Bol. Soc. Paran. Mat.
©SPM – E-ISSN-2175-1188
SPM: www.spm.uem.br/bspm

(3s.) **v. 2026 (44) 5** : 1–31.
ISSN-0037-8712
doi:10.5269/bspm.

Type-I Heavy-Tailed Topp-Leone Family of Distributions with Applications to Biomedical Data

Julian I. Mbegbu, Chinelo U. Chikwelu, Friday Ewere and Nzei C. Lawrence

ABSTRACT: In this study, a new family of probability distributions is proposed by integrating the tail flexibility of the Type I Heavy Tail family and the Topp Leone generated family of distributions and this is called the Type I Heavy Tailed Topp-Leone Generated (TIHTTL-G) family of distributions. The significant statistical properties of the TIHTTL-G family of distributions such as moments, generating functions, order statistics, stochastic ordering, and Rényi entropy are derived. The estimation of the parameters of a specific sub-model, the TIHTTL-Rayleigh (TIHTTLR), is performed employing ten different estimation techniques, both classical and Bayesian. Bayesian estimators are derived under three different loss functions namely: Square Error Loss Function, Linear Exponential Loss Function and Generalized Entropy Loss Function. Simulation experiments was conducted to examine parameter estimators from the different estimation procedures. Finally, to demonstrate the applicability of the proposed family, the TIHTTLR was fitted to three real-life datasets, and the results show its superior fit compared to competing models.

Keywords: Type I Heavy Tail family of distributions, Topp-Leone family of distributions, Type I Heavy Tailed Topp-Leone-G family of distributions, Maximum Likelihood estimation, Bayesian estimation.

Contents

2020 *Mathematics Subject Classification:* 35B40, 35L70.
Submitted January 03, 2025. Published February 14, 2026

1. Introduction

The usefulness of probability distributions lies in the fact that it is used to describe the likelihood of the occurrence of the outcome of a random variable in a probability experiment. Hence, the study of uncertainty in many real-life scenarios. A great number of standard probability distributions have been introduced and used for modeling data in several areas. In scientific, biological, and engineering surveys, empirical data frequently exhibit irregularities such as skewness, excessive kurtosis, tail heaviness, and outliers, or extreme values. The use of these standard distributions is hampered by such severe behavior, even though these distributions are preferred for their mathematical simplifications, yet they are useless when dealing with complicated real data. Some academics have created generalized and compound distributions by adding more shape factors or by employing transformation techniques in response to the shortcomings of standard distributions. One of the most encouraging innovations within this field is the transformed-transformer (T-X) generator, introduced by [27]. This method offers a formal means of modifying baseline distributions' tail behaviour without degrading or destroying their analytical properties. Since its formulation, this T-X generator has inspired the development of several new distribution families, such as the Type I Heavy-Tailed (TI-HT) family introduced by [3], the Heavy-Tailed Exponential by [5], the Type II Half-Logistical Odd Fréchet by [6], Type II heavy tailed family by [28].

The second general and widely used family of distributions is the Topp-Leone-G (TL-G) family introduced by [12], and other Topp-Leone extensions in the literature include the Power Topp-Leone Weibull proposed by [7], the Marshall-Olkin-Topp-Leone-Gompertz-G distribution introduced by [10], Topp-Leone-Harris-G distribution proposed by [9], Topp-Leone Dagum distribution introduced by [26], DUS Topp-Leone family by [8], the Odd Inverted Topp-Leone-H (OITL-H) family [25], the Type I Half-Logistics-Topp-Leone-G (TIHLTL-G) distribution [24], and the Topp-Leone Type I Heavy-Tailed -G Power [1]. Other relevant studies include [33], [34], [35], [36], [37], [38], [39], [40], [41], [42], [43], [44], [45], [46], [47], [48], [49], [50], [51], [52], [53] and [54].

The TL-G family is noted for its structural flexibility and ability to model a variety of distributional forms through transformations of a base distribution. While the TL-G family has rich modeling flexibility, the family does not admit heavy-tailed behavior or extreme-value characteristics in a natural way, limiting its application in domains where such properties are prominent. To bridge this gap, this study proposes a new and computationally efficient family of generalized distributions called the Type I Heavy-Tailed Topp-Leone-G (TI-HT-TL-G) family of distributions. The model integrates the tail-flexible of Type I heavy-tailed family developed by [3] and the Topp Leone (TL-G) family, introduced by [12]. The resulting distribution offers improved performance for data that have both moderate variability and extreme values and is therefore particularly targeted towards applications involving robust modeling of heavy-tailed, skewed, or irregularly shaped distributions.

