



## The Induced Lomax Distribution: Properties, Estimations, Simulation and Applications

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**ABSTRACT:** This paper presents the induced Lomax (ILO) distribution, a new one-parameter modification of the standard Lomax model, aimed at enhancing the modeling of lifetime and reliability data. The significance of the ILO distribution is in its adaptability to include heavy-tailed phenomena and various hazard rate configurations, rendering it appropriate for industrial and environmental datasets that encompass extreme events. We establish many mathematical features, encompassing moments, inverse moments, moment-generating functions, quantile functions, and Rényi and Tsallis entropies. Reliability functions, including survival, hazard rate, and reversed hazard rate, are also included. Six estimation techniques maximum likelihood, least squares, weighted least squares, Cramér–von Mises, Anderson–Darling, and percentile methods are utilized to estimate the model parameter. A thorough Monte Carlo simulation is performed to assess bias and mean square error, demonstrating that the Anderson–Darling estimator yields the most precise findings across various sample sizes. Ultimately, two empirical datasets from industrial and environmental contexts are examined to demonstrate the model’s enhanced fit relative to alternative distributions. The findings underscore the ILO distribution as a reliable and adaptable instrument for professionals engaged in dependability modeling and extreme value analysis.

**Keywords:** Lomax distribution, industrial and environmental datasets, weighted least squares estimation, moment generating function, odds function, order statistics, quantile function.

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### 1. Introduction

Real-world situations predominantly illustrate the utilization of statistical distributions in modeling lifetime data. These distributions are highly beneficial, their theoretical framework is extensively examined, and novel distributions are continually being developed. The objective of statistical parametric analysis is to identify the optimal model for a dataset obtained from experiments, observational studies, surveys, and various other sources. Most modeling solutions focus on identifying the most suitable probability distribution that elucidates the fundamental framework of the data collection. Nevertheless, no singular probability distribution is applicable to all data sets. Consequently, there has arisen a necessity to broaden or develop new traditional distributions (see [1]).

The exponential, gamma, and lognormal distributions can be employed to model monotonic hazard rates. These distributions, however, possess several shortcomings. Initially, none of their hazard rate functions exhibit bathtub shapes. These distributions exhibit solely monotonically rising, falling, or steady hazard rates. The bathtub-shaped hazard rate is the most accurate representation. This occurs in nearly all real-world systems. For instance, where a population is divided into subpopulations characterized by early failures, wear-out failures, and somewhat constant failures, similar patterns arise. An ideal bathtub comprises two changing points and a constant component situated between them. The practicality of bathtub designs is widely recognized throughout various domains. Numerous parametric probability distributions have been developed to analyze actual datasets exhibiting bathtub failure rates.

One method for generating new distributions is the concept of weighted distributions. The examination of weighted distributions is advantageous in distribution theory as it enhances comprehension of standard probability distributions and offers methodologies for augmenting these distributions to model lifetime data through the incorporation of an additional parameter, thereby introducing flexibility in their characteristics. Weighted distributions arise in the modeling of clustered sampling, heterogeneity, and extraneous variance within the dataset. The notion of weighted distributions was initially proposed by [2] to represent determination biases, subsequently formalized by [3] in a comprehensive theory addressing issues where observations occur in a non-experimental, non-replicated, and non-random fashion. When an investigator records observations in nature based on a certain stochastic model, the distribution of these recorded observations will deviate from the initial distribution unless each observation is afforded an equal probability of being captured.

Let  $X$  be a random variable with a probability density function (PDF)  $f(x)$ , and assume that the probability of observing  $X = x$  is proportional to a non-negative weight function  $w(x) \geq 0$ . Consequently, based on the principle of weighted probability, the PDF of the observed variable  $X$  is expressed as

$$f_w(x) = \frac{w(x)g(x)}{\int_{-\infty}^{\infty} w(x)g(x) dx}. \quad (1.1)$$

Weight functions vary depending on the situation and have been proposed by several authors. For instance, [4] investigated certain properties of bivariate weighted distributions. The length-biased weighted new quasi Lindley distribution was introduced by [5]. According to [6], the weighted Nwike distribution was proposed, while the weighted Zeghdoudi distribution was suggested by [7]. The Suja distribution was weighted in [8], and a comprehensive review of weighted distributions was provided by [9]. Furthermore, the weighted generalized quasi Lindley distribution was proposed by [10].

Singh and Das [11] examined a special case of weighted distributions known as the *induced distribution*, whose probability density function is defined as

$$f_I(x) = \frac{1 - G(x)}{E(X)}, \quad x \in \mathbb{R}, \quad (1.2)$$

where  $G(x)$  denotes the cumulative distribution function (CDF) and  $E(X)$  represents the expectation of the underlying random variable  $X$ . Singh and Das [11] proposed the induced Garima distribution as a modification of the Garima distribution using the induced distribution approach. More recently, Panta et al. [12] introduced an induced version of the Bilal distribution.

The analysis of heavy-tailed data plays a vital role in numerous engineering, actuarial, and medical applications. Pareto [13] conducted early work in this area to model income distributions. Subsequently, the Pareto family, particularly the Type II Lomax (LO) distribution, has received considerable attention (see [14]). The LO distribution has found wide applicability in engineering sciences, actuarial science, medicine, and economics. The PDF, CDF, and expectation of the LO distribution are given, respectively, by

$$g(x; \alpha, \lambda) = \frac{\alpha}{\lambda} \left(1 + \frac{x}{\lambda}\right)^{-(\alpha+1)}, \quad x \geq 0, \alpha > 0, \lambda > 0, \quad (1.3)$$

$$G(x; \alpha, \lambda) = 1 - \left(1 + \frac{x}{\lambda}\right)^{-\alpha}, \quad x \geq 0, \alpha > 0, \lambda > 0, \quad (1.4)$$

and

$$E(X) = \frac{\lambda}{\alpha - 1}, \quad \alpha > 1, \lambda > 0, \quad (1.5)$$

where  $\lambda$  is the scale parameter and  $\alpha$  is the shape parameter.

Numerous authors have proposed extensions of the Lomax distribution to better accommodate heavy-tailed data by generalizing its functional form through additional shape and scale parameters. Notable examples include the sine power Lomax distribution introduced by [15], the Kumaraswamy generalized power Lomax distribution proposed by [16], the Marshall–Olkin exponential Lomax distribution presented by [17], and the Nadarajah–Haghighi Lomax distribution discussed in [18]. Recently, a shifted Lomax– $X$  family of distributions with actuarial and financial applications was studied by [19], the exponentiated power Lomax distribution was proposed by [20], inference for a new four-parameter Lomax distribution was investigated by [21], and a novel sine– $G$  class of distributions with a Lomax illustration was introduced by [22].

The induced Lomax (ILO) distribution is a new model that is the subject of this research, which also analyzes its fundamental properties. It is significant for several reasons:

- The hazard rate function (HRF) may show up as decreasing, whereas the accompanying PDF may show decreasing, right skewness, or reversed J-shaped. These qualities are useful in a number of situations, such as uncertainty analysis, survival analysis, and reliability evaluation.
- The quantile function has a closed-form expression.
- Several estimating techniques for figuring out the ILO model's parameters. A pragmatic approach to assessing the effectiveness of different estimators is provided by simulation studies, given the complexity of cognitively assessing their performances. By examining several estimating techniques during simulation, this work improves understanding and application of the ILO model in statistical investigations.

The main contribution of this study is the introduction of the ILO distribution, a simple two-parameter model that extends the Lomax distribution while offering greater flexibility in hazard rate and density shapes. In addition, we derive its key reliability measures, investigate several estimation methods through Monte Carlo simulations, and validate its practical usefulness using two real-life datasets from environmental and engineering applications. Although the application of induced distributions to real-world lifetime data remains relatively underexplored in the literature, this study contributes by extending the Lomax family through the induced approach and demonstrating its usefulness in environmental and engineering contexts. To strengthen the scope of this research, we have provided a more comprehensive literature review, explicitly highlighted the research gap, and clarified the novelty of the proposed model. This ensures that the present work is well-grounded in existing studies, while offering meaningful advancement.

It has become increasingly crucial to model real-world data utilizing versatile statistical distributions, especially when the data show limited behavior, skewness, or truncation. One effective model that can capture these traits in a variety of sectors is the induced Lomax distribution. In this work, we investigate how well the ILO distribution models two different datasets taken from environment and industry sectors. We start by looking at a data set about snowfall rates, which frequently includes

variables with asymmetric distributions and little support. Improved fitting accuracy is made possible by the flexibility of the ILO distribution, which enables more reliable deduction of variables that affect specific climate changes. Second, we examine data on aircraft air conditioning system failures, where conventional distributions might not fully represent the data's characteristics. The ILO distribution, on the other hand, provides a more accurate depiction, facilitating a more thorough investigation of the reasons behind these system malfunctions.

Although many extensions of the Lomax and other classical lifetime distributions have been proposed, they often suffer from notable drawbacks. First, most models fail to capture the decreasing or bathtub-shaped hazards in hazard rates that are frequently observed in practice. Second, several generalizations rely on additional parameters, which increase complexity and reduce interpretability. Third, comparative studies of the estimation techniques for these models remain limited. To address these gaps, the present study introduces the ILO distribution, which preserves simplicity with only two parameters while allowing greater flexibility in hazard rate and density shapes. The model also has a closed-form quantile function and is evaluated under multiple estimation methods, ensuring both theoretical tractability and practical relevance.

The remainder of this article is organized as follows. Section 2 describes the construction of the ILO distribution. Reliability measures are presented in Section 3, while several statistical properties are discussed in Section 4. Parameter estimation methods are developed in Section 5, and their performance is assessed via Monte Carlo simulations in Section 6. Section 7 illustrates the applicability of the ILO model using two real datasets. Finally, concluding remarks are provided in Section 8.

## 2. Induced Lomax Distribution

In this section, we suggest a novel improvement of the LO distribution by considering the concept of induced distribution, which is defined in Eq. (1.1); hence the PDF and CDF of the ILO distribution can be determined by substituting Eqs. (1.3) and (1.4) in (1.1) as follows:

$$f(x; \alpha, \lambda) = \frac{\alpha - 1}{\lambda} \left(1 + \frac{x}{\lambda}\right)^{-\alpha}, \quad x \geq 0, \alpha > 1, \lambda > 0, \quad (2.1)$$

and the corresponding CDF is expressed as

$$F(x; \alpha, \lambda) = 1 - \left(1 + \frac{x}{\lambda}\right)^{-(\alpha-1)}, \quad x \geq 0, \alpha > 1, \lambda > 0. \quad (2.2)$$

The PDF and CDF shapes for the ILO model are depicted in Figure 1. This figure illustrates the potential forms of the ILO's PDF for the selected parameter values of  $\alpha$  and  $\lambda$ . It is clear from the plot of the PDF that it exhibits decreasing, right skewed, and reversed J-shaped. In addition, the CDF plot of the ILO model is an increasing function for all chosen values of parameters.

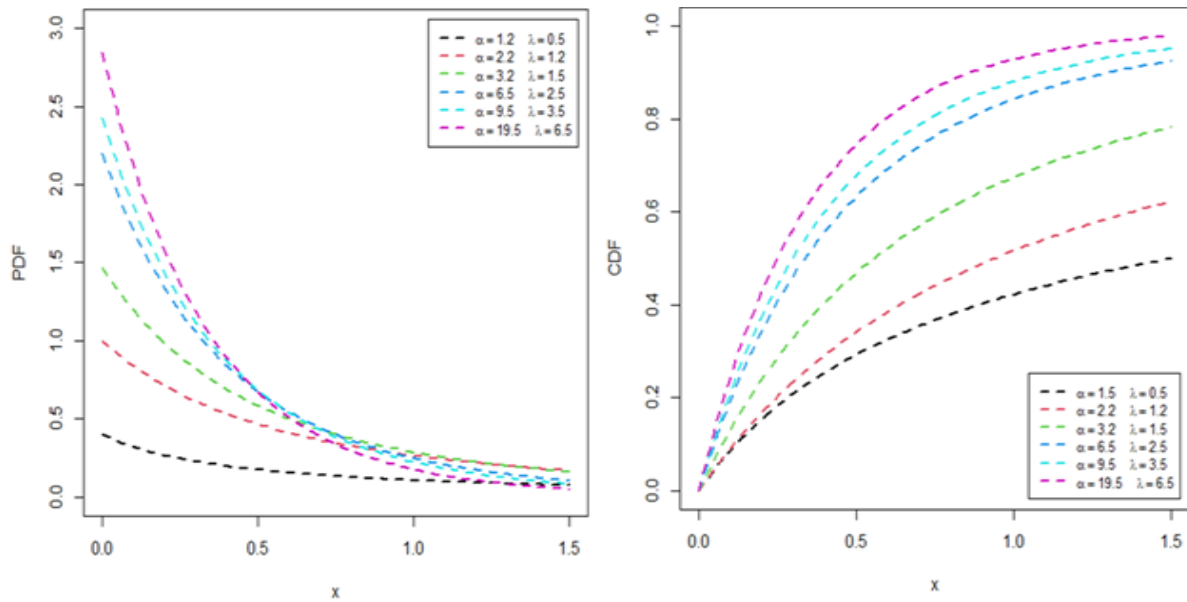


Figure 1: The plots of PDF and CDF for the ILO model

### 3. Reliability Measures

This section displays the survival, hazard rate, reversed hazard rate, and odds functions of the ILO distribution. These functions are useful for analyzing reliability.

#### 3.1. Survival Function

The survival function (SF), also known as the reliability function, indicates the probability of survival to at least  $x$  years of age. Reliability theory and survival analysis are crucial in the examination of  $S(x)$ . It is essential to characterize unit systems or assess the reliability of the system. The survival function of the ILO distribution can be determined as follows:

$$S(x; \alpha, \lambda) = 1 - F(x; \alpha, \lambda) = \left(1 + \frac{x}{\lambda}\right)^{-(\alpha-1)}. \tag{3.1}$$

#### 3.2. Hazard Rate Function

The hazard rate function (HRF) of a statistical distribution is analytically determined by dividing the survival function  $S(x)$  by the probability density function  $f(x)$ . Consequently, the HRF of the ILO distribution is delineated as follows:

$$h(x; \alpha, \lambda) = \frac{f(x; \alpha, \lambda)}{S(x; \alpha, \lambda)} = \frac{(\alpha - 1)}{(x + \lambda)}.$$

Figure 2 illustrates the behavior of HRF and SF for different values of  $\alpha$  and  $\lambda$  pertaining to the ILO distribution. The HRF seems to be consistently decreasing over time. In addition, the SF plot decreases as  $\alpha$  increases.

