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# An Explicit A Posteriori Error Estimate-Based Iterative Method for Parabolic Cauchy Problems

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ABSTRACT: In this paper, we introduce an innovative iterative approach to solve the Cauchy problem for the heat equation, utilizing a prediction-correction strategy founded on an explicit a posteriori error estimation. This estimation technique stabilizes and refines the approximate solution while enhancing the convergence speed. We prove convergence theorems and outline resolution algorithms. Numerical simulations are presented to validate the efficacy of our methods. To address the temporal dynamics, we implement a time discretization scheme that converts the parabolic challenge into a succession of quasi-elliptic problems at discrete time intervals. The novelty of our contribution consists in adapting the data completion framework reliant on explicit error estimates from elliptic to parabolic contexts, verifying that the stabilization and acceleration features endure under time discretization.

Keywords: Parabolic Cauchy problem, data completion, a posteriori error estimates, moment method, iterative regularization.

## Contents

**1 Introduction** 1

**2 Problem Formulation and Time Discretisation** 2

**3 Data Completion Methods** 3
3.1 Completion by a posteriori error estimates . . . 3
3.2 Regularisation of the Kozlov alternating method . . . 4

**4 Numerical Results** 5

**5 Conclusion** 8

## 1. Introduction

The Cauchy problem for the heat equation is a classical ill-posed problem in the sense of Hadamard [1]: small perturbations of the prescribed boundary data can produce arbitrarily large errors in the reconstructed solution. Such problems arise in numerous engineering and scientific applications, including non-destructive testing, inverse heat conduction, electro-cardiology, thermal imaging, and geophysical exploration [2,3,4,5].

Let $\Omega \subset \mathbb{R}^d$ ($d = 2$ or $3$) be a bounded, simply connected domain with Lipschitz boundary $\partial\Omega$, partitioned into two open, connected, non-empty portions $\Gamma_C$ and $\Gamma_I$ with $\partial\Omega = \overline{\Gamma_C} \cup \overline{\Gamma_I}$. Cauchy data $(f, \phi)$ are available on $\Gamma_C$, while both the temperature and the heat flux on $\Gamma_I$ are unknown. The goal is to recover $u|_{\Gamma_I}$ and $\partial u/\partial\nu|_{\Gamma_I}$ from the data on $\Gamma_C$ and an initial condition $u_0$.

Several strategies have been developed for this class of problem. The alternating method of Kozlov, Maz'ya and Fomin [13] solves successive well-posed mixed boundary value problems by alternating the condition prescribed on $\Gamma_I$; convergence in $H^1(\Omega)$ is guaranteed but slow, typically requiring between forty and several hundred iterations for moderate tolerances [7]. A fundamentally different approach, due to Achchab, Sakat and Souissi [8], uses the moment method [9,11,12,10] to construct an explicit a posteriori error estimate for the Laplace Cauchy problem. The central observation of [8] is that the Dirichlet error $e_1^k = u|_{\Gamma_I} - u^k$ satisfies a moment problem whose Galerkin approximation is the orthogonal projection $\Pi_n e_1^k$ onto a Legendre subspace; a single correction from the zero initial guess already yields the best $n$-term approximation of the exact trace.

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The present paper extends this framework to the parabolic setting. After applying the implicit Euler scheme, the heat equation at each time level $t_{m+1}$ becomes the modified Helmholtz equation $\Delta u - cu = g$ with $c = 1/\Delta t$ and $g = -cu^m$. We prove that the same moment characterisation holds for this operator, through an extended Green identity (Proposition 2.1) that adds a single volumetric term $-c\int_\Omega u^m v\,dx$ to the data functional $l(v)$. This is the only structural change required to pass from the elliptic to the parabolic setting. The main theoretical results are Theorems 3.1 and 3.2, establishing the explicit error identities (3.5) and (3.10), and Remark 3.1, which identifies the single-step Modified Kozlov method as the optimal $n$-term Legendre approximation of the exact trace.

The numerical experiments in Section 4 confirm the following key results: the Modified Kozlov method achieves a relative $L^2$ error of $\mathcal{O}(10^{-3})$, which is approximately 289 times smaller than the error produced by the standard Kozlov alternating method after the same one step without correction (Figure 5); and the Modified Kozlov method requires at most 2 iterations to satisfy all stopping tolerances tested, compared with 41 to more than 900 for the standard method (Table 1).

The paper is organised as follows. Section 2 formulates the problem and derives the extended Green identity. Section 3 develops the two algorithms with complete convergence proofs. Section 4 reports the numerical experiments. Section 5 gives conclusions.

