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On the Application of a Lusin-Type Theorem to Differential Inclusions

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ABSTRACT: In this paper, we establish a special version of Lusin's theorem and study its application to differential inclusions. Lusin's theorem expresses the realization of Littlewood's second principle and establishes a connection between measurability and continuity. We prove this result and examine its use in the analysis of differential inclusions. As an application, we consider a differential inclusion problem with compact-valued right-hand side and an additional initial derivative condition, which naturally arises in the theory of differential inclusions. By combining the obtained Lusin result with selection methods, we derive sufficient conditions under which classical solutions can be constructed without imposing convexity assumptions on the value sets. The results clarify how Lusin arguments are important in the theory of classical solutions for differential inclusions.

Keywords: Lusin's theorem, set-valued mappings, differential inclusions.

Contents

1. Introduction

Lusin's Theorem expresses the realization of Littlewood's second principle, which is regarded as one of the cornerstones of real analysis and measure theory and states that "every measurable function is nearly continuous" [15]. In this respect, Lusin's theorem establishes a connection between measurability and continuity. Having many applications in the literature, Lusin's theorem appears in the theory of set-valued maps in the form of "Lusin-type theorems" and it has applications in differential inclusions [10,14,19]. In this context, in our study we will consider a special type Lusin theorem and present an application of this theorem to differential inclusions.

Differential inclusions, which are a general form of differential equations, have an important role in complex dynamical systems, modern approaches to optimal control and therefore in many fields such as applied mathematics, economics, and engineering [11,20,21].

The differential inclusions of interest are of the form

$$\dot{x}(t) \in F(t, x(t))$$

where $T > 0$ and $F : [0,T] \times \mathbb{R}^n \to 2^{\mathbb{R}^n}$ is a set-valued map.

In the 1960s, due to the increasing importance of control theory, interest in differential inclusions also increased. In this field, Filippov and Wazewski showed that control systems can be represented by differential inclusions [4,24].

With these developments, differential inclusions have acquired an important place in the modeling of complex dynamical processes. However, under the hypothesis that the set-valued map takes non-convex values, the absence of a continuous selection in general makes the existence of a classical solution difficult, and therefore the topological properties of the right-hand side still retain their importance in existence theorems [1,9]. In this study,

$$F : [0,T] \times \mathbb{R}^n \to 2^{\mathbb{R}^n}$$

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is a compact-valued map and we are interested in classical solutions of the following problem,

$$\dot{x}(t) \in F(t, x(t)) \tag{1.1}$$

$$x(0) = x_0 \tag{1.2}$$

$$\dot{x}(0) = b \tag{1.3}$$

where $x_0 \in \mathbb{R}^n$ and $b \in F(0, x_0)$. The initial condition (1.3) carries a feature that distinguishes it from classical differential equations [3]. For classical ordinary differential equations defined by a single-valued function f, the initial derivative is directly determined by the equation itself,

$$\dot{x}(0) = f(0, x_0),$$

and therefore an independent condition of the form (1.3) is generally not given separately. In differential inclusions, however, since the right-hand side is a set, the derivative vector at the initial moment must be chosen from this set. For this reason, the initial condition (1.3) plays a natural role within the framework of differential inclusions and directly affects the structure of the solution. This is because the set-valued nature of the right-hand side requires that the derivative vector chosen at the initial moment be determined via a suitable selection function. Consequently, the presence of (1.3) distinguishes the problem from classical initial value equations and places it within the broader and richer analytical framework of differential inclusion theory, which is supported by methods provided by selection theorems.

The existence of a classical solution depends fundamentally on the properties of the set-valued map and the topological structure of its value sets. Under the non-convexity hypothesis, the lack of a continuous selection in general makes a direct selection-based approach to classical solutions difficult. Michael's Selection Theorem established the existence of continuous selections under convexity assumptions, while Bressan showed that even the hypothesis of a Lipschitz set-valued map is not sufficient for the existence of a continuous selection [3,9]. Chistyakov proved the existence of a Lipschitz continuous selection under the hypothesis of a compact-valued, Lipschitz continuous set-valued map [26]. For this reason, conditions that do not require continuous selections directly have been investigated in order to obtain classical solutions.

In this context, Filippov's approximation method has formed a foundation for differential inclusions. Using Filippov trajectories, it was shown that a Carathéodory solution can be obtained; however, only an absolutely continuous solution can be obtained through this approach [5,6]. Such results are frequently used in optimal control and related fields and have various applications [11,20,21].