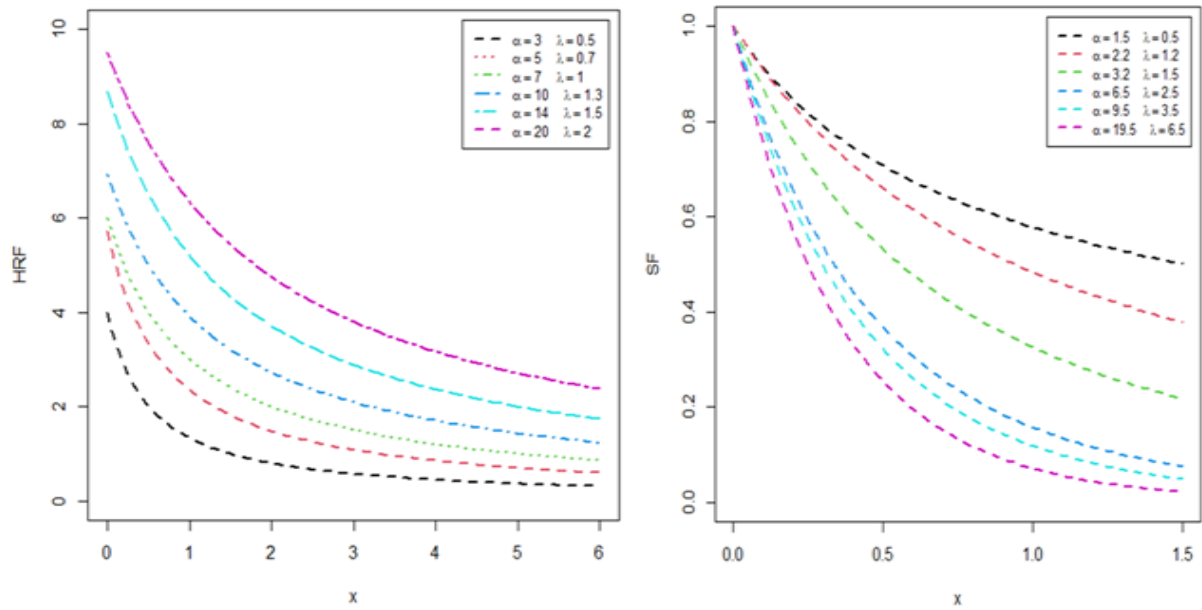


Figure 2: The plots of HRF and SF for the ILO model

### 3.3. Reversed Hazard Rate Function

The reversed hazard rate function (RHRF) is advantageous for reliability analysis, as mentioned in [23,24,25,26,27]. This is particularly beneficial for assessing left-censored lifespan statistics, concealed failures, and waiting periods. The RHRF is shown to be a decreasing function for substantial statistical distributions, making it suitable for maintenance engineering applications. The RHRF, defined as the ratio of the PDF to the CDF, mathematically represents the hazard rate function of a statistical distribution. It is supplied by:

$$\text{RHRF}(x; \alpha, \lambda) = \frac{f(x; \alpha, \lambda)}{F(x; \alpha, \lambda)} = \frac{\frac{(\alpha-1)}{\lambda} \left(1 + \frac{x}{\lambda}\right)^{-\alpha}}{1 - \left(1 + \frac{x}{\lambda}\right)^{-(\alpha-1)}}.$$

### 3.4. Odds Function

Odds functions are developed for discrete lifetime distributions in the analysis of lifetime data, providing information on aging features and reliability ideas [28]. Thus, odds functions are essential for various applications, including data analysis, decision-making, and model interpretability as cited in [29,30]. The odds function of the ILO model is calculated using the CDF described by (2.2) and the SF defined by (3.1) as follows:

$$O(x; \alpha, \lambda) = \frac{F(x; \alpha, \lambda)}{S(x; \alpha, \lambda)} = \left(1 + \frac{x}{\lambda}\right)^{(\alpha-1)} - 1.$$

## 4. Statistical Properties

This section elucidates the statistical and mathematical properties of the ILO model. These features are essential, especially when employing the distribution to analyze data from real-world sources.

### 4.1. Quantile Function

The quantile function (QF) is a commonly employed tool in statistics to determine the statistical features of a distribution and to pinpoint the essential percentiles. The quantile function is determined by the inverse of the distribution's CDF. Like the CDF, the QF provides an extensive examination of the statistical characteristics of the distribution. Therefore, it is vital to comprehend every aspect of it. It is essential to note that the CDF defined in Eq. (2.2) is continuous and strictly increasing; hence, the

quantile function of  $X$  is defined as  $Q_X(p) = F_X^{-1}(p)$ , where  $p \in [0, 1]$ . It corresponds to the solution of the following equation, as presented in the result below:

$$x_p = \lambda \left( (1 - p)^{\frac{-1}{\alpha-1}} - 1 \right). \tag{4.1}$$

Specifically, by substituting  $p = 0.25, 0.5$ , and  $0.75$ , we obtain the first, second (median), and third quantiles. Furthermore, predicated on the quantiles, Bowley’s skewness (BS), and Moor’s kurtosis (MK), are provided respectively by the following relations:

$$BS = \frac{Q(0.75) - 2Q(0.5) + Q(0.25)}{Q(0.75) - Q(0.25)},$$

and

$$MK = \frac{Q(0.875) - Q(0.625) - Q(0.375) + Q(0.125)}{Q(0.75) - Q(0.25)}.$$

These measurements provide significant insight into the modeling capabilities of skewness and kurtosis of the ILO distribution and possess the advantage of being applicable to all parameter values. Table 1 displays the possible quantile values, BS, and MK for a designated set of parameter values, that encompass the real and positive roots. Table 1 illustrates that as the parameters values increase, the results of the quantiles take an ascending pattern, while the values of BS and MK take a slightly downward trend as the value of  $\alpha$  increases.

Table 1: Numerical values for the ILO quartiles with BS and MK

$\alpha$	$\lambda$	$Q_1$	$Q_2$	$Q_3$	BS	MK
6	0.5	0.0296	0.0743	0.1597	0.3126	1.4231
	1.5	0.0888	0.2230	0.4793	0.3125	1.4234
	2.5	0.1481	0.3717	0.7988	0.3125	1.4234
	3.5	0.2073	0.5204	1.1183	0.3125	1.4233
7	0.5	0.0246	0.0612	0.1299	0.3045	1.4027
	1.5	0.0737	0.1837	0.3899	0.3042	1.4023
	2.5	0.1228	0.3062	0.6498	0.3041	1.4022
	3.5	0.1719	0.4286	0.9097	0.3041	1.4022
8	0.5	0.0209	0.0521	0.1095	0.2989	1.3888
	1.5	0.0629	0.1562	0.3285	0.2979	1.3877
	2.5	0.1049	0.2602	0.5475	0.2981	1.3876
	3.5	0.1468	0.3643	0.7665	0.2981	1.3876
9	0.5	0.0183	0.0453	0.0946	0.2931	1.3776
	1.5	0.0549	0.1358	0.2838	0.2937	1.3769
	2.5	0.0915	0.2263	0.4731	0.2937	1.3767
	3.5	0.1282	0.3168	0.6622	0.2936	1.3768

#### 4.2. Moments and Some Related Measures

Key features of a model, such as kurtosis, skewness, and dispersion, are delineated by the moments of distributions. Consequently, the  $r^{\text{th}}$  moment of the ILO distribution is offered by:

$$\mu'_r = \int_0^\infty x^r f(x; \alpha, \lambda) dx. \tag{4.2}$$

Using Eq. (2.1) in Eq. (4.2), we have

$$\mu'_r = \frac{\alpha - 1}{\lambda} \int_0^\infty x^r \left( 1 + \frac{x}{\lambda} \right)^{-\alpha} dx.$$

Let  $u = x/\lambda$ , then  $x = \lambda u$  and  $dx = \lambda du$ ,

$$\mu'_r = (\alpha - 1)\lambda^r \int_0^\infty u^r (1 + u)^{-\alpha} du.$$

By using the relation of the incomplete beta function as below:

$$B(\alpha, \beta) = \int_0^\infty u^{\beta-1} (1 + u)^{-(\alpha+\beta)} du,$$

then, the moments of the ILO model can be determined as follows:

$$\mu'_r = (\alpha - 1)\lambda^r B(\alpha - r - 1, r + 1), \quad \alpha > r + 1. \quad (4.3)$$

Therefore, the initial four moments of the ILO model can be derived by putting  $r = 1, 2, 3, 4$  into Eq. (4.2). The relations of the first four moments and variance can be determined respectively, as follows:

$$\begin{aligned} \mu'_1 &= (\alpha - 1)\lambda B(\alpha - 2, 2), \\ \mu'_2 &= (\alpha - 1)\lambda^2 B(\alpha - 3, 3), \\ \mu'_3 &= (\alpha - 1)\lambda^3 B(\alpha - 4, 4), \\ \mu'_4 &= (\alpha - 1)\lambda^4 B(\alpha - 5, 5), \\ \text{var} &= (\alpha - 1)\lambda^2 B(\alpha - 3, 3) - [\lambda(\alpha - 1)B(\alpha - 2, 2)]^2. \end{aligned}$$

Table 2 provides a numerical depiction of the initial four moments, variance ( $\sigma^2$ ), skewness ( $\omega_1$ ), kurtosis ( $\omega_2$ ), coefficient of variation ( $\omega_3$ ), and index of dispersion ( $\omega_4$ ) for different values of the parameters  $\alpha$  and  $\lambda$ . As the parameters increase, all moment measures rise, while the coefficients of  $\omega_1$  and  $\omega_2$  diminish. Also, we can note that the ILO model is right skewed, and the values of  $\omega_1$ ,  $\omega_2$ , and  $\omega_3$  don't change with the difference in the parameter  $\lambda$  within each fixed value of the parameter  $\alpha$ , which makes the distribution more focused and stable. In addition, the 3D plots of the measurements of  $\mu_1$ ,  $\sigma^2$ ,  $\omega_1$ ,  $\omega_2$ ,  $\omega_3$ , and  $\omega_4$  are presented in Figure 3 for further clarification and elucidation. Figure 3 presents three-dimensional graphs showing the variance and mean of the ILO distribution and provides the results in Table 2.

Table 2: Moments and some measures of the ILO model

$\alpha$	$\lambda$	$\mu'_1$	$\mu'_2$	$\mu'_3$	$\mu'_4$	$\sigma^2$	$\omega_1$	$\omega_2$	$\omega_3$	$\omega_4$
6	0.5	0.125	0.0417	0.0313	0.0625	0.0261	4.6476	73.81	1.291	0.2083
	1.5	0.375	0.375	0.8437	5.0625	0.2344	4.6476	73.81	1.291	0.6251
	2.5	0.625	1.0417	3.9063	39.0625	0.6511	4.6476	73.81	1.291	1.0417
	3.5	0.875	2.0417	10.7188	150.06	1.2761	4.6476	73.81	1.291	1.4583
7	0.5	0.1	0.025	0.0125	0.0125	0.015	3.8103	38.6667	1.2247	0.15
	1.5	0.3	0.225	0.3375	1.0125	0.135	3.8103	38.6667	1.2247	0.45
	2.5	0.5	0.625	1.5625	7.8125	0.375	3.8103	38.6667	1.2247	0.75
	3.5	0.7	1.225	4.2875	30.0125	0.735	3.8103	38.6667	1.2247	1.050
8	0.5	0.0833	0.0167	0.0063	0.0042	0.0097	3.3806	27.8751	1.1832	0.1167
	1.5	0.25	0.15	0.1688	0.3375	0.0875	3.3806	27.8751	1.1832	0.35
	2.5	0.4167	0.4167	0.7813	2.6042	0.2431	3.3806	27.8751	1.1832	0.5833
	3.5	0.5833	0.8167	2.1438	10.0042	0.4764	3.3806	27.8751	1.1832	0.8167
9	0.5	0.0714	0.0119	0.0036	0.0018	0.0068	3.1177	22.725	1.1547	0.0952
	1.5	0.2143	0.1071	0.0964	0.1446	0.0612	3.1177	22.725	1.1547	0.2857
	2.5	0.3571	0.2976	0.4464	1.1161	0.1701	3.1177	22.725	1.1547	0.4762
	3.5	0.5	0.5833	1.225	4.2875	0.3334	3.1177	22.725	1.1547	0.6667

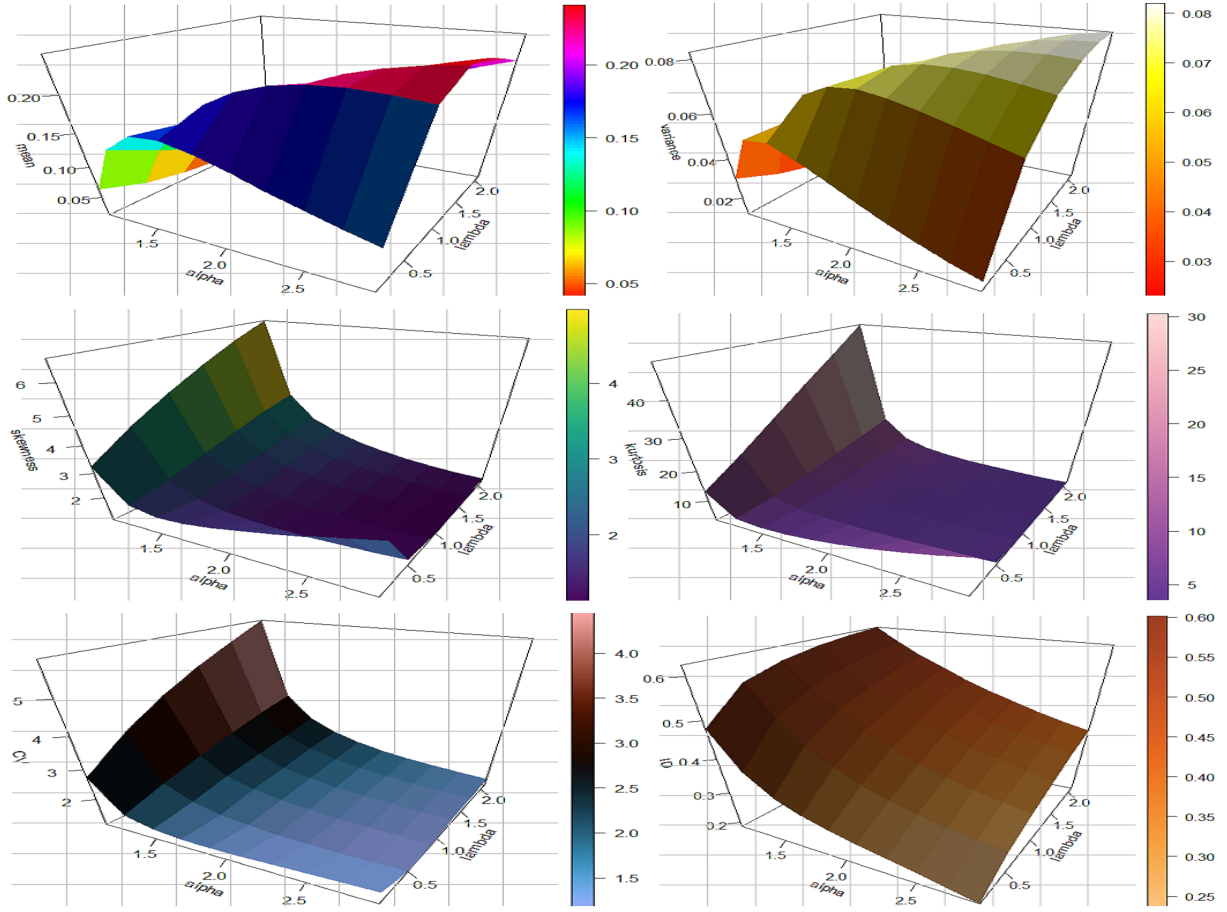


Figure 3: 3D Plots for some measures of the ILO model

For every positive integer  $r$ , the inverse  $r^{\text{th}}$  moment of  $X$  can be determined as follows:

$$\mu'_{-r} = \int_0^{\infty} x^{-r} f(x; \alpha, \lambda) dx = \frac{\alpha - 1}{\lambda} \int_0^{\infty} x^{-r} \left(1 + \frac{x}{\lambda}\right)^{-\alpha} dx = (\alpha - 1)\lambda^{-r} B(\alpha + r - 1, 1 - r), \quad \alpha > 1 - r.$$