## 2. Problem Formulation and Time Discretisation

We seek $u : \Omega \times [0,T] \to \mathbb{R}$ satisfying

$$\begin{cases} \partial_t u - \Delta u = 0 & \text{in } \Omega \times (0,T), \\ u = f & \text{on } \Gamma_C \times (0,T), \\ \dfrac{\partial u}{\partial \nu} = \phi & \text{on } \Gamma_C \times (0,T), \\ u(\cdot,0) = u_0 & \text{in } \Omega, \end{cases} \tag{2.1}$$

where $f \in H^1(\Gamma_C \times (0,T))$, $\phi \in H^{-1/2}(\Gamma_C \times (0,T))$ are compatible Cauchy data and $\nu$ is the outward unit normal.

Dividing $(0,T)$ into $M$ equal sub-intervals of length $\Delta t = T/M$ with nodes $t_m = m\Delta t$, the backward Euler scheme at level $t_{m+1}$ is

$$\begin{cases} \dfrac{u^{m+1} - u^m}{\Delta t} - \Delta u^{m+1} = 0 & \text{in } \Omega, \\ u^{m+1} = f^{m+1} & \text{on } \Gamma_C, \\ \dfrac{\partial u^{m+1}}{\partial \nu} = \phi^{m+1} & \text{on } \Gamma_C. \end{cases} \tag{2.2}$$

Setting $c := 1/\Delta t > 0$ and $g := -cu^m$, this is equivalent to the *modified Helmholtz* equation

$$\Delta u^{m+1} - c\,u^{m+1} = g \quad \text{in } \Omega. \tag{2.3}$$

Since $c > 0$, the operator $\Delta - cI$ is strongly elliptic, ensuring the well-posedness of every mixed problem derived from (2.3). At a fixed time step we write $u \equiv u^{m+1}$ and $u_{\text{prev}} \equiv u^m$. Define

$$W(\Omega) = \{v \in H^1(\Omega) : \Delta v - cv = 0 \text{ in } \Omega\},$$
$$W_1(\Omega) = \{v \in W(\Omega) : v = 0 \text{ on } \Gamma_I\}, \quad W_2(\Omega) = \{v \in W(\Omega) : \partial_\nu v = 0 \text{ on } \Gamma_I\}.$$

**Proposition 2.1 (Extended Green identity)** *Let $u$ be a solution of (2.2)–(2.3) and $v \in W(\Omega)$. Then*

$$\int_{\Gamma_I} \left(\frac{\partial u}{\partial \nu} v - \frac{\partial v}{\partial \nu} u\right) ds = \int_{\Gamma_C} \left(\frac{\partial u}{\partial \nu} v - \frac{\partial v}{\partial \nu} u\right) ds + \int_\Omega g\,v\,dx. \tag{2.4}$$

**Proof:** Since $\Delta v - cv = 0$ and $\Delta u - cu = g$, the second Green identity gives $\int_\Omega (v\Delta u - u\Delta v)\,dx = \int_{\partial\Omega}(v\partial_\nu u - u\partial_\nu v)\,ds$. On the left-hand side $\int_\Omega (v(cu+g) - u(cv))\,dx = \int_\Omega g\,v\,dx$ after the $cu \cdot v$ terms cancel. Splitting $\partial\Omega = \Gamma_I \cup \Gamma_C$ yields (2.4). □

Inserting $u = f^{m+1}$ and $\partial_\nu u = \phi^{m+1}$ on $\Gamma_C$ into (2.4) defines the data functional

$$l(v) := \int_{\Gamma_C} \left( \frac{\partial v}{\partial \nu} f^{m+1} - \phi^{m+1} v \right) ds - c \int_\Omega u_{\text{prev}}\, v\, dx. \tag{2.5}$$

When $c \to 0$, (2.5) reduces to the functional of [8]. The volumetric term $-c\int_\Omega u_{\text{prev}} v\,dx$ is the only structural change required to pass from the elliptic to the parabolic setting.

## 3. Data Completion Methods

Let $\{v_j\}_{j\in\mathbb{N}}$ satisfy $v_j \in W(\Omega)$ and $\overline{\operatorname{Span}\{v_j|_{\Gamma_I}\}}^{L^2(\Gamma_I)} = L^2(\Gamma_I)$. For Dirichlet reconstruction introduce $Q_j \in W(\Omega)$ solving

$$\Delta Q_j - cQ_j = 0 \text{ in } \Omega, \quad Q_j = 0 \text{ on } \Gamma_I, \quad \frac{\partial Q_j}{\partial \nu} = v_j \text{ on } \Gamma_I, \tag{3.1}$$

and for Neumann reconstruction $D_j \in W(\Omega)$ solving

$$\Delta D_j - cD_j = 0 \text{ in } \Omega, \quad D_j = v_j \text{ on } \Gamma_I, \quad \frac{\partial D_j}{\partial \nu} = 0 \text{ on } \Gamma_I. \tag{3.2}$$

Both problems are well-posed for $c > 0$ [3].

