φ of order ϑ and type $0 \leq \upsilon \leq 1$, is

$$^{\mathbb{H}}\mathbf{D}_{a^+}^{\vartheta, \upsilon; \psi} \varphi(\beta) = I_{a^+}^{\upsilon(n-\vartheta); \psi} \Big(\frac{1}{\psi'(\beta)} \frac{d}{d\beta} \Big)^n I_{a^+}^{(1-\upsilon)(n-\vartheta); \psi} \varphi(\beta).$$

Definition 2.4 [3] Consider ϕ is the collection of continuous operators $\mathfrak{F} : R^+ \times R^+ \to R^+$ so that:

1. $\mathfrak{F}(p, q) \leq p$;
2. $\mathfrak{F}(p, q) = p$ implies that either $p = 0$ or $q = 0$. Also $\mathfrak{F}(0, 0) = 0$.

Theorem 2.1 ([1], Schauder) Consider $\mathcal{H}$ be a non-empty, bounded, closed and convex ($\mathcal{NBCC}$) subset of $\mathbb{BS}$, $\mathbb{S}$. Then every continuous and compact mapping $\mathfrak{X} : \mathcal{H} \to \mathcal{H}$ has at least one fixed point in $\mathcal{H}$.

Theorem 2.2 ([6], Darbo) Suppose $\mathcal{Q}$ be a $\mathcal{NBCC}$ subset of Banach space, $\mathbb{S}$ and consider $\mathfrak{X} : \mathcal{Q} \to \mathcal{Q}$ be a continuous mapping and there is a constant $k \in [0, 1)$ along with $\mu(\mathfrak{X}\varsigma) \leq k\mu(\varsigma)$. Then $\mathfrak{X}$ has at least one $\mathbb{FP}$ in $\mathcal{Q}$.

3. New Results

In this segment, we try to expand the idea of the Darbo contraction principle.

Theorem 3.1 Consider $\mathbb{S}$ be a $\mathbb{BS}$ and $\vartheta \subseteq \mathbb{S}$ be $\mathcal{NBCC}$. Also, assume $\mathfrak{X} : \vartheta \longrightarrow \vartheta$ be a continuous mapping satisfying

$$\int_0^{\mu(\mathfrak{X}\varsigma)} \aleph(\mathfrak{n}) d\mathfrak{n} \leq \mathfrak{F}\Big[\Phi\Big(\int_0^{\mu(\varsigma)} \aleph(\mathfrak{n}) d\mathfrak{n} \Big), \varepsilon\Big(\int_0^{\mu(\varsigma)} \aleph(\mathfrak{n}) d\mathfrak{n} \Big) \Big], \tag{3.1}$$

for all $\varsigma \subseteq \vartheta$, where $\mathfrak{F} \in \phi$, $\Phi : R^+ \to R^+$ is a continuous map with $\lim_{n \to \infty} \Phi^n(\mathfrak{s}) = 0$, $\mathfrak{s} \geq 0$. Also, $\aleph : [0, \infty) \to (0, \infty)$ is a continuous and bounded map so that for any $t > 0$, $\int_0^t \aleph(\mathfrak{s}) d\mathfrak{s} > 0$ which is summable on each compact subset on R^+, and $\varepsilon : R^+ \to R^+$ is a continuous function. Moreover, μ is an arbitrary $\mathbb{MNC}$. Then $\mathfrak{X}$ possess at least one $\mathbb{FP}$ in ϑ.

Proof: Define a sequence $\{\varsigma_n\}$ with $\varsigma_0 = \varsigma$ and $\varsigma_{n+1} = Conv(\mathfrak{X}(\varsigma))$ for all $n \in \mathbb{N}$, where $\mathbb{N}$ is a set of natural numbers.

If there is $n \in \mathbb{N}$ satisfying $\mu(\varsigma_n) = 0$, then ς_n is compact set. Then by using Theorem 2.1, we can say that $\mathfrak{X}$ has a $\mathbb{FP}$ in ϑ. Let $\mu(\varsigma_n) > 0$, $n \in \mathbb{N}$.
So this specifies that $\varsigma_n \in \mathbb{N}$ sequence of $\mathcal{NBCC}$ along with

$$\varsigma_0 \supseteq \varsigma_1 \supseteq \dots \supseteq \varsigma_{n+1}.$$

Now, we deduce that

$$\int_0^{\mu(\varsigma_{n+1})} \aleph(\mathfrak{n})d\mathfrak{n} = \int_0^{\mu(conv(\mathfrak{X}\varsigma_n))} \aleph(\mathfrak{n})d\mathfrak{n} = \int_0^{\mu(\mathfrak{X}\varsigma_n)} \aleph(\mathfrak{n})d\mathfrak{n}$$

$$\leq \mathfrak{F}\Big[\Phi\Big(\int_0^{\mu(\varsigma_n)} \aleph(\mathfrak{n})d\mathfrak{n}\Big), \varepsilon\Big(\int_0^{\mu(\varsigma_n)} \aleph(\mathfrak{n})d\mathfrak{n}\Big)\Big]$$

$$\leq \Phi\Big(\int_0^{\mu(\varsigma_n)} \aleph(\mathfrak{n})d\mathfrak{n}\Big)$$

$$\leq \Phi^2\Big(\int_0^{\mu(\varsigma_{n-1})} \aleph(\mathfrak{n})d\mathfrak{n}\Big)$$

$$\vdots$$

$$\leq \Phi^n\Big(\int_0^{\mu(\varsigma)} \aleph(\mathfrak{n})d\mathfrak{n}\Big). \tag{3.2}$$

