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A Study on Leap Sigma Index and Multiplicative Leap Sigma Index of Some Nanostructures

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ABSTRACT: This paper introduces several novel topological indices derived from molecular structures including the Leap sigma index, Leap sigma multiplicative index, Reduced Sombor leap index, and Reduced Sombor leap multiplicative index, along with their exponential forms. We computed leap sigma indices and explore their applications in characterizing molecular properties. Additionally, we compute the Leap indices for various nanostructures and Polycyclic aromatic hydrocarbons (PAHn), highlighting their significance in understanding the structural characteristics and behaviour of these compounds. Our findings contribute to the ongoing research in molecular topology and its implications in chemistry, materials science, nanoscience, pharmacology, graph theory and others.

Keywords: Sigma index, nanostructures, leap index, aromatic hydrocarbons.

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1. Introduction

Topological indices(TI) which are numerical values associated with the structural properties of a graph, are widely used in mathematical chemistry to describe molecular graphs. These indices are derived from the topology of the graph and are invariant under graph isomorphism, meaning that they are not influenced by the drawing style of the graph.

The degree of a vertex v in a graph describes the number of edges connected to that vertex v denoted by $d(v)$. The distance $d(u, v)$ between two vertices u and v in a graph $G = (V, E)$ is the length of the shortest path connecting u and v. The neighborhood degree of a vertex v in a graph $G = (V, E)$ is the sum of the degrees of all vertices that are adjacent to v denoted by

$$ND(v) = \sum_{u \in N(v)} d(u)$$

where N(v) is the set of all neighbors of vertex v.

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A class of chemical molecules known as polycyclic aromatic hydrocarbons (PAHs) is made up of many aromatic rings that have been fused together in a planar shape. Their importance in organic synthesis, environmental research, and materials chemistry makes them an important field of study in chemistry.

A particular kind of molecular structure called an armchair polyhex nanotube $TUAC_6[2p, q]$ is made from a hexagonal (honeycomb) lattice that has been rolled into a tubular shape. It is frequently used in mathematical chemistry and nanotechnology to investigate its topological and physical characteristics.

The molecular structure known as the armchair polyhex nanotorus $TUAC_6[p, q]$, is made of a hexagonal (honeycomb) lattice that is bent and joined to create a toroidal (doughnut-like) shape. It has applications in materials science, theoretical chemistry, and nanotechnology.

The type of molecular structure known as a V-Phenylenic Nanotube VPHX[p,q] produced from a graph of benzene rings (phenyl groups) organized in a particular tubular orientation. Because of its special qualities, this structure is researched in theoretical chemistry, materials science, and nanotechnology.

The V-Phenylenic Nanotorus, denoted as VPHY[p,q], is a toroidal (doughnut-shaped) molecular structure constructed from phenyl units (benzene rings) arranged in a specific V-phenylenic pattern. This structure represents an intriguing combination of aromatic systems and toroidal geometry, making it a subject of interest in theoretical chemistry, nanotechnology, and materials science.

V-tetracenic nanotube is defined by the parameter where both are greater than 1 . The structure is made up of q rows and p cycles in a row, which combine to form a complicated tube-like arrangement of carbon atoms. This setup makes it possible to compute a number of topological indices, which are crucial for comprehending their chemical characteristics and behaviours.

H-Tetracenic nanotubes are formed from a rolled-up structure of tetracene, which consists of multiple fused aromatic rings. The parameters p and q define the specific arrangement of these rings within the nanotube. The structure is similar to other carbon nanotubes but has distinct topological features that influence its chemical and physical properties.

Naji et.al defined first ,second leap Zagreb indices [4,7,8,9,10,14,15,19,20,21,22,23,24] as ,

$$LM_1(G) = \sum (d_2(u) + d_2(v))$$

$$LM_2(G) = \sum (d_2(u)d_2(v))$$

The Sigma Index defined as [6],

$$\sigma(G) = \sum_{uv \in E(G)} (d(u) - d(v))^2$$

Sombor index defined by I Gutman as,

$$SO(G) = \sum \sqrt{d_u^2 + d_v^2}$$

The reduced Sombor index and its exponential is defined as [13],

$$SO_{\text{red}}(G) = \sum_{uv \in E(G)} \sqrt{(d(u) - 1)^2 + (d(v) - 1)^2}$$

$$SO_{\text{red}}(G, x) = \sum_{uv \in E(G)} x^{\sqrt{(d(u)-1)^2 + (d(v)-1)^2}}$$

Sombor leap index of a graph G and its exponentials defined by Kulli as,

$$SL(G) = \sum_{uv \in E(G)} \sqrt{(d_2(u))^2 + (d_2(v))^2}$$

$$SL(G, x) = \sum_{uv \in E(G)} x^{\sqrt{(d_2(u))^2 + (d_2(v))^2}}$$

In this study,Introduced leap sigma Index and its exponentials as,

$$L\sigma_1(G) = \sum_{uv \in E(G)} [d_2(u) - d_2(v)]^2$$

$$L\sigma_1(G,x) = \sum_{uv \in E(G)} x^{[d_2(u)-d_2(v)]^2}$$

In this work, Introduced leap sigma multiplicative Index and its exponentials as,

$$L\sigma_2(G) = \prod_{uv \in E(G)} [d_2(u) - d_2(v)]^2$$

$$L\sigma_2(G,x) = \prod_{uv \in E(G)} x^{[d_2(u)-d_2(v)]^2}$$

In this work,Introduced reduced sombor leap index and their corresponding exponentials of a graph as,

$$RSL_1(G) = \sum_{uv \in E(G)} \sqrt{[(d_2(u)-1)^2] + [(d_2(v)-1)^2]}$$

$$RSL_1(G,x) = \sum_{uv \in E(G)} x^{\sqrt{[(d_2(u)-1)^2]+[(d_2(v)-1)^2]}}$$

In this study,Introduced reduced sombor leap multiplicative index and their corresponding exponentials of a graph as,

$$RSL_2(G) = \prod_{uv \in E(G)} \sqrt{[(d_2(u)-1)^2] + [(d_2(v)-1)^2]}$$

$$RSL_2(G,x) = \prod_{uv \in E(G)} x^{\sqrt{[(d_2(u)-1)^2]+[(d_2(v)-1)^2]}}$$

Many article have been published related to leap indices. In this paper, determined Leap Sigma index, leap sigma multiplicative index,reduced sombor leap index , reduced sombor leap Multiplicative index and its exponentials for for some nanostructures and Polycyclic Aromatic Hydrocarbons of PAHn. [1,2,3, 5,11,12,16,17,18]

2. Results for Leap Sigma Indices of Some Nanostructures

In this section, Determined various leap sigma indices and their exponential for nanostructures such as armchair polyhex nanotube, armchair polyhex nanotorus, V-Phenylenic nanotube, V-Phenylenic nanotorus,V-tetracenic nanotube, V-tetracenic nanotorus, H-tetracenic nanotube, H-tetracenic nanotorus.

Table 1: EDGE SET PARTITION OF GRAPH (G_1)

$d_2(u), d_2(v), uv \in E(G_1)$	(3,3)	(3,5)	(5,5)	(5,6)	(6,6)
Number of Edges	$2p$	$4p$	$2p$	$4p$	$3pq - 14p$

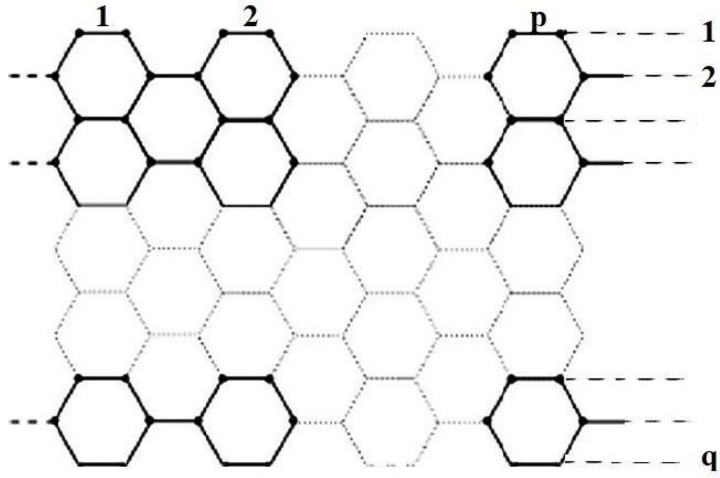

Figure 1: Molecular structure of Armchair Polyhex Nanotube(G_1)

Theorem 2.1 *Let G_1 be the armchair polyhex nanotube, denoted by $TUAC_6[2p, q]$.*

$$(i) \quad L\sigma_1(G_1) = 20p$$

$$(ii) \quad L\sigma_1(G_1, x) = 2p + 4px^4 + 2p + 4px + 3pq - 14p$$

$$(iii) \quad RSL_1(G_1) = 4p\sqrt{2} + 8p\sqrt{5} + 8p\sqrt{2} + 4p\sqrt{41} + 5\sqrt{2}(3pq - 14p)$$

$$(iv) \quad RSL_1(G_1, x) = 2px^{(4\sqrt{2})} + 2px^{(2\sqrt{5})} + 2px^{(4\sqrt{2})} + 4px^{(\sqrt{41})} + (3pq - 14p)x^{(5\sqrt{2})}$$

$$(v) \quad L\sigma_2(G_1) = 0$$

$$(vi) \quad L\sigma_2(G_1, x) = 64p^4x^2(3pq - 14q)$$

$$(vii) \quad RSL_2(G_1) = 64p^4\sqrt{10496000}(3pq - 14q)$$

$$(viii) \quad RSL_2(G_1, x) = [x^{\sqrt{8}}(2p)] \times [x^{\sqrt{20}}(4p)] \times [x^{\sqrt{32}}(2p)] \times [x^{\sqrt{41}}(4p)] \times [x^{\sqrt{50}}(3pq - 14q)]$$