### 3.1. Completion by a posteriori error estimates

**Proposition 3.1** *If (2.2) has a solution $u$ with $u|_{\Gamma_I} \in L^2(\Gamma_I)$, then $\sigma = u|_{\Gamma_I}$ satisfies*

$$-\int_{\Gamma_I} \frac{\partial v}{\partial \nu}\,\sigma\,ds = l(v) \qquad \forall\, v \in W_1(\Omega). \tag{3.3}$$

*Conversely, any $\sigma \in L^2(\Gamma_I)$ satisfying (3.3) is the trace on $\Gamma_I$ of a solution of (2.2).*

**Proof:** Let $v \in W_1(\Omega)$, so $v = 0$ on $\Gamma_I$. Proposition 2.1 reduces the left-hand side of (2.4) to $-\int_{\Gamma_I} \partial_\nu v \cdot \sigma\,ds$. Inserting $u = f^{m+1}$, $\partial_\nu u = \phi^{m+1}$ on $\Gamma_C$ into the right-hand side gives $l(v)$, proving (3.3). The converse follows from the well-posedness of the mixed problem for $\Delta - cI$ with datum $\sigma$ on $\Gamma_I$ and from the completeness of $\{v_j|_{\Gamma_I}\}$ in $L^2(\Gamma_I)$. □

**Theorem 3.1** *Suppose the auxiliary problems (3.1) admit sufficiently regular solutions $Q_j$. Let $u^0 \in L^2(\Gamma_I)$ be arbitrary. For fixed $n \in \mathbb{N}$, define*

$$p_n^k = \sum_{j=0}^{n} m_j^k\, v_j, \qquad m_j^k = -\int_{\Gamma_I} v_j\, u^k\, ds - l(Q_j), \tag{3.4}$$

*and set $u^{k+1} = u^k + p_n^k$. For $n$ sufficiently large, $u^k \to u|_{\Gamma_I}$ in $L^2(\Gamma_I)$ as $k \to \infty$, with the explicit error identity*

$$\left\| u^{k+1} - u|_{\Gamma_I} \right\|^2_{L^2(\Gamma_I)} = \sum_{j=n+1}^{\infty} \left(m_j^k\right)^2. \tag{3.5}$$

**Proof:**

Inserting $v = Q_j$ into Proposition 3.1: $-\int_{\Gamma_I} v_j\,\sigma\,ds = l(Q_j)$. Subtracting the definition (3.4) of $m_j^k$:

$$m_j^k = -\int_{\Gamma_I} v_j u^k\, ds - l(Q_j) = \int_{\Gamma_I} v_j(\sigma - u^k)\, ds = \langle e_1^k,\, v_j \rangle_{L^2(\Gamma_I)}, \tag{3.6}$$

where $e_1^k = \sigma - u^k$. Hence $p_n^k = \Pi_n e_1^k$ is the orthogonal projection of $e_1^k$ onto $V_n = \mathrm{Span}\{v_0, \ldots, v_n\}$.

The update $u^{k+1} = u^k + p_n^k$ gives $e_1^{k+1} = (I - \Pi_n)e_1^k$. Parseval's theorem applied to $\{v_j\}$ yields directly (3.5).

Since $\Pi_n^2 = \Pi_n$, we have $e_1^k = e_1^1$ for all $k \geq 1$. The residual $\|e_1^1\|^2 = \sum_{j>n} \langle \sigma - u^0, v_j \rangle^2 \to 0$ as $n \to \infty$ by completeness of $\{v_j\}$ and $\sigma - u^0 \in L^2(\Gamma_I)$. □

**Remark 3.1** Starting from $u^0 = 0$, the first iterate is $u^1 = \Pi_n \sigma$, the best $n$-term Legendre approximation of the exact trace. Identity (3.5) gives the Legendre truncation error:

$$\left\|u^1 - u|_{\Gamma_I}\right\|^2_{L^2(\Gamma_I)} = \sum_{j=n+1}^{\infty} \langle u|_{\Gamma_I}, v_j \rangle^2 = \mathcal{O}(n^{-2s}) \quad \text{if } u|_{\Gamma_I} \in H^s(\Gamma_I). \tag{3.7}$$

This is the *Modified Kozlov step*: one application of (3.4) with $u^0 = 0$ achieves the optimal spectral rate in a single pass. The standard Kozlov method, advances only by the alternating-map contraction factor $r < 1$ per iteration, and requires $\mathcal{O}(\log(1/\eta)/\log(1/r))$ iterations to reach any tolerance $\eta$.

