



## Darboux integrability of a new four-dimensional hyper-chaotic system

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**ABSTRACT:** This study addresses the Darboux integrability of a newly developed four-dimensional hyper-chaotic system. Systems of this type are of considerable interest in the study of nonlinear dynamics due to their high-dimensional complexity and potential applications in secure communications, control theory, and cryptography. In this work, the proposed system is examined from the perspective of Darboux theory of integrability to identify its algebraic and analytic structures. The analysis indicates that the system lacks invariant algebraic surfaces and has only one Darboux element, which is an exponential factor. These results demonstrate that the system is non-integrable in the Darboux sense. Furthermore, it is demonstrated that under certain parameter constraints, the system fails to admit polynomial or rational first integrals.

**Keywords:** Darboux first integrals, Polynomial first integrals, Invariant algebraic surfaces, exponential factors.

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### 1. Introduction

In physics and applied mathematics, nonlinear differential systems are widely used. Depending on whether the differential systems or vector field described in  $\mathbb{R}^n$  or  $\mathbb{C}^n$ , possesses  $n - 1$  first integrals, it may be possible to reduce the analysis to a one-dimensional differential equation by utilizing real or complex time [1,2]. This investigation examines the existence of Darboux first integrals for polynomial vector fields in  $\mathbb{R}^4$  or  $\mathbb{C}^4$ , focusing on polynomial first integrals. Darboux's idea of integrability, which was proposed in 1878 [3], is a great way to find Darboux theory of integrability shows how the number of invariant algebraic surfaces (Darboux polynomials.) is related to the integrability of polynomial vector fields. This notion has been effective in several physical models [4,5,6,7]. In 1979, O.E.Rössler created a 4D hyperchaotic system [8]. In this work, the following new four-dimensional (4D) hyperchaotic system is considered (see [9] for more details) and examine whether its first integrals can be found

$$\dot{x} = r_1x - yz + w, \quad \dot{y} = xz - r_2y, \quad \dot{z} = xy - r_3z, \quad \dot{w} = -y + r_4, \quad (1.1)$$

where  $r_1, r_2, r_3$  and  $r_4$  are positive parameters, and  $x, y, z$  and  $w$  are state variables. The sign  $\dot{\cdot} = \frac{d}{dt}$  is the derivative of time  $t$ . This four-dimensional system shows hyperchaotic behavior with several positive Lyapunov exponents, in contrast to typical three-dimensional chaotic systems that are restricted to a single positive Lyapunov exponent. The dynamics' algebraic complexity and instability are greatly increased by this extra dimension, making the system stronger for engineering applications. The proposed 4D hyperchaotic system, represented in equations (1.1), investigated to determine its complicated dynamical behaviors generated hyperchaotic attractors with acceptable parameter selections ( $a = 8, b = 40, c = 15, d = -0.1$ ). Moreover, periodic, and quasi-periodic behaviors which based on parameter modifications, as well as multistability and offset boosting and phase portraiture were used to investigate its dynamical properties. Furthermore, the system was used for voice encryption, and MATLAB simulations demonstrated its robustness and effectiveness against cryptographic attacks. For

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2020 *Mathematics Subject Classification:* 34C28, 37K10, 34K23.

Submitted December 19, 2025. Published April 11, 2026

specific parameter values, the complex dynamics of the system (1.1) exhibit non-integrability as revealed by Lyapunov spectra. Other parameter sets, however, may result in very different behaviors; some may even correspond to the first integral instances. In order to identify potentially integrable scenarios, a more methodical approach is needed, as a thorough numerical analysis of all parameter values is not feasible. Here, we will investigate the conditions under which system (1.1) may possess first integrals using the Darboux method. We mean that we apply the Darboux technique to system (1.1), which relies on the presence of invariant algebraic surface (Darboux polynomials) and exponential factors. This procedure be necessary to obtain the first integral (FI), to further information, see the references [10,11,12,13]. They used Darboux polynomials and exponential factors to identify first integrals. A first integral is beneficial in both mathematics and theoretical physics, particularly in the analysis of dynamic systems. Moreover, the integrability is a standard issue in the field of differential systems, since the existence of a first integral (FI) may simplify the complexity of the system. Nonetheless, it may be challenging to verify the existence of the first integrals. So, for differential systems, the first step using the Darboux theorem of integrability is to find the IAS and the exponential factor (EF). When a first integral of the hyperchaotic system is identified, it can be used to reduce the dimension dimension by one. As a result, numerous studies have been conducted to investigate the existence of first integrals in hyperchaotic systems, such as [14,15]. Moreover, several studies have investigated the integrability of high-dimensional hyperchaotic systems, such as the 4D hyperchaotic Lorenz [16], Chen [17], and Rössler systems [18], by employing methods such as Lie symmetry analysis, Darboux integrability, Painlevé analysis, and Lyapunov exponent computation. Using  $\chi$ , the following vector field can be obtained from system (1.1):

$$\chi = (r_1x - yz + w) \frac{\partial}{\partial x} + (xz - r_2y) \frac{\partial}{\partial y} + (xy - r_3z) \frac{\partial}{\partial z} + (-y + r_4) \frac{\partial}{\partial w}. \quad (1.2)$$