Proof: *The network $G_1 = TUAC_6[2p, q]$ has $3pq - 2p$ edges and $2pq$ vertices. With the use of Table 1 and definitions of the leap sigma index, reduced sombor index, and the related exponential and edge set partition of graph G_1, we obtain*

$$(i) \quad L\sigma(G_1) = \sum_{uv \in E(G)} [d_2(u) - d_2(v)]^2$$
$$= (3-3)^2(2p) + (3-5)^2(4p) + (5-5)^2(2p) + (5-6)^2(4p) + (6-6)^2(3pq - 14p)$$
$$= 0 + 4(4p) + 0 + 1(4p) + 0$$
$$= 16p + 4p$$
$$= 20p$$

$$(ii) \quad L\sigma(G_1, x) = \sum_{uv \in E(G)} x^{[d_2(u) - d_2(v)]^2}$$
$$= x^{(3-3)^2}(2p) + x^{(3-5)^2}(4p) + x^{(5-5)^2}(2p) + x^{(5-6)^2}(4p) + x^{(6-6)^2}(3pq - 14p)$$
$$= 2p + 4px^4 + 2p + 4px + 3pq - 14p$$

$$(iii) \quad RSL_1(G_1) = \sum_{uv\in E(G)} \left[[(d_2(u)-1)^2] + [(d_2(v)-1)^2]\right]^{1/2}$$
$$= \left[(3-1)^2+(3-1)^2\right]^{1/2}(2p) + \left[(3-1)^2+(5-1)^2\right]^{1/2}(4p)$$
$$+ \left[(5-1)^2+(5-1)^2\right]^{1/2}(2p) + \left[(5-1)^2+(6-1)^2\right]^{1/2}(4p)$$
$$+ \left[(6-1)^2+(6-1)^2\right]^{1/2}(3pq-14p)$$
$$= [4+4]^{1/2}(2p) + [4+16]^{1/2}(4p) + [16+16]^{1/2}(2p)$$
$$+ [16+25]^{1/2}(4p) + [25+25]^{1/2}(3pq-14p)$$
$$= [8]^{1/2}(2p) + [20]^{1/2}(4p) + [32]^{1/2}(2p) + [41]^{1/2}(4p)$$
$$+ [50]^{1/2}(3pq-14p)$$
$$= 4p\sqrt{2} + 8p\sqrt{5} + 8p\sqrt{2} + 4p\sqrt{41} + 5\sqrt{2}(3pq-14p)$$

$$(iv) \quad RSL_1(G_1,x) = \sum_{uv\in E(G)} x^{\left[[(d_2(u)-1)^2]+[(d_2(v)-1)^2]\right]^{1/2}}$$
$$= x^{\left[(3-1)^2+(3-1)^2\right]^{1/2}}(2p) + x^{\left[(3-1)^2+(5-1)^2\right]^{1/2}}(4p)$$
$$+ x^{\left[(5-1)^2+(5-1)^2\right]^{1/2}}(2p) + x^{\left[(5-1)^2+(6-1)^2\right]^{1/2}}(4p)$$
$$+ x^{\left[(6-1)^2+(6-1)^2\right]^{1/2}}(3pq-14p)$$
$$= x^{4\sqrt{2}}(2p) + x^{2\sqrt{5}}(4p) + x^{4\sqrt{2}}(2p) + x^{\sqrt{41}}(4p) + x^{5\sqrt{2}}(3pq-14p)$$
$$= 2px^{4\sqrt{2}} + 2px^{2\sqrt{5}} + 2px^{4\sqrt{2}} + 4px^{\sqrt{41}} + (3pq-14p)x^{5\sqrt{2}}$$

$$(v) \quad L\sigma_2(G_1) = \prod_{uv\in E(G)} [d_2(u)-d_2(v)]^2$$
$$= [(3-3)^2\cdot 2p] \times [(3-5)^2\cdot 4p] \times [(5-5)^2\cdot 2p]$$
$$\times [(5-6)^2\cdot 4p] \times [(6-6)^2\cdot (3pq-14p)]$$
$$= 0$$

$$(vi) \quad L\sigma_2(G_1,x) = \prod_{uv\in E(G)} x^{[d_2(u)-d_2(v)]^2}$$
$$= [x^{(3-3)^2}\cdot 2p] \times [x^{(3-5)^2}\cdot 4p] \times [x^{(5-5)^2}\cdot 2p]$$
$$\times [x^{(5-6)^2}\cdot 4p] \times [x^{(6-6)^2}\cdot (3pq-14p)]$$
$$= 64p^4x^2(3pq-14q)$$

$$(vii) \quad RSL_2(G_1) = \prod_{uv\in E(G)} \left[[(d_2(u)-1)^2] + [(d_2(v)-1)^2]\right]^{1/2}$$
$$= \left[[(3-1)^2+(3-1)^2]^{1/2}\cdot 2p\right] \times \left[[(3-1)^2+(5-1)^2]^{1/2}\cdot 4p\right]$$
$$\times \left[[(5-1)^2+(5-1)^2]^{1/2}\cdot 2p\right] \times \left[[(5-1)^2+(6-1)^2]^{1/2}\cdot 4p\right]$$
$$\times \left[[(6-1)^2+(6-1)^2]^{1/2}\cdot (3pq-14q)\right]$$
$$= [\sqrt{8}(2p)] \times [\sqrt{20}(4p)] \times [\sqrt{32}(2p)]$$
$$\times [\sqrt{41}(4p)] \times [\sqrt{50}(3pq-14q)]$$
$$= 64p^4\sqrt{10496000}(3pq-14q)$$

$$(viii)\quad RSL_2(G_1,x) = \prod_{uv\in E(G)} x^{\left[[(d_2(u)-1)^2]+[(d_2(v)-1)^2]\right]^{1/2}}$$
$$= \left[x^{\left[(3-1)^2+(3-1)^2\right]^{1/2}}\cdot 2p\right] \times \left[x^{\left[(3-1)^2+(5-1)^2\right]^{1/2}}\cdot 4p\right]$$
$$\times \left[x^{\left[(5-1)^2+(5-1)^2\right]^{1/2}}\cdot 2p\right] \times \left[x^{\left[(5-1)^2+(6-1)^2\right]^{1/2}}\cdot 4p\right]$$
$$\times \left[x^{\left[(6-1)^2+(6-1)^2\right]^{1/2}}\cdot (3pq-14q)\right]$$
$$= [x^{\sqrt{8}}(2p)] \times [x^{\sqrt{20}}(4p)] \times [x^{\sqrt{32}}(2p)]$$
$$\times [x^{\sqrt{41}}(4p)] \times [x^{\sqrt{50}}(3pq-14q)]$$

□

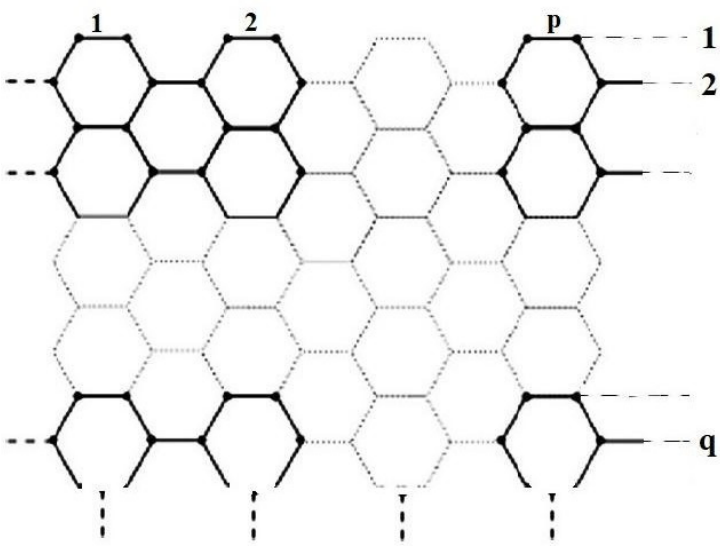

Figure 2: Molecular structure of Armchair Polyhex Nanotorus(G_2)

Table 2: EDGE SET PARTITION OF GRAPH (G_2)

$d_2(u), d_2(v), uv \in E(G_2)$	(6,6)
Number of Edges	$3pq$

Theorem 2.2 *G_2 is an armchair Polyhex Nanotorus $TUAC_6[p,q]$,then*

$$(i)\quad L\sigma_1(G_2) = 0$$
$$(ii)\quad L\sigma_1(G_2,x) = 3pq$$
$$(iii)\quad RSL_1(G_2) = 3\sqrt{50}pq$$
$$(iv)\quad RSL_1(G_2,x) = 3pqx^{\sqrt{50}}$$
$$(v)\quad L\sigma_2(G_2) = 0$$
$$(vi)\quad L\sigma_2(G_2,x) = 3pq$$
$$(vii)\quad RSL_2(G_2) = 3\sqrt{50}pq$$
$$(viii)\quad RSL_2(G_2,x) = x^{\sqrt{50}}\cdot 3pq$$