Due to its attractive characteristics, the new generalized distribution has numerous advantages. It is mathematically tractable and can be represented as an infinite linear combination of the exponentiated-G distribution. The new distribution is also versatile and can accommodate data having monotonic or non-monotonic hazard rate functions and hence suitable for modeling heavy-tailed and skewed data.

2. Derivation of the TI-HT-TL-G Distributions

Based on the Type I Heavy-Tailed-G (TIHTG) distribution introduced by [3] and the Topp-Leone-G family proposed by [12], this study introduces a new and more flexible class of distributions, referred to as the Type I Heavy-Tailed Topp-Leone-G (TI-HT-TL-G) family. This newly developed family is designed to extend the modeling capabilities of TIHTG and TLG by incorporating features from both frameworks. The cumulative distribution function (CDF) of the Type I Heavy-Tailed-G distribution, as discussed in [3], is:

$$G(x;\theta,\Delta) = 1 - \left(\frac{1-F(x;\Delta)}{1-(1-\theta)F(x;\Delta)}\right)^{\theta}, \tag{2.1}$$

While the probability density function (PDF) is:

$$g(x;\theta,\Delta) = \frac{\theta^2 f(x;\psi)\,[1-F(x;\Delta)]^{\theta-1}}{[1-(1-\theta)F(x;\Delta)]^{\theta+1}}, \tag{2.2}$$

for $\theta, x > 0$, where Δ denotes the parameter vector from the baseline distribution $F(.)$. Also the cdf of the Topp Leone-G family of distribution is

$$F(x) = \left[1-(1-T(x))^2\right]^{\alpha} \tag{2.3}$$

with a corresponding pdf of:

$$f(x) = 2\alpha\, t(x)\,[1-T(x)]\left[1-(1-T(x))^2\right]^{\alpha-1}, \quad \alpha > 0 \tag{2.4}$$

Replacing the baseline CDF in Eq. (2.1) with the CDF of the TL-G family in Eq. (2.3) an substituting $\alpha = 1$ to have the new family of distribution called TI-HT-TL-G with CDF

$$G(x;\theta,\Delta) = 1 - \left[\frac{1-\left[1-\left(\bar{T}(x)\right)^2\right]}{1-(1-\theta)\left(1-\left(\bar{T}(x)\right)^2\right)}\right]^{\theta} \tag{2.5}$$

Substitute Eq. (2.3) and (2.4) into Eq. (2.2) and taking $\alpha = 1$ to have the PDF of the TI-HT-TLG family of distributions

$$g(x;\theta,\Delta) = \frac{2\theta^2\, t(x)\,(\bar{T}(x))^{2\theta-1}}{\left[1-(1-\theta)\left(1-\left(\bar{T}(x)\right)^2\right)\right]^{\theta+1}} \tag{2.6}$$

Where $\bar{T}(x) = 1 - T(x)$ is the baseline survival function. $t(x)$ is the associated pdf of the baseline distribution and Δ is the parameter vector of the baseline distribution $F(.)$.

2.1. Reliability Measures

Consider a continuous random variable X characterized by a probability density function (PDF) $g(x)$, a cumulative distribution function (CDF) $G(x)$, and a survival function defined as $S(x) = 1 - G(x)$, several reliability functions are typically of interest. Some of them are the hazard rate, the mean residual life, and the reverse hazard rate functions. The hazard rate function, the instantaneous failure rate at time, can be defined as: $h(x) = \frac{g(x)}{S(x)}$ The mean residual life function is employed to establish the expected remaining lifetime upon survival to time, and the reverse hazard rate function indexes the instantaneous rate upon non-occurrence prior to x . As shown by [13], such functions possess analogous analytical features. In this paper, we focus on survival and hazard rate functions since they are directly relevant to the TIHTTL-G distribution. Using the previously derived CDF and PDF in Eq (2.5) and Eq (2.6), the survival function of the TI-HT-TLG distribution is:

$$S(x) = \left[\frac{\left(\bar{T}(x)\right)^2}{1-(1-\theta)\left\{1-\left(\bar{T}(x)\right)^2\right\}}\right]^{\theta} \tag{2.7}$$

and the hazard rate is:

$$h(x) = \frac{2\theta^2\, t(x)}{\left[1-(1-\theta)\left\{1-\left(\bar{T}(x)\right)^2\right\}\right]\bar{T}(x)} \tag{2.8}$$

2.2. Quantile Function

The quantile function is instrumental in simulation studies, especially for sampling random variates of a given distribution. Suppose X is a random variable having the TIHTTL-G distribution, and $U\ Uniform(0,1)$. To find the quantile corresponding to X, one has to solve the nonlinear equation:

Let

$$
\begin{aligned}
G(x) &= p \\
p &= 1 - \left(\frac{1 - \left(1 - (\bar{T}(x))^2\right)}{1 - (1-\theta)\left(1 - (\bar{T}(x))^2\right)} \right)^{\theta} \\
1 - \left(1 - (\bar{T}(x))^2\right) &= (\bar{T}(x))^2 \\
1 - p &= \left(\frac{(\bar{T}(x))^2}{1 - (1-\theta)\left(1 - (\bar{T}(x))^2\right)} \right)^{\theta} \\
(1-p)^{1/\theta} &= \frac{(\bar{T}(x))^2}{1 - (1-\theta)\left(1 - (\bar{T}(x))^2\right)} \\
\theta(1-p)^{1/\theta} &= (\bar{T}(x))^2 \left[1 - (1-\theta)(1-p)^{1/\theta}\right] \\
(\bar{T}(x))^2 &= \frac{\theta(1-p)^{1/\theta}}{1 - (1-\theta)(1-p)^{1/\theta}} \\
\bar{T}(x) &= \sqrt{\frac{\theta(1-p)^{1/\theta}}{1 - (1-\theta)(1-p)^{1/\theta}}} \\
Q(p) &= T^{-1}\left(1 - \sqrt{\frac{\theta(1-p)^{1/\theta}}{1 - (1-\theta)(1-p)^{1/\theta}}}\right)
\end{aligned}
\tag{2.9}
$$

Thereby, random variates from the TI-HT-TL-G distribution can be generated using Eq (2.9), where T represents the baseline cumulative distribution function.

2.3. Expansion of the density function

given the pdf in Eq (2.6), the density function can be expanded thus

$$
\begin{aligned}
g(x;\theta,\Delta) &= \frac{2\theta^2\, t(x)\, (\bar{T}(x))^{2\theta-1}}{\left[1 - (1-\theta)\left(1 - (\bar{T}(x))^2\right)\right]^{\theta+1}} \\
\left(1 - (1-\theta)\left(1 - (\bar{T}(x))^2\right)\right)^{-(\theta+1)} &= \sum_{j=0}^{\infty} \binom{\theta+1}{j} (1-\theta)^j \left(1 - (\bar{T}(x))^2\right)^j \\
\left(1 - (\bar{T}(x))^2\right)^j &= \sum_{k=0}^{j} (-1)^k \binom{j}{k} (\bar{T}(x))^{2k} \\
\left(\theta + (1-\theta)\left(1 - (\bar{T}(x))^2\right)\right)^{-(\theta+1)} &= \sum_{j=0}^{\infty}\sum_{k=0}^{j} (-1)^k \binom{\theta+1}{j} (1-\theta)^j \binom{j}{k} (\bar{T}(x))^{2k}
\end{aligned}
$$

$$
g(x;\theta,\Delta) = 2\theta^2\, t(x)\, (\bar{T}(x))^{2\theta-1} \sum_{j=0}^{\infty}\sum_{k=0}^{j} (-1)^k \binom{\theta+1}{j} (1-\theta)^j \binom{j}{k} (\bar{T}(x))^{2k},
$$

$$
g(x;\theta,\Delta) = 2\theta^2 \sum_{j=0}^{\infty}\sum_{k=0}^{j} (-1)^k \binom{\theta+1}{j} (1-\theta)^j \binom{j}{k} (\bar{T}(x))^{2(\theta+k)-1} t(x) \tag{2.10}
$$

2.4. Statistical Properties

Here you can get comprehensive points of interest with respect to the TI-HT-TL-G distribution family. This comprises their moment, moment-generating function, order-statistics, Rényi entropy.

2.4.1. Moments and Generating Function. The s-th moment is defined as

$$\mu_s = \mathbb{E}[X^s] = \int_{-\infty}^{\infty} x^s g(x;\theta,\Delta)\,dx$$

By employing the series representation of the probability density function given in Eq (10), we can derive the formula for the s^{th} raw moment of the TI-HT-TL-G distribution. Let $X \sim TI-HT-TL-G$ model. The s^{th} raw moment is expressed as:

$$\begin{aligned}\mu_s &= \int_{-\infty}^{\infty} x^s \left(2\theta^2 t(x) \sum_{j=0}^{\infty}\sum_{k=0}^{j} (-1)^k \binom{\theta+1}{j} (1-\theta)^j \binom{j}{k} \left(\bar{T}(x)\right)^{2(\theta+k)-1} \right) dx \\ &= 2\theta^2 \sum_{j=0}^{\infty}\sum_{k=0}^{j} (-1)^k \binom{\theta+1}{j} (1-\theta)^j \binom{j}{k} \int_{-\infty}^{\infty} x^s t(x) (\bar{T}(x))^{2(\theta+k)-1} dx \\ C_{s,j} &= \int_{-\infty}^{\infty} x^s t(x) \left(\bar{T}(x)\right)^{2(\theta+k)-1} dx\end{aligned}$$

$$\mu_s = 2\theta^2 \sum_{j=0}^{\infty}\sum_{k=0}^{j} (-1)^k \binom{\theta+1}{j} (1-\theta)^j \binom{j}{k} C_{s,j}$$