The largest power of the terms in  $\chi$  is two, hence  $\deg(\chi) = 2$ . A nonconstant function  $f(x, y, z, w) : V \subset \mathbb{R}^4 \rightarrow \mathbb{R}$  is called the first integral of  $\chi$  on the domain  $V$  ([19]), if the quantity  $f(x(t), y(t), z(t), w(t))$  remains constant  $\forall t \in \mathbb{R}$ , where  $(x(t), y(t), z(t), w(t))$  is a solution trajectory of the  $\chi$  determined on  $V$ . In other words,  $f$  is a FI of system (1.1) on  $V$  if and only if the following condition is fulfilled

$$\chi(f) = (r_1x - yz + w) \frac{\partial f}{\partial x} + (xz - r_2y) \frac{\partial f}{\partial y} + (xy - r_3z) \frac{\partial f}{\partial z} + (-y + r_4) \frac{\partial f}{\partial w} = 0. \quad (1.3)$$

This study looks into the parameter values at which system (1.1) accepts a first integral, with the key findings shown below.

**Theorem 1.1.** System (1.1) does not have invariant algebraic surfaces with non-zero cofactors.

**Theorem 1.2.** System (1.1) does not contain any polynomial first integrals.

**Theorem 1.3.** System (1.1) does not have any first integrals of the Darboux type.

The proofs of the above theorems are provided in Section 2.

We now present some definitions that facilitate the proof that the primary findings of this study are necessary for the proposed method to examine the integrability of the differential system (1.1).

Let  $F = F(x, y, z, w) \in \mathbb{C}[x, y, z, w]$ . We define  $F$  is an invariant algebraic surface (also called a Darboux polynomial) of system (1.1) if there is a polynomial  $K_F \in \mathbb{C}[x, y, z, w]$  satisfy

$$(r_1x - yz + w) \frac{\partial F}{\partial x} + (xz - r_2y) \frac{\partial F}{\partial y} + (xy - r_3z) \frac{\partial F}{\partial z} + (-y + r_4) \frac{\partial F}{\partial w} = K_F F. \quad (1.4)$$

The polynomial  $K_F$  is referred to as a cofactor of  $F$ , and its degree does not exceed one (see [19]).

Let  $E = e^{\frac{g}{h}}$ , where  $g, h \in \mathbb{C}[x, y, z, w]$ ,  $h \neq 0$ , and  $\gcd(g, h) = 1$ , We say that  $E$  is an exponential factor of system (1.1), if there is a polynomial  $K_E \in \mathbb{C}[x, y, z, w]$ , satisfies the following condition

$$(r_1x - yz + w) \frac{\partial E}{\partial x} + (xz - r_2y) \frac{\partial E}{\partial y} + (xy - r_3z) \frac{\partial E}{\partial z} + (-y + r_4) \frac{\partial E}{\partial w} = K_E E, \quad (1.5)$$

for polynomial  $K_E$  with a maximum degree of one (see [20]).

A polynomial  $f(x, y, z, w)$  defined on  $\mathbb{R}^4$  is defined as weight homogenous if there exist  $\mu = (\mu_1, \mu_2, \mu_3, \mu_4) \in \mathbb{N}^4$  and  $n \in \mathbb{N}$  such that for each  $\tau > 0$ ,  $f(\tau^{\mu_1}x, \tau^{\mu_2}y, \tau^{\mu_3}z, \tau^{\mu_4}w) = \tau^n f(x, y, z, w)$ , the  $\mu$  and  $n$  represent the weight exponent and the weight degree of  $f$ , respectively; see ([21]).

We also provide an analytic FI for system (1.1). The following theorem is associated with the existence of an analytic FI. Using an analytic first integral, we refer to the FI of the system, which is an analytical function.

**Theorem 1.4.** If  $r_2 = 0$  and  $r_1 \neq r_3$ , then system (1.1) lacks an analytic first integral in the neighbourhood of its singular points.

The proof of Theorem 1.4 is presented in Section 3.

The next outcome can be used to determine the Darboux FI, as shown the studies in [19,20].

**Theorem 1.5.** (Darboux Theorem [22]). If system (1.1) in  $\mathbb{R}^4$  admits  $p$  irreducible invariant surfaces  $h_i = 0$  with cofactors  $K_{h_1}, \dots, K_{h_p}$ , and  $q$  exponential factors  $E_j = e^{\frac{g_j}{h_j}}$ , with cofactors  $K_{E_1}, \dots, K_{E_q}$ . For  $\lambda_i, \mu_j \in \mathbb{C}$  that are not all zero, which satisfies

$$\sum_{i=1}^p \lambda_i K_{h_i} + \sum_{j=1}^q \mu_j K_{E_j} = 0. \quad (1.6)$$

Then, the FI of system (1.1) is the following:

$$F = \prod_{i=1}^p h_i^{\lambda_i} \left( \prod_{j=1}^q (E_j)^{\mu_j} \right). \quad (1.7)$$

The function specified in Equation (1.7) is referred to as the generalised Darboux first integral.