Now regarding the fact that for every $t > 0$, $\int_0^t \aleph(\mathfrak{n})d\mathfrak{n} > 0$ we can conclude that $\mu(\varsigma_n) \to 0$ as $n \to \infty$. From (vii) of Definition 2.1, we have $\varsigma_\infty = \bigcap_{n=1}^\infty \varsigma_n$, which is a non-empty, closed, convex subset of ϑ. Moreover, ς_∞ is $\mathfrak{X}$ invariant and $\varsigma_\infty \in \ker\mu$. Thus, by the Theorem 2.1, $\mathfrak{X}$ admits at least one $\mathbb{FP}$ in $\mathbb{S}$. □

Corollary 3.1 *Suppose $\mathbb{S}$ be a $\mathbb{BS}$ and $\vartheta \subseteq \mathbb{S}$ be $\mathcal{NBCC}$. Also, assume $\mathfrak{X} : \vartheta \longrightarrow \vartheta$ be a continuous mapping satisfying*

$$\mu(\mathfrak{X}\varsigma) \leq k\mu(\varsigma), \tag{3.3}$$

for all $\varsigma \subseteq \vartheta$, where $k \in [0,1)$. Moreover, μ is an arbitrary $\mathbf{MNC}$, then $\mathfrak{X}$ possess atleast one $\mathbb{FP}$ in ϑ.

Proof: Putting $\aleph = 1, \mathfrak{F}(\mathfrak{g},\mathfrak{h}) = \mathfrak{g} - \mathfrak{h}, \varepsilon \equiv 0$, and $\Phi(\mathfrak{t}) = k\mathfrak{t}, 0 \leq k < 1$ in the Theorem 3.1, we get

$$\int_0^{\mu(\mathfrak{X}\varsigma)} \aleph(\mathfrak{n})d\mathfrak{n} = \mu(\mathfrak{X}\varsigma) \leq \Phi\Big(\int_0^{\mu(\varsigma)} \aleph(\mathfrak{n})d\mathfrak{n}\Big) = k\mu(\varsigma),$$

hence we derive the Darbo's fixed point theorem. □

4. Application

We now introduce the following hybrid fractional differential equation involving ψ Hilfer fractional derivative [16]:

$$\begin{cases} {}^{\mathbb{H}}\mathbf{D}_{0^+}^{\vartheta,\upsilon;\psi} \dfrac{\varphi(\beta)}{D(\beta,\varphi(\beta))} = \chi(\beta,\varphi(\beta)), & \beta \in L, \\[3mm] \mathrm{I}_{0^+}^{1-\varsigma;\psi} \dfrac{\varphi(0)}{D(0,\varphi(0))} = \varphi_0, \end{cases} \tag{4.1}$$

where ${}^{\mathbb{H}}\mathbf{D}_{0^+}^{\vartheta,\upsilon;\psi}(\cdot)$ is the ψ fractional derivative and the order of the operator $0 < \vartheta < 1; 0 \leq \upsilon \leq 1; \vartheta \leq \varsigma = \vartheta + \upsilon - \vartheta\upsilon < 1$; $\mathrm{I}_{0^+}^{(1-\vartheta)(1-\upsilon);\psi}$ is the Reimann-Liouvillle fractional integral of type $(1-\varsigma)$. $D \in \mathcal{C}(L \times R, R \setminus \{0\})$ and $\chi \in \mathcal{C}(L \times R, R)$.

The following result, which derive in [22,24] and by using Lemma [22], establishes that the fractional differential equation (4.1) admits an equivalent formulation in the form of the integral equation (4.2) as follows:

$$\varphi(\beta) = D(\beta,\varphi(\beta))\Big[\frac{(\psi(\beta)-\psi(0))^{\varsigma-1}}{\Gamma(\varsigma)}\varphi_0 + \frac{1}{\Gamma(\vartheta)}\int_0^\beta \psi'(\zeta)(\psi(\beta)-\psi(\zeta))^{\vartheta-1}\chi(\zeta,\varphi(\zeta))d\zeta\Big], \quad \beta \in L. \tag{4.2}$$

Let $\mathfrak{X}_{g_0} = \{\varphi \in \mathcal{C}(L) : \|\varphi\| \leq g_0\}$.