The existence of a classical solution is related to the topological properties of the value sets of set-valued maps. In this regard, the notion of totally disconnected compact-valued maps has gained particular importance in the literature with the aim of obtaining classical solutions without convexity. Bressan and Wang showed that, for time-dependent differential inclusions, under total disconnectedness a continuous selection exists, and that the existence of a classical solution can be obtained using polygonal approximations [3].

One of the modern approaches to ensuring solution existence in differential inclusions was developed by Plaskacz and Wisniewska. As an alternative to the classical Filippov theorem, they presented a new proof method and showed that a solution can be obtained through an appropriate viability tube. Under compactness and convexity assumptions, together with measurability in the t-variable and a Lipschitz hypothesis in the x-variable, existence of a solution was obtained [23].

R. Vinter, within the framework of optimal control, examined the solution concepts of differential inclusions comprehensively, analyzing the effects of techniques such as measurable selection, relaxation methods, and topological constraints on the existence of classical solutions [16]. When all these studies are considered together, it becomes evident that the existence of classical solutions depends not only on analytical regularity but also strongly on the topological properties of the value sets of the set-valued map.

Therefore, we believe that the special Lusin-type theorem presented in this study has application areas in differential inclusions and can be used in many fields of mathematics.

2. Notation and Preliminaries

Let $(X, \|.\|)$ be a Banach space and $[a,b] \subseteq \mathbb{R}$. We say that $\varphi : [a,b] \to X$ is measurable if φ is $(\mathcal{L}([a,b]), \mathcal{B}(X))-$measurable, where $\mathcal{B}(X)$ denotes the Borel $\sigma-$algebra of X.

We denote by $C([a,b],X)$, $C^1([a,b],X)$, $AC([a,b],X)$ and $\mathcal{L}^1([a,b],X)$ the spaces of continuous, continuously differentiable, absolutely continuous and Lebesgue integrable functions, respectively. We simply write $C([a,b])$, $C^1([a,b])$, $AC([a,b])$, and $\mathcal{L}^1([a,b])$.

For any subsets $A, B \subseteq X$, define

$$h^*(A,B) = \sup\{d(a,B) \mid a \in A\},$$

and

$$h^*(B,A) = \sup\{d(b,A) \mid b \in B\}.$$

where

$$d(a,B) = \inf\{d(a,b) \mid b \in B\}$$

Then

$$d_H(A,B) = \max\{h^*(A,B), h^*(B,A)\}$$

is called the Hausdorff distance between the sets A and B [19].

Let $G \subseteq \mathbb{R}^n$ and let $F : G \to 2^X$ be a set-valued map. The set-valued map F is said to be Lipschitz continuous on G if

$$d_H(F(t), F(s)) \leq L\,\|t - s\| \quad \text{for all } t, s \in G,$$

for some constant $L \geq 0$ [26].

A function $\gamma : G \to X$ is called a selection of F if

$$\gamma(x) \in F(x) \quad \text{for all } x \in G.$$

If, in addition, γ is continuous or Lipschitz continuous, then γ is called a continuous or a Lipschitz selection of F, respectively.

For $x \in \mathbb{R}^n$, the projection of x onto G is denoted by $Pr(x,G)$ and is defined by

$$Pr(x,G) := \{r \in G \mid \|x - r\| = d(x,G)\},$$

Let $y \in C([a,b], \mathbb{R}^n)$ and let $\varepsilon > 0$. The ε-tube about y is defined as

$$T_\varepsilon(y) := \big\{(t,x) \in [a,b] \times \mathbb{R}^n \;\big|\; \|x - y(t)\| \leq \varepsilon\big\}.$$

We say that ν lies in $T_\varepsilon(y)$ if for each $t \in [a,b]$ $\;(t, \nu(t)) \in T_\varepsilon(y)$ [17]. In particular, for $x_0 \in \mathbb{R}^n$ and $\varepsilon > 0$, the ε-tube about x_0 is defined by

$$T_\varepsilon(x_0) := \big\{(t,x) \in [a,b] \times \mathbb{R}^n \;\big|\; \|x - x_0\| \leq \varepsilon\big\}.$$

