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Approximation of Linear Positive Fuzzy Operators

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ABSTRACT: This study extends the main results of classical approximation theory to fuzzy theory. We begin by defining fuzzy valued functions, exploring their properties and then applying the fuzzy Korovkin theorem to approximate them. The study delves into the approximation by various linear positive fuzzy operators – Fuzzy Bernstein, fuzzy Szász-Mirakyan and fuzzy Baskakov operators – utilizing them to approximate fuzzy-valued functions. Their Voronovskaya type asymptotic result is also proved using Taylor's theorem.

Key Words: Fuzzy valued functions, linear positive fuzzy operators, fuzzy rate of convergence, Bernstein polynomial, Szász-Mirakyan operators, Baskakov operators.

Contents

1. Introduction

In mathematical language, a fuzzy set is a notion used in fuzzy logic to represent a set where the membership of an element is described by a degree of belonging, ranging from 0 (not a member) to 1 (fully a member), rather than the traditional binary distinction of being either in the set or not. Unlike the classical set theory which focuses on black and white, fuzzy theory focuses on the grey portions. The inception of fuzzy sets dates back to Zadeh's groundbreaking study in 1965 [27]. Sugeno [21], in 1977, introduced a different concept of fuzziness where, for any object x in the universe X, a value $g_x(A) \in [0,1]$ is assigned to each nonfuzzy subset A of X representing the "grade of fuzziness" as a certainty measure of the assertion "x belongs to A". The framework formulated by Sugeno revolves around guessing whether x is a member of A, rather than addressing vagueness as characterized by Zadeh. Dubois and Prade in their study [12] extensively focused on a comprehensive exploration of mathematical concepts within the realm of fuzzy set theory. They thoroughly discussed the introduced mathematical notions, providing an in-depth analysis and presentation of the framework underlying fuzzy set theory. They defined various types of fuzzy sets, set operations, and properties and introduced the extension principle, applying it to mathematics and higher-order fuzzy sets. The findings of Dubois and Prade are considered crucial as they covered the definitions on fuzzy relations and fuzzy functions, emphasizing their extremum, integration, differentiation, fuzzy topology and categories of fuzzy objects.

From here on many researchers have worked upon this idea. Subsequently, numerous authors extended and explored the notions of fuzziness dealing in the area of real analysis. In 2000, Burgin [8] explored and examined the fuzzy limits of functions using two approaches: the first one relies on the concept of a fuzzy limit in the context of a sequence, while the other extends and generalizes the traditional

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ε-δ definition. In 1975, Kramosil and Michalek first introduced the concept of fuzzy metric spaces [17], derived from the expansion of probabilistic metric spaces to encompass fuzzy scenarios. They infused the traditional concepts of metrics and metric spaces with the idea of fuzziness and compared their adapted notions against those derived using other generalizations. Taking motivation from this work, Kaleva and Seikkala [18] expanded the notion of a metric space by defining the distance between two points as a non-negative fuzzy number. This introduced a more intuitive definition of fuzzy metric spaces. They presented various properties of fuzzy numbers, leading to the definition and exploration of fuzzy metric spaces. Their contribution extends to providing fixed point theorems specifically tailored for fuzzy metric spaces. For a comprehensive and intriguing exploration of these aspects, readers are encouraged to refer [4,13,24].

Fuzzy numbers are essential in a variety of fields with uncertainty, such as mathematics, engineering, economics, and artificial intelligence. Unlike their precise counterparts, these numbers have a particular modeling aspect that allows them to reflect the approximate values found in linguistic terms like "big", "small", "early" or "around." This attribute is extremely useful for modeling and analyzing systems or scenarios when it is difficult to obtain exact numerical numbers. Fuzzy numbers therefore aid in representing uncertainty in decision-making processes in a more realistic manner. Due to their flexibility, there has been a noticeable surge in the amount of research devoted to fuzzy number approximation over the past few years. In an important study towards fuzzy numbers in 2014, Ban and Coroianu [5] defined a set of real parameters correlated with a fuzzy number, demonstrating the existence of a unique trapezoidal fuzzy number preserving a fixed parameter. Roldan et al. [23] characterized fuzzy numbers based on their level sets extremes, establishing relationships between their images and usual operations while preserving its continuity.

Researchers in recent years have explored various approximations of fuzzy numbers, categorizing them into Euclidean and non-Euclidean distance types. Euclidean approximations can be formulaically calculated, while the non-Euclidean counterparts pose greater complexity. The study introduced by Yen and Chu [26] focused on LR-type fuzzy numbers for approximating fuzzy numbers, offering generalized approximations within the Euclidean class, specifically without constraints. Additionally, they introduced an efficient formula for calculation.

In our study, we will be dealing with the approximation of fuzzy numbered-valued functions. The fuzzy approximation theory was first dealt by George A. Anastassiou [3] in 2010, where he gave numerous applications, all consistently situated within the context of fuzzy mathematics. He extended his study to fuzzy differentiation and integration, covering topics such as fuzzy Taylor formulae and fuzzy Ostrowski inequalities. But what interests us more is his idea of fuzzy approximation using algebraic and trigonometric polynomials, wherein he developed a theoretical framework exploring how linear positive fuzzy operators approach the fuzzy unit operator with respect to convergence rates, thus giving rise to the fuzzy Korovkin theorem. Results regarding the approximation of fuzzy numbered-valued functions can be seen in [2,6,7,14,15,20,25].

The approximation of linear positive operators have been an important topic in the world of approximation theory. Owing to their real-world applications, the Bernstein polynomials constitute a focal point of intensive research, offering a wide scope of surprising findings; for more details, see [10,11,16,19,22]. We extend the work done on the classical Bernstein, Szász, and Baskakov operators to fuzzy-valued functions.

2. Preliminaries

In order to proceed with our paper, we must first lay the groundwork by introducing some basic definitions and concepts in fuzzy theory. Let $\tilde{p}$ be a fuzzy set with membership function $\tilde{p}(u) : \mathbb{R} \to [0,1]$ such that:

(i) Normality: There exists $u_0 \in \mathbb{R}$ such that $\tilde{p}(u_0) = 1$.

(ii) Convexity: $\forall\, s,t \in \mathbb{R}$ and $\forall\, \gamma \in [0,1]$, $\tilde{p}\left(\gamma s + (1-\gamma)\, t\right) \geqslant \min\left\{\tilde{p}\left(s\right), \tilde{p}\left(t\right)\right\}$.

(iii) Upper semi-continuity: $\forall\, u_0 \in \mathbb{R}$ and $\forall\, \varepsilon > 0$, $\exists$ a neighourhood $V\left(u_0\right) : \tilde{p}(u) \leqslant \tilde{p}(u_0) + \varepsilon$, $\forall\, u \in V\left(u\right)$.

(iv) Boundedness of $\overline{supp\,(\tilde{p})}$: Define $supp\,(\tilde{p}) = \{u \in \mathbb{R} : \tilde{p}\,(x) > 0\}$ as the support of $\tilde{p}$. Then the closure of the support of $\tilde{p}$, i.e. $\overline{supp\,(\tilde{p})}$ is bounded in $\mathbb{R}$.

Then $\tilde{p}$ is called a fuzzy real number. It may be interesting to note that a fuzzy set is not a collection of fuzzy numbers. Rather, a fuzzy number in itself is a fuzzy set with the above properties. Let $\mathbb{R}_{\mathcal{F}}$ represent the set of all fuzzy numbers $\tilde{p}$.

Firstly, let us recall the ℓ-level cut of $\tilde{p}$. For $0 \leqslant \ell \leqslant 1$ and $\tilde{p} \in \mathbb{R}_{\mathcal{F}}$ with membership function $\tilde{p}(u)$, the ℓ-level cut or simply ℓ-cut of $\tilde{p}$, denoted by $[\tilde{p}]^{\ell}$, is defined as,

$$[\tilde{p}]^{\ell} = \begin{cases} \{u : \tilde{p}\,(u) \geqslant \ell\}\,, & \text{if } 0 < \ell \leqslant 1 \\ \overline{\{u : \tilde{p}\,(u) > 0\}}, & \text{if } \ell = 0 \end{cases}$$

where $\overline{A}$ denotes the closure of set A.

For each $\ell \in [0,1]$, $[\tilde{p}]^{\ell} = [p_{-}^{(\ell)}, p_{+}^{(\ell)}]$ is a closed and bounded interval of $\mathbb{R}$. It is apparent that, if $p_{-}^{(\ell)} = p_{+}^{(\ell)}$ then $\tilde{p}$ will reduce to a crisp real number.

Definition 2.1 *[14] For any $\tilde{p}$ and $\tilde{q}$ belonging to the set $\mathbb{R}_{\mathcal{F}}$, and for any α in $\mathbb{R}$, we uniquely define the sum $\tilde{p} \oplus \tilde{q}$ and the product with real scalars $\alpha \odot \tilde{p}$ in $\mathbb{R}_{\mathcal{F}}$ by $\oplus : \mathbb{R}_{\mathcal{F}} \times \mathbb{R}_{\mathcal{F}} \to \mathbb{R}_{\mathcal{F}}$,*

$$(\tilde{p} \oplus \tilde{q})\,(u) = \sup_{v+w=u} \min\{\tilde{p}\,(v)\,, \tilde{q}\,(w)\}$$

and by $\odot : \mathbb{R} \times \mathbb{R}_{\mathcal{F}} \to \mathbb{R}_{\mathcal{F}}$,

$$(\alpha \odot \tilde{p})\,(u) = \begin{cases} \tilde{p}\left(\frac{u}{\alpha}\right) & \text{if } \alpha \neq 0 \\ \tilde{o} & \text{if } \alpha = 0 \end{cases}$$

where $\tilde{o} : \mathbb{R} \to [0,1]$ is $\tilde{o} = \chi_{\{0\}}$.

This can also be written as,

$$\begin{aligned} [\tilde{p} \oplus \tilde{q}]^{\ell} &= [\tilde{p}]^{\ell} + [\tilde{q}]^{\ell}, && \forall \ell \in [0,1] \\ [\alpha \odot \tilde{p}]^{\ell} &= \alpha[\tilde{p}]^{\ell}, && \forall \ell \in [0,1] \end{aligned}$$

Here, $[\tilde{p}]^{\ell} + [\tilde{q}]^{\ell}$ represents the standard addition of intervals considered as real subsets, while $\alpha[\tilde{p}]^{\ell}$ denotes the standard multiplication between a scalar and a real subset. The positivity of a fuzzy number is defined in the sense of its ℓ-cut for $\ell = 0$.

(i) $\tilde{p}$ is defined as positive, for $\tilde{p}_{-}^{0} \geqslant 0$.

(ii) $\tilde{p}$ is defined as negative, for $\tilde{p}_{+}^{0} \leqslant 0$.