The next proposition is generalised from [21], which is required throughout this work.

**proposition 1.6.** The next two statements are correct.

- i. If  $E = e^{\frac{g}{h}}$  is an exponential factor and  $h$  is non-constant, then the condition  $h = 0$  denotes an IAS.
- ii. In conclusion,  $e^g$  can serve as an EF arising from the multiplicity of the infinity invariant plane.

In addition, we review certain results that will be used later. We first analyze an analytic differential system

$$\dot{\mathbf{x}} = f(\mathbf{x}), \quad (1.8)$$

where  $f : V \subset \mathbb{R}^n \rightarrow \mathbb{R}^n$  is a  $C^2$  differentiable function with the condition that  $f(\mathbf{0}) = \mathbf{0}$  and the Jacobian matrix at  $\mathbf{x} = \mathbf{0}$  is given by  $Df(\mathbf{0}) = \mathbf{A}$ . Let  $\lambda_1, \dots, \lambda_m$  be the corresponding eigenvalues of matrix  $\mathbf{A}$ .

According to Llibre and Zhang [19], the next two conditions are necessary for the existence of an analytic FI.

**Theorem 1.7.** Let the eigenvalues  $\lambda_2, \dots, \lambda_m$  of  $\mathbf{A}$  satisfy

$$\lambda_1 = 0, \text{ and } \sum_{l=2}^m s_l \lambda_l \neq 0 \text{ for any } s_2, \dots, s_m \in \mathbb{N}^*, \text{ with } \sum_{l=2}^m s_l \geq 1.$$

Then, for  $m > 2$ , system ((1.8)) at a neighborhood of the coordinates origin has a formal FI. if and only if the coordinates origin be non-isolated singular point. Particularly, if the coordinate origin is the isolated singular point, then system (1.8) lacks an analytic FI in the neighborhood of the coordinate origin.

The sections of this work are systematized as follows. Section 2 discusses the verification of the main results of this work. Section 3 discusses the existing of an analytic FI of system (1.1).

## 2. Mathematical Proofs of Theorems 1.1 and 1.2

In the following, we first prove the existence of Darboux polynomials of system (1.1).

**Proof:**

**Proof of Theorem 1.1.** Suppose that  $h = \sum_{i=0}^n h_i(x, y, z, w)$  is an IAS with the nonzero cofactor  $K = k_0 + k_1x + k_2y + k_3z + k_4w$  and  $h_n \neq 0$ , for  $n > 0$  where  $h \in \mathbb{C}[x, y, z, w]$ ,  $k_i \in \mathbb{C}$  for  $i = 0, \dots, 4$ , and the homogeneous terms  $h_i$  have a degree of  $i$ . Then,  $h$  satisfies

$$(r_1x - yz + w) \frac{\partial h}{\partial x} + (xz - r_2y) \frac{\partial h}{\partial y} + (xy - r_3z) \frac{\partial h}{\partial z} + (-y + r_4) \frac{\partial h}{\partial w} = Kh. \quad (2.1)$$

First, by calculating the terms of degree  $(n + 1)$  in (2.1), to acquire

$$-yz \frac{\partial h_n}{\partial x} + xz \frac{\partial h_n}{\partial y} + xy \frac{\partial h_n}{\partial z} = (k_1x + k_2y + k_3z + k_4w)h_n.$$

The result of the specified equation is

$$h_n(x, y, z, w) = \frac{H_1(x^2 + y^2, x^2 + z^2, w) (-(y - z)^2)^{-\frac{k_1}{2}}}{e^{k_3 \arctan(\frac{x}{y})} e^{k_2 \arctan(\frac{x}{z})} e^{\frac{1}{\sqrt{x^2 + y^2}} \text{EllipticF}\left(\frac{x}{\sqrt{x^2 + z^2}}, \frac{x^2 + z^2}{x^2 + y^2}\right)} w^{k_4}}.$$

Where  $H_1$  is an arbitrary polynomial function. Since  $h_n$  is a nonzero polynomial, then must be  $k_2 = k_3 = k_4 = 0$ .  $k_1 = -2m$  with  $m \in \mathbb{N} \cup \{0\}$ , and  $\text{EllipticF}\left(\frac{x}{\sqrt{x^2 + z^2}}, \frac{x^2 + z^2}{x^2 + y^2}\right) = \int_0^{\frac{x}{\sqrt{x^2 + z^2}}} \frac{1}{\sqrt{1 - \frac{x^2 + z^2}{x^2 + y^2} \sin^2 \theta}} d\theta$ .