Proof: *The network $G_2 = TUAC_6[p,q]$ has $2pq$ vertices and $3pq$ edges. With the use of Table 2 and definitions of the leap sigma index, reduced sombor index, and the related exponential and edge set partition of graph G_2, we obtain*

$$(i) \quad L\sigma_1(G_2) = \sum_{uv\in E(G)} [d_2(u) - d_2(v)]^2 = (6-6)^2(3pq) = 0$$

$$(ii) \quad L\sigma_1(G_2, x) = \sum_{uv\in E(G)} x^{[d_2(u)-d_2(v)]^2} = x^{[6-6]^2} \cdot 3pq = 3pq$$

$$(iii) \quad RSL_1(G_2) = \sum_{uv\in E(G)} \left[[(d_2(u)-1)^2] + [(d_2(v)-1)^2]\right]^{1/2} = \left[(6-1)^2 + (6-1)^2\right]^{1/2}(3pq) = 3\sqrt{50}pq$$

$$(iv) \quad RSL_1(G_2, x) = \sum_{uv\in E(G)} x^{\left[[(d_2(u)-1)^2]+[(d_2(v)-1)^2]\right]^{1/2}} = x^{\left[(6-1)^2+(6-1)^2\right]^{1/2}}(3pq) = 3pq\cdot x^{\sqrt{50}}$$

$$(v) \quad L\sigma_2(G_2) = \prod_{uv\in E(G)} [d_2(u) - d_2(v)]^2 = [(6-6)^2 \cdot 3pq] = 0$$

$$(vi) \quad L\sigma_2(G_2, x) = \prod_{uv\in E(G)} x^{[d_2(u)-d_2(v)]^2} = [x^{(6-6)^2} \cdot 3pq] = 3pq$$

$$(vii) \quad RSL_2(G_2) = \prod_{uv\in E(G)} \left[[(d_2(u)-1)^2] + [(d_2(v)-1)^2]\right]^{1/2} = \left[(6-1)^2 + (6-1)^2\right]^{1/2} \cdot (3pq) = 3\sqrt{50}\cdot(3pq)$$

$$(viii) \quad RSL_2(G_2, x) = \prod_{uv\in E(G)} x^{\left[[(d_2(u)-1)^2]+[(d_2(v)-1)^2]\right]^{1/2}} = \left[x^{\left[(6-1)^2+(6-1)^2\right]^{1/2}} \cdot (3pq)\right] = x^{\sqrt{50}}\cdot(3pq)$$

□

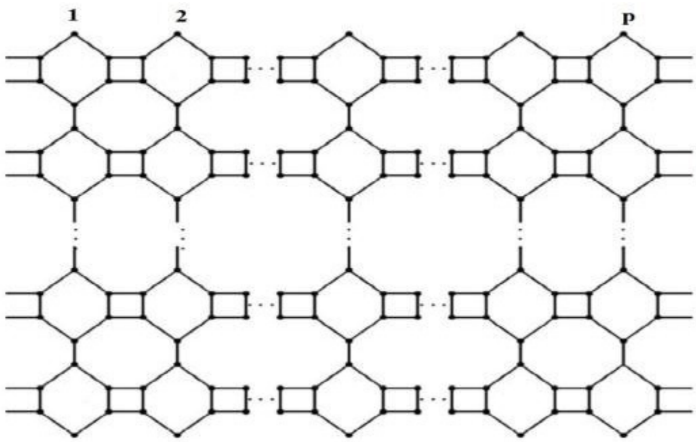

Figure 3: Molecular structure of V-Phenylenic Nanotube(G_3)

Table 3: EDGE SET PARTITION OF GRAPH G_3

$d_2(u), d_2(v), uv \in E(G_3)$	$(4,4)$	$(4,5)$	$(5,5)$	$(5,6)$	$(6,6)$
Number of Edges	$6p$	$4p$	$2p(2q-3)$	$4p(q-1)$	$p(q-1)$

Theorem 2.3 *G_3 is a V-Phenylenic Nanotube then*

$$(i) \quad L\sigma_1(G_3) = 4pq$$

$$(ii) \quad L\sigma_1(G_3, x) = 5pq + 4pqx - p$$

$$(iii) \quad RSL_1(G_3) = \sqrt{18}(6p) + \sqrt{25}(4p) + \sqrt{32}(2p)(2q-3) + \sqrt{41}(4p)(q-1) + \sqrt{50}(p)(q-1)$$

$$(iv) \quad RSL_1(G_3, x) = x^{\sqrt{18}}(6p) + x^{\sqrt{25}}(4p) + x^{\sqrt{32}}(2p)(2q-3) + x^{\sqrt{41}}(4p)(q-1) + x^{\sqrt{50}}(p)(q-1)$$

$$(v) \quad L\sigma_2(G_3) = 0$$

$$(vi) \quad L\sigma_2(G_3, x) = (6p)(4xp)(2p)(2q-3)(4xp)(q-1)(p)(q-1)$$

$$(vii) \quad RSL_2(G_3) = (\sqrt{18} \times 6p)(\sqrt{25} \times 4p)(\sqrt{32} \times 2p(2q-3))(\sqrt{41} \times 4p(q-1))$$
$$(\sqrt{50} \times p(q-1))$$

$$(viii) \quad RSL_2(G_3, x) = (x^{\sqrt{18}} \times 6p)(x^{\sqrt{25}} \times$$
$$4p)(x^{\sqrt{32}} \times 2p(2q-3))(x^{\sqrt{41}} \times$$
$$4p(q-1))(x^{\sqrt{50}} \times p(q-1))$$

Proof: *There are $6pq$ vertices and $9pq{-}p$ edges in the graph $G_3{=}[p,q]$. With the help of the definitions of the leap sigma index, reduced sombor index, and the associated exponential and edge set partition of* *graph G_3 provided in Table 3, we obtain,*

$$(i)\quad L\sigma_1(G_3) = \sum_{uv\in E(G)} [d_2(u) - d_2(v)]^2$$
$$= (4-4)^2(6p) + (4-5)^2(4p) + (5-5)^2(2p)(2q-3) + (5-6)^2(4p)(q-1)+$$
$$(6-6)^2(p)(q-1)$$
$$= 0\cdot(6p) + 1^2\cdot(4p) + 0\cdot(2p)(2q-3) + 1^2\cdot(4p)(q-1) + 0\cdot(p)(q-1)$$
$$= 4p + 4p(q-1)$$
$$= 4pq.$$

$$(ii)\quad L\sigma_1(G_3, x) = \sum_{uv\in E(G)} x^{[d_2(u)-d_2(v)]^2}$$
$$= x^{(4-4)^2}(6p) + x^{(4-5)^2}(4p) + x^{(5-5)^2}(2p)(2q-3) + x^{(5-6)^2}(4p)(q-1)+$$
$$x^{(6-6)^2}(p)(q-1)$$
$$= x^0(6p) + x^1(4p) + x^0(2p)(2q-3) + x^1(4p)(q-1) + x^0(p)(q-1)$$
$$= 6p + 4pqx + 0 + 4p(q-1)x + 0$$
$$= 5pq + 4pqx - p.$$

$$(iii)\quad RSL_1(G_3) = \sum_{uv\in E(G)} \sqrt{[(d_2(u)-1)^2] + [(d_2(v)-1)^2]}$$
$$= \sqrt{(4-1)^2 + (4-1)^2}(6p) + \sqrt{(4-1)^2 + (5-1)^2}(4p)+$$
$$\sqrt{(5-1)^2 + (5-1)^2}(2p)(2q-3) \quad + \sqrt{(5-1)^2 + (6-1)^2}(4p)(q-1)+$$
$$\sqrt{(6-1)^2 + (6-1)^2}(p)(q-1)$$
$$= \sqrt{18}(6p) + \sqrt{25}(4p) + \sqrt{32}(2p)(2q-3) + \sqrt{41}(4p)(q-1)+$$
$$\sqrt{50}(p)(q-1).$$

$$(iv)\quad RSL_1(G_3, x) = \sum_{uv\in E(G)} x^{\sqrt{[(d_2(u)-1)^2]+[(d_2(v)-1)^2]}}$$
$$= x^{\sqrt{(4-1)^2+(4-1)^2}}(6p) + x^{\sqrt{(4-1)^2+(5-1)^2}}(4p) + x^{\sqrt{(5-1)^2+(5-1)^2}}(2p)(2q-3)$$
$$+ x^{\sqrt{(5-1)^2+(6-1)^2}}(4p)(q-1) + x^{\sqrt{(6-1)^2+(6-1)^2}}(p)(q-1)$$
$$= x^{\sqrt{18}}(6p) + x^{\sqrt{25}}(4p) + x^{\sqrt{32}}(2p)(2q-3) + x^{\sqrt{41}}(4p)(q-1)+$$
$$x^{\sqrt{50}}(p)(q-1).$$

$$(v)\quad L\sigma_2(G_3) = \prod_{uv\in E(G)} [d_2(u) - d_2(v)]^2$$
$$= \big[(4-4)^2(6p)\big] \times \big[(4-5)^2(4p)\big] \times \big[(5-5)^2(2p)(2q-3)\big]$$
$$\times \big[(5-6)^2(4p)(q-1)\big] \times \big[(6-6)^2(p)(q-1)\big]$$
$$= 0.$$

$$
\begin{aligned}
(vi)\quad L\sigma_2(G_3,x) &= \prod_{uv\in E(G)} x^{[d_2(u)-d_2(v)]^2}\\
&= \left[x^{(4-4)^2}(6p)\right]\times\left[x^{(4-5)^2}(4p)\right]\times\left[x^{(5-5)^2}(2p)(2q-3)\right]\\
&\quad\times\left[x^{(5-6)^2}(4p)(q-1)\right]\times\left[x^{(6-6)^2}(p)(q-1)\right]\\
&= (6p)(4xp)(2p)(2q-3)(4xp)(q-1)(p)(q-1).
\end{aligned}
$$