Definition 2.2 (Hausdorff metric in $\mathbb{R}_{\mathcal{F}}$) *[14] The formula for Hausdorff distance between two fuzzy numbers is given by $D : \mathbb{R}_{\mathcal{F}} \times \mathbb{R}_{\mathcal{F}} \to \mathbb{R}^{+} \cup \{0\}$ as*

$$D\,(\tilde{p}, \tilde{q}) = \sup_{\ell \in [0,1]} \max\left\{\left|p_{-}^{(\ell)} - q_{-}^{(\ell)}\right|, \left|p_{+}^{(\ell)} - q_{+}^{(\ell)}\right|\right\},$$

where $[\tilde{p}]^{\ell} = [p_{-}^{(\ell)}, p_{+}^{(\ell)}]$ and $[\tilde{q}]^{\ell} = [q_{-}^{(\ell)}, q_{+}^{(\ell)}]$. Then, $(\mathbb{R}_{\mathcal{F}}, D)$ is a complete metric space in $\mathbb{R}_{\mathcal{F}}$, known as the Hausdorff metric space.

Some properties of $(\mathbb{R}_{\mathcal{F}}, D)$ are:

(i) $D\,(\tilde{p} \oplus \tilde{r}, \tilde{q} \oplus \tilde{r}) = D\,(\tilde{p}, \tilde{q})\,, \quad \forall\, \tilde{p}, \tilde{q}, \tilde{r} \in \mathbb{R}_{\mathcal{F}}$

(ii) $D\,(a \odot \tilde{p}, a \odot \tilde{q}) = |a|\, D\,(\tilde{p}, \tilde{q})\,, \quad \forall\, \tilde{p}, \tilde{q} \in \mathbb{R}_{\mathcal{F}},\ a \in \mathbb{R}$

(iii) $D(\tilde{p} \oplus \tilde{q}, \tilde{r} \oplus \tilde{s}) \leqslant D(\tilde{p}, \tilde{r}) + D(\tilde{q}, \tilde{s}), \quad \forall\, \tilde{p}, \tilde{q}, \tilde{r}, \tilde{s} \in \mathbb{R}_{\mathcal{F}}$

The symbol $\precsim$ represents a partial order on $\mathbb{R}_{\mathcal{F}}$ and is defined as follows:

$$\tilde{p} \precsim \tilde{q} \quad \text{iff} \quad p_{-}^{(\ell)} \leqslant q_{-}^{(\ell)}, \;\; p_{+}^{(\ell)} \leqslant q_{+}^{(\ell)},$$

for all fuzzy numbers $\tilde{p}$ and $\tilde{q}$ and $\ell \in [0,1]$. Herein, $\leqslant$ represents the partial order on the real number set.

Lemma 2.1 *[14] Let $\eta, \mu \in \mathbb{R}$ and $\tilde{p}, \tilde{q}, \tilde{r} \in \mathbb{R}_{\mathcal{F}}$. Let $\tilde{o}$ be the characteristic function of the set $\{0\}$, i.e. $\tilde{o} = \chi_{\{0\}}$. Then, the following conditions hold on $\mathbb{R}_{\mathcal{F}}$:*

(i) $\tilde{p} \oplus \tilde{q} = \tilde{q} \oplus \tilde{p}$, $\tilde{p} \oplus (\tilde{q} \oplus \tilde{r}) = (\tilde{p} \oplus \tilde{q}) \oplus \tilde{r}$.

(ii) The element $\tilde{o} \in \mathbb{R}_{\mathcal{F}}$ serves as the identity element for $\oplus$ and hence, $\tilde{p} \oplus \tilde{o} = \tilde{o} \oplus \tilde{p} = \tilde{p}$.

(iii) For any $\tilde{p} \in \mathbb{R}_{\mathcal{F}} \backslash \mathbb{R}$, there is no opposite element with respect to $\tilde{o}$ under the operation $\oplus$.

(iv) When $\eta\mu \geqslant 0$, the expression $(\eta + \mu) \odot \tilde{p} = \eta \odot \tilde{p} \oplus \mu \odot \tilde{p}$ holds true. However, this condition will not be satisfied for $\eta, \mu \in \mathbb{R}$.

(v) $\eta \odot (\tilde{p} \oplus \tilde{q}) = \eta \odot \tilde{p} \oplus \eta \odot \tilde{q}$, $\eta \odot (\mu \odot \tilde{p}) = (\eta\mu) \odot \tilde{p}$.

(vi) Let us define the usual norm on $\mathbb{R}_{\mathcal{F}}$ as $\|\tilde{p}\|_{\mathcal{F}} = D(\tilde{p}, \tilde{o})$. Then $\|\tilde{p}\|_{\mathcal{F}}$ has the following properties:

$$\begin{aligned} \|\tilde{p}\|_{\mathcal{F}} &= 0 \textit{ iff } \tilde{p} = \tilde{o} \\ \|\eta \odot \tilde{p}\|_{\mathcal{F}} &= |\eta|\, \|\tilde{p}\|_{\mathcal{F}} \\ \|\tilde{p} \oplus \tilde{q}\|_{\mathcal{F}} &\leqslant \|\tilde{p}\|_{\mathcal{F}} + \|\tilde{q}\|_{\mathcal{F}} \\ \|\tilde{p}\|_{\mathcal{F}} + \|\tilde{q}\|_{\mathcal{F}} &\leqslant D(\tilde{p}, \tilde{q}) \end{aligned}$$

(vii) $D(\eta \odot \tilde{p}, \mu \odot \tilde{p}) = |\eta - \mu|\, D(\tilde{o}, \tilde{p})$, for all $\eta\mu \geqslant 0$. This equation does not hold when η and μ are of opposite sign.

Having covered the fundamental arithmetic operations of fuzzy numbers, our focus now shifts to the examination of fuzzy valued functions.

3. Fuzzy Numbered-Valued Function

When domain of a function is the set of real numbers, and the co-domain is the set of fuzzy numbers, the function can have some unique properties and characteristics that reflect the inherent uncertainty and imprecision in the co-domain. Fuzzy numbers in the co-domain allow the function to associate each real number in the domain with a degree of membership in different fuzzy sets. This means the output is not a single real number but a fuzzy value that can represent a range of possible values. The function may not yield a unique result for a given input; instead, it can produce a range of values with varying degrees of certainty. This continuous variation allows the function to capture nuances and gradual transitions in the output. This reflects the imprecision and variability often encountered in real-world applications. Fuzzy function is useful in scenarios where precise values are hard to define, such as linguistic terms like "very hot" or "moderately cold" or data with inherent imprecision.

Chang and Zadeh [9], in their paper defined the class of fuzzy bunches of functions as a function with a fuzzy parameter. A fuzzy bunch of functions, generally speaking, is the fuzzy subset of a classical space of functions. To put it mathematically, a fuzzy bunch W of functions mapping from X to Y can be described as a fuzzy set on Y^X, that is, each function $w : X \to Y$ within the fuzzy bunch W is assigned a membership value $\mu(w)$.

However, we will not be dealing with fuzzy bunches of functions or their corresponding fuzzy sets at x. In our study, we are simply concerned with a function $\tilde{h}$, that takes inputs from the real interval $[a, b]$ and maps them to the fuzzy field $\mathbb{R}_{\mathcal{F}}$, having the representation:

$$\left[\tilde{h}\left(x\right)\right]^{\ell} = \left[h_{-}^{(\ell)}\left(x\right), h_{+}^{(\ell)}\left(x\right)\right],$$

for every $x \in [a, b]$ and ℓ within the range $[0, 1]$, where $h_{-}^{(\ell)}\left(x\right)$ and $h_{+}^{(\ell)}\left(x\right)$ represent the left and right endpoints of $\left[\tilde{h}\left(x\right)\right]^{\ell}$, respectively. Furthermore, $h_{-}^{(\ell)}$ and $h_{+}^{(\ell)}$ are real valued functions defined over the interval $[a, b]$.

Illustration 3.1 *In order to increase our understanding towards a fuzzy valued function, let us take an example. Let* $\tilde{h} : [0, 1] \to \mathbb{R}_{\mathcal{F}}$ *such that*

$$\tilde{h}\left(x\right) = \sin^{2}3x - \frac{1}{2}\cos 9x + x^{6},$$

where, each $\tilde{h}(x)$ *is a triangular fuzzy number with the membership function:*

$$\mu_{h}\left(t; \tilde{h}(x) - 1, \tilde{h}(x), \tilde{h}(x) + 1\right) = \max\left(\min\left(t - \tilde{h}(x) + 1, \tilde{h}(x) + 1 - t\right), 0\right).$$

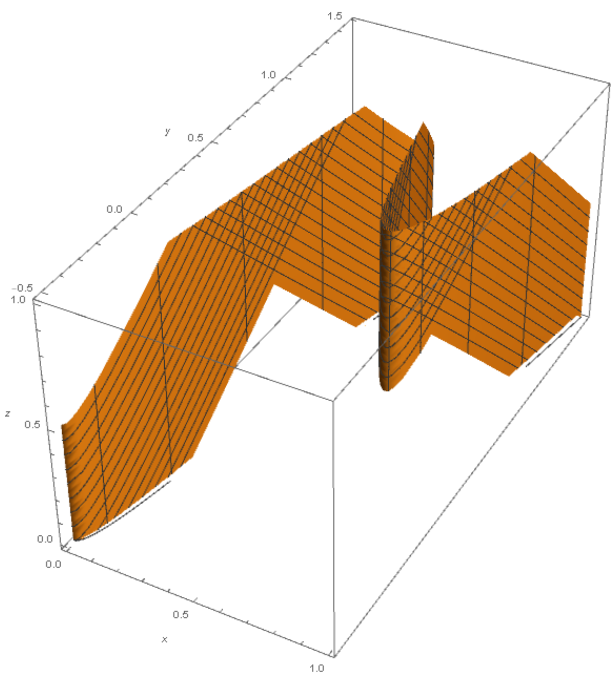
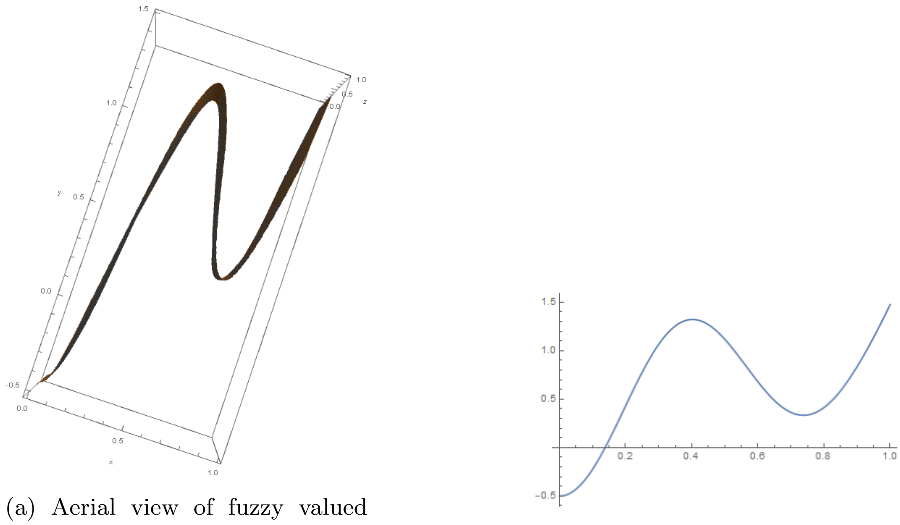


Figure 1: Graph of fuzzy number-valued function $\tilde{h}\left(x\right) = \sin^{2}3x - \frac{1}{2}\cos 9x + x^{6}$ with triangular membership function.