To establish the existence of a solution to (4.2), we require the following conditions:

1. $D : L \times R \to R$ is continuous, and there exits a constant $\overline{D} > 0$ satisfying

$$|D(\beta, \varphi_1(\beta_1)) - D(\beta, \varphi_2(\beta_2))| \leq \overline{D}|\varphi_1(\beta_1) - \varphi_2(\beta_2)|,$$

for all $\beta, \beta_1, \beta_2 \in L$ and $\varphi_1, \varphi_2 \in R$. Also, for all $\beta \in L$,

$$D(\beta, 0) = \mathfrak{v} \geq 0.$$

2. $\chi : L \times R \to R$ is continuous, and there exists a constant $\overline{\chi} > 0$ such that

$$|\chi(\beta, \varphi_1(\beta_1)) - \chi(\beta, \varphi_2(\beta_2))| \leq \overline{\chi}|\varphi_1(\beta_1) - \varphi_2(\beta_2)|,$$

for all $\beta, \beta_1, \beta_2 \in L$ and $\varphi_1, \varphi_2 \in R$.
Also, for all $\beta \in L$,

$$\chi(\beta, 0) = 0.$$

3. $\psi : L \to R$ and ψ is continuous and increasing function.

4. There exists a positive number g_0 so that

$$\left[\overline{D}g_0 + \mathfrak{v}\right] \frac{1}{\Gamma(\varsigma)} \big(\psi(\beta) - \psi(0)\big)^{\varsigma-1} |\varphi_0| + \frac{1}{\Gamma(\vartheta+1)}\left[\overline{D}g_0 + \mathfrak{v}\right] \big(\psi(\beta) - \psi(0)\big)^{\vartheta} \overline{\chi}g_0 \leq g_0.$$

Theorem 4.1 *Under assumptions (1)-(4), the equation (4.2) has at least one solution in $\mathcal{C}(L)$.*

Proof: Suppose the operator $\mathfrak{G} : \mathcal{C}(L) \to \mathcal{C}(L)$ defined as

$$(\mathfrak{G}\varphi)(\beta) = D(\beta, \varphi(\beta)) \left[\frac{\big(\psi(\beta) - \psi(0)\big)^{\varsigma-1}}{\Gamma(\varsigma)} \varphi_0 + \frac{1}{\Gamma(\vartheta)} \int_0^\beta \psi'(\zeta)\big(\psi(\beta) - \psi(\zeta)\big)^{\vartheta-1} \chi(\zeta, \varphi(\zeta)) d\zeta \right].$$

Phase 1: In this section, we will prove that the operator $\mathfrak{G}$ maps $\mathfrak{X}_{g_0}$ to $\mathfrak{X}_{g_0}$. Let $\varphi \in \mathfrak{X}_{g_0}$. Now, we have

$$|(\mathfrak{G}\varphi)(\beta)|$$

$$= \left| D(\beta, \varphi(\beta)) \left[\frac{\big(\psi(\beta) - \psi(0)\big)^{\varsigma-1}}{\Gamma(\varsigma)} \varphi_0 + \frac{1}{\Gamma(\vartheta)} \int_0^\beta \psi'(\zeta)\big(\psi(\beta) - \psi(\zeta)\big)^{\vartheta-1} \chi(\zeta, \varphi(\zeta)) d\zeta \right] \right|$$

$$\leq \left| \frac{D(\beta, \varphi(\beta))\big(\psi(\beta) - \psi(0)\big)^{\varsigma-1}}{\Gamma(\varsigma)} \varphi_0 \right| + \left| \frac{D(\rho, \varphi(\beta))}{\Gamma(\vartheta)} \int_0^\beta \psi'(\zeta)\big(\psi(\beta) - \psi(\zeta)\big)^{\vartheta-1} \chi(\zeta, \varphi(\zeta)) d\zeta \right|.$$

Now,

$$\left| \frac{D(\beta, \varphi(\beta))\big(\psi(\beta) - \psi(0)\big)^{\varsigma-1}}{\Gamma(\varsigma)} \varphi_0 \right|$$

$$\leq \left[|D(\beta, \varphi(\beta)) - D(\beta, 0)| + |D(\beta, 0)| \right] \left| \frac{1}{\Gamma(\varsigma)} \big(\psi(\beta) - \psi(0)\big)^{\varsigma-1} \varphi_0 \right|$$

$$\leq \left[\overline{D}\|\varphi(\beta)\| + \mathfrak{v} \right] \frac{1}{\Gamma(\varsigma)} \big(\psi(\beta) - \psi(0)\big)^{\varsigma-1} |\varphi_0|$$

$$\leq \left[\overline{D}g_0 + \mathfrak{v} \right] \frac{1}{\Gamma(\varsigma)} \big(\psi(\beta) - \psi(0)\big)^{\varsigma-1} |\varphi_0|.$$