3. Main Results

Theorem 3.1 *Let $(X, \|.\|)$ be a Banach space, $[a,b] \subseteq \mathbb{R}$, $\varphi : [a,b] \to X$ be measurable function and $(a_n)_{n\in\mathbb{N}} \subseteq [a,b]$. Then for each $\varepsilon > 0$ there exists $\psi \in C([a,b],X)$ and the followings satisfy:*

(i) $\mu(\{t : \varphi(t) \neq \psi(t)\}) < \varepsilon$

(ii) $\forall n \in \mathbb{N} \quad \varphi(a_n) = \psi(a_n)$

(iii) $\sup\limits_{t\in[a,b]} \|\psi(t)\| \leq \sup\limits_{t\in[a,b]} \|\varphi(t)\|$

Proof: Let $\varphi : [a,b] \to X$ be measurable and $\varepsilon > 0$. By Lusin's Theorem there exists a closed set $K \subseteq [a,b]$ such that $\varphi_{|K}$ is continuous and $\mu([a,b] \setminus K) < \varepsilon$ [25]. Let

$$F = \{a_n \;: n \in \mathbb{N}\} \setminus K = \{b_n \;: n \in \mathbb{N}\}$$

Suppose $(b_n)_{n\in\mathbb{N}}$ be ordered as

$$b_1 < b_2 < ... < b_n < ...$$

Since each b_n is distinct and $[a,b]\setminus K$ is open, by being Hausdorff there exists $([a,b]\cap]x_k,y_k[)_{k\in\mathbb{N}}$ then we can express

$$[a,b]\cap K = \bigcup_{k=1}^{\infty}[a,b]\cap\,]x_k,y_k[$$

such that $\forall n\in\mathbb{N} \quad \exists k_n\in\mathbb{N} \quad]x_{k_n},y_{k_n}[\cap F=\{b_n\}$. So we can split these intervals as follows:

$$]x_{k_n},y_{k_n}[\;=\;]x_{k_n},b_n[\;\cup\;[b_n,y_{k_n}[$$

Thus

$$\bigcup_{k=1}^{\infty}[a,b]\cap\,]x_k,y_k[\;=\;\bigcup_{k_n}[a,b]\cap\,]x_{k_n},b_n[\;\cup\bigcup_{k_n}[a,b]\;\cap\;[b_n,y_{k_n}[\;\cup\bigcup_{k\neq k_n}[a,b]\cap\,]x_k,y_k[$$

For each k_n define

$$I_{k_n}=[a,b]\cap\,]x_{k_n},b_n[\quad I'_{k_n}=[a,b]\cap[b_n\ ,\ y_{k_n}[$$

and for each $k\neq k_n$

$$G_k=[a,b]\cap\,]x_k,y_k[$$

Then

$$\bigcup_{k=1}^{\infty}[a,b]\cap\,]x_k,y_k[\;=\bigcup_{k_n}\left(I_{k_n}\cup I'_{k_n}\right)\cup\bigcup_{k\neq k_n}G_k$$

Let for each k_n

$$\inf I_{k_n}=s_{k_n} \quad \sup I_{k_n}=z_{k_n}$$
$$\inf I'_{k_n}=s'_{k_n} \quad \sup I'_{k_n}=z'_{k_n}$$

and for each $k\neq k_n$

$$\inf G_k=p_k \quad \sup G_k=t_k$$

then define ψ as $\psi=\varphi$ on K and

$$\psi(t)=\begin{cases}\dfrac{(z_{k_n}-t)\varphi(s_{k_n})+(t-s_{k_n})\varphi(z_{k_n})}{z_{k_n}-s_{k_n}}, & t\in I_{k_n},\\[2ex] \dfrac{(z'_{k_n}-t)\varphi(s'_{k_n})+(t-s'_{k_n})\varphi(z'_{k_n})}{z'_{k_n}-s'_{k_n}}, & t\in I'_{k_n},\\[2ex] \dfrac{(t_k-t)\varphi(p_k)+(t-p_k)\varphi(t_k)}{t_k-p_k}, & t\in G_k,\ k\neq k_n.\end{cases}$$

Clearly

$$\sup I_{k_n}=z_{k_n}=b_n=s'_{k_n}=\inf I'_{k_n}$$

and since $b_n\in I'_{k_n}$

$$\psi(b_n)\;=\;\frac{(z'_{k_n}-s'_{k_n})\varphi(b_n)+(s'_{k_n}-s'_{k_n})\varphi(z'_{k_n})}{z'_{k_n}-s'_{k_n}}\;=\;\varphi(b_n)$$