The graph of a fuzzy valued function is a 3-dimensional graph, as the Z-axis corresponds to the membership value corresponding to each $\tilde{h}(x)$. The above figure represents a function whose range is the triangular fuzzy number. Let us change this membership function and see what changes we get in our graph. Suppose each point in the range is a trapezoidal fuzzy number. Then the membership value



(a) Aerial view of fuzzy valued function

(b) Crisp function

Figure 2: Association of a fuzzy valued function with its corresponding crisp function.

corresponding to each $\tilde{h}(x)$ will be,

$$\mu_h\left(t;\tilde{h}(x)-1.5,\tilde{h}(x)-0.5,\tilde{h}(x)+\frac{1}{2},\tilde{h}(x)+1.5\right)$$
$$=\max\left(\min\left(t-\tilde{h}(x)+1.5,1,\tilde{h}(x)+1.5-t\right),0\right).$$

The aerial view will remain the same; only the degree of uncertainty, given by the membership function μ_h, changes along the z axis.

Consider two fuzzy numbered-valued functions, $\tilde{h}$ and $\tilde{g}$, defined on the interval $[a,b] \in \mathbb{R}$. We define the distance between $\tilde{h}$ and $\tilde{g}$ as follows:

$$D^*\left(\tilde{h},\tilde{g}\right) := \sup_{x\in[a,b]} D\left(\tilde{h}\left(x\right),\tilde{g}\left(x\right)\right).$$

We define $\tilde{h}$ as crisp when $\tilde{h}(x)$ assumes crisp values for all x in its domain. $\tilde{h}$ is defined to be a fuzzy continuous function if,

$$\lim_{x\to x_0} D\left(\tilde{h}(x),\tilde{h}(x_0)\right)=0,$$

holds for any $x_0 \in [a,b]$.

Represented by $C\,[a,b]$ and $C_{\mathcal{F}}\,[a,b]$ are the sets of continuous and fuzzy continuous functions on the interval $[a,b]$. If $\tilde{h}$ is a fuzzy continuous function over $[a,b]$, then the corresponding functions $h_-^{(r)}$ and $h_+^{(r)}$ become real-valued continuous functions defined on the same interval. Also, the space ($C_{\mathcal{F}}\,[a,b]$, D^*) forms a complete metric space. The operations of addition and scalar multiplication in $C_{\mathcal{F}}\,[a,b]$ are defined as follows:

$$\left(a\odot\tilde{g}\right)(x)=a\odot\tilde{g}\left(x\right)$$
$$\left(\tilde{h}\oplus\tilde{g}\right)(x)=\tilde{h}\left(x\right)\oplus\tilde{g}\left(x\right),$$

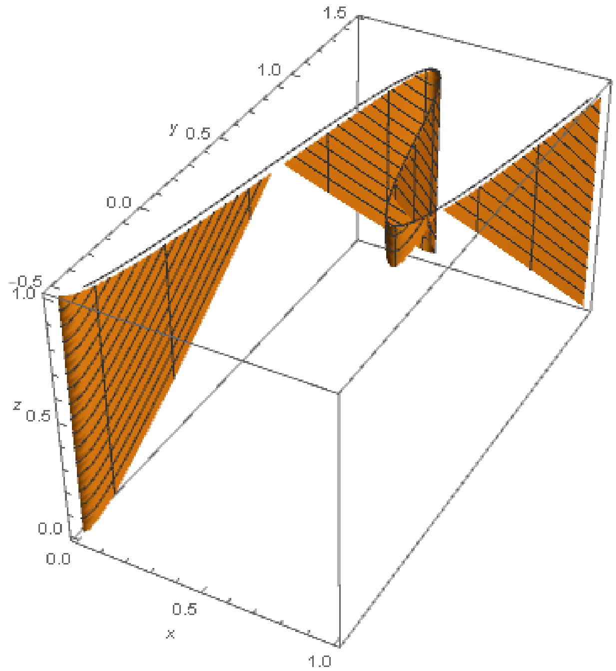


Figure 3: Graph of fuzzy number-valued function $\tilde{h}(x) = \sin^2 3x - \frac{1}{2}\cos 9x + x^6$ with trapezoidal membership function.

for every $x \in [a,b]$, $a \in \mathbb{R}$ and $\tilde{h}, \tilde{g} \in C_{\mathcal{F}}[a,b]$. Moreover, a fuzzy number-valued function $\tilde{0}$ exists, defined on the interval $[a,b]$, satisfying the condition $\tilde{0}(x) = \tilde{o}$ for every $x \in [a,b]$ where $\tilde{o}$ represents the neutral element with respect to the operation $\oplus$ in $\mathbb{R}_{\mathcal{F}}$. We can also define the norm of $\tilde{h}$ as

$$\left\|\tilde{h}\right\|_{\mathcal{F}} = \sup_{a \leqslant x \leqslant b} D\left(\tilde{o}, \tilde{h}(x)\right).$$

Based on the above definitions we obtain the following properties.

Lemma 3.1 *For any $\tilde{h}, \tilde{g}, \tilde{f} \in C_{\mathcal{F}}[a,b]$ and real constants $\eta, \nu \in \mathbb{R}$, we have the following properties:*

$\oplus$ is commutative and associative, that is,

$$\tilde{h} \oplus \tilde{g} = \tilde{g} \oplus \tilde{h},$$
$$\tilde{h} \oplus \left(\tilde{g} \oplus \tilde{f}\right) = \left(\tilde{h} \oplus \tilde{g}\right) \oplus \tilde{f}.$$

(ii) $\tilde{h} \oplus \tilde{0} = \tilde{0} \oplus \tilde{h}$.

(iii) *Consider the function space $C_{\mathcal{F}}[a,b]$ with neutral element $\tilde{0}(x)$. If for any function $\tilde{h}$ in this space, the range of $\tilde{h}$ over the interval $[a,b]$ has a non-empty intersection with the set of real numbers, then there is no opposite member with respect to the operation $\oplus$ in $C_{\mathcal{F}}[a,b]$.*

(iv) *For all $\eta\nu \geqslant 0$, $(\eta + \nu) \odot \tilde{h} = \left(\eta \odot \tilde{h}\right) \oplus \left(\nu \odot \tilde{h}\right)$.*
For general $\eta, \nu \in \mathbb{R}$, this property does not hold.

(v) *For any $\tilde{h}, \tilde{g} \in C_{\mathcal{F}}[a,b]$ and real constants $\eta, \nu \in \mathbb{R}$,*

$$\eta \odot \left(\tilde{h} \oplus \tilde{g}\right) = \left(\eta \odot \tilde{h}\right) \oplus (\eta \odot \tilde{g})$$

$$\eta \odot \left(\nu \odot \tilde{h}\right) = (\eta\nu) \odot \tilde{h}$$

(vi) For any $\tilde{h}, \tilde{g} \in C_{\mathcal{F}}[a,b]$,

$$\left\|\tilde{h}\right\|_{\mathcal{F}} = 0 \text{ iff } \tilde{h} = \tilde{0}$$

$$\left\|\eta \odot \tilde{h}\right\|_{\mathcal{F}} = |\eta| \left\|\tilde{h}\right\|_{\mathcal{F}}$$

$$\left\|\tilde{h} \oplus \tilde{g}\right\|_{\mathcal{F}} \leqslant \left\|\tilde{h}\right\|_{\mathcal{F}} + \|\tilde{g}\|_{\mathcal{F}}$$

$$\left\|\tilde{h}\right\|_{\mathcal{F}} - \|\tilde{g}\|_{\mathcal{F}} \leqslant D^*\left(\tilde{h}, \tilde{g}\right)$$

(vii) For $\eta\nu \geqslant 0$,

$$D^*\left(\eta \odot \tilde{h}, \nu \odot \tilde{h}\right) = |\eta - \nu| \, D^*\left(\tilde{0}, \tilde{h}\right)$$

(viii) For any $\tilde{h}, \tilde{g}, \tilde{f}, \tilde{e} \in C_{\mathcal{F}}[a,b]$ *and* $\eta \in \mathbb{R}$,

$$D^*\left(\tilde{h} \oplus \tilde{f}, \tilde{g} \oplus \tilde{f}\right) = D^*\left(\tilde{h}, \tilde{g}\right)$$

$$D^*\left(\eta \odot \tilde{h}, \eta \odot \tilde{g}\right) = |\eta| \, D^*\left(\tilde{h}, \tilde{g}\right)$$

$$D^*\left(\tilde{h} \oplus \tilde{g}, \tilde{f} \oplus \tilde{e}\right) \leqslant D^*\left(\tilde{h}, \tilde{f}\right) + D^*\left(\tilde{g}, \tilde{e}\right)$$

Definition 3.1 (Fuzzy modulus of continuity) *For a continuous fuzzy-valued function* $\tilde{h}$ *mapping from the interval* $[a,b]$ *to* $\mathbb{R}_{\mathcal{F}}$*, the first modulus of continuity is defined as follows:*

$$\omega_1^{\mathcal{F}}\left(\tilde{h}; \delta\right) := \sup_{\substack{u,v \in [a,b] \\ |u-v| \leqslant \delta}} D\left(\tilde{h}(u), \tilde{h}(v)\right)$$

Definition 3.2 (Fuzzy exponential modulus of continuity) *Consider a continuous fuzzy-valued function* $\tilde{h} : [0, \infty) \to \mathbb{R}_{\mathcal{F}}$*. We can define the exponential fuzzy modulus of continuity for* $\tilde{h}$ *in the following manner:*

$$\omega_*^{\mathcal{F}}\left(\tilde{h}; \delta\right) := \sup_{\substack{u,v \geqslant 0 \\ |e^{-u} - e^{-v}| < \delta}} D\left(\tilde{h}(u), \tilde{h}(v)\right)$$

4. Linear Positive Fuzzy Operators

In the case of operators $\tilde{\mathcal{L}}$ from $C_{\mathcal{F}}[a,b]$ to itself, their representation takes the form,

$$\left[\tilde{\mathcal{L}}\left(\tilde{h}\right)(x)\right]^{\ell} = \left[\left(\tilde{\mathcal{L}}\left(\tilde{h}\right)(x)\right)_-^{(\ell)}, \left(\tilde{\mathcal{L}}\left(\tilde{h}\right)(x)\right)_+^{(\ell)}\right].$$

Suppose $\tilde{\mathcal{L}}$ is an operator mapping the space of all continuous fuzzy-valued functions $C_{\mathcal{F}}[a,b]$ to itself, with the condition,

$$\tilde{\mathcal{L}}\left(a \odot \tilde{h}\right) = a \odot \tilde{\mathcal{L}}\left(\tilde{h}\right)$$

$$\tilde{\mathcal{L}}\left(\tilde{h} \oplus \tilde{g}\right) = \tilde{\mathcal{L}}\left(\tilde{h}\right) \oplus \tilde{\mathcal{L}}\left(\tilde{g}\right),$$

for any $a \in \mathbb{R}$ and $\tilde{h}, \tilde{g} \in C_{\mathcal{F}}[a,b]$. In this case, $\tilde{\mathcal{L}}$ is identified as a fuzzy linear operator. Consider $\tilde{\mathcal{L}}$ mapping $C_{\mathcal{F}}[a,b]$ to itself such that for any $\tilde{h}, \tilde{g} \in C_{\mathcal{F}}[a,b]$, $\tilde{\mathcal{L}}$ is linear and

$$\tilde{h} \precsim \tilde{g} \Rightarrow \tilde{\mathcal{L}}\left(\tilde{h}\right) \precsim \tilde{\mathcal{L}}\left(\tilde{g}\right).$$

Then $\tilde{\mathcal{L}}$ termed as a linear positive fuzzy operator.