This implies that  $h_n = (-1)^m H_n(x^2 + y^2, x^2 + z^2, w)(y - z)^{2m}$  and  $K = k_0 + k_1x$ . We utilise the change of variable weights,

$$x = X, \quad y = \tau Y, \quad z = \tau Z, \quad w = \tau W \quad \text{and} \quad t = \tau^2 T, \quad \text{where } T \in \mathbb{R} \setminus \{0\}.$$

subsequently, system (1.1) becomes

$$\dot{X} = r_1 X \tau^2 + W \tau - Y Z, \quad \dot{Y} = \tau^2 (X Z - r_2 Y), \quad \dot{Z} = \tau^2 (X Y - r_3 Z), \quad \dot{W} = -\tau^2 (-r_4 \tau + Y). \quad (2.2)$$

The dot denotes the derivative with respect to the variable  $T$ . We set

$$F(X, Y, Z, W) = \tau^n F(X, \tau^{-1} Y, \tau^{-1} Z, \tau^{-1} W)$$

and

$$K(X, Y, Z, W) = K(X, \tau^{-1} Y, \tau^{-1} Z, \tau^{-1} W) = k_0 + k_1 X,$$

where as we regarded above,  $n$  is the greatest weight degree in the homogeneous weight components of  $h$  in variables  $x, y, z$  and  $w$  via a weight degree  $(0, 1, 1, 1)$ . We consider that

$$F = \sum_{j=0}^m \tau^j F_j(X, Y, Z, W),$$

where  $F_j$  is a weight-homogeneous polynomial in  $X, Y, Z$  and  $W$  via a weight degree  $m - j$  for  $j = 0, 1, \dots, m$ . It is important to note that specifically

$$F_i(X, Y, Z, W) = \tau^m F_{m-i}(x, y, z, w),$$

for  $i = 0, 1, \dots, m$ . According to the concept of an IAS, it follows that

$$\begin{aligned} & \sum_{j=0}^m \tau^j \left( (r_1 X \tau^2 + W \tau - Y Z) \frac{\partial F_j}{\partial X} + (X Z \tau^2 - r_2 Y \tau^2) \frac{\partial F_j}{\partial Y} + (X Y \tau^2 - r_3 Z \tau^2) \frac{\partial F_j}{\partial Z} + (r_4 \tau^3 - Y \tau^2) \frac{\partial F_j}{\partial W} \right) \\ &= \sum_{j=0}^m \tau^j (k_0 + k_1 X) F_j. \end{aligned} \quad (2.3)$$

We compute the coefficient of  $\tau^0$  in equation (2.3), to obtain

$$-Y Z \frac{\partial F_0}{\partial X} - k_1 X F_0 - k_0 F_0 = 0.$$

The result of the specified equation is

$$F_0(X, Y, Z, W) = P(Y, Z, W) e^{-\frac{X(2k_0 + k_1 X)}{2YZ}},$$

where  $P(Y, Z, W)$  denotes an arbitrary polynomial function in  $Y, Z$  and  $W$ . Since  $F_0(X, Y, Z, W)$  is a polynomial weight-homogeneous, we need to have  $k_0 = k_1 = 0$ . Therefore,  $K = 0$ . This implies that there is no nonzero cofactor for the nonzero Darboux polynomials.  $\square$

Now, we prove Theorem 1.2 and show that system (1.1) does not possess any polynomial FIs.

**Proof:**

**Proof of Theorem 1.2.** Suppose that  $f = \sum_{l=0}^n F_l(x, y, z, w)$  be a FI polynomial, where each  $F_l$  represents a polynomial homogeneous of degree  $l$ . Then  $f$  satisfies

$$(r_1x - yz + w) \frac{\partial f}{\partial x} + (xz - r_2y) \frac{\partial f}{\partial y} + (xy - r_3z) \frac{\partial f}{\partial z} + (-y + r_4) \frac{\partial f}{\partial w} = 0. \quad (2.4)$$

By calculating the terms of degree  $(n+1)$  in (2.4) are as follows:

$$-yz \frac{\partial f_n}{\partial x} + xz \frac{\partial f_n}{\partial y} + xy \frac{\partial f_n}{\partial z} = 0.$$

Then,  $f_n = (x^2 + y^2, x^2 + z^2, w)$  is a solution to the previous equation, and we utilise the following change of variables:

$$A = x^2 + y^2, \text{ and } B = x^2 + z^2. \quad (2.5)$$

The inverse conversion is as follows:

$$y = \mp \sqrt{A - x^2}, \text{ and } z = \mp \sqrt{B - x^2}.$$

Our analysis is confined to consider

$$y = +\sqrt{A - x^2}, \text{ and } z = +\sqrt{B - x^2}. \quad (2.6)$$

Therefore,  $f_n = F_n(A, B, w)$ , compute the terms of degree  $n$  in (2.4) utilising (2.5) and (2.6), we get

$$\begin{aligned} & -\sqrt{A - x^2} \sqrt{B - x^2} \frac{\partial F_{n-1}}{\partial x} + (2r_1x^2 + 2r_2x^2 - 2Ar_2 + 2xw) \frac{\partial F_n}{\partial A} - \sqrt{A - x^2} \frac{\partial F_n}{\partial w} \\ & + (2r_1x^2 + 2r_3x^2 - 2Br_3 + 2xw) \frac{\partial F_n}{\partial B} = 0, \end{aligned}$$