$$
\begin{aligned}
(vii)\quad RSL_2(G_3) &= \prod_{uv\in E(G)} \sqrt{[(d_2(u)-1)^2]+[(d_2(v)-1)^2]}\\
&= \left[\sqrt{(4-1)^2+(4-1)^2}(6p)\right]\times\left[\sqrt{(4-1)^2+(5-1)^2}(4p)\right]\\
&\quad\times\left[\sqrt{(5-1)^2+(5-1)^2}(2p)(2q-3)\right]\\
&\quad\times\left[\sqrt{(5-1)^2+(6-1)^2}(4p)(q-1)\right]\\
&\quad\times\left[\sqrt{(6-1)^2+(6-1)^2}(p)(q-1)\right]\\
&= \sqrt{18}(6p)\times\sqrt{25}(4p)\times\sqrt{32}(2p)(2q-3)\\
&\quad\times\sqrt{41}(4p)(q-1)\times\sqrt{50}(p)(q-1).
\end{aligned}
$$

$$
\begin{aligned}
(viii)\quad RSL_2(G_3,x) &= \prod_{uv\in E(G)} x^{\sqrt{[(d_2(u)-1)^2]+[(d_2(v)-1)^2]}}\\
&= \left[x^{\sqrt{(4-1)^2+(4-1)^2}}(6p)\right]\times\left[x^{\sqrt{(4-1)^2+(5-1)^2}}(4p)\right]\\
&\quad\times\left[x^{\sqrt{(5-1)^2+(5-1)^2}}(2p)(2q-3)\right]\\
&\quad\times\left[x^{\sqrt{(5-1)^2+(6-1)^2}}(4p)(q-1)\right]\\
&\quad\times\left[x^{\sqrt{(6-1)^2+(6-1)^2}}(p)(q-1)\right]\\
&= (x^{\sqrt{18}}\cdot 6p)(x^{\sqrt{25}}\cdot 4p)(x^{\sqrt{32}}\cdot 2p(2q-3))\\
&\quad(x^{\sqrt{41}}\cdot 4p(q-1))(x^{\sqrt{50}}\cdot p(q-1)).
\end{aligned}
$$

□

Table 4: EDGE SET PARTITION OF GRAPH G_4

$d_2(u), d_2(v), uv \in E(G_4)$	$(5,5)$	$(5,6)$	$(6,6)$
Number of Edges	$4pq$	$4pq$	pq

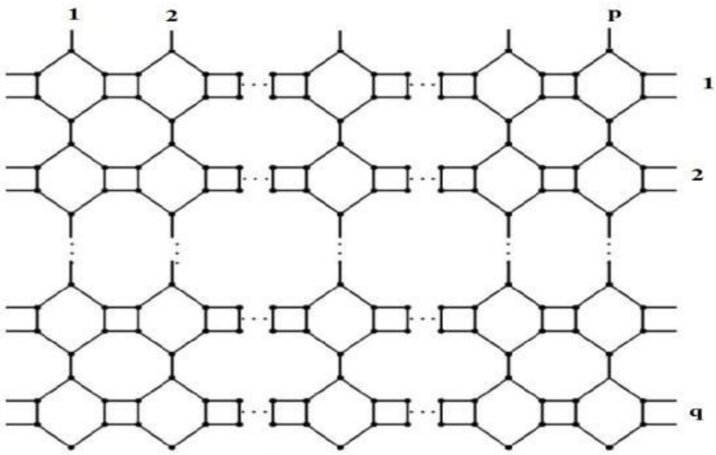

Figure 4: Molecular structure of V-Phenylenic Nanotorus(G_4)

Theorem 2.4 *G_4 is a V-phenylenic nanotorus [p,q], then*

$$\begin{aligned}
(i)\quad & L\sigma_1(G_4) = 4pq\\
(ii)\quad & L\sigma_1(G_4, x) = 5pq + 4pqx\\
(iii)\quad & RSL_1(G_4) = \sqrt{32}(4pq) + \sqrt{41}(4pq) + \sqrt{50}pq\\
(iv)\quad & RSL_1(G_4, x) = x^{\sqrt{32}}(4pq) + x^{\sqrt{41}}(4pq) + x^{\sqrt{50}}pq\\
(v)\quad & L\sigma_2(G_4) = 0\\
(vi)\quad & L\sigma_2(G_4, x) = 16p^3q^3x\\
(vii)\quad & RSL_2(G_4) = [\sqrt{32}(4pq)] \times [\sqrt{41}(4pq)] \times [\sqrt{50}pq]\\
(viii)\quad & RSL_2(G_4, x) = (x^{\sqrt{32}} \times 4pq)(x^{\sqrt{41}} \times 4pq)(x^{\sqrt{50}}pq)
\end{aligned}$$

Proof: *There are 6pq vertices and 9pq edges in the graph G_4 = [p, q]. With the help of the definitions of the leap sigma index, reduced sombor index, and the associated exponential and edge set partition of graph G4 provided in Table 4, we obtain*

$$\begin{aligned}
i.\ L\sigma_1(G_4) &= \sum_{uv\in E(G)} [d_2(u) - d_2(v)]^2\\
&= (5-5)^2(4pq) + (5-6)^2(4pq) + (6-6)^2pq\\
&= 0 \cdot (4pq) + 1^2 \cdot (4pq) + 0 \cdot pq\\
&= 4pq.
\end{aligned}$$

$$\begin{aligned}
ii.\ L\sigma_1(G_4, x) &= \sum_{uv\in E(G)} x^{[d_2(u)-d_2(v)]^2}\\
&= x^{(5-5)^2}(4pq) + x^{(5-6)^2}(4pq) + x^{(6-6)^2}pq\\
&= x^0(4pq) + x^1(4pq) + x^0(pq)\\
&= 4pq + 4pqx + pq\\
&= 5pq + 4pqx.
\end{aligned}$$

$$iii.\ RSL_1(G_4) = \sum_{uv\in E(G)} \sqrt{[(d_2(u)-1)^2]+[(d_2(v)-1)^2]}$$
$$= \sqrt{(5-1)^2+(5-1)^2}(4pq) + \sqrt{(5-1)^2+(6-1)^2}(4pq) + \sqrt{(6-1)^2+(6-1)^2}(pq)$$
$$= \sqrt{16+16}(4pq) + \sqrt{16+25}(4pq) + \sqrt{25+25}(pq)$$
$$= \sqrt{32}(4pq) + \sqrt{41}(4pq) + \sqrt{50}(pq).$$

$$iv.\ RSL_1(G_4,x) = \sum_{uv\in E(G)} x^{\sqrt{[(d_2(u)-1)^2]+[(d_2(v)-1)^2]}}$$
$$= x^{\sqrt{(5-1)^2+(5-1)^2}}(4pq) + x^{\sqrt{(5-1)^2+(6-1)^2}}(4pq) + x^{\sqrt{(6-1)^2+(6-1)^2}}(pq)$$
$$= x^{\sqrt{16+16}}(4pq) + x^{\sqrt{16+25}}(4pq) + x^{\sqrt{25+25}}(pq)$$
$$= x^{\sqrt{32}}(4pq) + x^{\sqrt{41}}(4pq) + x^{\sqrt{50}}(pq).$$

$$v.\ L\sigma_2(G_4) = \prod_{uv\in E(G)} [d_2(u)-d_2(v)]^2$$
$$= \left[(5-5)^2(4pq)\right] \times \left[(5-6)^2(4pq)\right] \times \left[(6-6)^2 pq\right]$$
$$= 0\cdot(4pq) + 1^2\cdot(4pq) + 0\cdot pq$$
$$= 0.$$

$$vi.\ L\sigma_2(G_4,x) = \prod_{uv\in E(G)} x^{[d_2(u)-d_2(v)]^2}$$
$$= \left[x^{(5-5)^2}(4pq)\right] \times \left[x^{(5-6)^2}(4pq)\right] \times \left[x^{(6-6)^2} pq\right]$$
$$= x^0(4pq) \times x^1(4pq) \times x^0(pq)$$
$$= (4pq)(4pq)x$$
$$= 16p^3q^3x.$$

$$vii.\ RSL_2(G_4) = \prod_{uv\in E(G)} \sqrt{[(d_2(u)-1)^2]+[(d_2(v)-1)^2]}$$
$$= \left[\sqrt{(5-1)^2+(5-1)^2}(4pq)\right] \times \left[\sqrt{(5-1)^2+(6-1)^2}(4pq)\right] \times \left[\sqrt{(6-1)^2+(6-1)^2}(pq)\right]$$
$$= \left[\sqrt{16+16}(4pq)\right] \times \left[\sqrt{16+25}(4pq)\right] \times \left[\sqrt{25+25}(pq)\right]$$
$$= [\sqrt{32}(4pq)] \times [\sqrt{41}(4pq)] \times [\sqrt{50}pq].$$

$$viii.\ RSL_2(G_4,x) = \prod_{uv\in E(G)} x^{\sqrt{[(d_2(u)-1)^2]+[(d_2(v)-1)^2]}}$$
$$= \left[x^{\sqrt{(5-1)^2+(5-1)^2}}(4pq)\right] \times \left[x^{\sqrt{(5-1)^2+(6-1)^2}}(4pq)\right] \times \left[x^{\sqrt{(6-1)^2+(6-1)^2}}(pq)\right]$$
$$= \left(x^{\sqrt{16+16}}(4pq)\right) \times \left(x^{\sqrt{16+25}}(4pq)\right) \times \left(x^{\sqrt{25+25}}(pq)\right)$$
$$= (x^{\sqrt{32}}\cdot 4pq)(x^{\sqrt{41}}\cdot 4pq)(x^{\sqrt{50}}pq).$$