Theorem 4.1 (Fuzzy analog of Shisha-Mond inequality) *[3] Consider a sequence $\left\{\tilde{\mathcal{L}}_m\right\}_{m\in\mathbb{N}}$ of positive fuzzy linear operators that map $C_{\mathcal{F}}[a,b]$ to itself. We posit the existence of a corresponding sequence $\{\mathcal{L}_m\}_{m\in\mathbb{N}}$, consisting of linear positive operators from $C[a,b]$ into itself, satisfying the property,*

$$\left(\tilde{\mathcal{L}}_m\left(\tilde{h}\right)\right)_{\pm}^{(\ell)} = \mathcal{L}_m\left(\tilde{h}_{\pm}^{(\ell)}\right)$$

respectively, $\forall \ell \in [0,1]$ and $\forall \tilde{h} \in C_{\mathcal{F}}[a,b]$. With the assumption that the sequence $\{\mathcal{L}_m(1)\}_{m\in\mathbb{N}}$ is bounded, we can deduce that, for any $m \in \mathbb{N}$,

$$D^*\left(\tilde{\mathcal{L}}_m\tilde{h},\tilde{h}\right) \leqslant \|\mathcal{L}_m(1)-1\|\, D^*\left(\tilde{h},\tilde{0}\right) + \|\mathcal{L}_m(1)+1\|\,\omega_1^{\mathcal{F}}\left(\tilde{h};\mu_m\right),$$

where $\mu_m = \left\|\mathcal{L}_m\left((t-x)^2\right)(x)\right\|^{1/2}$.

If $\mathcal{L}_m(1)=1$, then, $D^*\left(\tilde{\mathcal{L}}_m\tilde{h},\tilde{h}\right) \leqslant 2\omega_1^{\mathcal{F}}\left(\tilde{h};\mu_m\right)$.

The theorem discussed above enables us to demonstrate the fuzzy counterpart of the Korovkin theorem in the closed and bounded interval $[a,b]$.

Theorem 4.2 (Fuzzy Korovkin Theorem) *Consider a sequence $\left\{\tilde{\mathcal{L}}_m\right\}_{m\in\mathbb{N}}$ of linear positive fuzzy operators, mapping $C_{\mathcal{F}}[a,b]$ into itself. We posit the existence of a corresponding sequence $\{\mathcal{L}_m\}_{m\in\mathbb{N}}$, consisting of linear positive operators from $C[a,b]$ into itself, satisfying the property,*

$$\left(\tilde{\mathcal{L}}_m\left(\tilde{h}\right)\right)_{\pm}^{(\ell)} = \mathcal{L}_m\left(\tilde{h}_{\pm}^{(\ell)}\right)$$

respectively, $\forall \ell \in [0,1]$ and $\forall \tilde{h} \in C_{\mathcal{F}}([a,b])$. Furthermore, assume that, as $m\to\infty$

$$\begin{gathered}\mathcal{L}_m(1)\to 1\\ \mathcal{L}_m(t)\to x\\ \mathcal{L}_m\left(t^2\right)\to x^2,\end{gathered}$$

uniformly. Then,

$$D^*\left(\tilde{\mathcal{L}}_m\tilde{h},\tilde{h}\right)\to 0$$

as $m\to\infty$, for any $\tilde{h}\in C_{\mathcal{F}}[a,b]$. That is, we can say that,

$$\tilde{\mathcal{L}}_m\tilde{h} \xrightarrow{D^*} \tilde{h}.$$

This theorem proves an operator to approximate a function by using the test functions $1, t, t^2$. However, Altomare [1] in his study generalized the set of test functions, known as the Korovkin set. The following theorems are for using different test functions. In a real sense, we have the following theorem:

Theorem 4.3 *Given the metric space (X,d) and the linear space $C(X)$ of all continuous real-valued functions on X, take a subset E containing the constant function 1 and the functions d_x^2, where $d_x(y)=d(x,y)$ for $x,y\in X$. Assume $\{\mathcal{L}_m\}_{m\geqslant 1}$ is a sequence of linear positive operators mapping from E into $C(X)$. Then, for any uniformly continuous function h in E,*

$$|\mathcal{L}_m(h)(x) - h(x)\mathcal{L}_m(1)(x)| \leqslant \mathcal{L}_m(|h-h(x)|)(x) \leqslant \frac{2\|h\|_\infty}{\delta^2}\mathcal{L}_m\left(d_x^2\right)(x) + \varepsilon\mathcal{L}_m(1)(x).$$

Based on theorem 4.3, Altomare also stated the following remark.

Remark 4.1 *Assuming M is a subset of $C_0(X)$ and given that $f_0 \in C_0(X)$ is strictly positive, the set $\{f_0\} \cup \{f_0 M\} \cup \{f_0 M^2\}$ serves as a Korovkin set within $C_0(X)$.*

By setting $M = \{e^{-x}\}$ and $f_0 = 1$, we form the Korovkin set $\{1, e^{-x}, e^{-2x}\}$. It is necessary to broaden the findings established by Altomare to encompass fuzzy-valued functions. Consider a subset $E_{\mathcal{F}}$ of $F^{\mathcal{F}}(X)$ containing the fuzzy constant function 1 and all the fuzzy functions d_x^2, $x \in X$.

Theorem 4.4 *Consider a sequence $\left\{\tilde{\mathcal{L}}_m\right\}_{m\in\mathbb{N}}$ of linear positive fuzzy operators, mapping $C_{\mathcal{F}}(X)$ into itself, for the metric space (X,d). We posit the existence of a corresponding sequence $\{\mathcal{L}_m\}_{m\in\mathbb{N}}$, consisting of linear positive operators from $C(X)$ into itself, satisfying the property,*

$$\left(\tilde{\mathcal{L}}_m\left(\tilde{h}\right)\right)_{\pm}^{(\ell)} = \mathcal{L}_m\left(\tilde{h}_{\pm}^{(\ell)}\right),$$

respectively, $\forall \ell \in [0,1]$ and $\forall \tilde{h} \in C_{\mathcal{F}}(X)$. Then, we have

$$D^*\left(\tilde{\mathcal{L}}_m\left(\tilde{h}\right), \tilde{h}\mathcal{L}_m(1)\right) \leqslant \frac{2}{\delta^2}\mathcal{L}_m\left(d_x^2\right) D^*\left(\tilde{h}, \tilde{o}\right) + \varepsilon \mathcal{L}_m(1)$$

where $d_x(y) = d(x,y) \quad \forall x, y \in X$.

Proof: Let $\tilde{h} \in C_{\mathcal{F}}(X)$. Then,

$$\begin{aligned}
&D^*\left(\tilde{\mathcal{L}}_m\left(\tilde{h}\right), \tilde{h}\mathcal{L}_m(1)\right)\\
&= \sup_{x\in X} D\left(\tilde{\mathcal{L}}_m\left(\tilde{h}\right)(x), \tilde{h}(x)\mathcal{L}_m(1)\right)\\
&= \sup_{x\in X}\sup_{\ell\in[0,1]} \max\left\{\left|\mathcal{L}_m\left(\left(\tilde{h}\right)_{-}^{(\ell)}(x)\right) - \left(\tilde{h}\right)_{-}^{(\ell)}(x)\mathcal{L}_m(1)_{-}^{(\ell)}\right|,\right.\\
&\quad\left.\left|\mathcal{L}_m\left(\left(\tilde{h}\right)_{+}^{(\ell)}(x)\right) - \left(\tilde{h}\right)_{+}^{(\ell)}(x)\mathcal{L}_m(1)_{+}^{(\ell)}\right|\right\}\\
&= \sup_{\ell\in[0,1]} \max\left\{\left\|\mathcal{L}_m\left(\left(\tilde{h}\right)_{-}^{(\ell)}(x)\right) - \left(\tilde{h}\right)_{-}^{(\ell)}(x)\mathcal{L}_m(1)_{-}^{(\ell)}\right\|,\right.\\
&\quad\left.\left\|\mathcal{L}_m\left(\left(\tilde{h}\right)_{+}^{(\ell)}(x)\right) - \left(\tilde{h}\right)_{+}^{(\ell)}(x)\mathcal{L}_m(1)_{+}^{(\ell)}\right\|\right\}\\
&\leqslant \sup_{\ell\in[0,1]} \max\left\{2\frac{\left\|\left(\tilde{h}\right)_{-}^{(\ell)}\right\|}{\delta^2}\mathcal{L}_m\left(d_x^2\right) + \varepsilon\mathcal{L}_m(1), 2\frac{\left\|\left(\tilde{h}\right)_{+}^{(\ell)}\right\|}{\delta^2}\mathcal{L}_m\left(d_x^2\right) + \varepsilon\mathcal{L}_m(1)\right\}\\
&\leqslant \frac{2}{\delta^2}\mathcal{L}_m\left(d_x^2\right)\sup_{\ell\in[0,1]}\max\left\{\left\|\left(\tilde{h}\right)_{-}^{(\ell)}\right\|, \left\|\left(\tilde{h}\right)_{+}^{(\ell)}\right\|\right\} + \varepsilon\mathcal{L}_m(1)\\
&\leqslant \frac{2}{\delta^2}\mathcal{L}_m\left(d_x^2\right) D^*\left(\tilde{h}, \tilde{o}\right) + \varepsilon\mathcal{L}_m(1)
\end{aligned}$$

where, $\tilde{o} = \chi_{\{0\}}$ is the neutral element for $\oplus$. □

Corollary 4.1 *Further, if we assume that,*

(i) $\lim_{m\to\infty} D^*\left(\tilde{\mathcal{L}}_m(1), 1\right) = 0$ *uniformly on X*

(ii) $\lim_{m\to\infty} D^*\left(\tilde{\mathcal{L}}_m\left(d_x^2(x)\right), 0\right) = 0$ *uniformly on X*

Then for every fuzzy continuous $\tilde{h} \in C_{\mathcal{F}}(X)$,

$$\lim_{m\to\infty} D^*\left(\tilde{\mathcal{L}}_m\left(\tilde{h}\right), \tilde{h}\right) = 0$$

uniformly on X.