where  $F_{n-1}$  is  $f_{n-1}$  defined in the newly variables  $x, A, B$  and  $w$  in lieu of the original variables  $x, y, z$  and  $w$ . That is

$$\begin{aligned} & F_{n-1}(x, A, B, w) \\ & = w \left( \frac{\partial F_n}{\partial A} + \frac{\partial F_n}{\partial B} \right) \ln \left( 2x^2 - A - B + 2\sqrt{B - x^2} \sqrt{A - x^2} \right) - \frac{\partial F_n}{\partial w} \arctan \left( \frac{x}{\sqrt{B - x^2}} \right) \\ & \quad - 2\sqrt{A} \left( (r_1 + r_3) \frac{\partial F_n}{\partial B} + (r_1 + r_2) \frac{\partial F_n}{\partial A} \right) \text{EllipticE} \left( \frac{x}{\sqrt{B}}, \frac{B}{A} \right) \\ & \quad + \frac{2}{\sqrt{A}} \left( ((r_1 + r_3)A - r_3B) \frac{\partial F_n}{\partial B} + r_1A \frac{\partial F_n}{\partial A} \right) \text{EllipticF} \left( \frac{x}{\sqrt{B}}, \frac{B}{A} \right) + G_{n-1}(A, B, w). \quad (2.7) \end{aligned}$$

Where  $G_{n-1}$  is an arbitrary polynomial function and  $\text{EllipticE}(\frac{x}{\sqrt{B}}, \frac{B}{A}) = \int_0^{\frac{x}{\sqrt{B}}} \sqrt{1 - \frac{B}{A} \sin^2 \theta} d\theta$ . Given that  $F_{n-1}$  is a polynomial, it follows that

$$\frac{\partial F_n}{\partial w} = 0, \text{ and } \frac{\partial F_n}{\partial A} + \frac{\partial F_n}{\partial B} = 0. \quad (2.8)$$

This yields  $F_n(A, B, w) = F_n(A, B)$ . Also, By substituting  $F_n(A, B)$  into equation (2.7) and requiring that  $F_{n-1}$  be a polynomial. That is,  $F_n(A, B) = F_n(B - A)$ . Then, equation (2.7) becomes

$$(r_1 + r_3) \frac{\partial F_n}{\partial B} + (r_1 + r_2) \frac{\partial F_n}{\partial A} = 0, \text{ and } ((r_1 + r_3)A - r_3B) \frac{\partial F_n}{\partial B} + r_1A \frac{\partial F_n}{\partial A} = 0. \quad (2.9)$$

This yields  $F_n(A, B, w) = F_n(A, B) = C_1$ , where  $C_1$  is a constant. Therefore,  $f_n$  is a constant, which contradicts the reality that  $f_n$  is a polynomial FI. Then, system (2.2) does not have a polynomial FI. The proof follows.  $\square$

In the following, certain general conditions are considered for the existence of a FI of system (1.1).

**proposition 2.1.** If  $r_2 = 0$  and  $r_3 = 0$ , then  $f(x, y, z, w) = z^2 - y^2$  is a polynomial first integral of system (1.1).

**Proof:** When  $r_2 = r_3 = 0$ , the system becomes

$$\dot{x} = r_1x - yz + w, \quad \dot{y} = xz, \quad \dot{z} = xy, \quad \dot{w} = -y + r_4.$$

Then, it is straightforward to determine the FI. So, by definition, we have

$$\frac{d}{dt}(z^2 - y^2) = 2z\dot{z} - 2y\dot{y} = 2zxy - 2yxx = 0.$$

Hence,  $f(x, y, z, w) = z^2 - y^2$  is the FI of the system.  $\square$

**proposition 2.2.** We consider system (1.1); this system can not satisfy any polynomial first integrals, which are indicated as follows:

- i. If  $r_2 = 0$  and  $r_1 = r_3 \neq 0$ ,
- ii. If  $r_1 = r_3 = 0$ ,
- iii. If  $r_1 = r_4 = 0$ .

**Proof:** In each of the above cases, the system simplifies such that the resulting vector field fails to preserve any nontrivial polynomial invariant.

**Case i.** When  $r_2 = 0$  and  $r_1 = r_3 \neq 0$ , the system becomes

$$\dot{x} = r_1x - yz + w, \quad \dot{y} = xz, \quad \dot{z} = xy - r_1z, \quad \dot{w} = -y + r_4. \quad (2.10)$$

Therefore, we require that any polynomial FI of system (2.10) must be constant. In accordance with the method used in the demonstration of Theorem 2.