### 4.3. Moment Generating Function

One crucial tool in probability theory is the moment-generating function (MGF). Through the convolution property, it simplifies the study of sums of random variables and makes it easier to compute the moments of a random variable. Furthermore, the uniqueness property of the MGF ensures that the probability distributions of two random variables must be equal if they have the same MGFs (in the area of the origin where the MGF converges). Using the PDF (2.1) in Eq. (4.3) the MGF of the ILO distribution is built as follows:

$$M_X(t) = E(e^{tX}) = \int_0^{\infty} e^{tx} f(x; \alpha, \lambda) dx. \quad (4.4)$$

Then, using the exponential expansion, we obtain the following

$$M_X(t) = \sum_{r=0}^{\infty} \frac{t^r}{r!} \int_0^{\infty} x^r f(x; \alpha, \lambda) dx = \sum_{r=0}^{\infty} \frac{t^r}{r!} \mu'_r.$$

Using Eq. (4.2), the MGF of the ILO distribution is given as

$$M_X(t) = \sum_{r=0}^{\infty} \frac{t^r (\alpha - 1) \lambda^r B(\alpha - r - 1, r + 1)}{r!}, \quad \alpha > r + 1. \quad (4.5)$$

#### 4.4. Rényi Entropy

Entropy is a useful tool for determining the amount of knowledge (or uncertainty) that a random sample contains about the parent population. A high entropy level indicates greater unpredictability in the data. Numerous fields, including physics, probability and statistics, economics, communication theory, and others, depend on the idea of entropy. The Rényi entropy of the ILO distribution is defined as follows:

$$\begin{aligned} \text{RE}(\delta) &= \frac{1}{1-\delta} \log \left[ \int_0^\infty f^\delta(x; \alpha, \lambda) dx \right] \\ &= \frac{1}{1-\delta} \log \left[ \left( \frac{\alpha-1}{\lambda} \right)^\delta \int_0^\infty \left( \left( 1 + \frac{x}{\lambda} \right)^{-\alpha} \right)^\delta dx \right] \\ &= \frac{1}{1-\delta} [\delta \log(\alpha-1) + (1-\delta) \log(\lambda) - \log(\alpha\delta-1)], \quad \alpha\delta > 1. \end{aligned}$$

#### 4.5. Tsallis Entropy

Tsallis [31] introduced an entropy called the Tsallis entropy to generalize standard statistical mechanics. The Tsallis entropy of the ILO model can be calculated as follows:

$$\begin{aligned} \text{TE}(\rho) &= \frac{1}{1-\rho} \log \left[ 1 - \int_0^\infty f^\rho(x; \alpha, \lambda) dx \right] \\ &= \frac{1}{1-\rho} \log \left[ 1 - \left( \frac{\alpha-1}{\lambda} \right)^\rho \int_0^\infty \left( \left( 1 + \frac{x}{\lambda} \right)^{-\alpha} \right)^\rho dx \right] \\ &= \frac{1}{1-\rho} \left[ \log \left[ 1 - \left( \frac{\alpha-1}{\lambda} \right)^\rho \left( \frac{\lambda}{\rho\alpha-1} \right) \right] \right], \quad \rho\alpha > 1. \end{aligned}$$

#### 4.6. Order Statistics

There are several uses for order statistics in reliability analysis and statistical estimation. In the context of the ILO distribution, we present the fundamentals of order statistics. Using the PDF and CDF given in equations (2.1) and (2.2), respectively, let  $X_1, X_2, \dots, X_n$  be independent random variables distributed identically taken from the ILO distribution. Next, the following formula is used to calculate the density of the  $j^{\text{th}}$  order statistic:

$$f_{X_{(j)}}(x) = \frac{n!}{(j-1)!(n-j)!} f_X(x) [F_X(x)]^{j-1} [1 - F_X(x)]^{n-j},$$

for  $j = 1, \dots, n$ . The PDF of the  $j^{\text{th}}$  order statistic for the ILO model is determined as follows:

$$\begin{aligned} f_{X_{(j)}}(x) &= \frac{n!}{(j-1)!(n-j)!} \\ &\times \left[ \frac{\alpha-1}{\lambda} \left( 1 + \frac{x}{\lambda} \right)^{-\alpha} \right] \left[ 1 - \left( 1 + \frac{x}{\lambda} \right)^{-(\alpha-1)} \right]^{j-1} \left[ \left( 1 + \frac{x}{\lambda} \right)^{-(\alpha-1)} \right]^{n-j}, \end{aligned}$$

thus, the PDF of the largest order statistic  $X_{(n)}$  is expressed as follows:

$$f_{X_{(n)}}(x) = n \left[ \frac{\alpha-1}{\lambda} \left( 1 + \frac{x}{\lambda} \right)^{-\alpha} \right] \left[ 1 - \left( 1 + \frac{x}{\lambda} \right)^{-(\alpha-1)} \right]^{n-1},$$

and the PDF of the smallest order statistic  $X_{(1)}$  is determined as follows:

$$f_{X_{(1)}}(x) = n \left[ \frac{\alpha-1}{\lambda} \left( 1 + \frac{x}{\lambda} \right)^{-\alpha} \right] \left[ \left( 1 + \frac{x}{\lambda} \right)^{-(\alpha-1)} \right]^{n-1}.$$

## 5. Methods of Estimation

This section looks at six common methods for determining the ILO distribution's parameters  $\alpha$  and  $\lambda$ . These estimating methods maximize or reduce an objective function to produce the best estimates. These methods are the maximum likelihood ( $V_1$ ), least squares ( $V_2$ ), weighted least squares ( $V_3$ ), Cramér Von-Misses ( $V_4$ ), Anderson-Darling ( $V_5$ ), and percentile ( $V_6$ ).

### 5.1. Maximum Likelihood Estimation ( $V_1$ )

The parameters of the ILO distribution were estimated using the  $V_1$  method. Let  $x_1, x_2, \dots, x_n$  be a sample of  $n$  values chosen from the random variables  $X_1, X_2, \dots, X_n$ , which denote the lifespan of the data. As a result, the likelihood function  $L$  is expressed as follows:

$$L = \prod_{i=1}^n f(x_i; \alpha, \lambda) = \prod_{i=1}^n \frac{\alpha - 1}{\lambda} \left(1 + \frac{x_i}{\lambda}\right)^{-\alpha}.$$

Consequently, the log likelihood ( $\ell$ ) function can be determined as below:

$$\begin{aligned} \ell &= \sum_{i=1}^n \ln(f(x_i; \alpha, \lambda)) = \sum_{i=1}^n \ln\left(\frac{\alpha - 1}{\lambda} \left(1 + \frac{x_i}{\lambda}\right)^{-\alpha}\right) \\ &= \sum_{i=1}^n \left[ \ln(\alpha - 1) - \ln(\lambda) + \ln\left(\left(1 + \frac{x_i}{\lambda}\right)^{-\alpha}\right) \right] \\ &= n \ln(\alpha - 1) - n \ln(\lambda) - \alpha \sum_{i=1}^n \ln\left(1 + \frac{x_i}{\lambda}\right). \end{aligned}$$

By taking the first derivative of the function ( $\ell$ ) with respect to the parameters  $\alpha$  and  $\lambda$ , we can find:

$$\frac{\partial \ell}{\partial \alpha} = \frac{n}{\alpha - 1} - \sum_{i=1}^n \ln\left(1 + \frac{x_i}{\lambda}\right), \quad (5.1)$$

and

$$\frac{\partial \ell}{\partial \lambda} = -\frac{n}{\lambda} - \alpha \sum_{i=1}^n \frac{\partial}{\partial \lambda} \ln\left(1 + \frac{x_i}{\lambda}\right). \quad (5.2)$$

We can use computing resources to determine the  $V_1$  estimates of the parameters  $\alpha$  and  $\lambda$  by setting Eqs. (5.1) and (5.2) to zero and solving the resulting equations numerically.

### 5.2. Least Squares Estimation ( $V_2$ )

Our chosen model estimators,  $\hat{\alpha}$  and  $\hat{\lambda}$ , are obtained by minimizing the following equation utilizing the  $V_2$  estimation approach. More details on this approach can be found in [32]:

$$\text{LS} = \sum_{j=1}^n \left[ F(x_{(j)}) - \frac{j}{n+1} \right]^2 = \sum_{j=1}^n \left[ \left(1 - \left(1 + \frac{x_j}{\lambda}\right)^{-(\alpha-1)}\right) - \frac{j}{n+1} \right]^2.$$

### 5.3. Weighted Least Squares Estimation ( $V_3$ )

Our favorite model estimator,  $\hat{\alpha}$ , is obtained by minimizing the next expression employing the  $V_3$  estimation technique. More details on this approach can be found in [33]:

$$\text{WLS} = \sum_{j=1}^n \frac{(n+1)^2(n+2)}{n-j+1} \left[ F(x_{(j)}) - \frac{j}{n+1} \right]^2 = \sum_{j=1}^n \frac{(n+1)^2(n+2)}{n-j+1} \left[ \left(1 - \left(1 + \frac{x_j}{\lambda}\right)^{-(\alpha-1)}\right) - \frac{j}{n+1} \right]^2.$$

#### 5.4. Cramér Von-Misses Estimation ( $V_4$ )

A specific type of minimum distance estimator that relies on the difference between the expected and actual cumulative distribution function is the  $V_4$  method. Our suggested model estimators,  $\hat{\alpha}$  and  $\hat{\lambda}$ , are obtained by minimizing the following equation using the  $V_4$  technique. See [34] for further information on this approach:

$$\text{CVM} = \frac{1}{12n} + \sum_{j=1}^n \left[ F(x_{(j)}) - \frac{2j-1}{2n} \right]^2 = \frac{1}{12n} + \sum_{j=1}^n \left[ \left( 1 - \left( 1 + \frac{x_j}{\lambda} \right)^{-(\alpha-1)} \right) - \frac{2j-1}{2n} \right]^2.$$

#### 5.5. Anderson-Darling Estimation ( $V_5$ )

Our chosen model estimator,  $\hat{\alpha}$  and  $\hat{\lambda}$ , are obtained by minimizing the next equation applying the  $V_5$  estimating approach. For further information on this approach, see [35]:

$$\begin{aligned} \text{AD} &= -n - \frac{1}{n} \sum_{j=1}^n (2j-1) [\log F(x_{(j)}) + \log \bar{F}(x_{(n-j+1)})] \\ &= -n - \frac{1}{n} \sum_{j=1}^n (2j-1) \left[ \log \left( 1 - \left( 1 + \frac{x_j}{\lambda} \right)^{-(\alpha-1)} \right) + \log \left( \left( 1 + \frac{x_{n-j+1}}{\lambda} \right)^{-(\alpha-1)} \right) \right]. \end{aligned}$$

#### 5.6. Percentile Estimation ( $V_6$ )

A distribution parameter can be estimated by using a closed-form distribution function, which plots a straight line against the percentile points. Ref. [36] suggested this method for figuring out the parameters of the new version of geometric distribution. In the distribution function, provided  $n$  random samples  $x_1, x_2, \dots, x_n$  where  $x_{(1)} < \dots < x_{(n)}$  indicates ordered samples, the  $V_6$  estimates of the parameters  $\alpha$  and  $\lambda$  can be obtained by minimizing the following equation:

$$P(\alpha, \lambda) = \sum_{i=1}^n \left[ x_{(i)} - \lambda \left( (1 - p_i)^{\frac{-1}{\alpha-1}} - 1 \right) \right]^2, \quad (5.3)$$

where  $p_i = i/(n+1)$  be an unbiased estimator of  $F(x_{(i)}; \alpha, \lambda)$ . Hence, the estimate  $V_6$  can be derived by differentiating Eq. (5.3) with respect to  $\alpha$  and  $\lambda$ , and equating the resulting equations by zero, we can obtain the values of estimated parameters.

## 6. Simulation Study

This section assesses the efficacy of the estimation methods discussed in Section 5. The simulation datasets were generated using the ILO model, and the unknown values were determined using the estimation methods that were explained. Correlation performance was assessed using mean square errors (MSEs) and relative bias (RBIAS), two distinct metrics. How to collect random samples from the ILO distribution using the inversion strategy is explained below:

The ILO distribution produces random samples  $X_1, X_2, \dots, X_n$  of sizes  $n = 200, 300, 400$  and  $500$ . We examine the values of  $\alpha = 1.6, 1.8, 2, 2.5, 2.8, 3$  and  $\lambda = 0.2, 0.3, 0.4, 0.5, 0.6, 0.7$ . These parameters and sample sizes variables are used to evaluate the ILO model's estimates. For different parameter values, the estimation' biases and mean squared errors (MSEs) are computed. Tables 3 to 8 present the results of the empirical investigation. The following observations can be made based on the results of these tables':

- The ILO distribution is stable according to the mean square errors (MSEs) and reduced bias, which are monitored for their characteristics.
- As the sample size increases, the biases and MSEs of all estimates may occasionally decrease.
- As the sample size increases, the bias and MSE values of all estimators drop, suggesting improved accuracy of the model parameters for prediction.