Also,

$$\left| \frac{D(\beta, \varphi(\beta))}{\Gamma(\vartheta)} \int_0^\beta \psi'(\zeta) \big(\psi(\beta) - \psi(\zeta)\big)^{\vartheta-1} \chi(\zeta, \varphi(\zeta)) d\zeta \right|$$

$$\leq \frac{1}{\Gamma(\vartheta)} \big[|D(\beta, \varphi(\beta)) - D(\beta, 0)| + |D(\beta, 0)| \big]$$

$$\times \int_0^\beta \psi'(\zeta)\big(\psi(\beta) - \psi(\zeta)\big)^{\vartheta-1} \big[|\chi(\zeta, \varphi(\zeta)) - \chi(\zeta, 0)| + \chi(\zeta, 0)| \big] d\zeta$$

$$\leq \frac{1}{\vartheta \Gamma(\vartheta)} \big[\overline{D} \|\varphi(\beta)\| + \mathfrak{v} \big] \big(\psi(\beta) - \psi(0)\big)^\vartheta \overline{\chi} \|\varphi(\zeta)\|$$

$$\leq \frac{1}{\Gamma(\vartheta+1)} \big[\overline{D} g_0 + \mathfrak{v} \big] \big(\psi(\beta) - \psi(0)\big)^\vartheta \overline{\chi} g_0.$$

Hence, we get

$$|(\mathfrak{G}\varphi)(\beta)|$$
$$\leq \big[\overline{D} g_0 + \mathfrak{v} \big] \frac{1}{\Gamma(\varsigma)} \big(\psi(\beta) - \psi(0)\big)^{\varsigma-1} |\varphi_0| + \frac{1}{\Gamma(\vartheta+1)} \big[\overline{D} g_0 + \mathfrak{v} \big] \big(\psi(\beta) - \psi(0)\big)^\vartheta \overline{\chi} g_0,$$

hence, $\|\varphi\| \leq g_0$ which gives that

$$\|\mathfrak{G}\| \leq g_0.$$

From the condition(4), $\mathfrak{G}$ maps $\mathfrak{X}_{g_0}$ to $\mathfrak{X}_{g_0}$.

Phase 2: In this section, we discuss the continuity of $\mathfrak{G}$ on $\mathfrak{X}_{g_0}$. Suppose $\epsilon > 0$ and $\varphi, \overline{\varphi} \in \mathfrak{X}_{g_0}$ so that $\|\varphi - \overline{\varphi}\| < \epsilon$. Then,

$$|(\mathfrak{G}\varphi)(\beta) - (\mathfrak{G}\overline{\varphi})(\beta)|$$

$$= \left| D(\beta, \varphi(\beta)) \left[\frac{\big(\psi(\beta) - \psi(0)\big)^{\varsigma-1}}{\Gamma(\varsigma)} \varphi_0 + \frac{1}{\Gamma(\vartheta)} \int_0^\beta \psi'(\zeta)\big(\psi(\beta) - \psi(\beta)\big)^{\vartheta-1} \chi(\zeta, \varphi(\zeta)) d\zeta \right] - \right.$$

$$\left. D(\beta, \overline{\varphi}(\beta)) \left[\frac{\big(\psi(\beta) - \psi(0)\big)^{\varsigma-1}}{\Gamma(\varsigma)} \varphi_0 + \frac{1}{\Gamma(\vartheta)} \int_0^\beta \psi'(\zeta)\big(\psi(\beta) - \psi(\zeta)\big)^{\vartheta-1} \chi(\zeta, \overline{\varphi}(\zeta)) d\zeta \right] \right|$$

$$\leq |D(\beta, \varphi(\beta)) - D(\beta, \overline{\varphi}(\beta))| \frac{1}{\Gamma(\varsigma)} \big(\psi(\beta) - \psi(0)\big)^{\varsigma-1} |\varphi_0| + |D(\beta, \varphi(\beta)) - D(\beta, \overline{\varphi}(\beta))|$$

$$\left[\frac{1}{\Gamma(\vartheta)} \int_0^\beta \psi'(\zeta)\big(\psi(\beta) - \psi(\zeta)\big)^{\vartheta-1} |\chi(\zeta, \varphi(\zeta))| d\zeta \right] + |D(\beta, \overline{\varphi}(\beta))|$$

$$\left[\frac{1}{\Gamma(\vartheta)} \int_0^\beta \psi'(\zeta)\big(\psi(\beta) - \psi(\zeta)\big)^{\vartheta-1} |\chi(\zeta, \varphi(\zeta)) - \chi(\zeta, \overline{\varphi}(\zeta))| d\zeta \right]$$

$$\leq \overline{D} \|\varphi - \overline{\varphi}\| \frac{1}{\Gamma(\varsigma)} \big(\psi(\beta) - \psi(0))^{\varsigma-1} |\varphi_0| + \overline{D} \|\varphi - \overline{\varphi}\| \frac{1}{\Gamma(\vartheta+1)} \big(\psi(\beta) - \psi(0)\big)^\vartheta \overline{\chi} \|\varphi\| +$$