□

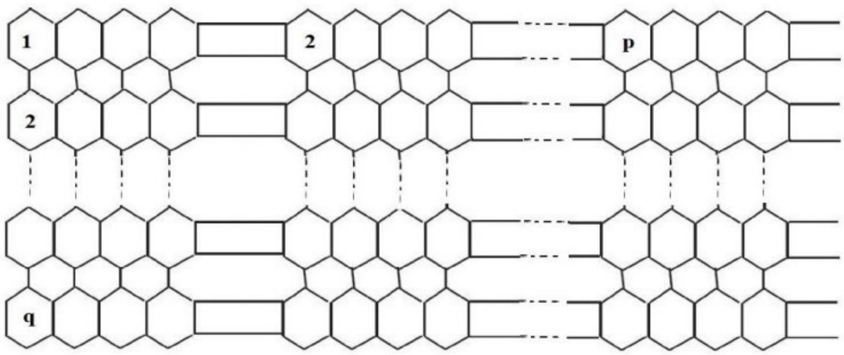

Figure 5: Molecular structure of V-Tetracenic Nanotube G_5

Table 5: EDGE SET PARTITION OF GRAPH G_5

$d_2(u), d_2(v), uv \in E(G)$	(4,4)	(4,5)	(4,6)	(5,6)	(6,6)
Number of Edges	18p	4p	6p	4p	9p(3q-4)

Theorem 2.5 *G_5 is a V-tetracenic nanotube [p, q], then*

(i) $L\sigma_1(G_5) = 32p$

(ii) $L\sigma_1(G_5, x) = 18p + 8px + x^4 \cdot 6p + 9p(3q-4)$

(iii) $RSL_1(G_5) = \sqrt{18} \cdot (18p) + \sqrt{25} \cdot (4p) + \sqrt{34} \cdot (6p) + \sqrt{41} \cdot (4p) + \sqrt{50} \cdot (9p)(3q-4)$

(iv) $RSL_1(G_5, x) = x^{\sqrt{18}}(18p) + x^{\sqrt{25}}(4p) + x^{\sqrt{34}}(6p) + x^{\sqrt{41}}(4p) + x^{\sqrt{50}}(9p)(3q-4)$

(v) $L\sigma_2(G_5) = 0$

(vi) $L\sigma_2(G_5, x) = (18p)(4xp)(x^4 \cdot 6p)(4xp)(9p(3q-4))$

(vii) $RSL_2(G_5) = (\sqrt{18} \times 18p)(\sqrt{25} \times 4p)(\sqrt{34} \times 6p)(\sqrt{41} \times 4p)(\sqrt{50} \times 9p(3q-4))$

(viii) $RSL_2(G_5, x) = (x^{\sqrt{18}} \times 18p)(x^{\sqrt{25}} \times 4p)(x^{\sqrt{34}} \times 6p)(x^{\sqrt{41}} \times 4p)(x^{\sqrt{50}} \times 9p(3q-4))$

Proof: *The network G_5 = [p, q] contains 27pq-4p edges and 18pq vertices. By using the definitions of the leap sigma index, reduced sombor index, and the related exponential and edge set partition of the graph G_5 provided in Table 5, we obtain,*

$$
\begin{aligned}
(i) \quad L\sigma_1(G_5) &= \sum_{uv \in E(G)} [d_2(u) - d_2(v)]^2 \\
&= (4-4)^2(18p) + (4-5)^2(4p) + (4-6)^2(6p) + (5-6)^2(4p) \\
+ &\quad (6-6)^2(9p)(3q-4) \\
&= 32p
\end{aligned}
$$

$$
\begin{aligned}
(ii) \quad L\sigma_1(G_5, x) &= \sum_{(u,v) \in E(G)} x^{[d_2(u) - d_2(v)]^2} \\
&= x^{(4-4)^2}(18p) + x^{(4-5)^2}(4p) + x^{(4-6)^2}(6p) + x^{(5-6)^2}(4p) \\
&\quad + x^{(6-6)^2}(9p)(3q-4) \\
&= 18p + 8px + 6px^4 + 9p(3q-4)
\end{aligned}
$$

$$
\begin{aligned}
(iii) \quad RSL_1(G_5) &= \sum_{(u,v)\in E(G)} \Big[\left(d_2(u)-1\right)^2 + \left(d_2(v)-1\right)^2\Big]^{1/2} \\
&= \sqrt{(4-1)^2+(4-1)^2}(18p) + \sqrt{(4-1)^2+(5-1)^2}(4p) \\
&\quad + \sqrt{(4-1)^2+(6-1)^2}(6p) + \sqrt{(5-1)^2+(6-1)^2}(4p) \\
&\quad + \sqrt{(6-1)^2+(6-1)^2}(9p)(3q-4) \\
&= \sqrt{18}(18p) + \sqrt{25}(4p) + \sqrt{34}(6p) + \sqrt{41}(4p) + \sqrt{50}(9p)(3q-4)
\end{aligned}
$$

$$
\begin{aligned}
(iv) \quad RSL_1(G_5,x) &= \sum_{(u,v)\in E(G)} x^{\left[(d_2(u)-1)^2+(d_2(v)-1)^2\right]^{1/2}} \\
&= x^{\sqrt{(4-1)^2+(4-1)^2}}(18p) + x^{\sqrt{(4-1)^2+(5-1)^2}}(4p) \\
&\quad + x^{\sqrt{(4-1)^2+(6-1)^2}}(6p) + x^{\sqrt{(5-1)^2+(6-1)^2}}(4p) \\
&\quad + x^{\sqrt{(6-1)^2+(6-1)^2}}(9p)(3q-4) \\
&= x^{\sqrt{18}}(18p) + x^{\sqrt{25}}(4p) + x^{\sqrt{34}}(6p) + x^{\sqrt{41}}(4p) + x^{\sqrt{50}}(9p)(3q-4)
\end{aligned}
$$

$$
\begin{aligned}
(v) \quad L\sigma_2(G_5) &= \prod_{(u,v)\in E(G)} \left[d_2(u)-d_2(v)\right]^2 \\
&= \big[(4-4)^2(18p)\big] \times \big[(4-5)^2(4p)\big] \times \big[(4-6)^2(6p)\big] \\
&\quad \times \big[(5-6)^2(4p)\big] \times \big[(6-6)^2(9p)(3q-4)\big] \\
&= 0
\end{aligned}
$$

$$
\begin{aligned}
(vi) \quad L\sigma_2(G_5,x) &= \prod_{(u,v)\in E(G)} x^{[d_2(u)-d_2(v)]^2} \\
&= \Big[x^{(4-4)^2}(18p)\Big] \times \Big[x^{(4-5)^2}(4p)\Big] \times \Big[x^{(4-6)^2}(6p)\Big] \\
&\quad \times \Big[x^{(5-6)^2}(4p)\Big] \times \Big[x^{(6-6)^2}(9p)(3q-4)\Big] \\
&= (18p) \times (4px) \times (6px^4) \times (4px) \times (9p(3q-4))
\end{aligned}
$$

$$
\begin{aligned}
(vii) \quad RSL_2(G_5) &= \prod_{(u,v)\in E(G)} \Big[\left(d_2(u)-1\right)^2 + \left(d_2(v)-1\right)^2\Big]^{1/2} \\
&= \Big[\sqrt{(4-1)^2+(4-1)^2}\cdot(18p)\Big] \times \\
&\quad \Big[\sqrt{(4-1)^2+(5-1)^2}\cdot(4p)\Big] \times \\
&\quad \Big[\sqrt{(4-1)^2+(6-1)^2}\cdot(6p)\Big] \times \\
&\quad \Big[\sqrt{(5-1)^2+(6-1)^2}\cdot(4p)\Big] \times \\
&\quad \Big[\sqrt{(6-1)^2+(6-1)^2}\cdot(9p)(3q-4)\Big] \\
&= (\sqrt{18}\cdot 18p)(\sqrt{25}\cdot 4p)(\sqrt{34}\cdot 6p)(\sqrt{41}\cdot 4p)(\sqrt{50}\cdot 9p(3q-4))
\end{aligned}
$$

$$(viii) \quad RSL_2(G_5,x) = \prod_{(u,v)\in E(G)} x^{\left[(d_2(u)-1)^2+(d_2(v)-1)^2\right]^{1/2}}$$
$$= \left[x^{\sqrt{(4-1)^2+(4-1)^2}}(18p)\right] \times$$
$$\left[x^{\sqrt{(4-1)^2+(5-1)^2}}(4p)\right] \times$$
$$\left[x^{\sqrt{(4-1)^2+(6-1)^2}}(6p)\right] \times$$
$$\left[x^{\sqrt{(5-1)^2+(6-1)^2}}(4p)\right] \times$$
$$\left[x^{\sqrt{(6-1)^2+(6-1)^2}}(9p)(3q-4)\right]$$
$$= (x^{\sqrt{18}} \cdot 18p)(x^{\sqrt{25}} \cdot 4p)(x^{\sqrt{34}} \cdot 6p)(x^{\sqrt{41}} \cdot 4p)(x^{\sqrt{50}} \cdot 9p(3q-4))$$