Theorem 4.5 (Fuzzy Korovkin Theorem for $\mathbb{R}^+$) *Consider a sequence* $\left\{\tilde{\mathcal{L}}_m\right\}_{m\in\mathbb{N}}$ *of linear positive fuzzy operators, mapping* $C_{\mathcal{F}}\left(\mathbb{R}^+\right)$ *into itself. We posit the existence of a corresponding sequence* $\{\mathcal{L}_m\}_{m\in\mathbb{N}}$, *consisting of linear positive operators from* $C\left(\mathbb{R}^+\right)$ *into itself, satisfying the property,*

$$\left(\tilde{\mathcal{L}}_m\left(\tilde{h}\right)\right)_{\pm}^{(r)} = \mathcal{L}_m\left(\tilde{h}_{\pm}^{(r)}\right),$$

for all $\ell \in [0,1]$ *and every* $\tilde{h} \in C_{\mathcal{F}}\left(\mathbb{R}^+\right)$, *respectively. Additionally, assume that, as* m *approaches* ∞,

$$\begin{aligned} \mathcal{L}_m(1) &\to 1 \\ \mathcal{L}_m\left(e^{-t}\right) &\to e^{-x} \\ \mathcal{L}_m\left(e^{-2t}\right) &\to e^{-2x}, \end{aligned}$$

uniformly. Then,

$$D^*\left(\tilde{\mathcal{L}}_m\tilde{h}, \tilde{h}\right) \to 0$$

as $m \to \infty$, *for any* $f \in C_{\mathcal{F}}\left(\mathbb{R}^+\right)$. *That is, we can say that,*

$$\tilde{\mathcal{L}}_m\tilde{h} \xrightarrow{D^*} \tilde{h}.$$

5. Fuzzy Bernstein Operators

Define the fuzzy Bernstein operators in the following way:

$$\left(B_m^{(\mathcal{F})}\tilde{h}\right)(x) = \sum_{k=0}^{m}{}^{*} \binom{m}{k} x^k(1-x)^{m-k} \odot \tilde{h}\left(\frac{k}{m}\right),$$

$\forall x \in [0,1]$, $\forall m \in \mathbb{N}$ and $\tilde{h} \in C_{\mathcal{F}}[0,1]$.

Then the ℓ-cut of the function $\tilde{h}$ and its operator $B_m^{(\mathcal{F})}\tilde{h}$ will be the closed intervals,

$$\begin{aligned} \left[\tilde{h}\left(\frac{k}{m}\right)\right]^{\ell} &= \left[\tilde{h}\left(\frac{k}{m}\right)_{-}^{(\ell)}, \tilde{h}\left(\frac{k}{m}\right)_{+}^{(\ell)}\right] \\ \left[B_m^{(\mathcal{F})}\tilde{h}\right]^{\ell} &= \left[\left(B_m^{(\mathcal{F})}\tilde{h}\right)_{-}^{(\ell)}, \left(B_m^{(\mathcal{F})}\tilde{h}\right)_{+}^{(\ell)}\right] \end{aligned}$$

Clearly, for any $\tilde{h} \in C_{\mathcal{F}}[0,1]$ and $\ell \in [0,1]$,

$$\left(B_m^{(\mathcal{F})}\tilde{h}\right)_{\pm}^{(\ell)} = B_m\left(\tilde{h}_{\pm}^{(\ell)}\right),$$

where B_m represents the traditional Bernstein operators.

Lemma 5.1 *The fuzzy Bernstein operators are characterized as linear positive fuzzy operators.*

Proof: Consider fuzzy continuous functions $\tilde{h}$ and $\tilde{g}$ defined on the interval $[0,1]$. We have,

$$\left(B_m^{(\mathcal{F})}\left(\tilde{h}\oplus\tilde{g};x\right)\right)_{\pm}^{(\ell)}=B_m\left(\left(\tilde{h}\oplus\tilde{g}\right)_{\pm}^{(\ell)};x\right)$$

for every $x\in[0,1]$ and $\ell\in[0,1]$, respectively. Given that $B_m^{(\mathcal{F})}\left(\tilde{h}\oplus\tilde{g}\right)\in C_{\mathcal{F}}[0,1]$, and taking into account the representation of $B_m^{(\mathcal{F})}$, we obtain $\left(B_m^{(\mathcal{F})}\left(\tilde{h}\oplus\tilde{g}\right)\right)_{-}^{\ell}, \left(B_m^{(\mathcal{F})}\left(\tilde{h}\oplus\tilde{g}\right)\right)_{+}^{\ell}\in C[0,1]$.

Taking into account theorem 3.1 and the linearity of $B_m^{(\mathcal{F})}$, we can say,

$$\begin{aligned}B_m\left(\left(\tilde{h}\oplus\tilde{g}\right)_{\pm}^{(\ell)};x\right)&=B_m\left(\tilde{h}_{\pm}^{(\ell)}+\tilde{g}_{\pm}^{(\ell)};x\right)\\&=B_m\left(\tilde{h}_{\pm}^{(\ell)};x\right)+B_m\left(\tilde{g}_{\pm}^{(\ell)};x\right)\end{aligned}$$

Thus we obtain,

$$\left(B_m^{(\mathcal{F})}\left(\tilde{h}\oplus\tilde{g};x\right)\right)_{\pm}^{(\ell)}=\left(B_m^{(\mathcal{F})}\left(\tilde{h};x\right)\right)_{\pm}^{(\ell)}+\left(B_m^{(\mathcal{F})}\left(\tilde{g};x\right)\right)_{\pm}^{(\ell)}$$

for each $x\in[0,1]$, and $\ell\in[0,1]$.

Using the above equation and taking into account the summation over the interval, we obtain,

$$\begin{aligned}&\left[B_m^{(\mathcal{F})}\left(\tilde{h}\oplus\tilde{g};x\right)\right]^{\ell}\\&=\left[\left(B_m^{(\mathcal{F})}\left(\tilde{h}\oplus\tilde{g};x\right)\right)_{-}^{(\ell)},\left(B_m^{(\mathcal{F})}\left(\tilde{h}\oplus\tilde{g};x\right)\right)_{+}^{(\ell)}\right]\\&=\left[\left(B_m^{(\mathcal{F})}\left(\tilde{h};x\right)\right)_{-}^{(\ell)}+\left(B_m^{(\mathcal{F})}\left(\tilde{g};x\right)\right)_{-}^{(\ell)},\left(B_m^{(\mathcal{F})}\left(\tilde{h};x\right)\right)_{+}^{(\ell)}+\left(B_m^{(\mathcal{F})}\left(\tilde{g};x\right)\right)_{+}^{(\ell)}\right]\\&=\left[\left(B_m^{(\mathcal{F})}\left(\tilde{h};x\right)\right)_{-}^{(\ell)},\left(B_m^{(\mathcal{F})}\left(\tilde{h};x\right)\right)_{+}^{(\ell)}\right]+\left[\left(B_m^{(\mathcal{F})}\left(\tilde{g};x\right)\right)_{-}^{(\ell)},\left(B_m^{(\mathcal{F})}\left(\tilde{g};x\right)\right)_{+}^{(\ell)}\right]\\&=\left[B_m^{(\mathcal{F})}\left(\tilde{h};x\right)\right]^{\ell}+\left[B_m^{(\mathcal{F})}\left(\tilde{g};x\right)\right]^{\ell}\\&=\left[B_m^{(\mathcal{F})}\left(\tilde{h};x\right)\oplus B_m^{(\mathcal{F})}\left(\tilde{g};x\right)\right]^{\ell}\\&=\left[\left(B_m^{(\mathcal{F})}\left(\tilde{h}\right)\oplus B_m^{(\mathcal{F})}\left(\tilde{g}\right);x\right)\right]^{\ell}\end{aligned}$$

for each $x\in[0,1]$ and $\ell\in[0,1]$. Thus,

$$B_m^{(\mathcal{F})}\left(\tilde{h}\oplus\tilde{g}\right)=B_m^{(\mathcal{F})}\left(\tilde{h}\right)\oplus B_m^{(\mathcal{F})}\left(\tilde{g}\right),\qquad \tilde{h},\tilde{g}\in C_F\left[0,1\right].$$

Suppose that $k\geqslant 0$ be any real number. And, for each $x\in[0,1]$ and $\ell\in[0,1]$,

$$\left(B_m^{(\mathcal{F})}\left(k\odot\tilde{h};x\right)\right)_{\pm}^{(\ell)}=B_m\left(\left(k\odot\tilde{h}\right)_{\pm}^{(\ell)};x\right)$$

for each $x\in[0,1]$ and $\ell\in[0,1]$, respectively. Since, $B_m^{(\mathcal{F})}\left(k\odot\tilde{h}\right)\in C_{\mathcal{F}}[0,1]$, and taking into account the representation of $B_m^{(\mathcal{F})}$, we obtain $\left(B_m^{(\mathcal{F})}\left(k\odot\tilde{h}\right)\right)_{-}^{(\ell)}, \left(B_m^{(\mathcal{F})}\left(k\odot\tilde{h}\right)\right)_{+}^{(\ell)}\in C[0,1]$.

Taking into account theorem 3.1 and the linearity of $B_m^{(\mathcal{F})}$, we can say,

$$\begin{aligned}B_m\left(\left(k\odot\tilde{h}\right)_{\pm}^{(\ell)};x\right)&=B_m\left(k\tilde{h}_{\pm}^{(\ell)};x\right)\\&=kB_m\left(\tilde{h}_{\pm}^{(\ell)};x\right)\end{aligned}$$

for every $x \in [0,1]$ and $\ell \in [0,1]$. Thus we obtain,

$$\left(B_m^{(\mathcal{F})}\left(k \odot \tilde{h}; x\right)\right)_{\pm}^{(\ell)} = k\left(B_m^{(\mathcal{F})}\left(\tilde{h}; x\right)\right)_{\pm}^{(\ell)} = \left(\left(k \odot B_m^{(\mathcal{F})}\right)\left(\tilde{h}; x\right)\right)_{\pm}^{(\ell)}$$

for every $x \in [0,1]$, and $\ell \in [0,1]$.