$$|D(\beta, \overline{\varphi}(\beta))| \frac{1}{\Gamma(\vartheta+1)} \big(\psi(\beta) - \psi(0)\big)^\vartheta \overline{\chi} \|\varphi - \overline{\varphi}\|$$

$$\leq \overline{D} \epsilon \frac{1}{\Gamma(\varsigma)} \big(\psi(\beta) - \psi(0))^{\varsigma-1} |\varphi_0| + \overline{D} \epsilon \frac{1}{\Gamma(\vartheta+1)} \big(\psi(\beta) - \psi(0)\big)^\vartheta \overline{\chi} g_0 +$$

$$|D(\beta, \overline{\varphi}(\beta))| \frac{1}{\Gamma(\vartheta+1)} \big(\psi(\beta) - \psi(0)\big)^\vartheta \overline{\chi} \epsilon.$$

As $\epsilon \to 0$ we get $|(\mathfrak{G}\varphi)(\beta) - (\mathfrak{G}\overline{\varphi})(\beta)| \to 0$, that is $|(\mathfrak{G}\varphi) - (\mathfrak{G}\overline{\varphi})| \to 0$. Hence, $\mathfrak{G}$ is continuous on $\mathfrak{X}_{g_0}$.

Phase 3: Now, we consider that $\Delta_\varphi \subseteq \mathfrak{X}_{g_0}$. Let $\epsilon > 0$ be arbitrary and let $\beta_1, \beta_2 \in L$ so that $|\beta_1 - \beta_2| \leq \epsilon$ with $\beta_1 > \beta_2$.

We have,

$$
|(\mathfrak{G}\varphi)(\beta_1) - (\mathfrak{G}\varphi)(\beta_2)|
$$

$$
= \left| D(\beta_1, \varphi(\beta_1)) \left[\frac{\big(\psi(\beta_1) - \psi(0)\big)^{\varsigma-1}}{\Gamma(\varsigma)} \varphi_0 + \frac{1}{\Gamma(\vartheta)} \int_0^{\beta_1} \psi'(\zeta)\big(\psi(\beta_1) - \psi(\zeta)\big)^{\vartheta-1} \chi(\zeta, \varphi(\zeta)) d\zeta \right] - \right.
$$

$$
\left. D(\beta_2, \varphi(\beta_2)) \left[\frac{\big(\psi(\beta_2) - \psi(0)\big)^{\varsigma-1}}{\Gamma(\varsigma)} \varphi_0 + \frac{1}{\Gamma(\vartheta)} \int_0^{\beta_2} \psi'(\zeta)\big(\psi(\beta_2) - \psi(\zeta)\big)^{\vartheta-1} \chi(\zeta, \varphi(\zeta)) d\zeta \right] \right|
$$

$$
\leq \left| D(\beta_1, \varphi(\beta_1)) \frac{\big(\psi(\beta_1) - \psi(0)\big)^{\varsigma-1}}{\Gamma(\varsigma)} \varphi_0 - D(\beta_2, \varphi(\beta_2)) \frac{\big(\psi(\beta_2) - \psi(0)\big)^{\varsigma-1}}{\Gamma(\varsigma)} \varphi_0 \right| +
$$

$$
\left| \frac{D(\beta_1, \varphi(\beta_1))}{\Gamma(\vartheta)} \int_0^{\beta_1} \psi'(\zeta)\big(\psi(\beta_1) - \psi(\zeta)\big)^{\vartheta-1} \chi(\zeta, \varphi(\zeta)) d\zeta - \frac{D(\beta_2, \varphi(\beta_2))}{\Gamma(\vartheta)} \int_0^{\beta_2} \psi'(\zeta)\big(\psi(\beta_2) - \psi(\zeta)\big)^{\vartheta-1} \right.
$$

$$
\left. \chi(\zeta, \varphi(\zeta)) d\zeta \right|.
$$

Now,

$$
\left| D(\beta_1, \varphi(\beta_1)) \frac{\big(\psi(\beta_1) - \psi(0)\big)^{\varsigma-1}}{\Gamma(\varsigma)} \varphi_0 - D(\beta_2, \varphi(\beta_2)) \frac{\big(\psi(\beta_2) - \psi(0)\big)^{\varsigma-1}}{\Gamma(\varsigma)} \varphi_0 \right|
$$

$$
\leq \left| D(\beta_1, \varphi(\beta_1)) \frac{\big(\psi(\beta_1) - \psi(0)\big)^{\varsigma-1}}{\Gamma(\varsigma)} \varphi_0 - D(\beta_1, \varphi(\beta_2)) \frac{\big(\psi(\beta_1) - \psi(0)\big)^{\varsigma-1}}{\Gamma(\varsigma)} \varphi_0 \right|
$$