□

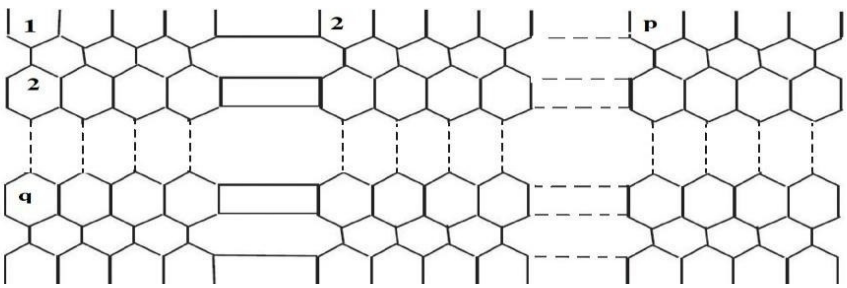

Figure 6: Molecular structure of H-Tetracenic Nanotube G_6

Table 6: EDGE SET PARTITION OF GRAPH G_6

$d_2(u), d_2(v), uv \in E(G_6)$	$(3,3)$	$(3,5)$	$(5,5)$	$(5,6)$	$(6,6)$
Number of Edges	pq	$2pq$	$3pq$	$2pq$	$19pq-2q$

Theorem 2.6 *G_6 is a H-tetracenic nanotube [p,q], then*

$$(i) \quad L\sigma_1(G_6) = 10pq$$
$$(ii) \quad L\sigma_1(G_6,x) = pq + (x^4 \times 2pq) + (3pq) + (2xpq) + (19pq-2q)$$
$$(iii) \quad RSL_1(G_6) = \sqrt{8}pq + (\sqrt{20}\times 2pq) + (\sqrt{32}\times 3pq) + (\sqrt{41}\times 2pq) + \sqrt{50}(19pq-2q)$$
$$(iv) \quad RSL_1(G_6,x) = (x^{\sqrt{18}} \times pq) + (x^{\sqrt{20}} \times 2pq)(x^{\sqrt{32}} \times 3pq)(x^{\sqrt{41}} \times 2pq)(x^{\sqrt{50}} \times (19pq-2q))$$
$$(v) \quad L\sigma_2(G_6) = 0$$
$$(vi) \quad L\sigma_2(G_6,x) = (pq)(x^4 \times 2pq)(3pq)(2xpq)(19pq-2q)$$
$$(vii) \quad RSL_2(G_6) = (\sqrt{32}\times pq)(\sqrt{20}\times 2pq)(\sqrt{32}\times 3pq)(\sqrt{41}\times 2pq)(\sqrt{50}\times(19pq-2q))$$
$$(viii) \quad RSL_2(G_6,x) = (x^{\sqrt{32}} \times pq)(x^{\sqrt{20}} \times 2pq)(x^{\sqrt{32}} \times 3pq)(x^{\sqrt{41}} \times 2pq)(x^{\sqrt{50}} \times (19pq-2q))$$

Proof: *There are 18pq vertices and 27pq-2q edges in the graph G_6 = [p, q]. With the use of Table 6 and definitions of the leap sigma index, reduced sombor index, and the related exponential and edge set* *partition of graph G_6, we obtain,*

$$(i)\quad L\sigma(G_6) = \sum_{(u,v)\in E(G)} [d_2(u) - d_2(v)]^2$$
$$= (3-3)^2(pq) + (3-5)^2(2pq) + (5-5)^2(3pq) + (5-6)^2(2pq) + (6-6)^2(19pq-2q)$$
$$= 0\cdot(pq) + (-2)^2(2pq) + 0\cdot(3pq) + (-1)^2(2pq) + 0\cdot(19pq-2q)$$
$$= 0 + 8pq + 0 + 2pq + 0$$
$$= 10pq$$

$$(ii)\quad L\sigma(G_6, x) = \sum_{(u,v)\in E(G)} x^{[d_2(u)-d_2(v)]^2}$$
$$= x^{(3-3)^2}(pq) + x^{(3-5)^2}(2pq) + x^{(5-5)^2}(3pq) + x^{(5-6)^2}(2pq) + x^{(6-6)^2}(19pq-2q)$$
$$= pq + (x^4\cdot 2pq) + (3pq) + (2xpq) + (19pq-2q)$$

$$(iii)\quad RSL_1(G_6) = \sum_{(u,v)\in E(G)} \left[(d_2(u)-1)^2 + (d_2(v)-1)^2\right]^{1/2}$$
$$= \left[(3-1)^2 + (3-1)^2\right]^{1/2}(pq) + \left[(3-1)^2 + (5-1)^2\right]^{1/2}(2pq)$$
$$+ \left[(5-1)^2 + (5-1)^2\right]^{1/2}(3pq) + \left[(5-1)^2 + (6-1)^2\right]^{1/2}(2pq)$$
$$+ \left[(6-1)^2 + (6-1)^2\right]^{1/2}(19pq-2q)$$
$$= \sqrt{8}pq + (\sqrt{20}\cdot 2pq) + (\sqrt{32}\cdot 3pq) + (\sqrt{41}\cdot 2pq) + (\sqrt{50}\cdot(19pq-2q))$$

$$(iv)\quad RSL_1(G_6, x) = \sum_{(u,v)\in E(G)} x^{\left[(d_2(u)-1)^2+(d_2(v)-1)^2\right]^{1/2}}$$
$$= x^{\left[(3-1)^2+(3-1)^2\right]^{1/2}}(pq) + x^{\left[(3-1)^2+(5-1)^2\right]^{1/2}}(2pq)$$
$$+ x^{\left[(5-1)^2+(5-1)^2\right]^{1/2}}(3pq) + x^{\left[(5-1)^2+(6-1)^2\right]^{1/2}}(2pq)$$
$$+ x^{\left[(6-1)^2+(6-1)^2\right]^{1/2}}(19pq-2q)$$
$$= (x^{\sqrt{18}}\cdot pq) + (x^{\sqrt{20}}\cdot 2pq) + (x^{\sqrt{32}}\cdot 3pq) + (x^{\sqrt{41}}\cdot 2pq) + (x^{\sqrt{50}}\cdot(19pq-2q))$$

$$(v)\quad L\sigma_2(G_6) = \prod_{(u,v)\in E(G)} [d_2(u) - d_2(v)]^2$$
$$= \left[(3-3)^2(pq)\right] \times \left[(3-5)^2(2pq)\right] \times \left[(5-5)^2(3pq)\right]$$
$$\times \left[(5-6)^2(2pq)\right] \times \left[(6-6)^2(19pq-2q)\right]$$
$$= 0$$

$$(vi)\quad L\sigma_2(G_6, x) = \prod_{(u,v)\in E(G)} x^{[d_2(u)-d_2(v)]^2}$$
$$= \left[x^{(3-3)^2}(pq)\right] \times \left[x^{(3-5)^2}(2pq)\right] \times \left[x^{(5-5)^2}(3pq)\right]$$
$$\times \left[x^{(5-6)^2}(2pq)\right] \times \left[x^{(6-6)^2}(19pq-2q)\right]$$
$$= (pq)(x^4\cdot 2pq)(3pq)(2xpq)(19pq-2q)$$

$$(vii) \quad RSL_2(G_6) = \prod_{(u,v)\in E(G)} \left[(d_2(u)-1)^2 + (d_2(v)-1)^2\right]^{1/2}$$
$$= \left[(3-1)^2 + (3-1)^2\right]^{1/2}(pq) \times \left[(3-1)^2 + (5-1)^2\right]^{1/2}(2pq)$$
$$\times \left[(5-1)^2 + (5-1)^2\right]^{1/2}(3pq) \times \left[(5-1)^2 + (6-1)^2\right]^{1/2}(2pq)$$
$$\times \left[(6-1)^2 + (6-1)^2\right]^{1/2}(19pq-2q)$$
$$= (\sqrt{8}\cdot pq)(\sqrt{20}\cdot 2pq)(\sqrt{32}\cdot 3pq)(\sqrt{41}\cdot 2pq)(\sqrt{50}\cdot(19pq-2q))$$

$$(viii) \quad RSL_2(G_6,x) = \prod_{(u,v)\in E(G)} x^{\left[(d_2(u)-1)^2+(d_2(v)-1)^2\right]^{1/2}}$$
$$= x^{\left[(3-1)^2+(3-1)^2\right]^{1/2}}(pq) \times x^{\left[(3-1)^2+(5-1)^2\right]^{1/2}}(2pq)$$
$$\times x^{\left[(5-1)^2+(5-1)^2\right]^{1/2}}(3pq) \times x^{\left[(5-1)^2+(6-1)^2\right]^{1/2}}(2pq)$$
$$\times x^{\left[(6-1)^2+(6-1)^2\right]^{1/2}}(19pq-2q)$$
$$= (x^{\sqrt{8}}\cdot pq)(x^{\sqrt{20}}\cdot 2pq)(x^{\sqrt{32}}\cdot 3pq)(x^{\sqrt{41}}\cdot 2pq)(x^{\sqrt{50}}\cdot(19pq-2q))$$

□

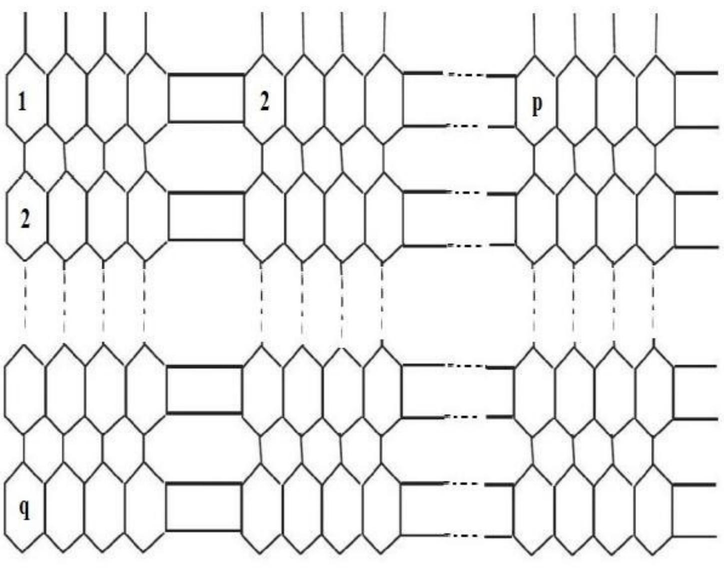

Figure 7: Molecular structure of Tetracenic Nanotorus G_7

Table 7: EDGE SET PARTITION OF GRAPH G_7

$d_2(u), d_2(v), uv \in E(G_7)$	$(5,5)$	$(5,6)$	$(6,6)$
Number of Edges	$4pq$	$4pq$	$19pq$