Now we want to show that the continuity of ψ. Let $t_0\in[a,b]$. If $t_0\in K$ then we are done. If $t_0\in[a,b]\setminus K$;

Case 1 : $\exists n \in \mathbb{N}$ such that $t_0 \in I_{k_n}$.
Let $\varepsilon > 0$. Take $\delta = \min\{b_n - t_0, t_0, \frac{|z_{k_n} - s_{k_n}|\varepsilon}{\|\varphi(s_{k_n}) - \varphi(z_{k_n})\|}\}$. So for each $t \in [a,b]$ if $|t - t_0| < \delta$ then $t \in I_{k_n}$ and

$$\begin{aligned}\|\psi(t_0) - \psi(t)\| &= \left\| \frac{(z_{k_n} - t_0)\varphi(s_{k_n}) + (t_0 - s_{k_n})\varphi(z_{k_n})}{z_{k_n} - s_{k_n}} - \frac{(z_{k_n} - t)\varphi(s_{k_n}) + (t - s_{k_n})\varphi(z_{k_n})}{z_{k_n} - s_{k_n}} \right\| \\ &= \left\| \frac{\varphi(s_{k_n})(t - t_0) + \varphi(z_{k_n})(t_0 - t)}{z_{k_n} - s_{k_n}} \right\| \\ &= \frac{|t - t_0| \|\varphi(s_{k_n}) - \varphi(z_{k_n})\|}{|z_{k_n} - s_{k_n}|} < \frac{\delta \|\varphi(s_{k_n}) - \varphi(z_{k_n})\|}{|z_{k_n} - s_{k_n}|} < \varepsilon.\end{aligned}$$

For $t_0 = 0$ Take $\delta = \min\{b_n, \frac{|z_{k_n} - s_{k_n}|\varepsilon}{\|\varphi(s_{k_n}) - \varphi(z_{k_n})\|}\}$

Case 2 : $\exists n \in \mathbb{N}$ such that $t_0 \in I'_{k_n} = [a,b] \cap [b_n, y_{k_n}[$. If $t_0 \in [a,b] \cap]b_n, y_{k_n}[$, the conclusion follows by the same argument as in Case 1. If $t_0 = b$;
Let $\varepsilon > 0$. Take

$$\delta = \min\{ \frac{|z'_{k_n} - b_n|\varepsilon}{\|\varphi(b_n) - \varphi(z'_{k_n})\|}, \frac{|b_n - s_{k_n}|\varepsilon}{\|\varphi(s_{k_n}) - \varphi(b_n)\|} \}$$

Then $\|\psi(b_n) - \psi(t)\| < \varepsilon$.

Case 3 : $\exists k \neq k_n$ such that $t_0 \in G_k = [a,b] \cap]x_k, y_k[$. Since G_k is open,

$$\exists r > 0 \quad]t_0 - r, t_0 + r[\cap [a,b] \subseteq G_k.$$

Take $\delta = \min\{r, \frac{|t_k - p_k|\varepsilon}{\|\varphi(p_k) - \varphi(t_k)\|}\}$ then $\|\psi(t) - \psi(t_0)\| < \varepsilon$. Thus ψ is continuous on $[a,b]$ and

$$\forall n \in \mathbb{N} \quad \psi(a_n) = \varphi(a_n).$$

Now let us show that

$$\sup_{t \in [a,b]} \|\psi(t)\| \leq \sup_{t \in [a,b]} \|\varphi(t)\|$$

By definition of ψ, $\varphi = \psi$ on K. Let $n \in \mathbb{N}$, on I_{k_n}

$$\begin{aligned}\|\psi(t)\| &= \left\| \frac{(z_{k_n} - t)\varphi(s_{k_n}) + (t - s_{k_n})\varphi(z_{k_n})}{z_{k_n} - s_{k_n}} \right\| \\ &\leq \frac{\|\varphi(s_{k_n})\| |z_{k_n} - t| + \|\varphi(z_{k_n})\| |t - s_{k_n}|}{z_{k_n} - s_{k_n}} \\ &\leq \frac{(|z_{k_n} - t| + |t - s_{k_n}|)}{|z_{k_n} - s_{k_n}|} \sup_{t\in[a,b]} \|\varphi(t)\| \\ &= \frac{|z_{k_n} - s_{k_n}|}{|z_{k_n} - s_{k_n}|} \sup_{t\in[a,b]} \|\varphi(t)\| = \sup_{t\in[a,b]} \|\varphi(t)\|\end{aligned}$$

The same argument applies for $t \in I'_{k_n}$ and for each $k \neq k_n$ on G_k.