The purpose of the simulation study is to determine the optimal estimation approach for the suggested model. Table 9 shows the partial and total grades for each estimation technique. The results in Table 9 demonstrate that all parameter estimation methods for the proposed model are highly accurate and closely match the actual values. The calculated metrics often drop as the sample size  $n$  increases.  $V_5$  has been found to be the most effective method for estimating the parameters, with an overall score of 28.5. The  $V_1$  technique can be regarded as a rival to the  $V_5$  approach, given its total score of 51.

Table 3: Numerical outcomes of simulation analysis for the ILO model at  $\alpha = 2, \lambda = 0.4$

$n$	Methods	For $\alpha$			For $\lambda$			$\Sigma$ Ranks
		Average	MSE	RBIAS	Average	MSE	RBIAS	
200	$V_1$	2.0333	0.0263 <sup>1</sup>	0.0167 <sup>4</sup>	0.4244	0.0128 <sup>1</sup>	0.0609 <sup>4</sup>	10 <sup>2</sup>
	$V_2$	2.0219	0.0425 <sup>4</sup>	0.0109 <sup>1</sup>	0.4172	0.0161 <sup>4</sup>	0.1073 <sup>5</sup>	14 <sup>4</sup>
	$V_3$	2.0244	0.0317 <sup>3</sup>	0.0122 <sup>3</sup>	0.4192	0.0179 <sup>5</sup>	0.0431 <sup>1</sup>	12 <sup>3</sup>
	$V_4$	2.0457	0.0481 <sup>5</sup>	0.0229 <sup>5</sup>	0.4313	0.0146 <sup>3</sup>	0.0481 <sup>3</sup>	16 <sup>5</sup>
	$V_5$	2.0239	0.0301 <sup>2</sup>	0.0119 <sup>2</sup>	0.4186	0.0141 <sup>2</sup>	0.0464 <sup>2</sup>	8 <sup>1</sup>
	$V_6$	1.9255	0.1876 <sup>6</sup>	0.0373 <sup>6</sup>	0.5146	0.3073 <sup>6</sup>	0.2866 <sup>6</sup>	24 <sup>6</sup>
300	$V_1$	2.0184	0.0162 <sup>1</sup>	0.0092 <sup>4</sup>	0.4131	0.0077 <sup>1</sup>	0.0328 <sup>4</sup>	10 <sup>2.5</sup>
	$V_2$	2.0077	0.0286 <sup>4</sup>	0.0039 <sup>1</sup>	0.4072	0.0121 <sup>4</sup>	0.0181 <sup>1</sup>	10 <sup>2.5</sup>
	$V_3$	2.0113	0.0212 <sup>3</sup>	0.0057 <sup>3</sup>	0.4092	0.0096 <sup>3</sup>	0.0229 <sup>3</sup>	12 <sup>4</sup>
	$V_4$	2.0231	0.0308 <sup>5</sup>	0.0115 <sup>5</sup>	0.4162	0.0131 <sup>5</sup>	0.0406 <sup>5</sup>	20 <sup>5</sup>
	$V_5$	2.0099	0.0203 <sup>2</sup>	0.0050 <sup>2</sup>	0.4081	0.0093 <sup>2</sup>	0.0203 <sup>2</sup>	8 <sup>1</sup>
	$V_6$	1.9144	0.1613 <sup>6</sup>	0.0428 <sup>6</sup>	0.5081	0.2942 <sup>6</sup>	0.2702 <sup>6</sup>	24 <sup>6</sup>
400	$V_1$	2.0173	0.0119 <sup>1</sup>	0.0087 <sup>4</sup>	0.4132	0.0054 <sup>1</sup>	0.0331 <sup>4</sup>	10 <sup>2.5</sup>
	$V_2$	2.0081	0.0195 <sup>4</sup>	0.0041 <sup>1</sup>	0.4074	0.0078 <sup>4</sup>	0.0186 <sup>1</sup>	10 <sup>2.5</sup>
	$V_3$	2.0116	0.0146 <sup>3</sup>	0.0058 <sup>3</sup>	0.4098	0.0063 <sup>3</sup>	0.0245 <sup>3</sup>	12 <sup>4</sup>
	$V_4$	2.0195	0.0206 <sup>5</sup>	0.0097 <sup>5</sup>	0.4141	0.0082 <sup>5</sup>	0.0353 <sup>5</sup>	20 <sup>5</sup>
	$V_5$	2.0101	0.0139 <sup>2</sup>	0.0050 <sup>2</sup>	0.4086	0.0060 <sup>2</sup>	0.0216 <sup>2</sup>	8 <sup>1</sup>
	$V_6$	1.9343	0.1505 <sup>6</sup>	0.0328 <sup>6</sup>	0.5261	0.2932 <sup>6</sup>	0.3153 <sup>6</sup>	24 <sup>6</sup>
500	$V_1$	2.0143	0.0099 <sup>1</sup>	0.0072 <sup>4</sup>	0.4101	0.0045 <sup>1</sup>	0.0251 <sup>4</sup>	10 <sup>2</sup>
	$V_2$	2.0126	0.0171 <sup>4</sup>	0.0063 <sup>2</sup>	0.4090	0.0069 <sup>4</sup>	0.0226 <sup>2</sup>	12 <sup>3.5</sup>
	$V_3$	2.0127	0.0124 <sup>3</sup>	0.0064 <sup>3</sup>	0.4092	0.0054 <sup>3</sup>	0.0231 <sup>3</sup>	12 <sup>3.5</sup>
	$V_4$	2.0217	0.0179 <sup>5</sup>	0.0109 <sup>5</sup>	0.4144	0.0072 <sup>5</sup>	0.0359 <sup>5</sup>	20 <sup>5</sup>
	$V_5$	2.0114	0.0119 <sup>2</sup>	0.0057 <sup>1</sup>	0.4083	0.0052 <sup>2</sup>	0.0208 <sup>1</sup>	6 <sup>1</sup>
	$V_6$	1.9192	0.1510 <sup>6</sup>	0.0404 <sup>6</sup>	0.5204	0.3104 <sup>6</sup>	0.3011 <sup>6</sup>	24 <sup>6</sup>

### 7. Real-Life Applications

During an examination of actual datasets, the paper explores the applicability and importance of the ILO distribution in this part. In order to demonstrate the model’s applicability, non-parametric data plots can be more appealing because they offer a visual illustration that shows how well the ILO model fits the data as compared to other models. The examination of environmental, and failures of air condition system datasets demonstrates the practical significance of the ILO model and demonstrates its superiority over current competing models. Numerous fields, including engineering, economics, and medical sciences, will be able to use the proposed ILO model. The remaining items provide a summary of the comprehensive overview of the dataset:

- The first dataset pertains to 30 records relating to daily snowfall quantities expressed in inches of water, obtained from unit experiments that were not seeded and executed at [37]. Using real data, this analysis highlights the usefulness of the ILO distribution in simulating complex events involving certain environmental phenomena, offering insightful information to numerous institutions that depend on weather forecasts. The data is presented as follows: 0.030, 0.020, 0.015, 0.045, 0.100, 0.100, 0.125, 0.190, 0.390, 0.110, 0.070, 0.010, 0.055, 0.220, 0.080, 0.005, 0.125, 0.035, 0.085, 0.060, 0.010, 0.065, 0.020, 0.260, 0.030, 0.015, 0.025, 0.010, 0.495, 0.085.

Table 4: Numerical outcomes of simulation analysis for the ILO model at  $\alpha = 2.5$ ,  $\lambda = 0.3$ 

$n$	Methods	For $\alpha$			For $\lambda$			$\sum$ Ranks
		Average	MSE	RBIAS	Average	MSE	RBIAS	
200	V <sub>1</sub>	2.5906	0.1151 <sup>1</sup>	0.0362 <sup>2</sup>	0.3273	0.0111 <sup>1</sup>	0.0909 <sup>3</sup>	7 <sup>1.5</sup>
	V <sub>2</sub>	2.6031	0.2961 <sup>4</sup>	0.0412 <sup>4</sup>	0.3311	0.0245 <sup>4</sup>	0.1038 <sup>5</sup>	17 <sup>4</sup>
	V <sub>3</sub>	2.5921	0.1879 <sup>3</sup>	0.0368 <sup>3</sup>	0.3283	0.0167 <sup>3</sup>	0.0943 <sup>4</sup>	13 <sup>3</sup>
	V <sup>4</sup>	2.6594	0.3681 <sup>5</sup>	0.0638 <sup>6</sup>	0.3461	0.0302 <sup>5</sup>	0.1537 <sup>6</sup>	22 <sup>6</sup>
	V <sub>5</sub>	2.5801	0.1497 <sup>2</sup>	0.0321 <sup>1</sup>	0.3245	0.0136 <sup>2</sup>	0.0818 <sup>2</sup>	7 <sup>1.5</sup>
	V <sub>6</sub>	2.3861	0.4053 <sup>6</sup>	0.0455 <sup>5</sup>	0.2924	0.0602 <sup>6</sup>	0.0254 <sup>1</sup>	18 <sup>5</sup>
300	V <sub>1</sub>	2.5419	0.0594 <sup>1</sup>	0.0168 <sup>4</sup>	0.3134	0.0056 <sup>1</sup>	0.0447 <sup>4</sup>	10 <sup>2.5</sup>
	V <sub>2</sub>	2.5349	0.1280 <sup>4</sup>	0.0139 <sup>3</sup>	0.3115	0.0105 <sup>4</sup>	0.0384 <sup>3</sup>	14 <sup>4</sup>
	V <sub>3</sub>	2.5341	0.0860 <sup>3</sup>	0.0136 <sup>2</sup>	0.3114	0.0076 <sup>3</sup>	0.0381 <sup>2</sup>	10 <sup>2.5</sup>
	V <sup>4</sup>	2.5676	0.1428 <sup>5</sup>	0.0270 <sup>5</sup>	0.3201	0.0117 <sup>5</sup>	0.0671 <sup>6</sup>	21 <sup>5</sup>
	V <sub>5</sub>	2.5284	0.0787 <sup>2</sup>	0.0114 <sup>1</sup>	0.3097	0.0070 <sup>2</sup>	0.0324 <sup>1</sup>	6 <sup>1</sup>
	V <sub>6</sub>	2.3683	0.3241 <sup>6</sup>	0.0527 <sup>6</sup>	0.2842	0.0520 <sup>6</sup>	0.0526 <sup>5</sup>	23 <sup>6</sup>
400	V <sub>1</sub>	2.5306	0.0404 <sup>1</sup>	0.0122 <sup>2</sup>	0.3089	0.0037 <sup>1</sup>	0.0299 <sup>2</sup>	6 <sup>1.5</sup>
	V <sub>2</sub>	2.5356	0.0878 <sup>4</sup>	0.0142 <sup>4</sup>	0.3104	0.0071 <sup>4</sup>	0.0347 <sup>4</sup>	16 <sup>4</sup>
	V <sub>3</sub>	2.5313	0.0576 <sup>3</sup>	0.0125 <sup>3</sup>	0.3093	0.0050 <sup>3</sup>	0.0310 <sup>3</sup>	12 <sup>3</sup>
	V <sup>4</sup>	2.5598	0.0955 <sup>5</sup>	0.0239 <sup>5</sup>	0.3168	0.0077 <sup>5</sup>	0.0558 <sup>5</sup>	20 <sup>5</sup>
	V <sub>5</sub>	2.5263	0.0543 <sup>2</sup>	0.0105 <sup>1</sup>	0.3078	0.0048 <sup>2</sup>	0.0261 <sup>1</sup>	6 <sup>1.5</sup>
	V <sub>6</sub>	2.3137	0.3039 <sup>6</sup>	0.0745 <sup>6</sup>	0.2571	0.0463 <sup>6</sup>	0.1431 <sup>6</sup>	24 <sup>6</sup>
500	V <sub>1</sub>	2.5346	0.0313 <sup>1</sup>	0.0138 <sup>4</sup>	0.3102	0.0030 <sup>1</sup>	0.0339 <sup>4</sup>	10 <sup>2</sup>
	V <sub>2</sub>	2.5285	0.0623 <sup>4</sup>	0.0114 <sup>2</sup>	0.3086	0.0053 <sup>4</sup>	0.0287 <sup>2</sup>	12 <sup>3.5</sup>
	V <sub>3</sub>	2.5288	0.0412 <sup>3</sup>	0.0115 <sup>3</sup>	0.3087	0.0038 <sup>3</sup>	0.0290 <sup>3</sup>	12 <sup>3.5</sup>
	V <sup>4</sup>	2.5474	0.0667 <sup>5</sup>	0.0189 <sup>5</sup>	0.3136	0.0056 <sup>5</sup>	0.0452 <sup>5</sup>	20 <sup>5</sup>
	V <sub>5</sub>	2.5252	0.0396 <sup>2</sup>	0.0101 <sup>1</sup>	0.3076	0.0037 <sup>2</sup>	0.0254 <sup>1</sup>	6 <sup>1</sup>
	V <sub>6</sub>	2.3595	0.2641 <sup>6</sup>	0.0562 <sup>6</sup>	0.2697	0.0425 <sup>6</sup>	0.1009 <sup>6</sup>	24 <sup>6</sup>

Table 5: Numerical outcomes of simulation analysis for the ILO model at  $\alpha = 3, \lambda = 0.5$