Theorem 2.7 *G_7 is a tetracenic nanotorus [p, q], then*

$$\begin{aligned}
(i)\quad & L\sigma_1(G_7) = 4pq\\
(ii)\quad & L\sigma_1(G_7,x) = 4pq + 4xpq + 19pq\\
(iii)\quad & RSL_1(G_7) = (\sqrt{32}\cdot 4pq) + (\sqrt{41}\cdot 4pq) + \sqrt{50}(19pq)\\
(iv)\quad & RSL_1(G_7,x) = (x^{\sqrt{32}}\cdot 4pq) + (x^{\sqrt{41}}\cdot 4pq) + (x^{\sqrt{50}}\cdot 19pq)\\
(v)\quad & L\sigma_2(G_7) = 0\\
(vi)\quad & L\sigma_2(G_7,x) = (4pq)\times(4xpq)\times(19pq)\\
(vii)\quad & RSL_2(G_7) = (\sqrt{32}\cdot 4pq)(\sqrt{41}\cdot 4pq)(\sqrt{50}\cdot 19pq)\\
(viii)\quad & RSL_2(G_7,x) = (x^{\sqrt{32}}\cdot 4pq)(x^{\sqrt{41}}\cdot 4pq)(x^{\sqrt{50}}\cdot 19pq)
\end{aligned}$$

Proof: *There are 18pq vertices and 27pq edges in the graph G_7 = [p, q]. With the use of Table 7 and definitions of the leap sigma index, reduced sombor index, and the related exponential and edge set partition of graph G_7, we obtain*

$$\begin{aligned}
(i)\quad L\sigma_1(G_7) &= \sum_{(u,v)\in E(G)} [d_2(u) - d_2(v)]^2\\
&= (5-5)^2(4pq) + (5-6)^2(4pq) + (6-6)^2(19pq)\\
&= 0\cdot(4pq) + (-1)^2(4pq) + 0\cdot(19pq)\\
&= 0 + 4pq + 0\\
&= 4pq
\end{aligned}$$

$$\begin{aligned}
(ii)\quad L\sigma_1(G_7,x) &= \sum_{(u,v)\in E(G)} x^{[d_2(u)-d_2(v)]^2}\\
&= x^{(5-5)^2}(4pq) + x^{(5-6)^2}(4pq) + x^{(6-6)^2}(19pq)\\
&= x^0(4pq) + x^1(4pq) + x^0(19pq)\\
&= 4pq + 4xpq + 19pq
\end{aligned}$$

$$\begin{aligned}
(iii)\quad RSL_1(G_7) &= \sum_{(u,v)\in E(G)} \left[(d_2(u)-1)^2 + (d_2(v)-1)^2\right]^{1/2}\\
&= \left[(5-1)^2 + (5-1)^2\right]^{1/2}(4pq) + \left[(5-1)^2 + (6-1)^2\right]^{1/2}(4pq)\\
&\quad + \left[(6-1)^2 + (6-1)^2\right]^{1/2}(19pq)\\
&= (\sqrt{32}\cdot 4pq) + (\sqrt{41}\cdot 4pq) + (\sqrt{50}\cdot 19pq)
\end{aligned}$$

$$\begin{aligned}
(iv)\quad RSL_1(G_7,x) &= \sum_{(u,v)\in E(G)} x^{\left[(d_2(u)-1)^2+(d_2(v)-1)^2\right]^{1/2}}\\
&= x^{\left[(5-1)^2+(5-1)^2\right]^{1/2}}(4pq) + x^{\left[(5-1)^2+(6-1)^2\right]^{1/2}}(4pq)\\
&\quad + x^{\left[(6-1)^2+(6-1)^2\right]^{1/2}}(19pq)\\
&= (x^{\sqrt{32}}\cdot 4pq) + (x^{\sqrt{41}}\cdot 4pq) + (x^{\sqrt{50}}\cdot 19pq)
\end{aligned}$$

$$
\begin{aligned}
(v)\quad L\sigma_2(G_7) &= \prod_{(u,v)\in E(G)} [d_2(u)-d_2(v)]^2 \\
&= \left[(5-5)^2(4pq)\right] \times \left[(5-6)^2(4pq)\right] \times \left[(6-6)^2(19pq)\right] \\
&= 0\cdot(4pq) + (-1)^2(4pq) + 0\cdot(19pq) \\
&= 0
\end{aligned}
$$

$$
\begin{aligned}
(vi)\quad L\sigma_2(G_7,x) &= \prod_{(u,v)\in E(G)} x^{[d_2(u)-d_2(v)]^2} \\
&= \left[x^{(5-5)^2}(4pq)\right] \times \left[x^{(5-6)^2}(4pq)\right] \times \left[x^{(6-6)^2}(19pq)\right] \\
&= (4pq)\times(4xpq)\times(19pq)
\end{aligned}
$$

$$
\begin{aligned}
(vii)\quad RSL_2(G_7) &= \prod_{(u,v)\in E(G)} \left[(d_2(u)-1)^2+(d_2(v)-1)^2\right]^{1/2} \\
&= \left[(5-1)^2+(5-1)^2\right]^{1/2}(4pq) \times \left[(5-1)^2+(6-1)^2\right]^{1/2}(4pq) \\
&\quad \times \left[(6-1)^2+(6-1)^2\right]^{1/2}(19pq) \\
&= (\sqrt{32}\cdot 4pq)(\sqrt{41}\cdot 4pq)(\sqrt{50}\cdot 19pq)
\end{aligned}
$$

$$
\begin{aligned}
(viii)\quad RSL_2(G_7,x) &= \prod_{(u,v)\in E(G)} x^{\left[(d_2(u)-1)^2+(d_2(v)-1)^2\right]^{1/2}} \\
&= \left[x^{\left[(5-1)^2+(5-1)^2\right]^{1/2}}(4pq)\right] \times \left[x^{\left[(5-1)^2+(6-1)^2\right]^{1/2}}(4pq)\right] \\
&\quad \times \left[x^{\left[(6-1)^2+(6-1)^2\right]^{1/2}}(19pq)\right] \\
&= (x^{\sqrt{32}}\cdot 4pq)(x^{\sqrt{41}}\cdot 4pq)(x^{\sqrt{50}}\cdot 19pq)
\end{aligned}
$$

□

3. Applications of the Studied Nanostructures

3.1. Tubulene Unit Cell Architectures (TUAC6)

TUAC6-based nanotubes exhibit excellent mechanical strength and high aspect ratios, making them suitable for composite reinforcement in aerospace and automotive industries. Their high electrical conductivity also supports applications in nanoelectronic interconnects and transparent conductive films.

3.2. V-Phenylenic Nanotubes and Torus Structures

V-phenylenic frameworks are characterized by high porosity and tunable electronic properties. These features are exploited in gas sensing, molecular filtration, and as structural components in organic field-effect transistors (OFETs). The toroidal variants may be used for nano-ring resonators in photonic devices due to their cyclic symmetry.

3.3. Tetracenic Nanotubes (Vertical and Horizontal)

Tetracenic systems possess extended π-conjugation, enabling efficient charge transport. Their vertical and horizontal configurations are of interest for organic semiconductors, photovoltaic devices, and as electrode materials in supercapacitors. The toroidal tetracenic nanostructures also present potential in supramolecular host–guest chemistry, enabling selective encapsulation of small molecules or ions.

Relevance of Leap Sigma Index in Applications

The Leap Sigma index is sensitive to degree irregularity, which correlates with structural stability and defect distribution in these nanostructures. Optimizing $L\sigma_1(G)$ for a given topology can therefore inform the design of nanomaterials with improved performance in the above-mentioned applications.

4. Results For Polycyclic Aromatic Hydrocarbons

We present the Leap Sigma index, its related graph exponentials, the reduced sombar leap index, and their corresponding graph exponentials in this work. Additionally, we calculate the lower sombar leap index, the associated exponentials of a graph, the newly defined Leap Sigma index, and their related exponentials index for a few polycyclic aromatic hydrocarbon results.We look at the PAHn family, which stands for polycyclic aromatic hydrocarbons. We provide the formulae for the leap indices of a few nanostructures for the first three members of the PAHn family in Figure 8 of a graph.

Let G_8 be $PAHn$'s molecular graph. By calculation, G_8 has $6n^2 + 6n$ vertices and $9n^2 + 3n$ edges. For an edge $uv \in E(G_8)$, the 2-distance degree of a vertex u and a vertex v are respectively denoted by $d_2(u)$ and $d_2(v)$. By calculation we obtain that the edge partition of G_8 with respect to 2-distance degree of vertices is given in Table 8.