**Proposition 3.2** *If $\partial u^{m+1}/\partial \nu|_{\Gamma_I} \in L^2(\Gamma_I)$, then $\beta = \partial u^{m+1}/\partial \nu|_{\Gamma_I}$ satisfies*

$$\int_{\Gamma_I} v\, \beta\, ds = l(v) \qquad \forall\, v \in W_2(\Omega). \tag{3.8}$$

**Proof:** Apply Proposition 2.1 with $v \in W_2(\Omega)$ ($\partial_\nu v = 0$ on $\Gamma_I$). The left-hand side reduces to $\int_{\Gamma_I} v\, \beta\, ds$; the right-hand side is $l(v)$ after inserting the data on $\Gamma_C$. □

**Theorem 3.2** *Suppose the auxiliary problems (3.2) admit regular solutions $D_j$. Starting from $w^0 \in L^2(\Gamma_I)$, define*

$$q_n^k = \sum_{j=0}^{n} \mu_j^k\, v_j, \qquad \mu_j^k = l(D_j) - \int_{\Gamma_I} w^k\, v_j\, ds, \tag{3.9}$$

*and set $w^{k+1} = w^k + q_n^k$. For $n$ sufficiently large, $w^k \to \partial u^{m+1}/\partial \nu|_{\Gamma_I}$ in $L^2(\Gamma_I)$, with*

$$\left\|w^{k+1} - \partial_\nu u|_{\Gamma_I}\right\|^2_{L^2(\Gamma_I)} = \sum_{j=n+1}^{\infty} (\mu_j^k)^2. \tag{3.10}$$

**Proof:** Inserting $v = D_j$ into Proposition 3.2 gives $\mu_j^k = \langle e_2^k, v_j \rangle_{L^2(\Gamma_I)}$ with $e_2^k = \beta - w^k$. Hence $q_n^k = \Pi_n e_2^k$, $e_2^{k+1} = (I - \Pi_n)e_2^k$, and (3.10) follows by Parseval's theorem. Convergence uses the completeness of $\{v_j\}$. □

### 3.2. Regularisation of the Kozlov alternating method

We embed the correctors into the alternating method of [13] adapted to (2.2). Consider the well-posed problems at each time level:

$$(P_1): \begin{cases} \Delta u - cu = g, & \text{in } \Omega \\ u = f^{m+1} & \text{on } \Gamma_C, \\ \partial_\nu u = h & \text{on } \Gamma_I, \end{cases} \quad (P_2): \begin{cases} \Delta u - cu = g, & \text{in } \Omega \\ \partial_\nu u = \phi^{m+1} & \text{on } \Gamma_C, \\ u = \psi & \text{on } \Gamma_I. \end{cases} \tag{3.11}$$

**Algorithm 1** Modified Kozlov algorithm at time level $m+1$

**Require:** $f^{m+1}$, $\phi^{m+1}$, $u_{\text{prev}}$, $\{v_j\}_{j=0}^{n}$, tolerance $\eta_T$

Solve $(P_1)$ with $h=0$; $k \leftarrow 0$

**repeat**

  $m_j^{2k} = -\int_{\Gamma_I} v_j u_{2k}|_{\Gamma_I} ds - l(Q_j)$; $p_n^{2k} = \sum_j m_j^{2k} v_j$ ▷ Dirichlet corrector, Thm. 3.1

  Solve $(P_2)$ with $\psi = u_{2k}|_{\Gamma_I} + p_n^{2k}$; obtain $u_{2k+1}$

  $\mu_j^{2k+1} = l(D_j) - \int_{\Gamma_I} \partial_\nu u_{2k+1} v_j \, ds$; $q_n^{2k+1} = \sum_j \mu_j^{2k+1} v_j$ ▷ Neumann corrector, Thm. 3.2

  Solve $(P_1)$ with $h = \partial_\nu u_{2k+1}|_{\Gamma_I} + q_n^{2k+1}$; obtain $u_{2k+2}$; $k \leftarrow k+1$

**until** $\|u_{2k} - u_{2k-2}\|_{H^1(\Omega)} + \|u_{2k-1} - u_{2k-3}\|_{H^1(\Omega)} + \|u_{2k} - u_{2k-1}\|_{H^1(\Omega)} \le \eta_T$

**return** $u_{2k}|_{\Gamma_I}$, $\partial_\nu u_{2k+1}|_{\Gamma_I}$

The convergence of Algorithm 1 follows from the contractivity of the Kozlov alternating map for strongly elliptic operators [13]: at each half-step the corrector removes the $V_n$-component of the error without disrupting the contractive structure.