Using the above equation we have,

$$\begin{aligned} &\left[B_m^{(\mathcal{F})}\left(k \odot \tilde{h}; x\right)\right]^{\ell} \\ &= \left[\left(B_m^{(\mathcal{F})}\left(k \odot \tilde{h}; x\right)\right)_{-}^{(\ell)}, \left(B_m^{(\mathcal{F})}\left(k \odot \tilde{h}; x\right)\right)_{+}^{(\ell)}\right] \\ &= \left[\left(\left(k \odot B_m^{(\mathcal{F})}\right)\left(\tilde{h}; x\right)\right)_{-}^{(\ell)}, \left(\left(k \odot B_m^{(\mathcal{F})}\right)\left(\tilde{h}; x\right)\right)_{+}^{(\ell)}\right] \\ &= \left[\left(k \odot B_m^{(\mathcal{F})}\right)\left(\tilde{h}; x\right)\right]^{\ell} \end{aligned}$$

Therefore,

$$B_m^{(\mathcal{F})}\left(k \odot \tilde{h}\right) = \left(k \odot B_m^{(\mathcal{F})}\right)\left(\tilde{h}\right), \quad k \geqslant 0, \tilde{h} \in C_{\mathcal{F}}[0,1].$$

Similarly, we can prove for $k < 0$. Thus, the fuzzy Bernstein operators are fuzzy linear operators. Consider fuzzy continuous functions $\tilde{h}$ and $\tilde{g}$ defined on the interval $[0,1]$ with $\tilde{h} \precsim \tilde{g}$, where $\precsim$ is a partial order on $C_{\mathcal{F}}[0,1]$ previously defined. Then, $\tilde{h}_-^{(\ell)} \leqslant \tilde{g}_-^{(\ell)}$ and $\tilde{h}_+^{(\ell)} \leqslant \tilde{g}_+^{(\ell)}$, where $\leqslant$ is a partial order on $C[0,1]$.

Since, $\tilde{h}_-^{(\ell)}$,$\tilde{h}_+^{(\ell)}$,$\tilde{g}_-^{(\ell)}$ and $\tilde{g}_+^{(\ell)} \in C[0,1]$ and by the positivity of B_m, we have,

$$B_m\left(\tilde{h}_{\pm}^{(\ell)}\right) \leqslant B_m\left(\tilde{g}_{\pm}^{(\ell)}\right), \quad \ell \in [0,1].$$

Considering the above equation and theorem 3.1, we obtain,

$$\left(B_m^{(\mathcal{F})}\left(\tilde{h}\right)\right)_{\pm}^{(\ell)} \leqslant \left(B_m^{(\mathcal{F})}\left(\tilde{g}\right)\right)_{\pm}^{(\ell)}, \quad \ell \in [0,1].$$

Thus,

$$B_m^{(\mathcal{F})}\left(\tilde{h}; x\right) \precsim B_m^{(\mathcal{F})}\left(\tilde{g}; x\right), \quad x \in [0,1],\ \ell \in [0,1]$$

This gives the positivity of $B_m^{(\mathcal{F})}$. □

Since,

$$\begin{aligned} B_m(1) &\to 1 \\ B_m(t) &\to x \\ B_m\left(t^2\right) &\to x^2, \end{aligned}$$

Thus, by fuzzy Korovkin theorem we can say that, $D^*\left(B_m^{(\mathcal{F})}\tilde{h}(t), \tilde{h}(x)\right) \to 0$, as $m \to \infty$.

Theorem 5.1 *Let $\tilde{h}(x)$ is a continuous, bounded and twice differential fuzzy function with membership value for each $\tilde{h}(x)$ as $\mu_h(x)$, defined for all $x \in [0,1]$. Then,*

$$\lim_{m\to\infty} mD^*\left(B_m^{(\mathcal{F})}\tilde{h}(t), \tilde{h}(x)\right) \leqslant \frac{1}{2}x(1-x)D^*\left(\tilde{h}''(x), \tilde{o}\right)$$

where $\tilde{o}$ is the zero element of $\mathbb{R}_{\mathcal{F}}$.

Proof: Consider,

$$\begin{aligned}
D^*\left(B_m^{(\mathcal{F})}\tilde{h}\left(\frac{k}{m}\right),\tilde{h}(x)\right) &= \sup_{x\in[0,1]} D\left(B_m^{(\mathcal{F})}\tilde{h}\left(\frac{k}{m}\right),\tilde{h}(x)\right)\\
&= \sup_{x\in[0,1]} D\left(\sum_{k=0}^{m}{}^{*} p_{m,k}(x)\odot\tilde{h}\left(\frac{k}{m}\right),\sum_{k=0}^{m}{}^{*} p_{m,k}(x)\odot\tilde{h}(x)\right)\\
&\leqslant \sup_{x\in[0,1]}\sum_{k=0}^{m} p_{m,k}(x)\, D\left(\tilde{h}\left(\frac{k}{m}\right),\tilde{h}(x)\right)
\end{aligned}$$

Now,

$$D\left(\tilde{h}\left(\frac{k}{m}\right),\tilde{h}(x)\right)=\sup_{r\in[0,1]}\max\left\{\left|\tilde{h}\left(\frac{k}{m}\right)_{-}^{(\ell)}-\tilde{h}(x)_{-}^{(\ell)}\right|,\left|\tilde{h}\left(\frac{k}{m}\right)_{+}^{(\ell)}-\tilde{h}(x)_{+}^{(\ell)}\right|\right\}$$

From Representation theorem/ Characteristic theorem,

$$\tilde{h}\left(\frac{k}{m}\right)_{\pm}^{(\ell)}=\tilde{h}(x)_{\pm}^{(\ell)}+\left(\frac{k}{m}-x\right)\tilde{h}'(x)_{\pm}^{(\ell)}+\frac{1}{2!}\left(\frac{k}{m}-x\right)^2\tilde{h}''(x)_{\pm}^{(\ell)}+\frac{1}{2!}\sum_{x}^{k/m}\left(\frac{k}{m}-x\right)^2\tilde{h}'''(x)_{\pm}^{(\ell)}$$

Thus,

$$\begin{aligned}
\left[\tilde{h}\left(\frac{k}{m}\right)\right]^{\ell} &= \left[\tilde{h}(x)\right]^{\ell}+\left(\frac{k}{m}-x\right)\left[\tilde{h}'(x)\right]^{\ell}\\
&\quad+\frac{1}{2!}\left(\frac{k}{m}-x\right)^2\left[\tilde{h}''(x)\right]^{\ell}+\frac{1}{2!}\left[\sum_{x}^{k/m}{}^{*}\left(\frac{k}{m}-x\right)^2\odot\tilde{h}'''(x)\right]^{\ell}
\end{aligned}$$

$$\begin{aligned}
&\Rightarrow \left[\tilde{h}\left(\frac{k}{m}\right)\right]^{\ell}-\left[\tilde{h}(x)\right]^{\ell}\\
&\quad=\left(\frac{k}{m}-x\right)\left[\tilde{h}'(x)\right]^{\ell}+\frac{1}{2!}\left(\frac{k}{m}-x\right)^2\left[\tilde{h}''(x)\right]^{\ell}+\frac{1}{2!}\left[\sum_{x}^{k/m}{}^{*}\left(\frac{k}{m}-x\right)^2\odot\tilde{h}'''(x)\right]^{\ell}\\
&\Rightarrow \left[\tilde{h}\left(\frac{k}{m}\right)_{-}^{(\ell)},\tilde{h}\left(\frac{k}{m}\right)_{+}^{(\ell)}\right]-\left[\tilde{h}(x)_{-}^{(\ell)},\tilde{h}(x)_{+}^{(\ell)}\right]\\
&\quad=\left(\frac{k}{m}-x\right)\left[\tilde{h}'(x)_{-}^{(\ell)},\tilde{h}'(x)_{+}^{(\ell)}\right]+\frac{1}{2!}\left(\frac{k}{m}-x\right)^2\left[\tilde{h}''(x)_{-}^{(\ell)},\tilde{h}''(x)_{+}^{(\ell)}\right]\\
&\quad+\frac{1}{2!}\left[\sum_{x}^{k/m}\left(\frac{k}{m}-x\right)^2\tilde{h}'''(x)_{-}^{(\ell)},\sum_{x}^{k/m}\left(\frac{k}{m}-x\right)^2\tilde{h}'''(x)_{+}^{(\ell)}\right]\\
&\Rightarrow \left[\tilde{h}\left(\frac{k}{m}\right)_{-}^{(\ell)}-\tilde{h}(x)_{-}^{(\ell)},\tilde{h}\left(\frac{k}{m}\right)_{+}^{(\ell)}-\tilde{h}(x)_{+}^{(\ell)}\right]\\
&\quad=\Bigg[\left(\frac{k}{m}-x\right)\tilde{h}'(x)_{-}^{(\ell)}+\frac{1}{2!}\left(\frac{k}{m}-x\right)^2\tilde{h}''(x)_{-}^{(\ell)}+\frac{1}{2!}\sum_{x}^{k/m}\left(\frac{k}{m}-x\right)^2\tilde{h}'''(x)_{-}^{(\ell)},\\
&\quad\left(\frac{k}{m}-x\right)\tilde{h}'(x)_{+}^{(\ell)}+\frac{1}{2!}\left(\frac{k}{m}-x\right)^2\tilde{h}''(x)_{+}^{(\ell)}+\frac{1}{2!}\sum_{x}^{k/m}\left(\frac{k}{m}-x\right)^2\tilde{h}'''(x)_{+}^{(\ell)}\Bigg]\\
&=[A,B]
\end{aligned}$$

Thus, we have,

$$\begin{aligned} mD^{*}\left(B_{m}^{(\mathcal{F})}\tilde{h}\left(\frac{k}{m}\right),\tilde{h}\left(x\right)\right) &= \sup_{x\in[0,1]} mD\left(\sum_{k=0}^{m}{}^{*}p_{m,k}\left(x\right)\odot\tilde{h}\left(\frac{k}{m}\right),\sum_{k=0}^{m}{}^{*}p_{m,k}\left(x\right)\odot\tilde{h}\left(x\right)\right) \\ &\leqslant \sup_{x\in[0,1]} m\sum_{k=0}^{m}p_{m,k}\left(x\right)D\left(\tilde{h}\left(\frac{k}{m}\right),\tilde{h}\left(x\right)\right) \\ &= \sup_{x\in[0,1]} m\sum_{k=0}^{m}p_{m,k}\left(x\right)\sup_{\ell\in[0,1]}\max\left\{A,B\right\} \\ &= \sup_{x\in[0,1]}\sup_{\ell\in[0,1]}\max\left\{m\sum_{k=0}^{m}Ap_{m,k}(x),n\sum_{k=0}^{m}Bp_{m,k}(x)\right\} \end{aligned}$$

Now,

$$\begin{aligned} &m\sum_{k=0}^{m}p_{m,k}(x)\left[\left(\frac{k}{m}-x\right)\tilde{h}'\left(x\right)_{-}^{(\ell)}+\frac{1}{2!}\left(\frac{k}{m}-x\right)^{2}\tilde{h}''\left(x\right)_{-}^{(\ell)}+\frac{1}{2!}\sum_{x}^{k/m}\left(\frac{k}{m}-x\right)^{2}\tilde{h}'''\left(x\right)_{-}^{(\ell)}\right] \\ &=\sum_{k=0}^{m}m\left(\frac{k}{m}-x\right)p_{m,k}(x)\tilde{h}'\left(x\right)_{-}^{(\ell)}+\frac{1}{2!}\sum_{k=0}^{m}m\left(\frac{k}{m}-x\right)^{2}p_{m,k}(x)\tilde{h}''\left(x\right)_{-}^{(\ell)}+mR_{m}\left(x\right) \\ &=\frac{1}{2}x\left(1-x\right)\tilde{h}''\left(x\right)_{-}^{(\ell)} \end{aligned}$$