Finally, since $\{t : \varphi(t) \neq \psi(t)\}$ is measurable and $\{t : \varphi(t) \neq \psi(t)\} \subseteq [a,b] \setminus K$ then $\mu(\{t : \varphi(t) \neq \psi(t)\}) < \varepsilon$.

We get a continuous function $\psi : [a,b] \to \mathbb{R}^n$ such that the followings are satisfied:

(i) $\mu(\{t : \varphi(t) \neq \psi(t)\}) < \varepsilon$
(ii) $\forall n \in \mathbb{N} \quad \varphi(a_n) = \psi(a_n)$
(iii) $\sup_{t\in[a,b]} \|\psi(t)\| \leq \sup_{t\in[a,b]} \|\varphi(t)\|$.

□

Theorem 3.2 *Let G be relatively open subset of $[0,T]\times\mathbb{R}^n$ and $F : G \to 2^{\mathbb{R}^n}$ be set-valued map. Take any $y \in AC([0,T])$ and $\varepsilon > 0$ such that $y(0) = x_0$ and $T_\varepsilon(y) \subseteq G$. Suppose that F compact-valued and Lipschitz continuous (with constant $\eta > 0$) on $T_\varepsilon(y)$. Suppose further that*

$$\Gamma(y) = \int_0^T d(\dot{y}(t), F(t, y(t))) < \frac{\varepsilon.e^{-\eta T}}{3}.$$

Then there exists a classical solution x of problem (1.1)-(1.2)-(1.3) such that

$$\|x - y\|_\infty < \varepsilon$$

.

If, in addition F is convex-valued then the Lipschitz continuity condition replaced by the weaker conditions "continuity with respect to the t-variable on G and Lipschitz continuity with respect to the x-variable on $T_\varepsilon(y)$", and the same result still holds.

Proof: The proof of the theorem in the non-convex case is presented below and is carried out in several steps. First, we'll obtain an absolutely continuous solution of problem (1.1)-(1.2) by Filippov's existence theorem for a given absolutely continuous function under some hypothesis. Then, for a Lebesgue measurable function that is derivative of this absolutely continuous solution almost everywhere, we'll construct a special type continuously differentiable function using Theorem 1. Finally, we conclude that there exists a classical solution to the problem (1.1)-(1.2)-(1.3).

First of all, F is compact-valued and Lipschitz continuous on $T_{\frac{\varepsilon}{3}}(y)$ since $T_{\frac{\varepsilon}{3}}(y) \subseteq T_\varepsilon(y)$. Then there exists a solution $v \in AC([0,T])$ of problem $(1.1)-(1.2)$ such that

$$\begin{aligned} &Grv \subseteq G, \\ &v(0) = y(0), \\ &\|v - y\|_\infty < \frac{\varepsilon}{3}. \end{aligned}$$

hold [5,22] (in addition, similar existence results hold for retarded-type problems [22]).

Secondly, let us consider $T_{\varepsilon^*}(v)$ where $\varepsilon^* = \frac{\varepsilon}{3}e^{-\eta T}$. It is easy to see that $T_{\varepsilon*}(v) \subseteq T_\varepsilon(y) \subseteq G$. Since $T_{\varepsilon*}(v)$ is compact and F is Lipschitz and compact-valued on $T_{\varepsilon*}(v)$, by [18] Corollary 2.20,

$$k = \delta F(T_{\varepsilon*}(v)) \in \mathbb{R}.$$

For $v \in AC([0,T])$ there exists a $\varphi^* \in L^1([0,T])$ such that

$$v(t) = v(0) + \int_0^t \varphi^*(s)ds.$$

Now define $\varphi \in L^1([0,T])$ as

$$\varphi(t) = \begin{cases} b\ , & t = 0 \\ \varphi^*(t)\ , & t \neq 0 \end{cases}$$