$n$	Methods	For $\alpha$			For $\lambda$			$\sum$ Ranks
		Average	MSE	RBIAS	Average	MSE	RBIAS	
200	V <sub>1</sub>	3.1733	0.3591 <sup>1</sup>	0.0578 <sup>3</sup>	0.5602	0.0432 <sup>1</sup>	0.1204 <sup>3</sup>	8 <sup>1.5</sup>
	V <sub>2</sub>	3.2579	0.7459 <sup>4</sup>	0.0966 <sup>5</sup>	0.5862	0.2203 <sup>5</sup>	0.1724 <sup>5</sup>	19 <sup>5</sup>
	V <sub>3</sub>	3.1791	0.6719 <sup>3</sup>	0.0597 <sup>4</sup>	0.5619	0.0736 <sup>3</sup>	0.1239 <sup>4</sup>	14 <sup>4</sup>
	V <sup>4</sup>	3.4162	0.8959 <sup>5</sup>	0.1287 <sup>6</sup>	0.6343	0.5728 <sup>6</sup>	0.2686 <sup>6</sup>	23 <sup>6</sup>
	V <sub>5</sub>	3.1472	0.4749 <sup>2</sup>	0.0491 <sup>2</sup>	0.5513	0.0541 <sup>2</sup>	0.1026 <sup>2</sup>	8 <sup>1.5</sup>
	V <sub>6</sub>	2.8938	0.9049 <sup>6</sup>	0.0354 <sup>1</sup>	0.4799	0.1560 <sup>4</sup>	0.0403 <sup>1</sup>	12 <sup>3</sup>
300	V <sub>1</sub>	3.1007	0.1723 <sup>1</sup>	0.0336 <sup>3</sup>	0.5352	0.0214 <sup>1</sup>	0.0704 <sup>2</sup>	7 <sup>2</sup>
	V <sub>2</sub>	3.1376	0.6685 <sup>5</sup>	0.0459 <sup>4</sup>	0.5483	0.0778 <sup>4</sup>	0.0966 <sup>5</sup>	18 <sup>4</sup>
	V <sub>3</sub>	3.1006	0.2920 <sup>3</sup>	0.0335 <sup>2</sup>	0.5359	0.0353 <sup>3</sup>	0.0718 <sup>3</sup>	11 <sup>3</sup>
	V <sup>4</sup>	3.2033	0.8604 <sup>6</sup>	0.0678 <sup>6</sup>	0.5689	0.1003 <sup>5</sup>	0.1378 <sup>6</sup>	23 <sup>6</sup>
	V <sub>5</sub>	3.0852	0.2379 <sup>2</sup>	0.0284 <sup>1</sup>	0.5306	0.0290 <sup>2</sup>	0.0611 <sup>1</sup>	6 <sup>1</sup>
	V <sub>6</sub>	2.8428	0.5922 <sup>4</sup>	0.0524 <sup>5</sup>	0.4531	0.1078 <sup>6</sup>	0.0937 <sup>4</sup>	19 <sup>5</sup>
400	V <sub>1</sub>	3.0864	0.1184 <sup>1</sup>	0.0288 <sup>1</sup>	0.5306	0.0152 <sup>1</sup>	0.0611 <sup>1</sup>	4 <sup>1</sup>
	V <sub>2</sub>	3.1389	0.4286 <sup>4</sup>	0.0463 <sup>4</sup>	0.5479	0.0486 <sup>4</sup>	0.0959 <sup>4</sup>	16 <sup>4</sup>
	V <sub>3</sub>	3.1078	0.2196 <sup>3</sup>	0.0359 <sup>3</sup>	0.5380	0.0264 <sup>3</sup>	0.0760 <sup>3</sup>	12 <sup>3</sup>
	V <sup>4</sup>	3.1854	0.4930 <sup>6</sup>	0.0618 <sup>5</sup>	0.5624	0.0556 <sup>5</sup>	0.1249 <sup>5</sup>	21 <sup>5</sup>
	V <sub>5</sub>	3.0928	0.1852 <sup>2</sup>	0.0309 <sup>2</sup>	0.5329	0.0227 <sup>2</sup>	0.0659 <sup>2</sup>	8 <sup>2</sup>
	V <sub>6</sub>	2.7998	0.4682 <sup>5</sup>	0.0667 <sup>6</sup>	0.4325	0.0864 <sup>6</sup>	0.1349 <sup>6</sup>	23 <sup>6</sup>
500	V <sub>1</sub>	3.0484	0.0853 <sup>1</sup>	0.0161 <sup>2</sup>	0.5171	0.0105 <sup>1</sup>	0.0341 <sup>2</sup>	6 <sup>1.5</sup>
	V <sub>2</sub>	3.0665	0.2193 <sup>4</sup>	0.0222 <sup>4</sup>	0.5232	0.0245 <sup>4</sup>	0.0464 <sup>4</sup>	16 <sup>4</sup>
	V <sub>3</sub>	3.0520	0.1309 <sup>3</sup>	0.0173 <sup>3</sup>	0.5186	0.0157 <sup>3</sup>	0.0373 <sup>3</sup>	12 <sup>3</sup>
	V <sup>4</sup>	3.0999	0.2399 <sup>5</sup>	0.0333 <sup>5</sup>	0.5335	0.0267 <sup>5</sup>	0.0671 <sup>5</sup>	20 <sup>5</sup>
	V <sub>5</sub>	3.0439	0.1202 <sup>2</sup>	0.0146 <sup>1</sup>	0.5158	0.0145 <sup>2</sup>	0.0316 <sup>1</sup>	6 <sup>1.5</sup>
	V <sub>6</sub>	2.7798	0.4314 <sup>6</sup>	0.0734 <sup>6</sup>	0.4175	0.0775 <sup>6</sup>	0.1649 <sup>6</sup>	24 <sup>6</sup>

Table 6: Numerical outcomes of simulation analysis for the ILO model at  $\alpha = 1.8$ ,  $\lambda = 0.6$ 

$n$	Methods	For $\alpha$			For $\lambda$			$\sum$ Ranks
		Average	MSE	RBIAS	Average	MSE	RBIAS	
200	V <sub>1</sub>	1.8249	0.0127 <sup>1</sup>	0.0138 <sup>4</sup>	0.6423	0.0289 <sup>1</sup>	0.0705 <sup>4</sup>	10 <sup>2.5</sup>
	V <sub>2</sub>	1.8134	0.0193 <sup>4</sup>	0.0075 <sup>1</sup>	0.6280	0.0366 <sup>4</sup>	0.0467 <sup>1</sup>	10 <sup>2.5</sup>
	V <sub>3</sub>	1.8171	0.0148 <sup>3</sup>	0.0095 <sup>3</sup>	0.6331	0.0313 <sup>3</sup>	0.0551 <sup>3</sup>	12 <sup>4</sup>
	V <sup>4</sup>	1.8295	0.0214 <sup>5</sup>	0.0164 <sup>6</sup>	0.6475	0.0408 <sup>5</sup>	0.0792 <sup>5</sup>	21 <sup>5</sup>
	V <sub>5</sub>	1.8169	0.0141 <sup>2</sup>	0.0094 <sup>2</sup>	0.6323	0.0302 <sup>2</sup>	0.0539 <sup>2</sup>	8 <sup>1</sup>
	V <sub>6</sub>	1.7718	0.1305 <sup>6</sup>	0.0157 <sup>5</sup>	0.8586	0.4235 <sup>6</sup>	0.1253 <sup>6</sup>	23 <sup>6</sup>
300	V <sub>1</sub>	1.8169	0.0080 <sup>1</sup>	0.0094 <sup>4</sup>	0.6238	0.0160 <sup>1</sup>	0.0396 <sup>4</sup>	10 <sup>2.5</sup>
	V <sub>2</sub>	1.8091	0.0121 <sup>4</sup>	0.0051 <sup>1</sup>	0.6141	0.0209 <sup>4</sup>	0.0234 <sup>1</sup>	10 <sup>2.5</sup>
	V <sub>3</sub>	1.8113	0.0093 <sup>3</sup>	0.0063 <sup>2</sup>	0.6171	0.0175 <sup>3</sup>	0.0285 <sup>3</sup>	11 <sup>4</sup>
	V <sup>4</sup>	1.8196	0.0129 <sup>5</sup>	0.0109 <sup>5</sup>	0.6265	0.0225 <sup>5</sup>	0.0442 <sup>5</sup>	20 <sup>5</sup>
	V <sub>5</sub>	1.8113	0.0091 <sup>2</sup>	0.0064 <sup>3</sup>	0.6168	0.0171 <sup>2</sup>	0.0279 <sup>2</sup>	9 <sup>1</sup>
	V <sub>6</sub>	1.7606	0.1149 <sup>6</sup>	0.0219 <sup>6</sup>	0.8145	0.0945 <sup>6</sup>	0.1124 <sup>6</sup>	24 <sup>6</sup>
400	V <sub>1</sub>	1.8057	0.0058 <sup>1</sup>	0.0031 <sup>4</sup>	0.6121	0.0115 <sup>1</sup>	0.0201 <sup>4</sup>	10 <sup>2.5</sup>
	V <sub>2</sub>	1.8022	0.0091 <sup>4</sup>	0.0012 <sup>1</sup>	0.6082	0.0161 <sup>4</sup>	0.0137 <sup>1</sup>	10 <sup>2.5</sup>
	V <sub>3</sub>	1.8033	0.0070 <sup>3</sup>	0.0018 <sup>3</sup>	0.6094	0.0133 <sup>3</sup>	0.0157 <sup>3</sup>	12 <sup>4</sup>
	V <sup>4</sup>	1.8099	0.0095 <sup>5</sup>	0.0055 <sup>5</sup>	0.6174	0.0169 <sup>5</sup>	0.0291 <sup>5</sup>	20 <sup>5</sup>
	V <sub>5</sub>	1.8027	0.0068 <sup>2</sup>	0.0015 <sup>2</sup>	0.6085	0.0129 <sup>2</sup>	0.0141 <sup>2</sup>	8 <sup>1</sup>
	V <sub>6</sub>	1.7523	0.0980 <sup>6</sup>	0.0265 <sup>6</sup>	0.8547	0.1026 <sup>6</sup>	0.1345 <sup>6</sup>	24 <sup>6</sup>
500	V <sub>1</sub>	1.8095	0.0041 <sup>1</sup>	0.0053 <sup>4</sup>	0.6154	0.0082 <sup>1</sup>	0.0257 <sup>4</sup>	10 <sup>2.5</sup>
	V <sub>2</sub>	1.8079	0.0069 <sup>4</sup>	0.0044 <sup>1</sup>	0.6135	0.0120 <sup>4</sup>	0.0224 <sup>1</sup>	10 <sup>2.5</sup>
	V <sub>3</sub>	1.8086	0.0051 <sup>3</sup>	0.0048 <sup>3</sup>	0.6145	0.0098 <sup>3</sup>	0.0242 <sup>3</sup>	12 <sup>4</sup>
	V <sup>4</sup>	1.8142	0.0072 <sup>5</sup>	0.0079 <sup>5</sup>	0.6208	0.0126 <sup>5</sup>	0.0347 <sup>5</sup>	20 <sup>5</sup>
	V <sub>5</sub>	1.8084	0.0050 <sup>2</sup>	0.0046 <sup>2</sup>	0.6139	0.0097 <sup>2</sup>	0.0233 <sup>2</sup>	8 <sup>1</sup>
	V <sub>6</sub>	1.7464	0.1001 <sup>6</sup>	0.0297 <sup>6</sup>	0.8978	0.0925 <sup>6</sup>	0.1028 <sup>6</sup>	24 <sup>6</sup>

Table 7: Numerical outcomes of simulation analysis for the ILO model at  $\alpha = 2.8$ ,  $\lambda = 0.7$ 

$n$	Methods	For $\alpha$			For $\lambda$			$\sum$ Ranks
		Average	MSE	RBIAS	Average	MSE	RBIAS	
200	V <sub>1</sub>	2.9063	0.1998 <sup>1</sup>	0.0379 <sup>2</sup>	0.7637	0.0634 <sup>1</sup>	0.0911 <sup>3</sup>	7 <sup>1.5</sup>
	V <sub>2</sub>	2.9581	0.8361 <sup>5</sup>	0.0565 <sup>5</sup>	0.7916	0.2446 <sup>5</sup>	0.1308 <sup>5</sup>	20 <sup>5</sup>
	V <sub>3</sub>	2.9083	0.3479 <sup>3</sup>	0.0387 <sup>3</sup>	0.7657	0.1051 <sup>3</sup>	0.0939 <sup>4</sup>	13 <sup>3.5</sup>
	V <sup>4</sup>	3.0493	1.2449 <sup>6</sup>	0.0891 <sup>6</sup>	0.8374	0.3663 <sup>6</sup>	0.1963 <sup>6</sup>	24 <sup>6</sup>
	V <sub>5</sub>	2.8896	0.2721 <sup>2</sup>	0.0321 <sup>1</sup>	0.7552	0.0829 <sup>2</sup>	0.0789 <sup>2</sup>	7 <sup>1.5</sup>
	V <sub>6</sub>	2.6727	0.5052 <sup>4</sup>	0.0455 <sup>4</sup>	0.6685	0.2417 <sup>4</sup>	0.0449 <sup>1</sup>	13 <sup>3.5</sup>
300	V <sub>1</sub>	2.8855	0.1253 <sup>1</sup>	0.0305 <sup>2</sup>	0.7504	0.0425 <sup>1</sup>	0.0721 <sup>2</sup>	6 <sup>1.5</sup>
	V <sub>2</sub>	2.9036	0.2868 <sup>4</sup>	0.0369 <sup>4</sup>	0.7599	0.0841 <sup>4</sup>	0.0855 <sup>5</sup>	17 <sup>4</sup>
	V <sub>3</sub>	2.8866	0.1761 <sup>3</sup>	0.0309 <sup>3</sup>	0.7513	0.0557 <sup>3</sup>	0.0733 <sup>3</sup>	12 <sup>3</sup>
	V <sup>4</sup>	2.9528	0.3312 <sup>5</sup>	0.0546 <sup>6</sup>	0.7842	0.0967 <sup>5</sup>	0.1203 <sup>6</sup>	22 <sup>6</sup>
	V <sub>5</sub>	2.8724	0.1530 <sup>2</sup>	0.0259 <sup>1</sup>	0.7434	0.0487 <sup>2</sup>	0.0619 <sup>1</sup>	6 <sup>1.5</sup>
	V <sub>6</sub>	2.6605	0.5375 <sup>6</sup>	0.0498 <sup>5</sup>	0.6477	0.2667 <sup>6</sup>	0.0747 <sup>4</sup>	21 <sup>5</sup>
400	V <sub>1</sub>	2.8557	0.0809 <sup>1</sup>	0.0199 <sup>2</sup>	0.7289	0.0251 <sup>1</sup>	0.0414 <sup>2</sup>	6 <sup>1.5</sup>
	V <sub>2</sub>	2.8798	0.2077 <sup>4</sup>	0.0285 <sup>4</sup>	0.7415	0.0573 <sup>4</sup>	0.0592 <sup>4</sup>	16 <sup>4</sup>
	V <sub>3</sub>	2.8627	0.1286 <sup>3</sup>	0.0224 <sup>3</sup>	0.7328	0.0375 <sup>3</sup>	0.0469 <sup>3</sup>	12 <sup>3</sup>
	V <sup>4</sup>	2.9154	0.2308 <sup>5</sup>	0.0412 <sup>5</sup>	0.7588	0.0635 <sup>5</sup>	0.0841 <sup>5</sup>	20 <sup>5</sup>
	V <sub>5</sub>	2.8528	0.1164 <sup>2</sup>	0.0189 <sup>1</sup>	0.7275	0.0341 <sup>2</sup>	0.0393 <sup>1</sup>	6 <sup>1.5</sup>
	V <sub>6</sub>	2.6475	0.4201 <sup>6</sup>	0.0545 <sup>6</sup>	0.6324	0.2039 <sup>6</sup>	0.0965 <sup>6</sup>	24 <sup>6</sup>
500	V <sub>1</sub>	2.8336	0.0621 <sup>1</sup>	0.0119 <sup>4</sup>	0.7191	0.0198 <sup>1</sup>	0.0274 <sup>4</sup>	10 <sup>2.5</sup>
	V <sub>2</sub>	2.8246	0.1392 <sup>4</sup>	0.0088 <sup>3</sup>	0.7148	0.0391 <sup>4</sup>	0.0212 <sup>3</sup>	14 <sup>4</sup>
	V <sub>3</sub>	2.8244	0.0882 <sup>3</sup>	0.0087 <sup>2</sup>	0.7147	0.0267 <sup>3</sup>	0.0211 <sup>2</sup>	10 <sup>2.5</sup>
	V <sup>4</sup>	2.8508	0.1495 <sup>5</sup>	0.0182 <sup>5</sup>	0.7277	0.0419 <sup>5</sup>	0.0395 <sup>5</sup>	20 <sup>5</sup>
	V <sub>5</sub>	2.8199	0.0836 <sup>2</sup>	0.0071 <sup>1</sup>	0.7122	0.0255 <sup>2</sup>	0.0174 <sup>1</sup>	6 <sup>1</sup>
	V <sub>6</sub>	2.6151	0.3702 <sup>6</sup>	0.0661 <sup>6</sup>	0.6058	0.1761 <sup>6</sup>	0.1345 <sup>6</sup>	24 <sup>6</sup>