Similarly,

$$\begin{aligned} &m\sum_{k=0}^{m}p_{m,k}(x)\left[\left(\frac{k}{m}-x\right)\tilde{h}'\left(x\right)_{+}^{(\ell)}+\frac{1}{2!}\left(\frac{k}{m}-x\right)^{2}\tilde{h}''\left(x\right)_{+}^{(\ell)}+\frac{1}{2!}\sum_{x}^{k/m}\left(\frac{k}{m}-x\right)^{2}\tilde{h}'''\left(x\right)_{+}^{(\ell)}\right] \\ &=\frac{1}{2}x\left(1-x\right)\tilde{h}''\left(x\right)_{+}^{(\ell)} \end{aligned}$$

That is,

$$\begin{aligned} &\lim_{m\to\infty} mD^{*}\left(B_{m}^{(\mathcal{F})}\tilde{h}\left(\frac{k}{m}\right),\tilde{h}\left(x\right)\right) \\ &\quad\leqslant \sup_{x\in[0,1]}\sup_{\ell\in[0,1]}\max\left\{\frac{1}{2}x\left(1-x\right)\tilde{h}''\left(x\right)_{-}^{(\ell)},\frac{1}{2}x\left(1-x\right)\tilde{h}''\left(x\right)_{+}^{(\ell)}\right\} \\ &\quad= \sup_{x\in[0,1]}\frac{1}{2}x\left(1-x\right)\sup_{\ell\in[0,1]}\max\left\{\tilde{h}''\left(x\right)_{-}^{(\ell)},\tilde{h}''\left(x\right)_{+}^{(\ell)}\right\} \\ &\quad= \sup_{x\in[0,1]}\frac{1}{2}x\left(1-x\right)D\left(\tilde{h}''\left(x\right),\tilde{o}\right) \\ &\quad\leqslant \frac{1}{2}x\left(1-x\right)D^{*}\left(\tilde{h}''\left(x\right),\tilde{o}\right) \end{aligned}$$

where $\tilde{o}$ is the zero element of $\mathbb{R}_{\mathcal{F}}$. □

6. Fuzzy Szász-Mirakyan Operators

The fuzzy Szász-Mirakyan operators are defined as,

$$\left(S_{m}^{(\mathcal{F})}\tilde{h}\right)(x)=e^{-mx}\sum_{k=0}^{\infty}{}^{*}\frac{(mx)^{k}}{k!}\odot\tilde{h}\left(\frac{k}{m}\right),$$

where, $x \in [0, \infty)$ and $\tilde{h}$ is a fuzzy valued function from $\mathbb{R}^+$ to $\mathbb{R}_F^+$.

From theorems 2.1 and 3.1, we can say that, there exists a corresponding sequence $S_m(h;x)$ of linear positive operators from $C[0,\infty)$ into itself with property,

$$\left(S_m^{(\mathcal{F})}\left(\tilde{h}\right)\right)_{\pm}^{(\ell)} = S_m\left(\tilde{h}_{\pm}^{(\ell)}\right)$$

respectively, $\forall \ell \in [0,1]$ and $\forall \tilde{h} \in C_{\mathcal{F}}[0,\infty)$, where, $S_m(h;x)$ are the classical known Szász-Mirakyan operators.

By fuzzy Korovkin theorem, we can say that,

$$D^*\left(S_m^{(\mathcal{F})}\left(\tilde{h}\right), \tilde{h}\right) \to 0, \quad \text{as } m \to \infty.$$

Lemma 6.1 *The moment generating function of the Szász-Mirakyan operators is,*

$$S_m\left(e^{-\alpha t};x\right) = e^{-\alpha x \frac{1-e^{-\alpha/m}}{\alpha/m}}. \tag{6.1}$$

Based on (6.1), we can claim the following exponential moments of Szász operators:

(i) $S_m(1;x) = 1$

(ii) $S_m(e^{-t};x) = e^{-mx\left(1-e^{-1/m}\right)} \to e^{-x}$

(iii) $S_m\left(e^{-2t};x\right) = e^{-mx\left(1-e^{-2/m}\right)} \to e^{-2x}$

Since, $\left\{1, e^{-x}, e^{-2x}\right\}$ is a Korovkin set, the above lemma also confirms the Korovkin theorem. By simple calculation we can show that

$$\left|S_m\left(e^{-\alpha t};x\right) - e^{-\alpha x}\right| = e^{-\alpha x}\left|\left(e^{-x}\right)^{m-\alpha-me^{-\alpha/m}} - 1\right| \tag{6.2}$$

Define a_m, b_m and c_m as the central moments corresponding to the Korovkin set $\left\{1, e^{-x}, e^{-2x}\right\}$ obtained by putting $\alpha = 0, 1, 2$, respectively in (6.2). Then, it is easy to verify that $a_m, b_m, c_m \to 0$ as $m \to \infty$.

Theorem 6.1 *If $f \in C_F[0,\infty)$, then for $m \geqslant 1$,*

$$D^*\left(S_m^{(\mathcal{F})}\left(\tilde{h}\right), \tilde{h}(x)\right) \leqslant 2\omega_*^{\mathcal{F}}\left(\tilde{h}; \frac{1}{\sqrt{m}}\right)$$

where, $\omega_*^{\mathcal{F}}$ is the fuzzy exponential type modulus of continuity.

Proof: Let, $s_{m,k}(x) = e^{-mx}\frac{(mx)^k}{k!} \Rightarrow \sum\limits_{k=0}^{\infty} s_{m,k}(x) = 1$. We can write,

$$\tilde{h}(x) = \left[\sum_{k=0}^{\infty} s_{m,k}(x)\right] \odot \tilde{h}(x) = \sum_{k=0}^{\infty}{}^{*}\left(s_{m,k}(x) \odot \tilde{h}(x)\right).$$

Thus,

$$\begin{aligned}
D^*\left(S_m^{(\mathcal{F})}\left(\tilde{h}\right), \tilde{h}\left(x\right)\right) &= D^*\left(\sum_{k=0}^{\infty}{}^* s_{m,k}\left(x\right) \odot \tilde{h}\left(\frac{k}{m}\right), \sum_{k=0}^{\infty}{}^* s_{m,k}\left(x\right) \odot \tilde{h}\left(x\right)\right) \\
&\leqslant \sum_{k=0}^{\infty} s_{m,k}\left(x\right) D^*\left(\tilde{h}\left(\frac{k}{m}\right), \tilde{h}\left(x\right)\right) \\
&\leqslant \sum_{k=0}^{\infty} s_{m,k}\left(x\right)\left[1 + \frac{\left(e^{-k/m} - e^{-x}\right)^2}{\delta^2}\right]\omega_*^{\mathcal{F}}\left(\tilde{h};\delta\right) \\
&= \omega_*^{\mathcal{F}}\left(\tilde{h};\delta\right) + \sum_{k=0}^{\infty} s_{m,k}\left(x\right)\frac{\left(e^{-k/m} - e^{-x}\right)^2}{\delta^2}\omega_*^{\mathcal{F}}\left(\tilde{h};\delta\right) \\
&\leqslant \omega_*^{\mathcal{F}}\left(\tilde{h};\delta\right) + \frac{a_m + 2b_m + c_m}{\delta^2}\omega_*^{\mathcal{F}}\left(\tilde{h};\delta\right)
\end{aligned}$$

Now, for every $m \geqslant 1$,

$$e^{-x\gamma_m} - e^{-x} < \frac{x_m}{2e}$$

where, $\gamma_m = \frac{1-e^{-x_m}}{x_m}$ and $x_m > 0$, for every $m \geqslant 1$. Hence, we get,

$$\left|S_m\left(e^{-\alpha t}; x\right) - e^{-\alpha x}\right| \leqslant \frac{\alpha}{2em}$$

It follows that,

$$b_m \leqslant \frac{1}{2em} \text{ and } c_m \leqslant \frac{1}{em}, \text{ for } m \geqslant 1,$$

which implies that,

$$a_m + 2b_m + c_m \leqslant \frac{1}{2em} + \frac{1}{em} \leqslant \frac{1}{m}, \text{ for } m \geqslant 1.$$

Taking $\delta = 1/\sqrt{m}$, we arrive at our result. □

Theorem 6.2 *Let $\tilde{h}(x)$ is a continuous, bounded and twice differential fuzzy function with membership value for each $\tilde{h}(x)$ as $\mu_h(x)$, defined for all $x \in [0, \infty)$. Then, for the fuzzy Szász-Mirakyan operators, we have the following Voronskya type asymptotic result:*

$$\lim_{m\to\infty} mD^*\left(S_m^{(\mathcal{F})}\left(\tilde{h}\right), \tilde{h}\left(x\right)\right) \leqslant \frac{1}{2}xD^*\left(\tilde{h}''\left(x\right), \tilde{o}\right)$$

where $\tilde{o}$ is the zero element of $\mathbb{R}_{\mathcal{F}}$.

Proof: Following the proof of theorem 5.1, we arrive at,

$$\begin{aligned}
&m\sum_{k=0}^{\infty} s_{m,k}\left(x\right)\left[\left(\frac{k}{m} - x\right)\tilde{h}'\left(x\right)_{\pm}^{(\ell)} + \frac{1}{2!}\left(\frac{k}{m} - x\right)^2\tilde{h}''\left(x\right)_{\pm}^{(\ell)} + R_m\left(x\right)\right] \\
&= \sum_{k=0}^{\infty} m\left(\frac{k}{m} - x\right)s_{m,k}\left(x\right)\tilde{h}'\left(x\right)_{\pm}^{(\ell)} + \frac{1}{2}\sum_{k=0}^{\infty} m\left(\frac{k}{m} - x\right)^2 s_{m,k}\left(x\right)\tilde{h}''\left(x\right)_{\pm}^{(\ell)} + mR_m\left(x\right) \\
&= \frac{1}{2}x\tilde{h}''\left(x\right)_{\pm}^{(\ell)}
\end{aligned}$$

Thus,

$$\begin{aligned}
\lim_{m\to\infty} mD^*\left(S_m^{(\mathcal{F})}\tilde{h}\left(\frac{k}{m}\right), \tilde{h}\left(x\right)\right) &= \sup_{x\in[0,1]}\sup_{r\in[0,1]}\max\left\{\frac{1}{2}x\tilde{h}''\left(x\right)_{-}^{(\ell)}, \frac{1}{2}x\tilde{h}''\left(x\right)_{+}^{(\ell)}\right\} \\
&= \sup_{x\in[0,1]}\frac{1}{2}xD\left(\tilde{h}''\left(x\right), \tilde{o}\right) \\
&\leqslant \frac{1}{2}xD^*\left(\tilde{h}''\left(x\right), \tilde{o}\right)
\end{aligned}$$

where $\tilde{o}$ is the zero element of $\mathbb{R}_{\mathcal{F}}$. □

7. Fuzzy Baskakov Operators

$$V_m^{(\mathcal{F})}\left(\tilde{h}\left(t\right);x\right)=\sum_{k=0}^{\infty}{}^{*}\left(\begin{array}{c}m+k-1\\k\end{array}\right)\frac{x^k}{\left(1+x\right)^{m+k}}\odot\tilde{h}\left(\frac{k}{m}\right)$$

Again, using theorems 2.1 and 3.1 we can claim that, there exists a corresponding sequence $V_m\left(h;x\right)$ of linear positive operators from $C[0,\infty)$ into itself with property,

$$\left(V_m^{(\mathcal{F})}\left(\tilde{h}\right)\right)_{\pm}^{(\ell)}=V_m\left(\tilde{h}_{\pm}^{(\ell)}\right)$$

respectively, $\forall\ell\in[0,1]$ and $\forall\tilde{h}\in C_{\mathcal{F}}[0,\infty)$, where, $V_m\left(h;x\right)$ are the classical known Baskakov operators.