$$
+ \left| D(\beta_1, \varphi(\beta_2)) \frac{\big(\psi(\beta_1) - \psi(0)\big)^{\varsigma-1}}{\Gamma(\varsigma)} \varphi_0 - D(\beta_1, \varphi(\beta_2)) \frac{\big(\psi(\beta_2) - \psi(0)\big)^{\varsigma-1}}{\Gamma(\varsigma)} \varphi_0 \right|
$$

$$
+ \left| D(\beta_1, \varphi(\beta_2)) \frac{\big(\psi(\beta_2) - \psi(0)\big)^{\varsigma-1}}{\Gamma(\varsigma)} \varphi_0 - D(\beta_2, \varphi(\beta_2)) \frac{\big(\psi(\beta_2) - \psi(0)\big)^{\varsigma-1}}{\Gamma(\varsigma)} \varphi_0 \right|
$$

$$
\leq |D(\beta_1, \varphi(\beta_1)) - D(\beta_1, \varphi(\beta_2))| \frac{\big(\psi(\beta_1) - \psi(0)\big)^{\varsigma-1}}{\Gamma(\varsigma)} \varphi_0 + |D(\beta_1, \varphi(\beta_2))|
$$

$$
\left[\frac{\big(\psi(\beta_1) - \psi(0)\big)^{\varsigma-1}}{\Gamma(\varsigma)} \varphi_0 - \frac{\big(\psi(\beta_2) - \psi(0)\big)^{\varsigma-1}}{\Gamma(\varsigma)} \varphi_0 \right] + |D(\beta_1, \varphi(\beta_2)) - D(\beta_2, \varphi(\beta_2))|
$$

$$
\frac{\big(\psi(\beta_2) - \psi(0)\big)^{\varsigma-1}}{\Gamma(\varsigma)} \varphi_0
$$

$$
\leq \overline{D} |\varphi(\beta_1) - \varphi(\beta_2)| \frac{\big(\psi(\beta_1) - \psi(0)\big)^{\varsigma-1}}{\Gamma(\varsigma)} \varphi_0 + |D(\beta_1, \varphi(\beta_2))| \left[\frac{\big(\psi(\beta_1) - \psi(0)\big)^{\varsigma-1}}{\Gamma(\varsigma)} \varphi_0 \right.
$$

$$
\left. \frac{\big(\psi(\beta_2) - \psi(0)\big)^{\varsigma-1}}{\Gamma(\varsigma)} \varphi_0 \right] + |D(\beta_1, \varphi(\beta_2)) - D(\beta_2, \varphi(\beta_2))| \frac{\big(\psi(\beta_2) - \psi(0)\big)^{\varsigma-1}}{\Gamma(\varsigma)} \varphi_0
$$

$$
\leq \overline{D} \varpi(\varphi, \epsilon) \frac{\big(\psi(\beta_1) - \psi(0)\big)^{\varsigma-1}}{\Gamma(\varsigma)} \varphi_0 + |D(\beta_1, \varphi(\beta_2))| \left[\frac{\big(\psi(\beta_1) - \psi(0)\big)^{\varsigma-1}}{\Gamma(\varsigma)} \varphi_0 \right.
$$

$$
\left. - \frac{\big(\psi(\beta_2) - \psi(0)\big)^{\varsigma-1}}{\Gamma(\varsigma)} \varphi_0 \right] + \varpi_D(g_0, \epsilon) \frac{\big(\psi(\beta_2) - \psi(0)\big)^{\varsigma-1}}{\Gamma(\varsigma)} \varphi_0.
$$

Also,

$$\left| \frac{D(\beta_1, \varphi(\beta_1))}{\Gamma(\vartheta)} \int_0^{\beta_1} \psi'(\zeta)\big(\psi(\beta_1) - \psi(\zeta)\big)^{\vartheta-1} \chi(\zeta, \varphi(\zeta)) d\zeta - \frac{D(\beta_2, \varphi(\beta_2))}{\Gamma(\vartheta)} \int_0^{\beta_2} \psi'(\zeta)\big(\psi(\beta_2) - \psi(\zeta)\big)^{\vartheta-1} \right.$$
$$\left. \chi(\zeta, \varphi(\zeta)) d\zeta \right|$$