Table 8: Numerical outcomes of simulation analysis for the ILO model at  $\alpha = 1.6$ ,  $\lambda = 0.2$ 

$n$	Methods	For $\alpha$			For $\lambda$			$\sum$ Ranks
		Average	MSE	RBIAS	Average	MSE	RBIAS	
200	V <sub>1</sub>	1.6113	0.0053 <sup>1</sup>	0.0071 <sup>5</sup>	0.2081	0.0027 <sup>1</sup>	0.0407 <sup>4</sup>	11 <sup>2.5</sup>
	V <sub>2</sub>	1.6076	0.0081 <sup>4</sup>	0.0048 <sup>2</sup>	0.2055	0.0035 <sup>4</sup>	0.0277 <sup>1</sup>	11 <sup>2.5</sup>
	V <sub>3</sub>	1.6091	0.0064 <sup>3</sup>	0.0057 <sup>4</sup>	0.2066	0.0029 <sup>3</sup>	0.0332 <sup>3</sup>	13 <sup>4</sup>
	V <sup>4</sup>	1.6175	0.0088 <sup>5</sup>	0.0109 <sup>6</sup>	0.2115	0.0038 <sup>5</sup>	0.0573 <sup>5</sup>	21 <sup>6</sup>
	V <sub>5</sub>	1.6086	0.0061 <sup>2</sup>	0.0054 <sup>3</sup>	0.2062	0.0028 <sup>2</sup>	0.0312 <sup>2</sup>	9 <sup>1</sup>
	V <sub>6</sub>	1.6074	0.0866 <sup>6</sup>	0.0046 <sup>1</sup>	0.5239	0.1423 <sup>6</sup>	0.0989 <sup>6</sup>	19 <sup>5</sup>
300	V <sub>1</sub>	1.6082	0.0032 <sup>1</sup>	0.0051 <sup>4</sup>	0.2058	0.0016 <sup>1</sup>	0.0288 <sup>4</sup>	10 <sup>2.5</sup>
	V <sub>2</sub>	1.6042	0.0047 <sup>4</sup>	0.0026 <sup>1</sup>	0.2034	0.0021 <sup>4</sup>	0.0168 <sup>1</sup>	10 <sup>2.5</sup>
	V <sub>3</sub>	1.6059	0.0037 <sup>3</sup>	0.0037 <sup>3</sup>	0.2044	0.0018 <sup>3</sup>	0.0221 <sup>3</sup>	12 <sup>4</sup>
	V <sup>4</sup>	1.6107	0.0049 <sup>5</sup>	0.0067 <sup>5</sup>	0.2072	0.0022 <sup>5</sup>	0.0361 <sup>5</sup>	20 <sup>5</sup>
	V <sub>5</sub>	1.6057	0.0036 <sup>2</sup>	0.0036 <sup>2</sup>	0.2043	0.0017 <sup>2</sup>	0.0213 <sup>2</sup>	8 <sup>1</sup>
	V <sub>6</sub>	1.6124	0.0794 <sup>6</sup>	0.0078 <sup>6</sup>	0.5749	0.1647 <sup>6</sup>	0.0991 <sup>6</sup>	24 <sup>6</sup>
400	V <sub>1</sub>	1.6046	0.0025 <sup>1</sup>	0.0029 <sup>5</sup>	0.2033	0.0011 <sup>1</sup>	0.0167 <sup>4</sup>	11 <sup>2.5</sup>
	V <sub>2</sub>	1.6022	0.0037 <sup>4</sup>	0.0014 <sup>2</sup>	0.2019	0.0014 <sup>4</sup>	0.0094 <sup>1</sup>	11 <sup>2.5</sup>
	V <sub>3</sub>	1.6032	0.0029 <sup>3</sup>	0.0019 <sup>4</sup>	0.2025	0.0012 <sup>2.5</sup>	0.0125 <sup>3</sup>	12.5 <sup>4</sup>
	V <sup>4</sup>	1.6071	0.0038 <sup>5</sup>	0.0044 <sup>6</sup>	0.2047	0.0015 <sup>5</sup>	0.0236 <sup>5</sup>	21 <sup>6</sup>
	V <sub>5</sub>	1.6029	0.0028 <sup>2</sup>	0.0018 <sup>3</sup>	0.2023	0.0012 <sup>2.5</sup>	0.0116 <sup>2</sup>	9.5 <sup>1</sup>
	V <sub>6</sub>	1.5979	0.0716 <sup>6</sup>	0.0012 <sup>1</sup>	0.5987	0.1689 <sup>6</sup>	0.1004 <sup>6</sup>	19 <sup>5</sup>
500	V <sub>1</sub>	1.6037	0.0020 <sup>1</sup>	0.0023 <sup>4</sup>	0.2036	0.0010 <sup>2</sup>	0.0182 <sup>4</sup>	11 <sup>4</sup>
	V <sub>2</sub>	1.6010	0.0027 <sup>4</sup>	0.0006 <sup>1</sup>	0.2019	0.0012 <sup>4.5</sup>	0.0097 <sup>1</sup>	10.5 <sup>2.5</sup>
	V <sub>3</sub>	1.6023	0.0022 <sup>2.5</sup>	0.0015 <sup>3</sup>	0.2028	0.0010 <sup>2</sup>	0.0139 <sup>3</sup>	10.5 <sup>2.5</sup>
	V <sup>4</sup>	1.6048	0.0028 <sup>5</sup>	0.0030 <sup>5</sup>	0.2042	0.0012 <sup>4.5</sup>	0.0211 <sup>5</sup>	19.5 <sup>5</sup>
	V <sub>5</sub>	1.6021	0.0022 <sup>2.5</sup>	0.0013 <sup>2</sup>	0.2026	0.0010 <sup>2</sup>	0.0130 <sup>2</sup>	8.5 <sup>1</sup>
	V <sub>6</sub>	1.6089	0.0686 <sup>6</sup>	0.0056 <sup>6</sup>	0.6423	0.1701 <sup>6</sup>	0.1123 <sup>6</sup>	24 <sup>6</sup>

Table 9: Partial and comprehensive rankings of all estimation methods for the ILO model

Parameters	$n$	$V_1$	$V_2$	$V_3$	$V^4$	$V_5$	$V_6$
$\alpha = 2, \lambda = 0.4$	200	2	4	3	5	1	6
	300	2.5	2.5	4	5	1	6
	400	2.5	2.5	4	5	1	6
	500	2	3.5	3.5	5	1	6
$\alpha = 2.5, \lambda = 0.3$	200	1.5	4	3	6	1.5	5
	300	2.5	4	2.5	5	1	6
	400	1.5	4	3	5	1.5	6
	500	2	3.5	3.5	5	1	6
$\alpha = 3, \lambda = 0.5$	200	1.5	5	4	6	1.5	3
	300	2	4	3	6	1	5
	400	1	4	3	5	2	6
	500	1.5	4	3	5	1.5	6
$\alpha = 1.8, \lambda = 0.6$	200	2.5	2.5	4	5	1	6
	300	2.5	2.5	4	5	1	6
	400	2.5	2.5	4	5	1	6
	500	2.5	2.5	4	5	1	6
$\alpha = 2.8, \lambda = 0.7$	200	1.5	5	3.5	6	1.5	3.5
	300	1.5	4	3	6	1.5	5
	400	1.5	4	3	5	1.5	6
	500	2.5	4	2.5	5	1	6
$\alpha = 1.6, \lambda = 0.2$	200	2.5	2.5	4	6	1	5
	300	2.5	2.5	4	5	1	6
	400	2.5	2.5	4	6	1	5
	500	4	2.5	2.5	5	1	6
$\sum$ Ranks		51	82	82	127	28.5	133.5
Overall Rank		2	3.5	3.5	5	1	6

- The second data set, which was examined by Linhart and Zucchini [38], takes into account 30 airplane air conditioning system failures (measured in hours). Bantan et al. [39] recently adjusted this data set to produce data between 0 and 1. The dataset is presented as follows: 0.018867925, 0.045283019, 0.086792453, 0.984905660, 0.328301887, 0.026415094, 0.452830189, 0.052830189, 0.233962264, 0.196226415, 0.358490566, 0.177358491, 0.849056604, 0.267924528, 0.928301887, 0.079245283, 0.158490566, 0.075471698, 0.452830189, 0.041509434, 0.011320755, 0.052830189, 0.041509434, 0.052830189, 0.267924528, 0.041509434, 0.060377358, 0.339622642, 0.003773585, 0.060377358.

Table 10 primarily examines the two datasets. A few visual representations of these datasets are shown in Figures 4 and 5, respectively. These include quantile-quantile (QQ) charts, violin plots, box plots, total time on test (TTT) graphs, histograms, and kernel density estimations. The first dataset, as illustrated in Figure 4, is asymmetric with multiple outliers, highly right-skewed with a large right tail, and the HRF shows an increasing pattern. Additionally, figure 5 shows that the second data set is asymmetric, right-skewed, and likely multi-modal, with multiple outliers. The TTT-plot also reveals an increasing HRF.

Table 10: Statistical summary of the two datasets.

Dataset	n	Mean	Median	Variance	Skewness	Kurtosis	Range	Min	Max
I	30	0.0962	0.0625	0.0131	2.110	7.1582	0.49	0.005	0.495
II	30	0.2249	0.0830	0.0736	1.6936	4.9667	0.9811	0.0038	0.9849

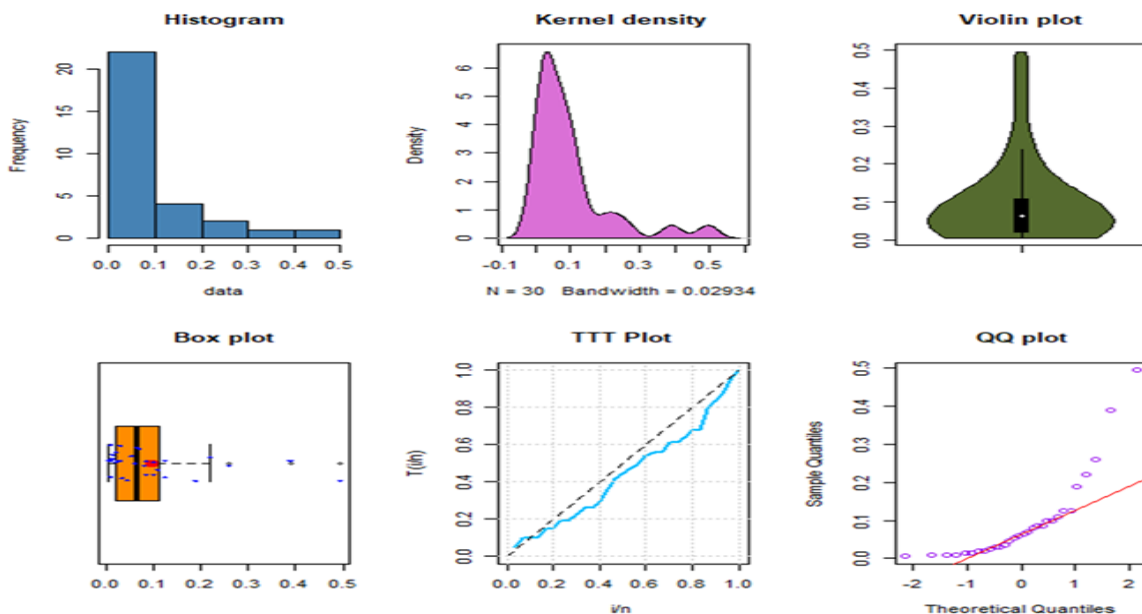


Figure 4: Some nonparametric plots for dataset I

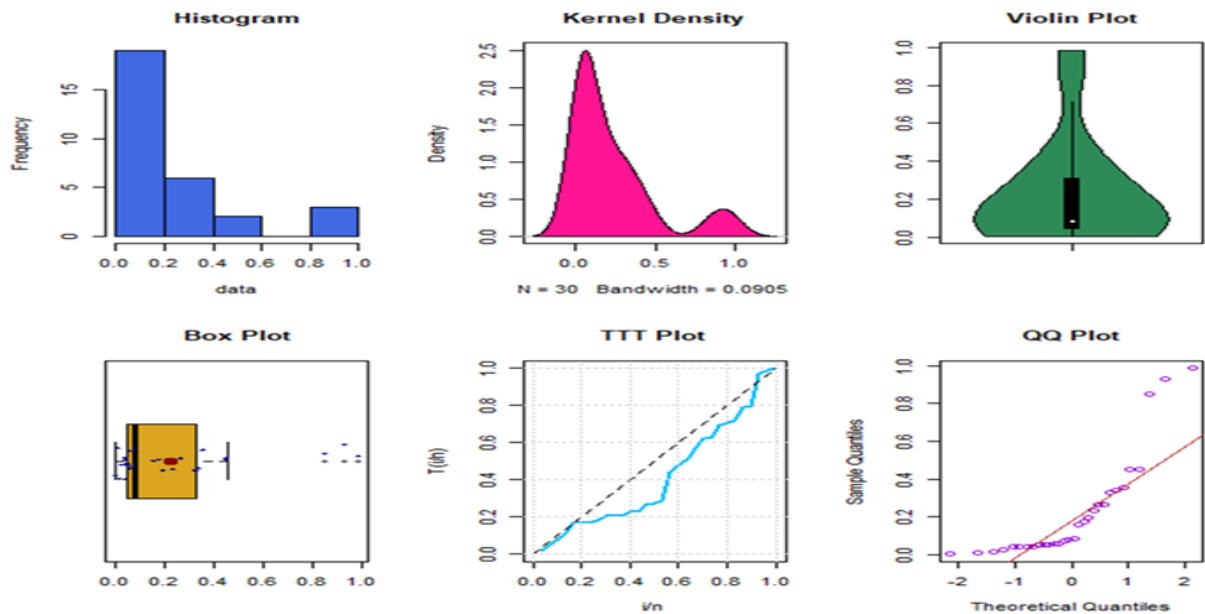


Figure 5: Some nonparametric plots for dataset II

The ILO model is thoroughly compared to other competing models, such as the generalized exponential (GIE<sub>x</sub>), beta, gamma (Ga), Kumaraswamy (Kum), and inverse exponential (IE) distributions, in this comparison analysis. The programming language, R, was used to estimate parameters and evaluate goodness-of-fit using the maximum likelihood approach. The findings for parameters estimate and the associated standard errors are shown in Tables 11 and 12. Given its lowest values in various goodness-of-fit metrics, such as the Akaike information criterion ( $S_1$ ), Bayesian information criterion ( $S_2$ ), corrected Akaike information criterion ( $S_3$ ), Hannan-Quinn information criterion ( $S_4$ ), Kolmogorov–Smirnov distance ( $S_5$ ) with associated P-value ( $S_6$ ), and Cramér–von Mises test ( $S_7$ ) for each distribution. The ILO model performs better than all other examined distributions, according to Tables 13–14’s findings, as evidenced by its lowest values in several goodness-of-fit metrics, such as  $S_1$ ,  $S_2$ ,  $S_3$ ,  $S_4$ ,  $S_5$ ,  $S_7$ , and the highest  $S_6$ . According to Tables 15 and 16, which provide the estimated values for the different ILO model estimating methods for these datasets, the  $V_1$  method performs better than the others for the first dataset, while the  $V_4$  method works better for the second dataset, as shown by the highest  $S_6$ . Furthermore, given numerical data, Figures 6 and 7 provide graphical depictions of densities, empirical CDFs, and P-P plots for each competing model, allowing for a visual evaluation of model effectiveness. Additionally, Figures 8–13 provide a visual comparison of the two datasets shapes using the fitted PDFs, CDFs, and P-P plots.