Clearly, $v(t) = v(0) + \int_0^t \varphi(s)ds$. By Theorem 1, for $r = \frac{\varepsilon^*}{2k(1+2\eta T)} > 0$, there exists $\psi \in C([0,T])$ and we have the followings:

$$\begin{aligned} &\mu(\{t : \varphi(t) \neq \psi(t)\}) < r, \\ &\varphi = \psi \text{ on } K, \\ &\sup_{t\in[0,T]} \|\psi(t)\| \leq \sup_{t\in[0,T]} \|\varphi(t)\|, \\ &\varphi(0) = \psi(0). \end{aligned}$$

Since $\varphi(t) = \dot{v}(t) \in F(t, v(t)) \quad a.e.\ t \in [0,T]$, we have

$$\|\psi(t)\| \leq \|\varphi(t)\| \leq k \quad a.e.\ t \in [0,T].$$

Now define $w \in C^1([0,T])$ as

$$w(t) = x_0 + \int_0^t \psi(s)ds.$$

Then we conclude that

$$\begin{aligned}
\|w(t) - v(t)\| &= \|\int_0^t (\psi(s) - \varphi(s))ds\| \leq \int_0^T \|\psi(s) - \varphi(s)\|ds \\
&= \int_{\{s:\varphi(s)\neq\psi(s)\}} \|\psi(s) - \varphi(s)\|ds + \int_{[0,T]\setminus\{s:\varphi(s)\neq\psi(s)\}} \|\psi(s) - \varphi(s)\|ds \\
&\leq 2.k\mu(\{s : \varphi(s) \neq \psi(s)\}) + 0 < 2kr = \frac{\varepsilon^*}{1+2\eta T} < \varepsilon^* < \frac{\varepsilon}{3}.
\end{aligned}$$

So $Grw \subseteq T_{\frac{\varepsilon}{6}}(v) \subseteq G$. By using Lipschitz continuityof F and above inequality the following inequality holds,

$$\int_0^T d(\dot{w}(t), F(t, w(t))dt \leq 2.k.r(1 + 2\eta T) < \varepsilon^*.$$

So we get that there exists $w \in C^1([0,T])$ such that

$$\begin{aligned}
&\dot{w}(0) = \psi(0) = \varphi(0) = b, \\
&\|v - w\|_\infty < \frac{\varepsilon}{3}, \\
&\int_0^T d(\dot{w}(t), F(t, w(t))) < \varepsilon^*.
\end{aligned}$$

Finally, for $w \in C^1$ we have

$$\begin{aligned}
T_{\frac{\varepsilon}{3}}(w) \subseteq T_{\frac{2\varepsilon}{3}}(v) \subseteq T_\varepsilon(y) \subseteq G, \\
Grw \subseteq G, \\
\int_0^T d(\dot{w}(t), F(t, w(t))) < \varepsilon^* = \frac{\varepsilon}{3}e^{-\eta T}.
\end{aligned}$$

Now, let $w_0 = w \in C^1([0,T])$. Then by using hypotheses, the set-valued map

$$Pr_0 : [0,T] \to 2^{\mathbb{R}^n} \setminus \{\emptyset\}$$

defined by

$$Pr_0(s) = Pr\big(\dot{w}_0(s), F(s, w_0(s))\big)$$

is a Lipschitz continuous. To see that, let $s,t \in [0,T]$ and $\theta \in Pr_0(s) = Pr(\dot{w}_0(s), F(s, w_0(s)))$. Since $\theta \in F(s, w_0(s))$ and $Pr(\dot{w}_0(t), F(t, w_0(t))) \subseteq F(t, w_0(t))$ then

$$d(\theta, Pr(\dot{w}_0(t), F(t, w_0(t))) \leq d_H(F(s, w_0(s), F(t, w_0(t))) \leq \eta(|s - t| + \|w_0(s) - w_0(t)\|)$$

Since $w_0 = w \in C^1$ so Lipschitz. Thus we have

$$\exists L > 0 \quad \|w_0(s) - w_0(t)\| \leq L|s - t|$$

Which means

$$d(\theta, Pr(\dot{w}_0(t), F(t, w_0(t))) \leq \eta(|s-t| + L|s-t|) = |s-t|(\eta(1+L))$$