By fuzzy Korovkin theorem, we can say that,

$$D^{*}\left(V_m^{(\mathcal{F})}\left(\tilde{h}\right),\tilde{h}\right)\to 0,\quad\text{as } m\to\infty.$$

We know the moment generating function of the classical Baskakov operators is:

$$V_m\left(e^{-\alpha t};x\right)=\left(1+x\left(1-e^{-\alpha/m}\right)\right)^{-m}. \tag{7.1}$$

Lemma 7.1 *Using (7.1), we can claim the following exponential moments of Baskakov operators:*

(i) $V_m\left(1;x\right)=1$

(ii) $V_m\left(e^{-t};x\right)=\left(1+x-xe^{-1/m}\right)^{-m}\to e^{-x}$

(iii) $V_m\left(e^{-2t};x\right)=\left(1+x-xe^{-2/m}\right)^{-m}\to e^{-2x}$

Since, $\left\{1,e^{-x},e^{-2x}\right\}$ is a Korovkin set, the above lemma also confirms the Korovkin theorem. By simple calculation we can show that

$$\left|V_m\left(e^{-\alpha t};x\right)-e^{-\alpha x}\right|=e^{-\alpha x}\left|e^{-m\ln\left(1+x\left(1-e^{-\alpha/m}\right)\right)+\alpha x}-1\right| \tag{7.2}$$

Define a_m,b_m and c_m as the expressions obtained by putting $\alpha=0,1,2$, respectively in (7.2). Clearly, $a_m,b_m,c_m\to 0$ as $m\to\infty$.

Theorem 7.1 *Let* $\tilde{h}\in C_{\mathcal{F}}[0,\infty)$. *Then, for* $m\geqslant 2$,

$$D^{*}\left(V_m^{(\mathcal{F})}\left(\tilde{h}\right),\tilde{h}\right)\leqslant 2\omega_{*}^{\mathcal{F}}\left(\tilde{h};\frac{5}{2\sqrt{m}}\right).$$

where, $\omega_{*}^{\mathcal{F}}$ *is the fuzzy exponential type modulus of continuity.*

Proof: Let $v_{m,k}\left(x\right)=\left(\begin{array}{c}m+k-1\\k\end{array}\right)\frac{x^k}{(1+x)^{m+k}}$.

We arrive at,

$$D^{*}\left(V_m^{(\mathcal{F})}\tilde{h}\left(\frac{k}{m}\right),\tilde{h}\left(x\right)\right)\leqslant\omega_{*}^{\mathcal{F}}\left(\tilde{h};\delta\right)+\frac{a_m+2b_m+c_m}{\delta^2}\omega_{*}^{\mathcal{F}}\left(\tilde{h};\delta\right) \tag{7.3}$$

Now, using the inequality: $e^t-1\leqslant e^t$, for

$$t=-m\ln\left(1+x\left(1-e^{-\alpha/m}\right)\right)+\alpha x\geqslant -mx\left(1-e^{-\alpha/m}\right)+\alpha x\geqslant -mx\frac{\alpha}{m}+\alpha x=0.$$

$$\left|V_m\left(e^{-\alpha t};x\right)-e^{-\alpha x}\right| = e^{-\alpha x}\left|e^{-m\ln\left(1+x\left(1-e^{-\alpha/m}\right)\right)+\alpha x}-1\right|$$
$$\leqslant \left[-m\ln\left(1+x\left(1-e^{-\alpha/m}\right)\right)+\alpha x\right]e^{-m\ln\left(1+x\left(1-e^{-\alpha/m}\right)\right)}$$

Because $\ln(1+t)\geqslant \frac{t}{1+t}$, for every $t\geqslant 0$, we obtain

$$\left|V_m\left(e^{-\alpha t};x\right)-e^{-\alpha x}\right| \leqslant \frac{-mx\left(1-e^{-\alpha/m}\right)+\alpha x+\alpha x^2\left(1-e^{-\alpha/m}\right)}{\left(1+x\left(1-e^{-\alpha/m}\right)\right)^{m+1}}$$
$$\leqslant \frac{-mx\left(1-e^{-\alpha/m}\right)+\alpha x+\alpha x^2\left(1-e^{-\alpha/m}\right)}{1+(m+1)x\left(1-e^{-\alpha/m}\right)+\frac{m(m+1)}{2}x^2\left(1-e^{-\alpha/m}\right)^2}.$$

Because $1-e^{-\alpha/m}\geqslant \frac{\alpha}{m}-\frac{\alpha^2}{2m^2}$, we get from the above inequality

$$\sup_{x\geqslant 0}\left|V_m\left(e^{-\alpha t};x\right)-e^{-\alpha x}\right| \leqslant \frac{2\alpha}{m(m+1)\left(1-e^{-\alpha/m}\right)}.$$

Using the same inequality, we obtain

$$b_m=\sup_{x\geqslant 0}\left|V_m\left(e^{-t};x\right)-e^{-x}\right| \leqslant \frac{2}{m(m+1)\left(\frac{1}{m}-\frac{1}{2m^2}\right)}\leqslant \frac{2}{m},\quad \text{for } m\geqslant 1$$

and using $1-e^{-2/m}\geqslant \frac{2}{m}-\frac{2}{m^2}+\frac{4}{3m^3}-\frac{2}{3m^4}$, we have

$$c_m=\sup_{x\geqslant 0}\left|V_m\left(e^{-2t};x\right)-e^{-2x}\right| \leqslant \frac{4}{m(m+1)\left(\frac{2}{m}-\frac{2}{m^2}+\frac{4}{3m^3}-\frac{2}{3m^4}\right)}=\frac{\zeta(m)}{m},$$

where, $\zeta(t)=\frac{6t^4}{(t+1)(3t^3-3t^2+2t-1)}$

$$\Rightarrow \zeta'(t)=\frac{6t^3}{(t+1)^2(3t^3-3t^2+2t-1)}\left(-2t^2+3t-4\right)<0,\quad t\geqslant 1,$$

we obtain $\zeta(m)\leqslant \frac{32}{15}$, for $m\geqslant 2$. Finally, we obtain

$$\sqrt{a_m+2b_m+c_m}\leqslant \frac{1}{\sqrt{m}}\sqrt{4+\frac{32}{15}}\leqslant \frac{5}{2\sqrt{m}}.$$

Taking $\delta=\sqrt{a_m+2b_m+c_m}\leqslant \frac{5}{2\sqrt{m}}$, then for $m\geqslant 2$, (7.3) becomes

$$D^*\left(V_m^{(\mathcal{F})}\left(\tilde{h}\right),\tilde{h}\right)\leqslant 2\omega_*^{\mathcal{F}}\left(\tilde{h};\frac{5}{2\sqrt{m}}\right).$$

□

Theorem 7.2 *For the fuzzy Baskakov operators and continuous fuzzy valued function f, whose ℓ^{th} level cut has continuous double derivative $\forall \ell\in[0,1]$,*

$$\lim_{m\to\infty} mD^*\left(V_m^{(\mathcal{F})}\left(\tilde{h}\right),\tilde{h}(x)\right)=\frac{1}{2}x(1+x)D^*\left(\tilde{h}''(x),\tilde{o}\right)$$

where $\tilde{o}$ is the zero element of $\mathbb{R}_{\mathcal{F}}$.

Proof: Following the proof of theorem 5.1, we get,

$$\begin{aligned}
&m\sum_{k=0}^{\infty} v_{m,k}\left(x\right)\left[\left(\frac{k}{m}-x\right)\tilde{h}'\left(x\right)_{\pm}^{(\ell)}+\frac{1}{2!}\left(\frac{k}{m}-x\right)^{2}\tilde{h}''\left(x\right)_{\pm}^{(\ell)}+R_m\left(x\right)\right]\\
&=\sum_{k=0}^{\infty} m\left(\frac{k}{m}-x\right)v_{m,k}\left(x\right)\tilde{h}'\left(x\right)_{\pm}^{(\ell)}+\frac{1}{2}\sum_{k=0}^{\infty} m\left(\frac{k}{m}-x\right)^{2}v_{m,k}\left(x\right)\tilde{h}''\left(x\right)_{\pm}^{(\ell)}+mR_m\left(x\right)\\
&=\frac{1}{2}x\left(1+x\right)\tilde{h}''\left(x\right)_{\pm}^{(\ell)}
\end{aligned}$$

Thus, we get,

$$\begin{aligned}
&\lim_{m\to\infty} mD^{*}\left(V_m^{(\mathcal{F})}\tilde{h}\left(\frac{k}{m}\right),\tilde{h}\left(x\right)\right)\\
&\qquad=\sup_{x\in[0,1]}\sup_{r\in[0,1]}\max\left\{\frac{1}{2}x\left(1+x\right)\tilde{h}''\left(x\right)_{-}^{(\ell)},\frac{1}{2}x\left(1+x\right)\tilde{h}''\left(x\right)_{+}^{(\ell)}\right\}\\
&\qquad=\sup_{x\in[0,1]}\frac{1}{2}x\left(1+x\right)D\left(\tilde{h}''\left(x\right),\tilde{o}\right)\\
&\qquad\leqslant\frac{1}{2}x\left(1+x\right)D^{*}\left(\tilde{h}''\left(x\right),\tilde{o}\right)
\end{aligned}$$

where $\tilde{o}$ is the zero element of $\mathbb{R}_{\mathcal{F}}$. □

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Declaration

Authors' contributions

Kanita was involved in conceptualization, methodology and in the original draft preparation. Naokant Deo contributed to review editing, investigation and supervision. All authors read and agreed to the final manuscript.

Data availability

Data sharing not applicable to this article as no datasets were generated or analyzed during the current study.

Ethical Approval

Not applicable.

Conflict of interest

There are no competing interests. The authors have no relevant competing interests to disclose, either financial or otherwise.

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