Table 11: Results of MLE estimates with standard errors for data set I

Models	Est. of $\alpha$	SE( $\hat{\alpha}$ )	Est. of $\lambda$	SE( $\hat{\lambda}$ )
ILO	6.4081	6.5756	0.4259	0.6067
GIE <sub>x</sub>	0.9517	0.2301	0.0275	0.0068
Beta	0.8576	0.1925	7.8057	2.2431
Ga	0.9687	0.2195	0.0993	0.0291
Kum	0.8615	0.1376	6.8355	2.3235
IE	0.0285	0.0052	–	–

Table 12: Results of MLE estimates with standard errors for data set II

Models	Est. of $\alpha$	SE( $\hat{\alpha}$ )	Est. of $\lambda$	SE( $\hat{\lambda}$ )
ILO	4.2967	3.0761	0.5332	0.6359
GIE <sub>x</sub>	0.6493	0.1505	0.0307	0.0087
Beta	0.5142	0.1118	1.3431	0.3643
Ga	0.8119	0.1811	0.2771	0.0835
Kum	0.5452	0.1148	1.3839	0.3362
IE	0.0422	0.0076	–	–

Table 13: Results of goodness of-fit of the ILO model for data set I

Models	S <sub>1</sub>	S <sub>2</sub>	S <sub>3</sub>	S <sub>4</sub>	S <sub>5</sub>	S <sub>6</sub>	S <sub>7</sub>
ILO	-79.168	-77.767	-79.025	-78.721	0.085	0.983	0.028
GIE <sub>x</sub>	-75.228	-72.425	-74.783	-74.331	0.155	0.468	0.124
Beta	-75.122	-72.319	-74.677	-74.225	0.129	0.702	0.086
Ga	-76.520	-73.718	-76.076	-75.624	0.106	0.892	0.058
Kum	-75.595	-72.793	-75.151	-74.699	0.121	0.774	0.065
IE	-77.185	-75.785	-77.043	-76.737	0.163	0.406	0.141

Table 14: Results of goodness of-fit of the ILO model for data set II

Models	S <sub>1</sub>	S <sub>2</sub>	S <sub>3</sub>	S <sub>4</sub>	S <sub>5</sub>	S <sub>6</sub>	S <sub>7</sub>
ILO	-27.109	-24.306	-26.664	-26.212	0.142	0.585	0.093
GIE <sub>x</sub>	-16.398	-13.596	-15.953	-15.501	0.211	0.139	0.225
Beta	-22.493	-19.691	-22.048	-21.596	0.196	0.201	0.289
Ga	-26.449	-23.647	-26.005	-25.553	0.169	0.356	0.117
Kum	-23.078	-20.275	-22.633	-22.181	0.188	0.241	0.246
IE	-14.659	-13.259	-14.517	-14.212	0.233	0.077	0.486

Table 15: Results of comparison between estimation methods of the ILO model for data set I

Methods	$\alpha$	$\lambda$	S <sub>5</sub>	S <sub>6</sub>
V <sub>1</sub>	6.4151	0.4267	0.0846	0.9828
V <sub>2</sub>	4.6715	0.2741	0.0899	0.9684
V <sub>3</sub>	4.6267	0.2748	0.0901	0.9682
V <sub>4</sub>	7.6661	0.5253	0.0848	0.9823
V <sub>5</sub>	5.8836	0.3799	0.0858	0.9799
V <sub>6</sub>	6.5369	0.5134	0.1325	0.6677

Table 16: Results of comparison between estimation methods of the ILO model for data set II

Methods	$\alpha$	$\lambda$	S <sub>5</sub>	S <sub>6</sub>
V <sub>1</sub>	4.2795	0.5332	0.1416	0.5847
V <sub>2</sub>	2.3371	0.1665	0.1241	0.7451
V <sub>3</sub>	2.7469	0.2407	0.1173	0.8036
V <sub>4</sub>	2.5785	0.2071	0.1171	0.8046
V <sub>5</sub>	2.9579	0.2763	0.1191	0.7885
V <sub>6</sub>	5.3037	0.8789	0.2001	0.1808