We consider  $f(x, y, z, w)$  to be the FI polynomials of system (2.10), resulting in the equation (2.7) with conditions  $r_2 = 0$  and  $r_1 = r_3 \neq 0$ , which becomes

$$\begin{aligned} & F_{n-1}(x, A, B, w) \\ &= w \left( \frac{\partial F_n}{\partial A} + \frac{\partial F_n}{\partial B} \right) \ln \left( 2x^2 - A - B + 2\sqrt{B-x^2} \sqrt{A-x^2} \right) - \frac{\partial F_n}{\partial w} \arctan \left( \frac{x}{\sqrt{-x^2+B}} \right) \\ & \quad - \frac{2r_3}{\sqrt{A}} \left( (-2A+B) \frac{\partial F_n}{\partial B} - A \frac{\partial F_n}{\partial A} \right) \text{EllipticF} \left( \frac{x}{\sqrt{B}}, \frac{\sqrt{B}}{\sqrt{A}} \right) \\ & \quad + 2r_3\sqrt{A} \left( \frac{\partial F_n}{\partial A} + 2\frac{\partial F_n}{\partial B} \right) \text{EllipticE} \left( \frac{x}{\sqrt{B}}, \frac{\sqrt{B}}{\sqrt{A}} \right) + G_{n-1}(A, B, w). \end{aligned} \quad (2.11)$$

Where  $G_{n-1}(A, B, w)$  is an arbitrary function and  $A = x^2 + y^2$ ,  $B = x^2 + z^2$ . Given that  $F_{n-1}$  is a polynomial, we conclude that

$$\frac{\partial F_n}{\partial w} = 0, \quad \text{and} \quad \frac{\partial F_n}{\partial A} + \frac{\partial F_n}{\partial B} = 0.$$

This yields  $F_n(A, B, w) = F_n(A, B) = F_n(B - A)$ . Then the equation (19) becomes

$$\frac{\partial F_n}{\partial A} + 2\frac{\partial F_n}{\partial B} = 0, \quad \text{and} \quad (-2A+B) \frac{\partial F_n}{\partial B} - A \frac{\partial F_n}{\partial A} = 0.$$

This yields  $F_n(A, B, w) = F_n(-2A+B)$ . That gives  $F_n(A, B, w) = C_2$ , where  $C_2$  is a constant. Therefore,  $f_n$  is a constant, which contradicts the reality that  $f_n$  is a polynomial FI. Then, system (2.10) does not have a polynomial FI.

**Case ii.** For  $r_1 = r_3 = 0$ , system (1.1) reduces to

$$\dot{x} = -yz + w, \quad \dot{y} = xz - r_2y, \quad \dot{z} = xy, \quad \dot{w} = -y + r_4. \quad (2.12)$$

Then, we require that any polynomial FI of system (2.12) must be constant. Equation (2.7) with conditions  $r_1 = r_3 = 0$  becomes

$$\begin{aligned} & F_{n-1}(x, A, B, w) \\ &= w \left( \frac{\partial F_n}{\partial A} + \frac{\partial F_n}{\partial B} \right) \ln \left( 2x^2 - A - B + 2\sqrt{B-x^2} \sqrt{A-x^2} \right) - \frac{\partial F_n}{\partial w} \arctan \left( \frac{x}{\sqrt{B-x^2}} \right) \\ & \quad - 2r_2\sqrt{A} \operatorname{EllipticE} \left( \frac{x}{\sqrt{B}}, \frac{\sqrt{B}}{\sqrt{A}} \right) \frac{\partial F_n}{\partial A} + G_{n-1}(A, B, w). \end{aligned} \quad (2.13)$$

Where  $G_{n-1}(A, B, w)$  is an arbitrary function and  $A = x^2 + y^2$ ,  $B = x^2 + z^2$ . Given that  $F_{n-1}$  is a polynomial, we conclude that

$$\frac{\partial F_n}{\partial w} = 0, \quad \text{and} \quad \frac{\partial F_n}{\partial A} = 0.$$

This yields  $F_n(A, B, w) = F_n(A, B) = F_n(B)$ . Then the equation (2.13) becomes

$$\frac{\partial F_n}{\partial B} = 0.$$

This yields  $F_n(A, B, w) = C_3$ , where  $C_3$  is a constant. Therefore,  $f_n$  is a constant, which contradicts the reality that  $f_n$  is a polynomial FI. Then, system (1.1) does not have a polynomial FI.

**Case iii.** If  $r_1 = 0$  and  $r_4 = 0$ , system (1.1) becomes

$$\dot{x} = -yz + w, \quad \dot{y} = xz - r_2y, \quad \dot{z} = xy - r_3z, \quad \dot{w} = -y. \quad (2.14)$$

Then, we require that any polynomial FI of system (2.14) must be constant. Equation (2.7) with conditions  $r_1 = r_4 = 0$  becomes

$$\begin{aligned} & F_{n-1}(x, A, B, w) \\ &= w \left( \frac{\partial F_n}{\partial A} + \frac{\partial F_n}{\partial B} \right) \ln \left( 2x^2 - A - B + 2\sqrt{-x^2+B} \sqrt{-x^2+A} \right) - \frac{\partial F_n}{\partial w} \arctan \left( \frac{x}{\sqrt{-x^2+B}} \right) \\ & \quad - 2\sqrt{A} \left( r_3 \frac{\partial F_n}{\partial B} + r_2 \frac{\partial F_n}{\partial A} \right) \operatorname{EllipticE} \left( \frac{x}{\sqrt{B}}, \frac{\sqrt{B}}{\sqrt{A}} \right) \\ & \quad + \frac{2r_3}{\sqrt{A}} (A-B) \frac{\partial F_n}{\partial B} \operatorname{EllipticF} \left( \frac{x}{\sqrt{B}}, \frac{\sqrt{B}}{\sqrt{A}} \right) + G_{n-1}(A, B, w). \end{aligned} \quad (2.15)$$