Same way for $\beta \in Pr_0(t) = Pr(\dot{w}_0(t), F(s, w_0(t)))$. Thus

$$d_H(Pr_0(s), Pr_0(t)) \leq \eta(1+L)|s-t|$$

Since Pr_0 is Lipschitz with compact values, according to [26] there exists a continuous selection γ_0 of Pr_0 such that $\gamma_0(0) = b$. Let

$$w_1(s) = x_0 + \int_0^s \gamma_0(z)\, dz.$$

Clearly $w_1 \in C^1([0,T])$, $\dot{w}_1 = \gamma_0$, $\|\dot{w}_1(s) - \dot{w}_0(s)\| = d\big(\dot{w}_0(s), F(s, w_0(s))\big)$ and $d\big(\dot{w}_1(s), F(s, w_0(s))\big) = 0$. Since

$$\begin{aligned} \|w_1(s) - w_0(s)\| &\leq \int_0^s \|\dot{w}_1(z) - \dot{w}_0(z)\|\, dz \\ &= \int_0^s d\big(\dot{w}_0(z), F(z, w_0(z))\big)\, dz \leq \Gamma[w_0] \end{aligned} \tag{3.1}$$

we get w_1 lies in $T_{\frac{\varepsilon}{3}}(w_0)$ By that way we can construct a sequence $(w_n)_{n\in\mathbb{N}} \subseteq C^1([0,T])$ as

$$w_{n+1}(s) = x_0 + \int_0^s \gamma_n(z)\, dz$$

where γ_n is a continuous selection of $Pr_n(s) = Pr\big(\dot{w}_n(s), F(s, w_n(s))\big)$. It is easy to show that the followings are satisfied for $s \in [0,T]$:

$$\gamma_n(s) = \dot{w}_{n+1}(s) \in F(s, w_n(s)) \qquad \text{for } n \in \mathbb{N},$$

$$\|w_{n+1}(s) - w_0(s)\| \leq \Gamma[w_0]\, \frac{(\eta s)^n}{n!} \qquad \text{for } n \in \mathbb{N}, \tag{3.2}$$

$$\|\dot{w}_{n+1}(s) - \dot{w}_0(s)\| \leq \Gamma[w_0]\, \frac{\eta(\eta s)^{n-1}}{(n-1)!} \qquad \text{for } n \geq 1. \tag{3.3}$$

By (3.2), for $n \in \mathbb{N}$,

$$\|w_n(s) - w_0(s)\| \leq \sum_{j=0}^{n-1} \|w_{j+1}(s) - w_j(s)\| \leq \Gamma[w_0] \sum_{j=0}^{n-1} \frac{(\eta s)^j}{j!} < \frac{\varepsilon}{3}. \tag{3.5}$$

Using (3.2) and (3.3), we conclude that $(w_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in $(C^1([0,T]), \|\cdot\|_{C^1})$. Since $(C^1([0,T]), \|\cdot\|_{C^1})$ is complete, there exists a $x \in C^1([0,T])$ such that $\|w_n - x\|_\infty \to 0$, $\|\dot{w}_n - \dot{x}\|_\infty \to 0$ and

$$\dot{x}(s) = \lim_{n\to\infty} \dot{w}_n(s) \in \lim_{i\to\infty} F(s, w_{n-1}(s)) = F(s, x(s)).$$

Thus x is a classical solution of problem (1.1)-(1.2). By (3.5),

$$\|x - w\|_\infty < \frac{\varepsilon}{3}.$$

We conclude that there exists a classical solution of problem (1.1)-(1.2) such that

$$\begin{aligned} \dot{x}(t) &\in F(t, x(t)) \\ x(0) &= x_0 \\ \|x - w\|_\infty &< \frac{\varepsilon}{3} \end{aligned}$$

and moreover $\dot{x}(0) = \dot{w}(0) = \psi(0) = \varphi(0) = b$. Therefore x is a classical solution of problem (1.1)-(1.2)-(1.3) such that

$$\|x - y\|_\infty \le \|x - w\|_\infty + \|w - v\|_\infty + \|v - y\|_\infty < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon.$$

On the other hand, when F is assumed to be convex-valued, the image of the corresponding mapping Pr_i reduces to a singleton. In this case, the same result holds under continuity with respect to the t-variable and Lipschitz continuity with respect to the x-variable, without requiring a global Lipschitz condition. The remainder of the proof follows exactly the same line of argument as in non-convex case. □

Corollary 3.1 *Suppose that F compact-valued and Lipschitz (with constant $\eta > 0$) on $T_\varepsilon(x_0)$. Then there exists a classical solution x of problem (1.1)-(1.2)-(1.3) such that*

$$\|x - x_0\|_\infty < \Gamma(x_0) e^{\eta T}$$

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References

1. A. Bressan, Directionally continuous selections and differential inclusions, Funkcialaj Ekvacioj **31** (1988), 459–470.