$$\leq \left| \frac{D(\beta_1, \varphi(\beta_1))}{\Gamma(\vartheta)} \int_0^{\beta_1} \psi'(\zeta)\big(\psi(\beta_1) - \psi(\zeta)\big)^{\vartheta-1} \chi(\zeta, \chi(\zeta)) d\zeta - \frac{D(\beta_1, \varphi(\beta_2))}{\Gamma(\vartheta)} \int_0^{\beta_1} \psi'(\zeta)\big(\psi(\beta_1) - \psi(\zeta)\big)^{\vartheta-1} \right.$$
$$\left. \chi(\zeta, \varphi(\zeta)) d\zeta \right| + \left| \frac{D(\beta_1, \varphi(\beta_2))}{\Gamma(\vartheta)} \int_0^{\beta_1} \psi'(\zeta)\big(\psi(\beta_1) - \psi(\zeta)\big)^{\vartheta-1} \chi(\zeta, \varphi(\zeta)) d\zeta - \right.$$
$$\frac{D(\beta_1, \varphi(\beta_2))}{\Gamma(\vartheta)} \int_0^{\beta_2} \psi'(\zeta)\big(\psi(\beta_2) - \psi(\zeta)\big)^{\vartheta-1} \chi(\zeta, \varphi(\zeta)) d\zeta \bigg| + \left| \frac{D(\beta_1, \varphi(\beta_2))}{\Gamma(\vartheta)} \int_0^{\beta_2} \psi'(\zeta)\big(\psi(\beta_2) - \psi(\zeta)\big)^{\vartheta-1} \right.$$
$$\left. \chi(\zeta, \varphi(\zeta)) d\zeta - \frac{D(\beta_2, \varphi(\beta_2))}{\Gamma(\vartheta)} \int_0^{\beta_2} \psi'(\zeta)\big(\psi(\beta_1) - \psi(\zeta)\big)^{\vartheta-1} \chi(\zeta, \varphi(\zeta)) d\zeta \right|$$

$$\leq |D(\beta_1, \varphi(\beta_1)) - D(\beta_1, \varphi(\beta_2))| \frac{1}{\Gamma(\vartheta)} \int_0^{\beta_1} \psi'(\zeta)\big(\psi(\beta_2) - \psi(\zeta)\big)^{\vartheta-1} |\chi(\zeta, \varphi(\zeta))| d\zeta +$$

$$|D(\beta_1, \varphi(\beta_2))| \left| \left[\frac{1}{\Gamma(\vartheta)} \int_{\beta_2}^{\beta_1} \psi'(\zeta)\big(\psi(\beta_1) - \psi(\zeta)\big)^{\vartheta-1} \chi(\zeta, \varphi(\zeta)) d\zeta + \right. \right.$$

$$\left. \frac{1}{\Gamma(\vartheta)} \int_0^{\beta_2} \psi'(\zeta)\big\{ \big(\psi(\beta_1) - \psi(\zeta)\big)^{\vartheta-1} - \big(\psi(\beta_2) - \psi(\zeta)\big)^{\vartheta-1} \big\} \chi(\zeta, \chi(\zeta)) d\zeta \right] \right| +$$

$$|D(\beta_1, \varphi(\beta_2)) - D(\beta_2, \varphi(\beta_2))| \frac{1}{\Gamma(\vartheta)} \int_0^{\beta_2} \psi'(\zeta)\big(\psi(\beta_2) - \psi(\zeta)\big)^{\vartheta-1} |\chi(\zeta, \varphi(\zeta))| d\zeta$$

$$\leq \overline{D} |\varphi(\beta_1) - \varphi(\beta_2)| \frac{1}{\Gamma(\vartheta+1)} \big(\psi(\beta_1) - \psi(0)\big)^{\vartheta} \overline{\chi} g_0 + |D(\beta_1, \varphi(\beta_2))| \left[\frac{\overline{\chi} g_0}{\Gamma(\vartheta)} \int_{\beta_2}^{\beta_1} \psi'(\zeta)\big(\psi(\beta_1) - \psi(\zeta)\big)^{\vartheta-1} d\zeta \right.$$

$$\left. + \frac{\overline{\chi} g_0}{\Gamma(\vartheta)} \int_0^{\beta_2} \psi'(\zeta)\big\{ \big(\psi(\beta_1) - \psi(\zeta)\big)^{\vartheta-1} - \big(\psi(\beta_2) - \psi(\zeta)\big)^{\vartheta-1} \big\} d\zeta \right] +$$

$$|D(\beta_1, \varphi(\beta_2)) - D(\beta_2, \varphi(\beta_2))| \frac{1}{\Gamma(\vartheta+1)} \big(\psi(\beta_2) - \psi(0)\big)^{\vartheta} \overline{\chi} g_0$$

$$\leq \overline{D} \varpi(\varphi, \epsilon) \frac{1}{\Gamma(\vartheta+1)} \big(\psi(\beta_1) - \psi(0)\big)^{\vartheta} \overline{\chi} g_0 + |D(\beta_1, \varphi(\beta_2))| \left[\frac{\overline{\chi} g_0}{\Gamma(\vartheta+1)} \big(\psi(\beta_1) - \psi(\beta_2)\big)^{\vartheta} \right.$$

$$\left. + \frac{\overline{\chi} g_0}{\Gamma(\vartheta+1)} \big\{ \big(\psi(\beta_1) - \psi(0)\big)^{\vartheta} - \big(\psi(\beta_2) - \psi(0)\big)^{\vartheta} - \big(\psi(\beta_1) - \psi(\beta_2)\big)^{\vartheta} \big\} d\zeta \right] +$$