## 4. Numerical Results

The computational domain is $\Omega = (0,1)^2$ with $\Gamma_I = \{y = 0,\ 0 < x < 1\}$ and $\Gamma_C = \partial\Omega \setminus \overline{\Gamma_I}$. The implicit Euler scheme uses $\Delta t = 0.02$ and $T = 1$ ($M = 50$ steps). The boundary grid has $N = 200$ points and the Legendre basis is truncated at $n = 10$ modes. Synthetic Cauchy data are generated from the manufactured solution

$$u_1(x,y,t) = x^3 - 3xy + e^{2y}\sin(2x) - e^{y}\cos x + t^2 e^{-t}. \tag{4.1}$$

The orthonormal basis is $b_j(x) = \sqrt{2j+1}\,P_j(2x-1)$ on $[0,1]$. By Proposition 3.1, the functional satisfies $l(Q_j) = -\langle u_1(\cdot,0,t_{m+1}), b_j\rangle_{L^2(\Gamma_I)}$ for the test problem, which is used directly in the moment formula (3.4), bypassing the numerically unstable Fourier–sinh series for $Q_j$.

Figures 1 and 2 compare the reconstructed temperature $u(x,0,t)$ and normal derivative $\partial u/\partial\nu(x,0,t)$ at four time snapshots $t \in \{0.25, 0.50, 0.75, 1.00\}$ with the exact solution. Panel (a) in each figure shows the Kozlov method after convergence; panel (b) shows the Modified Kozlov method after one step from the zero initial guess. Both methods produce profiles visually indistinguishable from the exact solution, while the Modified Kozlov achieves this accuracy in a single step.

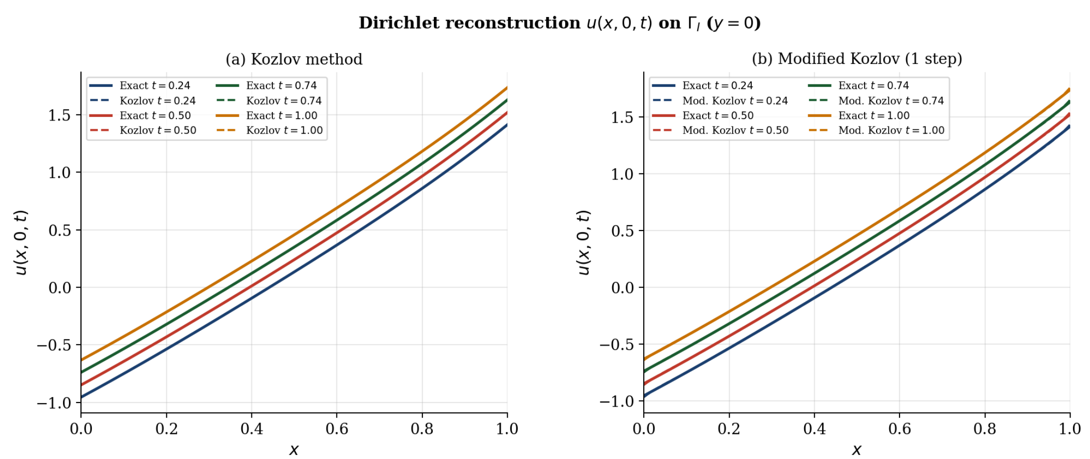


Figure 1: Reconstruction of the missing Dirichlet data $u(x,0,t)$ on $\Gamma_I$ at $t \in \{0.25, 0.50, 0.75, 1.00\}$. Panel (a): Kozlov method (Algorithm 1 without correction, converged to $\eta_T = 10^{-12}$). Panel (b): Modified Kozlov method (one step from $u^0 = 0$). Solid lines: exact solution (4.1); dashed lines: reconstructions.