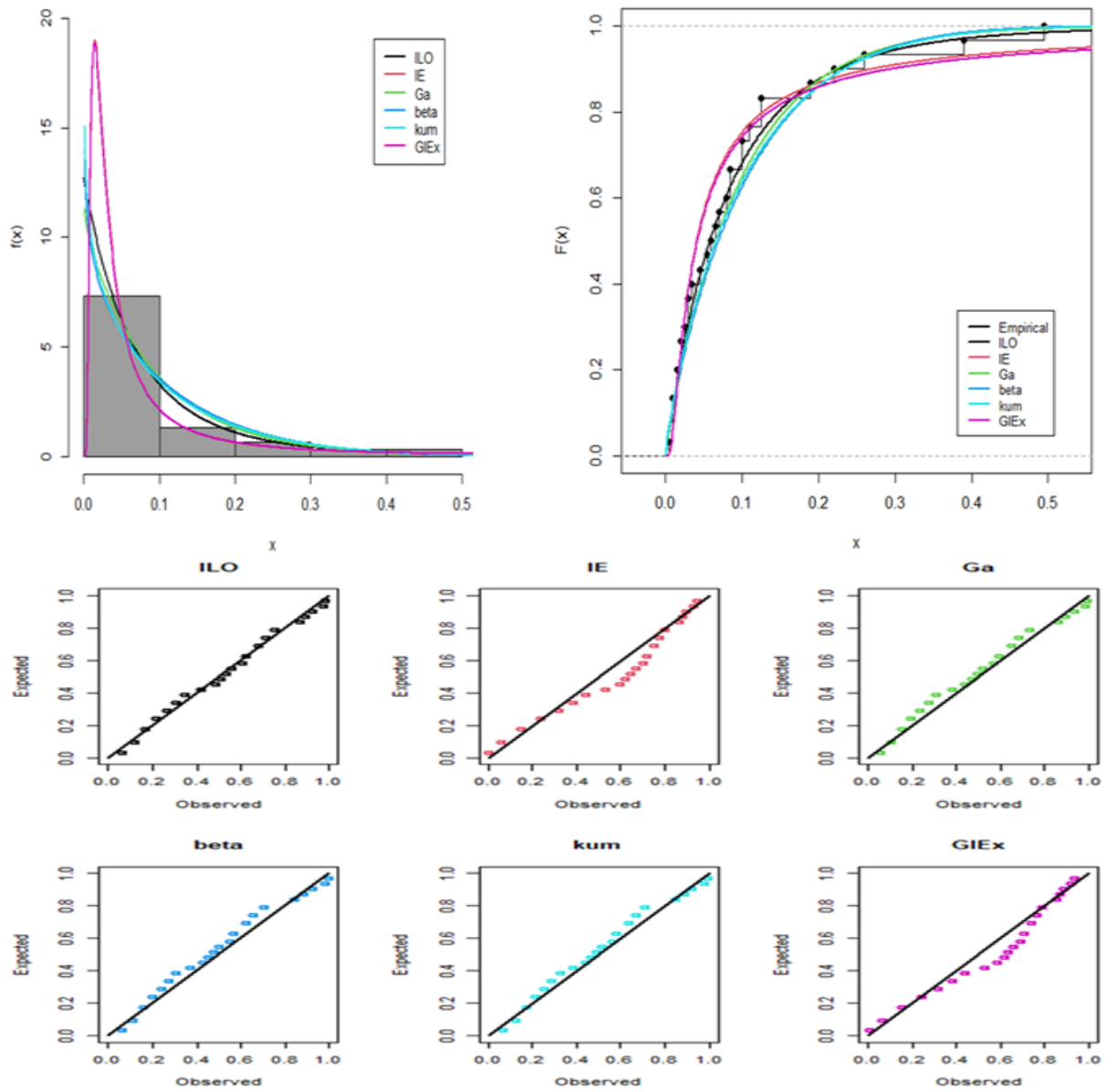


Figure 6: Plots of densities, CDF and P-P Plots for dataset I

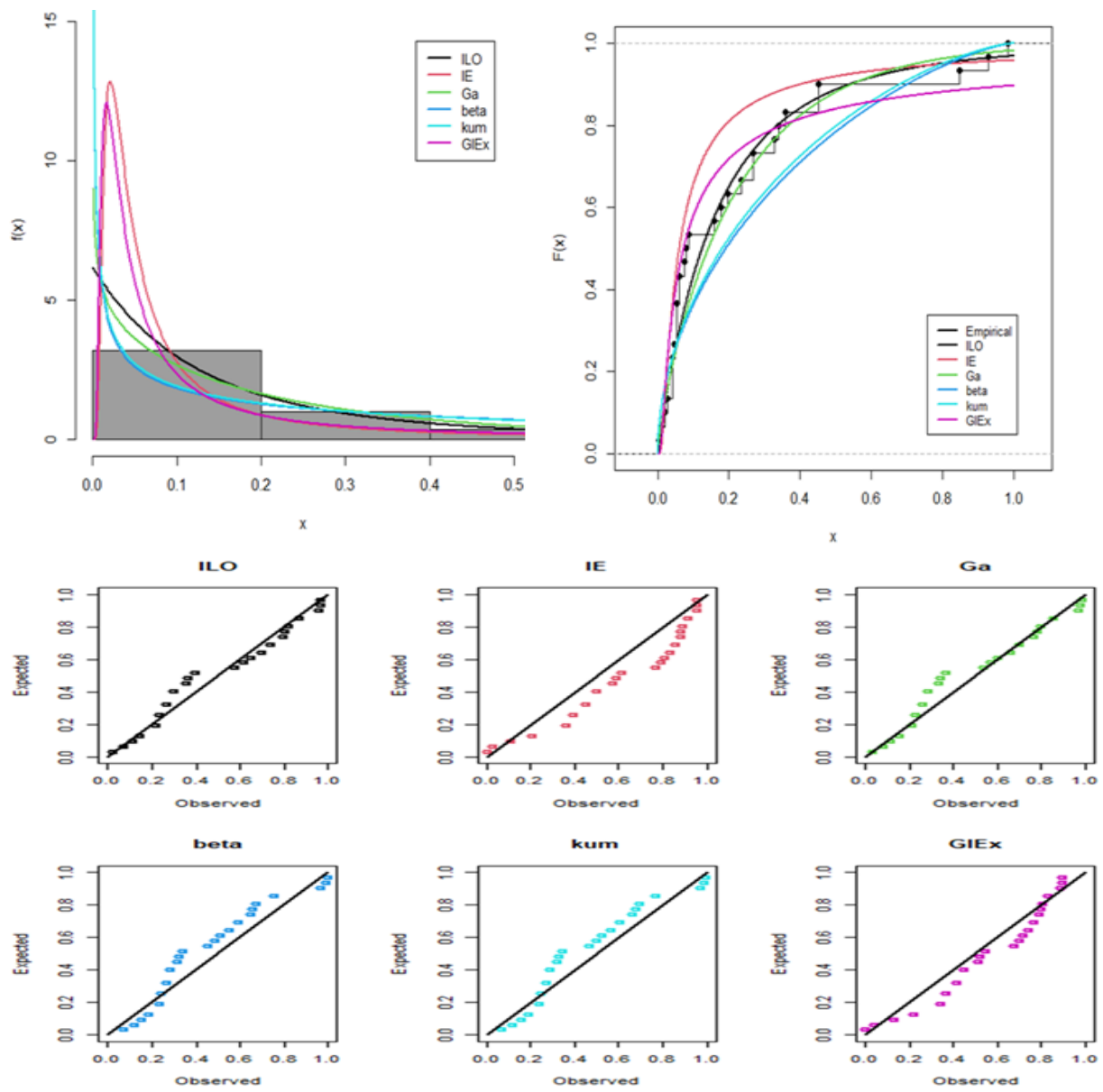


Figure 7: Plots of densities, CDF and P-P Plots for dataset II

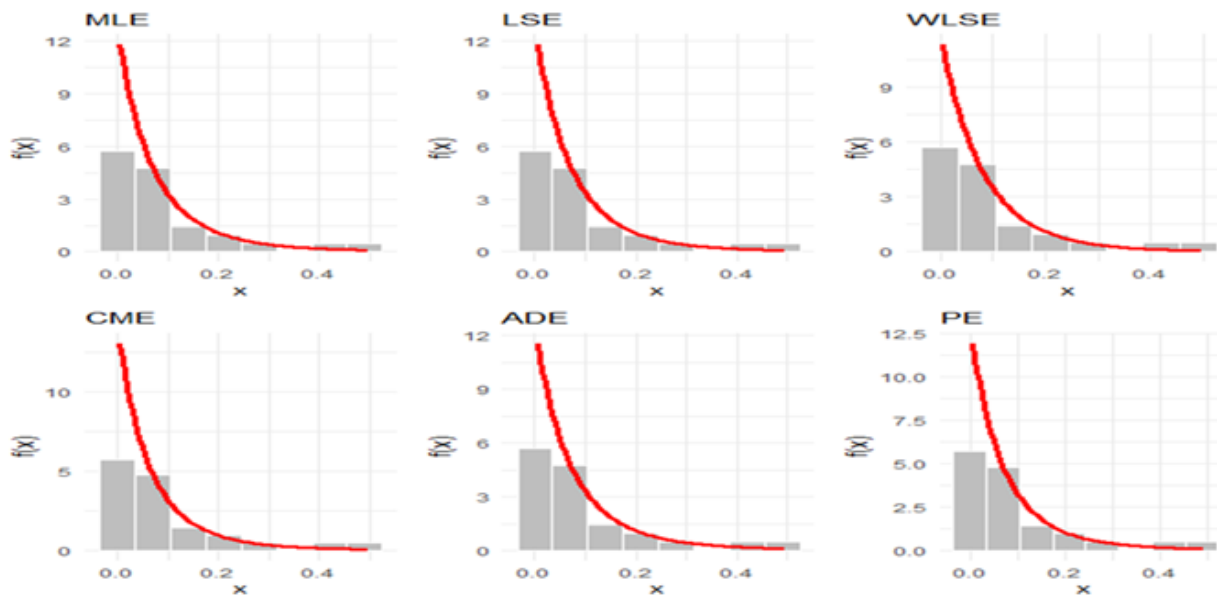


Figure 8: Plots of estimated PDFs for dataset I over estimation methods

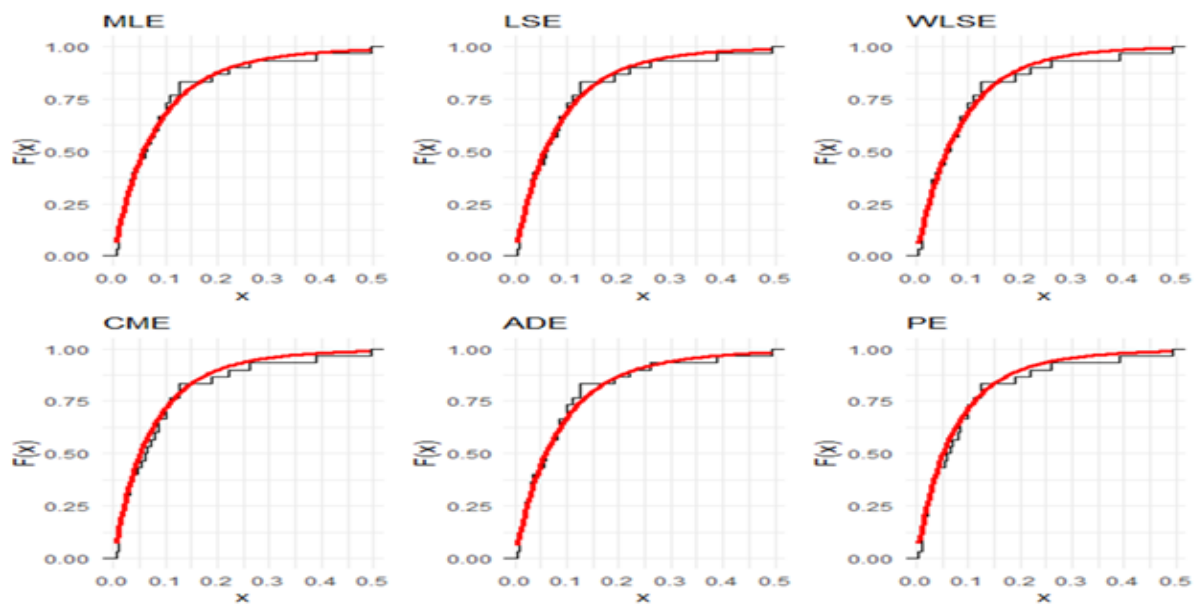


Figure 9: Plots of estimated CDFs for dataset I over estimation methods

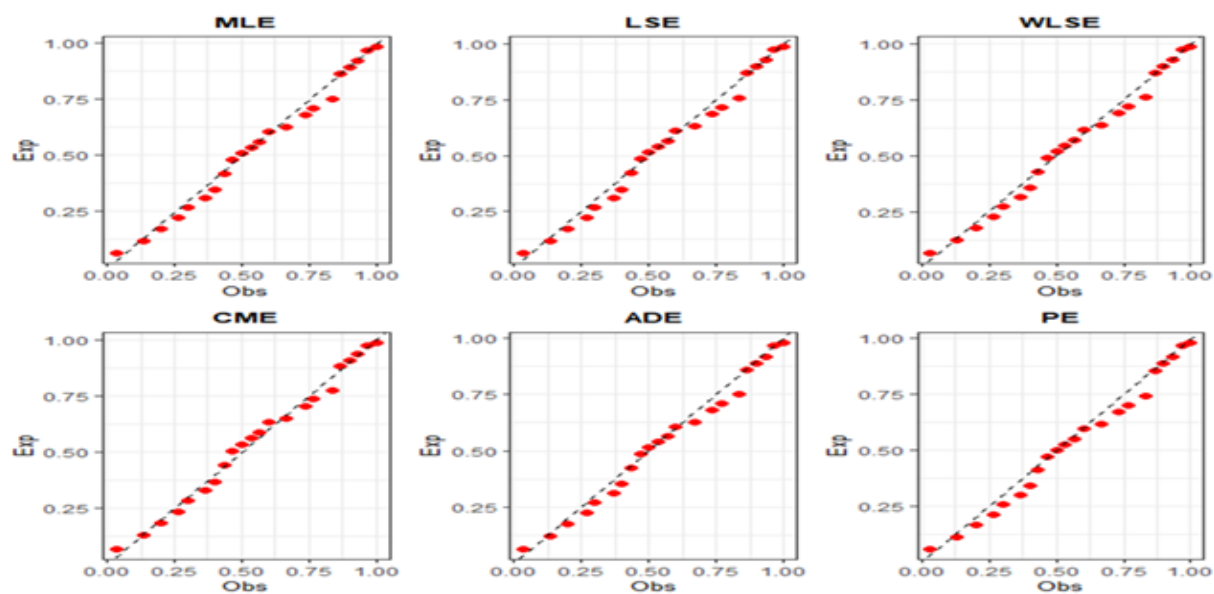


Figure 10: P-P plots for data set I over estimation methods

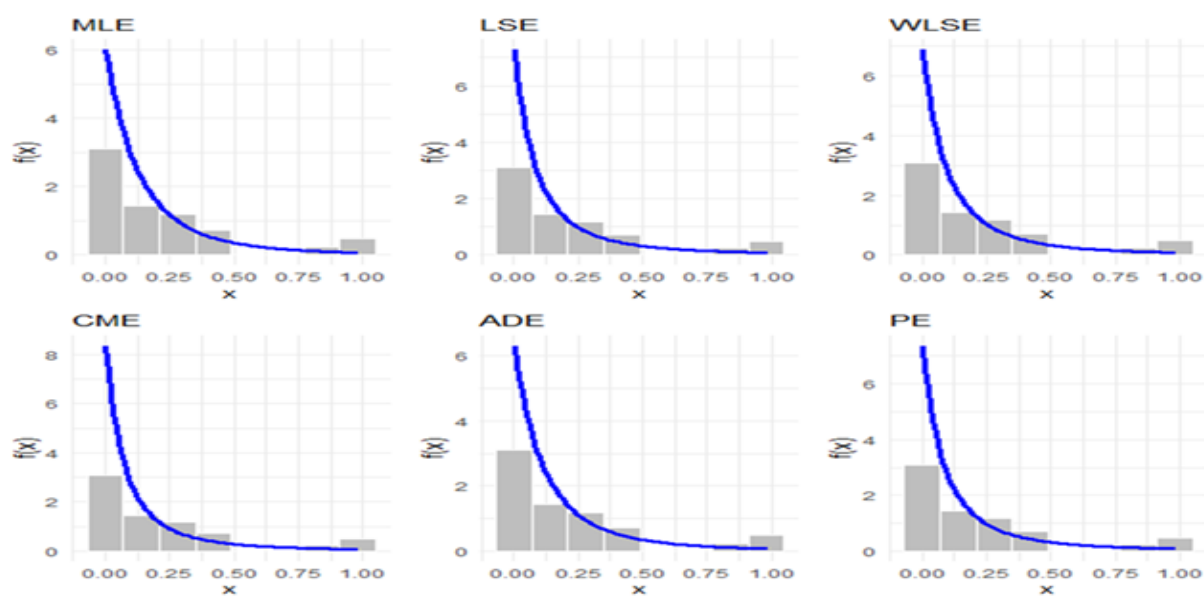


Figure 11: Plots of estimated PDFs for dataset II over estimation methods

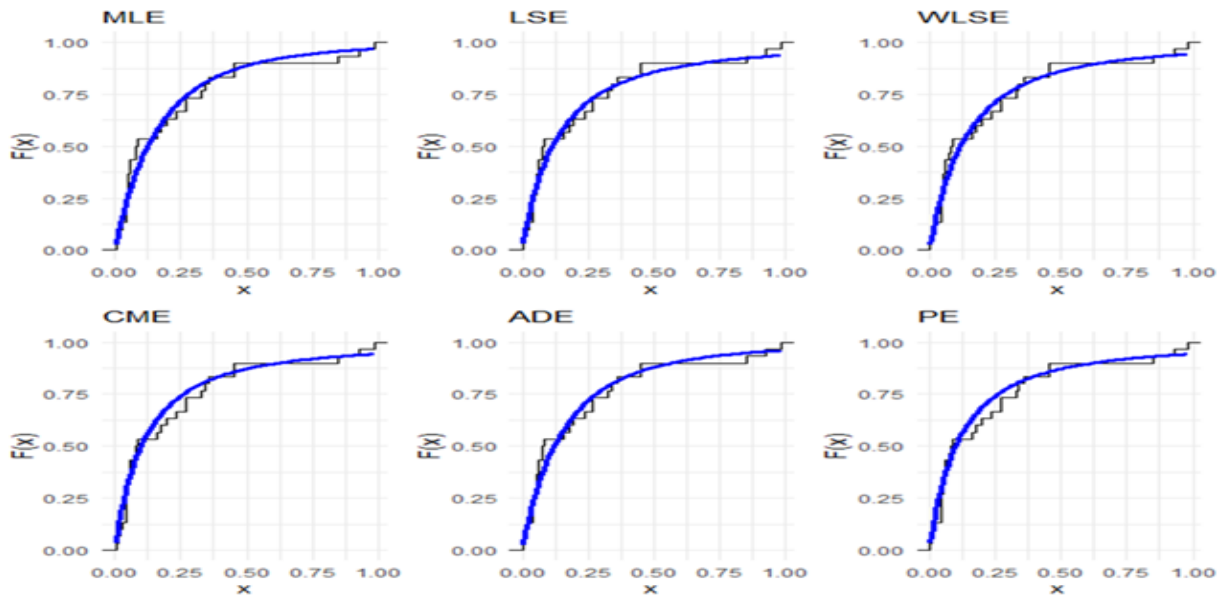


Figure 12: Plots of estimated CDFs for dataset II over estimation methods

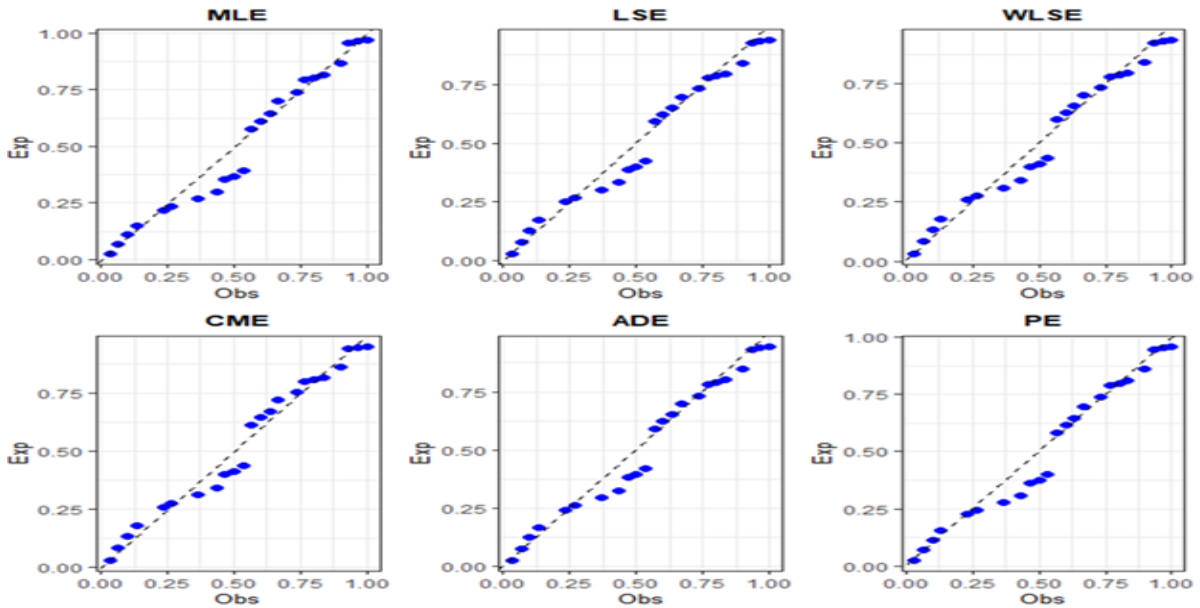


Figure 13: P-P plots for data set II over estimation methods

### 8. Concluding Remarks

The induced Lomax distribution introduced in this paper offers a strong and adaptable alternative to conventional Lomax models for lifetime and reliability data. We calculated and examined many statistical features, including moments, the quantile function, and essential reliability measures, demonstrating that the model can encapsulate declining hazard rates and right-skewed density distributions. Parameters were evaluated utilizing six established methods (MLE, LS, WLS, CVM, AD, and Percentile), and an extensive Monte Carlo simulation was performed to evaluate their efficacy regarding bias and MSE. Results repeatedly demonstrated that the Anderson–Darling estimator provides the most precise and

reliable estimates across diverse sample sizes and parameter configurations. Additionally, two empirical datasets from environmental and industrial contexts were examined, illustrating the enhanced goodness-of-fit of the proposed model relative to alternative distributions. The findings validate the practical utility of the induced Lomax distribution and underscore its potential as a preferred option for dependability modeling, survival analysis, and other fields involving heavy-tailed data. Subsequent investigations may examine its multivariate expansion and applications in more intricate real-world contexts.

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