$$\varpi_D(g_0, \epsilon) \frac{1}{\Gamma(\vartheta+1)} \big(\psi(\beta_2) - \psi(0)\big)^{\vartheta} \overline{\chi} g_0,$$

where
$$\varpi(\varphi, \epsilon) = \sup\{ |\varphi(\beta_1) - \varphi(\beta_2)|; |\beta_1 - \beta_2| \leq \epsilon; \beta_1, \beta_2 \in L \},$$
$$\varpi_D(g_0, \epsilon) = \sup\{ |D(\beta_1, \varphi(\beta_2)) - D(\beta_2, \varphi(\beta_2))| : \beta_1, \beta_2 \in L, |\beta_1 - \beta_2| \leq \epsilon, \|\varphi\| \leq g_0 \}.$$

Since D is continuous, $\varpi_D(g_0, \epsilon) \to 0$ as $\epsilon \to 0$.
As $\epsilon \to 0$, taking $\sup_{\varphi \in \Delta_\varphi}$, so we get

$$\varpi_0(\mathfrak{G} \Delta_\varphi) \leq \varpi_0(\Delta_\varphi) \left[\overline{D} \frac{\big(\psi(\beta_1) - \psi(0)\big)^{\varsigma-1}}{\Gamma(\varsigma)} \varphi_0 + \overline{D} \frac{1}{\Gamma(\vartheta+1)} \big(\psi(\beta_1) - \psi(0)\big)^{\vartheta} \overline{\chi} g_0 \right].$$

Thus, by Corollary 3.1, we conclude that $\mathfrak{G}$ has a $\mathbb{FP}$ for $\Delta_\varphi \in \mathfrak{X}_{g_0}$, and hence the above equation (4.2) has a solution in $\mathcal{C}(L)$.

□

5. Example

We examine the following ψ Hilfer fractional hybrid differential equation:

$$\begin{cases} \mathbb{H}\mathbf{D}_{0^+}^{\vartheta,\upsilon;\psi} \dfrac{\varphi(\beta)}{D(\beta,\varphi(\beta))} = \chi(\beta,\varphi(\beta)), & \beta \in L, \\[3mm] \mathrm{I}_{0^+}^{1-\varsigma;\psi} \dfrac{\varphi(0)}{D(0,\varphi(0))} = \dfrac{3}{2}. \end{cases} \tag{5.1}$$

Here,

$$D(\beta,\varphi(\beta)) = \frac{\varphi(\beta)}{21(1+\beta^2)} \ , \chi(\beta,\varphi(\beta)) = \frac{\beta^2\varphi(\beta)}{1+\beta^2} \ , L = [0,1],$$
$$\psi(\beta) = \sinh^{-1}\beta, \varsigma = \frac{1}{3}, \vartheta = \frac{1}{2}.$$

It is trivial that D is continuous and

$$|D(\beta,\varphi_1(\beta_1)) - D(\beta,\varphi_2(\beta_2))| \le \overline{D}|\varphi_1(\beta_1) - \varphi_2(\beta_2)|,$$

for all $\beta,\beta_1,\beta_2 \in L$ and $\varphi_1,\varphi_2 \in R$. So, $\overline{D} = 1$ and $D(\beta,0) = 0$. Also, χ is continuous, such that

$$|\chi(\beta,\varphi_1(\beta_1)) - \chi(\beta,\varphi_2(\beta_2))| \le \overline{\chi}|\varphi_1(\beta_1) - \varphi_2(\beta_2)|,$$

for all $\beta,\beta_1,\beta_2 \in L$ and $\varphi_1,\varphi_2 \in R$. So, $\overline{\chi} = 1$ and $\chi(\beta,0) = 0$.
Now, from the inequality in condition (4), we have

$$g_0 \frac{1}{\Gamma(\frac{1}{3})}[\sinh^{-1}(1)]^{\frac{-2}{3}}\frac{3}{2} + \frac{1}{\Gamma(\frac{3}{2})}g_0[\sinh^{-1}(1)]^{\frac{1}{2}}g_0 \le g_0$$
$$\implies 0.55 + 1.15g_0 \le 1$$
$$\implies 1.15g_0 \le 0.45$$
$$\implies g_0 \le 0.39.$$

Accordingly, the criteria of Theorem 4.1 are satisfied, and hence we deduce that the given fractional hybrid differential problem has a solution in $\mathcal{C}(L)$.

6. Conclusion

In this work, we introduce a new fixed point theorem based on a class of functions used to define a novel type of condensing operator. The theorem we obtain serves as a generalization of Darbo's classical fixed point theorem. We further demonstrate its applicability by constructing the existence of solutions to a existence results for the solutions to the first order hybrid fractional differential equation involving the ψ-Hilfer fractional derivative. To support our theoretical results, we present a concrete illustrative example.

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