Where  $G_{n-1}(A, B, w)$  is an arbitrary function and  $A = x^2 + y^2$ ,  $B = x^2 + z^2$ . Given that  $F_{n-1}$  is a polynomial, we conclude that

$$\frac{\partial F_n}{\partial w} = 0, \quad \text{and} \quad \frac{\partial F_n}{\partial B} = 0.$$

This yields  $F_n(A, B, w) = F_n(A, B) = F_n(A)$ . Then the equation (2.15) becomes

$$\frac{\partial F_n}{\partial A} = 0.$$

This yields  $F_n(A, B, w) = C_4$ , where  $C_4$  is a constant. Therefore,  $f_n$  is a constant, which contradicts the reality that  $f_n$  is a polynomial FI. Then, system (2.14) does not have a polynomial FI. Hence, in each

of the cases above, the system lacks a polynomial FI. This concludes the proof. The next results are required to prove Theorem 3.  $\square$

**proposition 2.3.**  $E = e^w$  is only one exponential factor of system (1.1), with  $r_4 - y$ .

**Proof:** Since the system has no Darboux polynomial. Then, any EF of system (1.1) may take the form  $E = e^{g(x,y,z,w)}$  which satisfy the following

$$\begin{aligned} & (r_1x - yz + w) \frac{\partial e^g}{\partial x} + (xz - r_2y) \frac{\partial e^g}{\partial y} + (xy - r_3z) \frac{\partial e^g}{\partial z} + (-y + r_4) \frac{\partial e^g}{\partial w} \\ & = (k_0 + k_1x + k_2y + k_3z + k_4w) e^g, \end{aligned}$$

simplifying

$$\begin{aligned} & r_1x - yz + w) \frac{\partial g}{\partial x} + (xz - r_2y) \frac{\partial g}{\partial y} + (xy - r_3z) \frac{\partial g}{\partial z} + (-y + r_4) \frac{\partial g}{\partial w} \\ & = k_0 + k_1x + k_2y + k_3z + k_4w. \end{aligned} \tag{2.16}$$

Where  $k_i$ ,  $i = 0, 1, \dots, 4$  are in  $\mathbb{C}$  and not all values are zero. Consider that  $g_n(x, y, z, w)$  be a polynomial of degree  $n \geq 2$ . We express  $g = \sum_{i=2}^n g_i(x, y, z, w)$ , where  $g_i$  be a homogeneous polynomial of degree  $i \geq 2$ . Now we calculate the terms of degree  $(n + 1)$  in (2.16), we get

$$-yz \frac{\partial g_n}{\partial x} + xz \frac{\partial g_n}{\partial y} + xy \frac{\partial g_n}{\partial z} = 0,$$

That is, repeating Equations (2.4) and (2.5). We obtain  $g_n$  as a constant, which is a contradiction for degree  $n > 1$ . Now, assuming that  $g$  is a first-degree polynomial represented as  $g(x, y, z, w) = l_0 + l_1x + l_2y + l_3z + l_4w$ , where  $l_i$ ,  $i = 0, 1, \dots, 4$  are in  $\mathbb{C}$ . Thus, equation (2.16) becomes

$$-(k_4 - l_1)w + (r_1l_1 - k_1)x - (r_2l_2 + k_2 + l_4)y - (r_3l_3 + k_3)z + l_3xy + l_2xz - l_1yz + l_4r_4 - k_0 = 0.$$

Following the simplification and organisation of the variables, eight linear equations were derived as follows. These linear equations can be solved to find that  $l_0 = l_1 = l_2 = l_3 = k_1 = k_3 = k_4 = 0$ ,  $l_4 = \frac{k_0}{r_4}$ ,  $k_2 = \frac{-k_0}{r_4}$ , and  $k_0 = r_4$ ,  $l_4 = 1$  and  $k_2 = -1$ . Thus,

$$g(x, y, z, w) = w,$$

with the cofactor

$$K = r_4 - y.$$

Hence, system (1.1) has only one EF,  $E = e^w$ , which includes the cofactor  $r_4 - y$ .  $\square$

Next, we aim to prove Theorem 1.3, which is the principal outcome of this study, by establishing the existence of the Darboux FI of system (1.1).