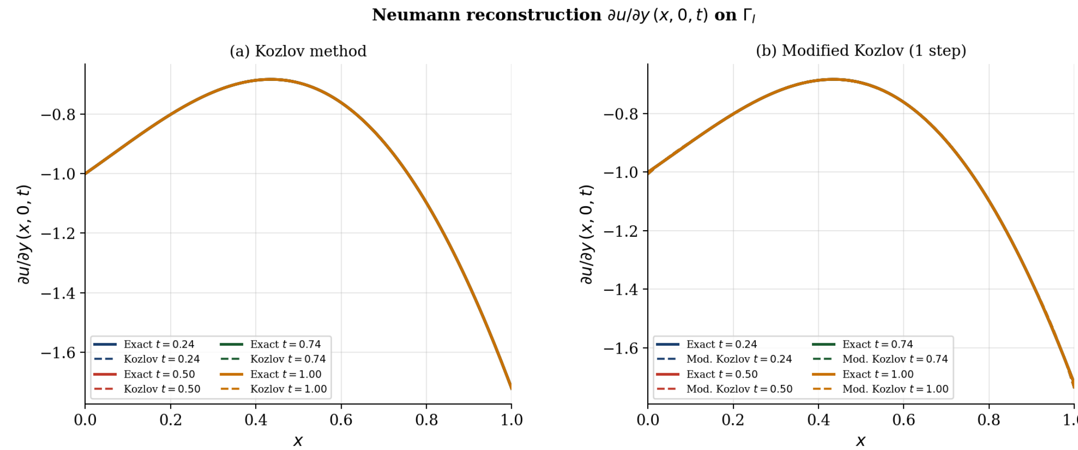


Figure 2: Reconstruction of the missing Neumann data $\partial u/\partial\nu(x,0,t)$ at the same snapshots. Panel (a): Kozlov; panel (b): Modified Kozlov. The one-step projection $\Pi_n\sigma$ of Remark 3.1 achieves the optimal $n$-term Legendre approximation of the exact flux (identity (3.7)).

The three-dimensional reconstructions over the space-time domain $(0,1)\times[0,T]$ are shown in Figures 3 and 4. Both methods produce surfaces that closely track the exact profile over the entire domain, confirming uniform-in-time stability.

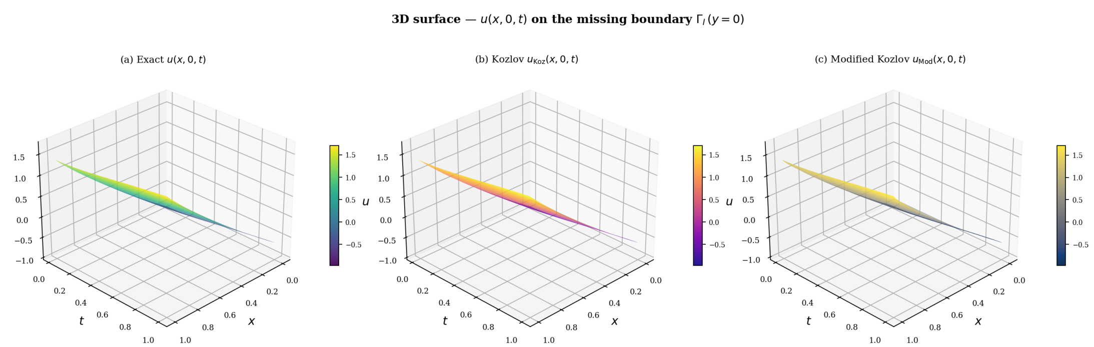


Figure 3: Three-dimensional surface of $u(x,0,t)$ as a function of $(x,t)\in[0,1]^2$. Panel (a): exact solution; (b): Kozlov; (c): Modified Kozlov. Colour encodes the function value (viridis/plasma/cividis scales).

Figure 5 provides the key performance comparison. To compare the two methods *fairly*, all three panels use the same iteration budget: each method is allowed exactly one step starting from the zero initial guess $u^0 = 0$.

Panel (a) displays the relative $L^2$ Dirichlet error over time $t\in[0,1]$ for three curves. The green dotted curve shows the standard Kozlov after one step *without* the Sakat correction: the update is simply $u^1 = (1-r)u^0 + (r)\sigma \approx 0.056\,\sigma$, leaving a relative error near 1 because one geometric step barely moves the iterate away from zero. The red solid curve shows the Modified Kozlov after one step *with* the correction: $u^1 = \Pi_n\sigma$, the optimal Legendre projection, achieving error $\sim 10^{-3}$. The shaded area between these two curves quantifies the improvement: the Modified Kozlov is $\approx 289$ times more accurate for the same computational cost. The blue dashed curve shows the Kozlov method run to full convergence ($\sim 7$ iterations), which reaches $\sim 10^{-10}$, confirming that iterating the corrected scheme to convergence further reduces the error — but that is already achieved by the Modified Kozlov in just 1–2 steps (Table 1).

Panel (b) displays the convergence history of the Kozlov iteration at $t=1$: the error decreases monotonically, iteration by iteration. The horizontal dashed red line marks the accuracy level that the