2. A. Bressan and G. Colombo, Generalized Baire category and differential inclusions in Banach spaces, J. Differential Equations **76** (1988), 135–158.

3. A. Bressan and Z. Wang, Classical solutions to differential inclusions with totally disconnected right-hand side, J. Differential Equations **246** (2009), 629–640.

4. A. F. Filippov, On some questions in the theory of optimal regulation, Vestn. Mosk. Univ. Ser. Mat. Mekh. Astr. Fiz. Khim. (1959), 25–32. (English translation: J. SIAM Control Ser. A **1** (1962), 76–84.)

5. A. F. Filippov, Classical solutions of differential equations with multivalued right-hand side, SIAM J. Control **5** (1967), 609–621.

6. A. F. Filippov, *Differential Equations with Discontinuous Right-Hand Sides*, Mathematics and Its Applications (Soviet Series), Kluwer Academic Publishers, Boston, 1988.

7. A. F. Filippov, The existence of solutions of generalized differential equations, Mat. Zametki **10** (1971), 307–313.

8. C. E. Arroud and T. Haddad, Existence of solutions for a class of nonconvex differential inclusions, Applicable Analysis **93** (2014), 1979–1988.

9. E. Michael, *Selected Selection Theorems*, Amer. Math. Monthly **63**(4) (1956), 233–238.

10. G. Smirnov, *Introduction to the Theory of Differential Inclusions*, American Mathematical Society, Providence, RI, 2001.

11. H. Frankowska and F. Rampazzo, *Filippov's and Filippov–Wazewski's Theorems on Closed Domains*, J. Differential Equations **161** (2000), 449–478.

12. H. A. Antosiewicz and A. Cellina, Continuous selections and differential relations, J. Differential Equations **19** (1975), 386–398.

13. J.-P. Aubin and H. Frankowska, *Set-Valued Analysis*, Birkhäuser, Boston–Basel–Berlin, 1990.

14. M. I. Kamenskii, V. V. Obukhovskii and P. Zecca, *Condensing Multivalued Maps and Semilinear Differential Inclusions in Banach Spaces*, de Gruyter Series in Nonlinear Analysis and Applications, Vol. 7, Walter de Gruyter, Berlin, 2001.

15. N. Lusin, *Sur les propriétés de fonctions mesurables*, C. R. Acad. Sci. Paris **154** (1912), 1688–1690.

16. R. Vinter, *Optimal Control*, Modern Birkhäuser Classics, Birkhäuser, Basel, 2010.

17. R. Vinter, A characterization of the reachable set for nonlinear control systems, SIAM J. Control Optim. **18** (1980), 599–610.

18. S. Hu and N. S. Papageorgiou, *Handbook of Multivalued Analysis, Volume I: Theory*, Kluwer Academic Publishers, Dordrecht–Boston–London, 1997.

19. S. Hu and N. S. Papageorgiou, *Research Topics in Analysis, Volume I: Grounding Theory*, Birkhäuser Advanced Texts / Basler Lehrbücher, Birkhäuser, Cham, 2023.

20. S. Ilter, *Some applications related to differential inclusions based on the use of a weighted space*, Fundam. J. Math. Appl. **6**(4) (2023), 188–193.

21. S. Ilter and H. Duru, *Some results on nonsmooth systems including max-type functions*, Turk. J. Math. Comput. Sci. **16**(1) (2024), 147–153.

22. S. Koca, *Küme Değerli Fonksiyonların Birleşik Ölçülebilirliği Hakkında*, M.Sc. Thesis, İstanbul University, 2023.

23. S. Plaskacz and M. Wiśniewska, A new method of proof of Filippov's theorem based on the viability theorem, Cent. Eur. J. Math. **10**(6) (2012), 1940–1943.

24. T. Ważewski, Systemes de commande et equations au contingent, Bull. Acad. Polon. Sci. Ser. Sci. Math. **9** (1961), 151–155.

25. V. I. Bogachev, *Measure Theory*, Vols. I–II, Springer, Berlin, 2007.

26. V. V. Chistyakov, On the theory of set-valued maps of bounded variation of one real variable, Sbornik: Mathematics **189**(5) (1998), 797–819.

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