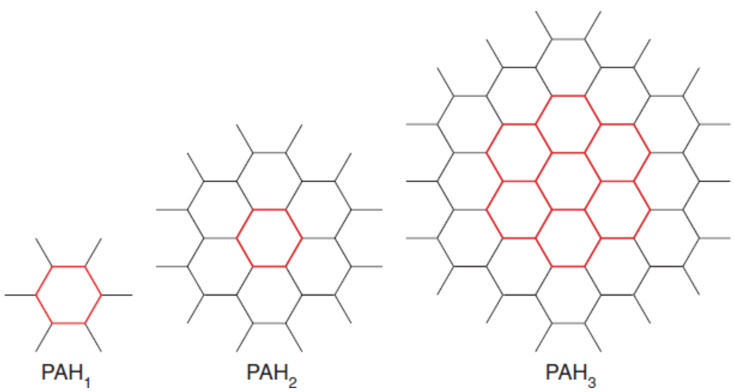

Figure 8: Molecular structure of The first three members of PAHn G_8

Table 8: Edge Set Partition of Graph G_8

$d_2(u), d_2(v), uv \in E(G_8)$	$(2,4)$	$(4,4)$	$(4,6)$	$(6,6)$
Number of Edges	$6n$	6	$2n^2 + 2n$	$7n^2 - 48n - 6$

Theorem 4.1 *If G_8 is a Polycyclic Aromatic Hydrocarbons of PAHn, then*

$$(i) \quad L\sigma_1(G_8) = 24n + 4(2n^2 + 2n)$$
$$(ii) \quad L\sigma_1(G_8, x) = x^4(6n) + 6 + x^4(2n^2 + 2n) + (7n^2 - 48n - 6)$$
$$(iii) \quad RSL_1(G_8) = \sqrt{10}(6n) + \sqrt{18}(6) + \sqrt{34}(2n^2 + 2n) + \sqrt{50}(7n^2 - 48n - 6)$$
$$(iv) \quad RSL_1(G_8, x) = x^{\sqrt{10}}(6n) + x^{\sqrt{18}}(6) + x^{\sqrt{34}}(2n^2 + 2n) + x^{\sqrt{50}}(7n^2 - 48n - 6)$$
$$(v) \quad L\sigma_2(G_8) = 0$$
$$(vi) \quad L\sigma_2(G_8, x) = x^4(6n) \times 6 \times x^4(2n^2 + 2n) \times (7n^2 - 48n - 6)$$
$$(vii) \quad RSL_2(G_8) = \sqrt{10}(6n) \times \sqrt{18}(6) \times \sqrt{34}(2n^2 + 2n) \times \sqrt{50}(7n^2 - 48n - 6)$$
$$(viii) \quad RSL_2(G_8, x) = x^{\sqrt{10}}(6n) \times x^{\sqrt{18}}(6) \times x^{\sqrt{34}}(2n^2 + 2n) \times x^{\sqrt{50}}(7n^2 - 48n - 6)$$

Proof: *There are 18pq vertices and 27pq edges in the graph* $G_8 = [p,\ q]$. *With the use of Table 8 and definitions of the leap sigma index, reduced sombor index, and the related exponential and edge set partition of graph* G_8, *we obtain*

$$(i)\quad L\sigma(G_8) = \sum_{(u,v)\in E(G)} [d_2(u) - d_2(v)]^2$$
$$= (2-4)^2(6n) + (4-4)^2(6) + (4-6)^2(2n^2+2n) + (6-6)^2(7n^2-48n-6)$$
$$= 24n + 4(2n^2+2n)$$

$$(ii)\quad L\sigma(G_8, x) = \sum_{(u,v)\in E(G)} x^{[d_2(u)-d_2(v)]^2}$$
$$= x^{(2-4)^2}(6n) + x^{(4-4)^2}(6) + x^{(4-6)^2}(2n^2+2n) + x^{(6-6)^2}(7n^2-48n-6)$$
$$= x^4(6n) + 6 + x^4(2n^2+2n) + (7n^2-48n-6)$$

$$(iii)\quad RSL(G_8) = \sum_{(u,v)\in E(G)} \left[(d_2(u)-1)^2 + (d_2(v)-1)^2\right]^{1/2}$$
$$= \left[(2-1)^2 + (4-1)^2\right]^{1/2}(6n) + \left[(4-1)^2 + (4-1)^2\right]^{1/2}(6)$$
$$+ \left[(4-1)^2 + (6-1)^2\right]^{1/2}(2n^2+2n) + \left[(6-1)^2 + (6-1)^2\right]^{1/2}$$
$$(7n^2-48n-6)$$
$$= \sqrt{10}(6n) + \sqrt{18}(6) + \sqrt{34}(2n^2+2n) + \sqrt{50}(7n^2-48n-6)$$

$$(iv)\quad RSL(G_8, x) = \sum_{(u,v)\in E(G)} x^{\left[(d_2(u)-1)^2+(d_2(v)-1)^2\right]^{1/2}}$$
$$= x^{\left[(2-1)^2+(4-1)^2\right]^{1/2}}(6n) + x^{\left[(4-1)^2+(4-1)^2\right]^{1/2}}(6)$$
$$+ x^{\left[(4-1)^2+(6-1)^2\right]^{1/2}}(2n^2+2n) + x^{\left[(6-1)^2+(6-1)^2\right]^{1/2}}(7n^2-48n-6)$$
$$= x^{\sqrt{10}}(6n) + x^{\sqrt{18}}(6) + x^{\sqrt{34}}(2n^2+2n) + x^{\sqrt{50}}(7n^2-48n-6)$$

$$(v)\quad L\sigma_2(G_8) = \prod_{(u,v)\in E(G)} [d_2(u) - d_2(v)]^2$$
$$= (2-4)^2(6n) \times (4-4)^2(6) \times (4-6)^2(2n^2+2n) \times (6-6)^2(7n^2-48n-6)$$
$$= 0$$

$$(vi)\quad L\sigma_2(G_8, x) = \prod_{(u,v)\in E(G)} x^{[d_2(u)-d_2(v)]^2}$$
$$= x^{(2-4)^2}(6n) \times x^{(4-4)^2}(6) \times x^{(4-6)^2}(2n^2+2n) \times x^{(6-6)^2}(7n^2-48n-6)$$
$$= x^4(6n) \times 6 \times x^4(2n^2+2n) \times (7n^2-48n-6)$$

$$(vii)\quad RSL_2(G_8) = \prod_{(u,v)\in E(G)} \left[(d_2(u)-1)^2 + (d_2(v)-1)^2\right]^{1/2}$$
$$= \left[(2-1)^2 + (4-1)^2\right]^{1/2}(6n) \times \left[(4-1)^2 + (4-1)^2\right]^{1/2}(6)$$
$$\times \left[(4-1)^2 + (6-1)^2\right]^{1/2}(2n^2+2n) \times \left[(6-1)^2 + (6-1)^2\right]^{1/2}$$
$$(7n^2-48n-6)$$
$$= \sqrt{10}(6n) \times \sqrt{18}(6) \times \sqrt{34}(2n^2+2n) \times \sqrt{50}(7n^2-48n-6)$$

$$(viii) \quad RSL_2(G_8, x) = \prod_{(u,v)\in E(G)} x^{\left[(d_2(u)-1)^2+(d_2(v)-1)^2\right]^{1/2}}$$
$$= x^{\left[(2-1)^2+(4-1)^2\right]^{1/2}}(6n) \times x^{\left[(4-1)^2+(4-1)^2\right]^{1/2}}(6)$$
$$\times x^{\left[(4-1)^2+(6-1)^2\right]^{1/2}}(2n^2+2n) \times x^{\left[(6-1)^2+(6-1)^2\right]^{1/2}}(7n^2-48n-6)$$
$$= x^{\sqrt{10}}(6n) \times x^{\sqrt{18}}(6) \times x^{\sqrt{34}}(2n^2+2n) \times x^{\sqrt{50}}(7n^2-48n-6)$$

□

5. Applications of Polycyclic Aromatic Hydrocarbons (PAHs)

Polycyclic aromatic hydrocarbons (PAHs) are a class of fused aromatic compounds that form the structural basis for graphene fragments and other two-dimensional carbon materials. Their extended conjugated π-systems impart exceptional electronic, optical, and chemical properties, making them valuable in various fields of nanoscience and technology.

5.1. Organic Electronics

Functionalized PAHs exhibit high charge carrier mobility and stability, which makes them suitable for organic field-effect transistors (OFETs), light-emitting diodes (OLEDs), and organic photovoltaic devices. Their planar structures facilitate π–π stacking, enhancing charge transport in thin films.

5.2. Energy Conversion and Storage

The π-conjugation in PAHs enables efficient light absorption and electron transfer, supporting their use as photoactive layers in solar cells and as redox-active components in rechargeable batteries and supercapacitors.

5.3. Chemical Sensing and Molecular Recognition

PAHs can be chemically tailored to detect specific ions, gases, or organic molecules through changes in their fluorescence or electronic properties. Their rigid aromatic frameworks provide predictable binding sites for host–guest chemistry.

5.4. Biomedical and Environmental Applications

Certain functionalized PAHs serve as carriers for drug delivery systems due to their ability to intercalate with biomolecules. Additionally, PAHs are employed as model systems to study the environmental behavior and toxicity of aromatic pollutants.

Relevance of the Leap Sigma Index

For PAHs, the Leap Sigma index $L\sigma_1(G)$ is sensitive to variations in molecular topology arising from heteroatom substitution, defect introduction, or edge functionalization. These topological changes directly affect electronic delocalization, stability, and reactivity, which in turn influence the performance of PAHs in the above applications.

6. Conclusion

In this article, Introduced and calculated Leap Sigma index, leap sigma multiplicative index,reduced sombor leap index , reduced sombor leap Multiplicative index and its exponentials for some nanostructures and polycyclic aromatic hydrocarbons.

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