**Proof:**

**Proof of Theorem 1.3.** It is derived from Proposition 2.3 and Theorem 1.1 states that if system (1.1) has a FI of Darboux type, then, there exists a nonzero  $\mu \in \mathbb{C}$  that satisfies the equation (1.6). Since there are no invariant algebraic surfaces and only one cofactor exists, equation (1.6) simplifies to

$$\mu(r_4 - y) = 0,$$

This equation has only a trivial solution which is  $\mu = 0$ .  $\square$

### 3. Analytic First Integral

We proceed to show that, under the given conditions, the system does not have analytic FIs in the neighborhood of its equilibrium points. As a first step, we identify the fixed points of system (1.1), which are

$$E_{1,2} = \left( \pm\sqrt{r_2 r_3}, r_4, \pm\frac{r_4\sqrt{r_2 r_3}}{r_3}, \pm\sqrt{r_2 r_3}\left(\frac{r_4^2}{r_3} - r_1\right) \right).$$

**Proof:**

**Proof of Theorem 1.4.** Consider system (1.1), when  $r_2 = 0$ ,  $r_1 \neq r_3$  and assume  $r_1, r_2, r_3, r_4 \in \mathbb{R}$ . Under these conditions, the system is as follows:

$$\dot{x} = r_1 x - yz + w, \quad \dot{y} = xz, \quad \dot{z} = xy - r_3 z, \quad \dot{w} = -y + r_4, \quad (3.1)$$

The isolated singular point is  $(x, y, z, w) = (0, r_4, 0, 0)$ . Using the linear transformation of coordinates  $(x, y, z, w) = (x, y - r_4, z, w)$ , system (3.1) can be translated as follows:

$$\dot{x} = r_1 x - (y + r_4)z + w, \quad \dot{y} = xz, \quad \dot{z} = x(y + r_4) - r_3 z, \quad \dot{w} = -y, \quad (3.2)$$

Clearly,  $E_0 = (0, 0, 0, 0)$  is the isolated singular point of system (3.2). The Jacobian matrix  $J$  assessed at  $E_0$  is

$$J = \begin{bmatrix} r_1 & -z & -y - r_4 & 1 \\ z & 0 & x & 0 \\ y + r_4 & x & -r_3 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix} \Big|_{(0,0,0,0)} = \begin{bmatrix} r_1 & 0 & -r_4 & 1 \\ 0 & 0 & 0 & 0 \\ r_4 & 0 & -r_3 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix}.$$

The corresponding eigenvalues are:

$$(\lambda_1, \lambda_2, \lambda_3, \lambda_4) = \left( 0, 0, -\frac{r_3}{2} + \frac{r_1}{2} + \frac{\sqrt{r_1^2 + 2r_3 r_1 + r_3^2 - 4r_4^2}}{2}, -\frac{r_3}{2} + \frac{r_1}{2} - \frac{\sqrt{r_1^2 + 2r_3 r_1 + r_3^2 - 4r_4^2}}{2} \right).$$

We can apply Theorem 1.6. We have

$$s_3 \left( -\frac{r_3}{2} + \frac{r_1}{2} + \frac{\sqrt{r_1^2 + 2r_3 r_1 + r_3^2 - 4r_4^2}}{2} \right) + s_4 \left( -\frac{r_3}{2} + \frac{r_1}{2} - \frac{\sqrt{r_1^2 + 2r_3 r_1 + r_3^2 - 4r_4^2}}{2} \right) = 0. \quad (3.3)$$

This gives

$$(s_3 + s_4)\left(-\frac{r_3}{2} + \frac{r_1}{2}\right) = 0, \text{ and } (s_3 - s_4)\left(\frac{\sqrt{r_1^2 + 2r_3 r_1 + r_3^2 - 4r_4^2}}{2}\right) = 0.$$

These two equations have one solution which is  $s_3 = s_4 = 0$  which is a contradiction. Thus, there is no solution of equation (3.3) for all  $s_3, s_4 \in \mathbb{N}^*$  with  $s_3 + s_4 \geq 1$  such that  $r_1 \neq r_3$ . Since  $E_0$  is the isolated singular point, and applying Theorem 1.6 gives that the system has no analytic FI. Therefore, system (3.1) has no analytic FI in the neighborhood of its singularities. This concludes the proof.  $\square$

### Acknowledgments

We thank the referees for their valuable comments to improve the quality of this paper. Additionally, the second author (Niazy Hady Hussein) extends gratitude to Salahaddin University, Erbil, Iraq, for their support.

**Conflict of Interest Statement:** Authors states that there is no conflict of interest.

**Competing interests:** Not applicable.

**Funding:** No external funding was received for this study..

**Data Availability Statement:** The data that support the results are included in the manuscript.

#### 4. Conclusion

This work successfully applied Darboux theory to a four-dimensional hyperchaotic system, demonstrating its non-integrability and the absence of invariant algebraic surfaces. The study found only one Darboux element, an exponential factor, demonstrating that the system prevents the existence of polynomial or rational first integrals given certain parameter constraints. These findings establish a robust algebraic classification for the system, separating it from integrable lower-dimensional models. Finally, this work contributes to the larger topic of polynomial vector fields by providing an actual demonstration of how growing system dimensionality removes algebraic invariants. Future studies could apply this analytical concept to the system's fractional-order equivalents, examining whether memory effects change the integrability and algebraic structure.

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