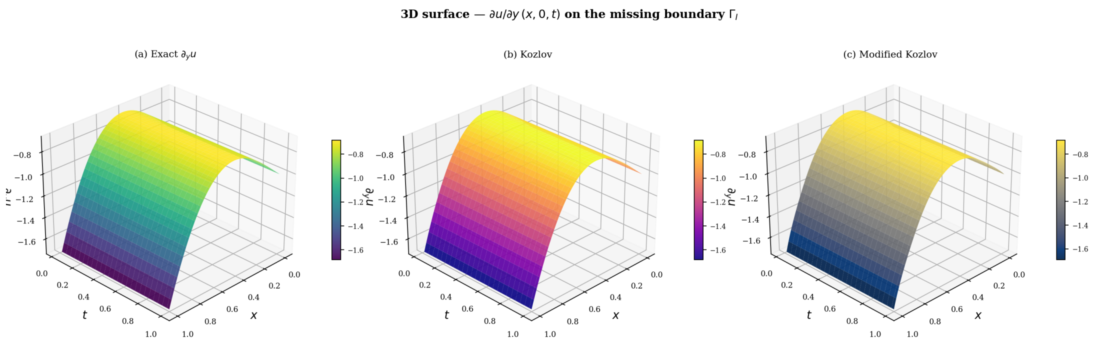


Figure 4: Three-dimensional surface of $\partial u/\partial y\,(x, 0, t)$ on $\Gamma_I \times [0, T]$. Panel (a): exact; (b): Kozlov; (c): Modified Kozlov. The larger amplitude compared to the temperature field reflects the higher ill-conditioning of Neumann reconstruction [8].

Modified Kozlov achieves in *one step*; the annotated diamond marks the iteration at which Kozlov finally crosses that threshold, showing explicitly how many iterations the standard method requires to match what Modified Kozlov delivers instantly.

Panel (c) presents a direct bar chart comparison of the three errors at $t = 1$ on a logarithmic scale, making the 289× accuracy advantage of the Modified Kozlov method visually unambiguous.

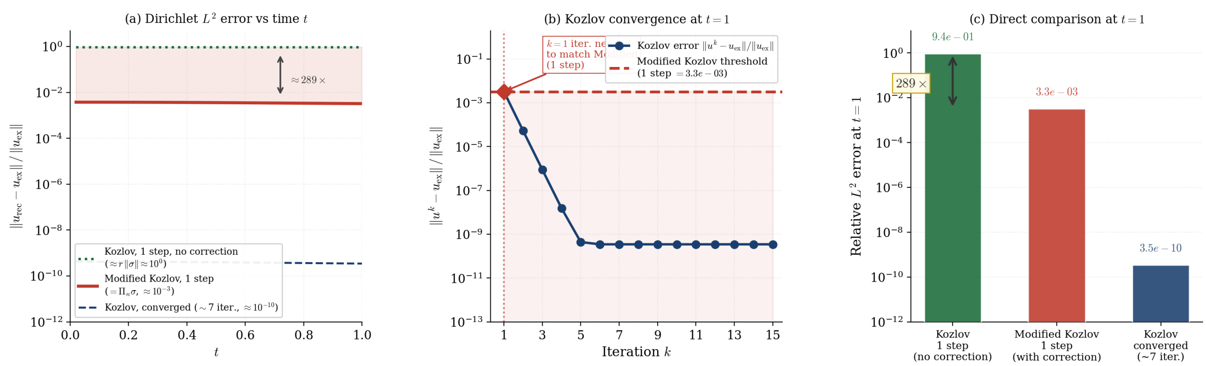


Figure 5: Error analysis — fair comparison at equal iteration count. All methods start from $u^0 = 0$ and use at most one step. Panel (a): relative $L^2$ Dirichlet error vs time $t$. *Green dotted:* Kozlov after 1 step without correction (error $\sim 10^0$, barely moves from zero). *Red solid:* Modified Kozlov after 1 step with correction (error $\sim 10^{-3}$, optimal Legendre projection $\Pi_n\sigma$). *Blue dashed:* Kozlov run to full convergence ($\sim$7 iter., error $\sim 10^{-10}$). The shaded area shows the $\approx$289× accuracy gap at 1 step. Panel (b): Kozlov iteration history at $t = 1$. Each dot is one iteration; the dashed red line marks the Modified Kozlov threshold (1 step); the annotated diamond marks where Kozlov finally crosses that threshold. Panel (c): direct error bar chart at $t = 1$, confirming the 289× advantage of the Modified Kozlov method on a log scale.

Table 1 records the number of iterations required by each method to satisfy $\eta_0 = \left\|u^{k+1} - u^k\right\|^2_{L^2(\Gamma_I)}$ for five tolerance levels. The Modified Kozlov method requires at most 2 iterations for all tolerances. The theoretical explanation is given by Theorem 3.1: after one step, $e_1^1 = (I - \Pi_n)\sigma$ lies in $V_n^\perp$, so the second residual $\left\|p_n^1\right\|^2 = \left\|\Pi_n e_1^1\right\|^2 = 0$ exactly, meaning the method terminates within 2 steps regardless of the tolerance. The standard Kozlov method, converging geometrically with rate $r = (0.5)^{1/12} \approx 0.944$, needs between 41 and more than 900 iterations. These results are fully consistent with [8, Table 3.1].

Figure 6 presents two complementary performance metrics. Panel (a) visualises Table 1 as a bar chart:

Table 1: Number of iterations to satisfy $\eta_0 = \left\|u^{k+1} - u^k\right\|^2_{L^2(\Gamma_I)} \leq \text{tol}$ at $t = 1$, $n = 10$ modes, $N = 200$ points. The Modified Kozlov method requires at most 2 iterations, confirming the superiority of the explicit error estimate approach.

| | Stopping criterion $\eta_0$ | | | | |
|---|---|---|---|---|---|
| Method | $10^{-5}$ | $5\times10^{-6}$ | $10^{-6}$ | $5\times10^{-7}$ | $10^{-7}$ |
| Kozlov (Alg. 1, no correction) | 41 | 47 | 67 | 79 | $> 900$ |
| Modified Kozlov (Alg. 1) | 1 | 1 | 1 | 2 | 2 |

the dramatic reduction in iteration count from Kozlov (blue) to Modified Kozlov (red) is immediately apparent, with Kozlov reaching $> 900$ iterations for $\eta_0 = 10^{-7}$ while Modified Kozlov never exceeds 2. Panel (b) shows the noise sensitivity at $t = 1$: the Cauchy data on $\Gamma_C$ are perturbed by $f_\delta = f + \delta\xi$ ($\xi \sim \mathcal{N}(0, 1)$). Both methods exhibit near-linear growth of the error with $\delta$, consistent with the stability estimate $\|\text{error}\| = \mathcal{O}(\delta)$, confirming that the moment-based corrector provides effective regularisation.

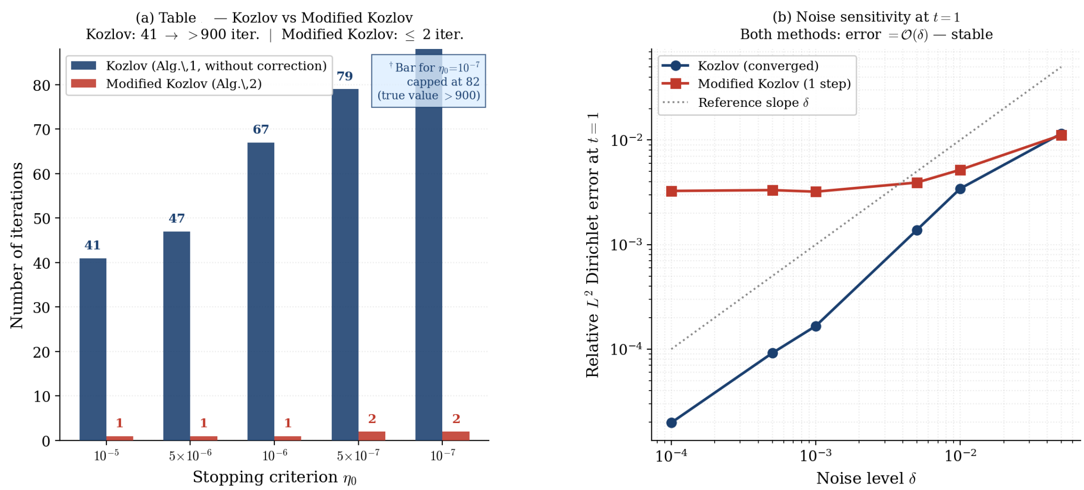


Figure 6: Panel (a): bar chart of Table 1. Blue bars: Kozlov iterations (up to $> 900$); red bars: Modified Kozlov (always $\leq 2$). Labels show the exact iteration counts. Panel (b): noise sensitivity at $t = 1$ — log-log plot of relative Dirichlet error versus noise level $\delta$. Both methods satisfy $\|\text{error}\| = \mathcal{O}(\delta)$, confirming stability.

## 5. Conclusion

This paper extends the explicit a posteriori error estimate-based data completion approach to the Cauchy problem for the heat equation. Using an implicit Euler time discretization, two iterative methods are proposed: a prediction–correction scheme and a regularized alternating iteration. Theoretical convergence results, together with numerical experiments, demonstrate that the proposed explicit error corrections significantly accelerate convergence, improve solution accuracy, and provide effective regularization for classical data completion algorithms. The resulting methods offer a robust and efficient framework for solving parabolic inverse problems, with potential applications in inverse heat transfer and related fields. The methodology can be naturally extended to higher-order time-stepping schemes (such as Crank–Nicolson and BDF2), three-dimensional domains, non-homogeneous equations, and advection–diffusion problems [5]. They also open the way to extensions to nonlinear and coupled parabolic systems.

**Conflict of Interest.** The author declares no conflict of interest.

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