The Moment Generating Function (MGF) is

$$M_X(t) = \mathbb{E}[e^{tX}] = \int_0^{\infty} e^{tx} g(x;\theta,\Delta)\,dx.$$

Substituting the pdf expansion in Eq (10): we have the MGF given as

$$\begin{aligned}M_X(t) &= 2\theta^2 \sum_{j=0}^{\infty}\sum_{k=0}^{j} (-1)^k \binom{\theta+1}{j} (1-\theta)^j \binom{j}{k} \int_0^{\infty} e^{tx} t(x) (\bar{T}(x))^{2(\theta+k)-1} dx \\ U_{t,j} &= \int_0^{\infty} e^{tx} t(x) \left(\bar{T}(x)\right)^{2(\theta+k)-1} dx \\ M_X(t) &= 2\theta^2 \sum_{j=0}^{\infty}\sum_{k=0}^{j} (-1)^k \binom{\theta+1}{j} (1-\theta)^j \binom{j}{k} U_{t,j}\end{aligned}$$

2.4.2. Renyi entropy.

$$\begin{aligned}\xi_\alpha(g) &= \frac{1}{1-\alpha} \log\left(\int_0^{\infty} g(x)^\alpha\,dx \right), \qquad \alpha \geq 0,\ \alpha \neq 1 \\ g(x)^\alpha &= \left(2\theta^2 t(x)\right)^\alpha \sum_{j=0}^{\infty}\sum_{k=0}^{j} (-1)^k \binom{\alpha\theta+\alpha+j-1}{j} (1-\theta)^j \binom{j}{k} \left(\bar{T}(x)\right)^{\alpha(2\theta-1)+2k} \\ \int_0^{\infty} g(x)^\alpha\,dx &= \int_0^{\infty} \left(2\theta^2 t(x)\right)^\alpha \sum_{j=0}^{\infty}\sum_{k=0}^{j} (-1)^k \binom{\alpha\theta+\alpha+j-1}{j} (1-\theta)^j \binom{j}{k} \left(\bar{T}(x)\right)^{\alpha(2\theta-1)+2k} dx \\ &= (2\theta^2)^\alpha \sum_{j=0}^{\infty}\sum_{k=0}^{j} (-1)^k \binom{\alpha\theta+\alpha+j-1}{j} (1-\theta)^j \binom{j}{k} \int_0^{\infty} t(x)^\alpha \left(\bar{T}(x)\right)^{\alpha(2\theta-1)+2k} dx \\ \xi_\alpha(g) &= \frac{1}{1-\alpha} \log\left[(2\theta^2)^\alpha \sum_{j=0}^{\infty}\sum_{k=0}^{j} (-1)^k \binom{\alpha\theta+\alpha+j-1}{j} (1-\theta)^j \binom{j}{k} \int_0^{\infty} t(x)^\alpha \left(\bar{T}(x)\right)^{\alpha(2\theta-1)+2k} dx \right]\end{aligned}$$

Define

$$A_{j,k} = (2\theta^2)^{\alpha} \sum_{j=0}^{\infty} \sum_{k=0}^{j} (-1)^k \binom{\alpha\theta + \alpha + j - 1}{j} (1-\theta)^j \binom{j}{k} \int_0^{\infty} t(x)^{\alpha} \left(\bar{T}(x)\right)^{\alpha(2\theta-1)+2k} dx,$$

so that

$$\xi_{\alpha}(g) = \frac{1}{1-\alpha} \log A_{j,k}.$$

2.4.3. Order Statistics. Order statistics is a key branch of the theory of probability and statistical inference. Consider a sample of independent and identically distributed random variables $X_1, X_2, \ldots, X_n$ from the TI-HT-TL-G distribution. The pdf of the s^{th} order statistic of this sample is as follows:

$$g_{s:n}(x) = \frac{n!}{(s-1)!(n-s)!} g(x) \left[G(x)\right]^{s-1} \left[1 - G(x)\right]^{n-s}$$

$$g_{(s)}(x) = \frac{n!}{(s-1)!(n-s)!} \frac{2\theta^2 t(x) \bar{T}(x)^{2\theta(n-s+1)-1}}{\left[1-(1-\theta)(1-\bar{T}(x)^2)\right]^{\theta(n-s+1)+1}} \left(1 - \left[\frac{\bar{T}(x)^2}{1-(1-\theta)(1-\bar{T}(x)^2)}\right]^{\theta}\right)^{s-1}$$

3. Sub-Model of TI-HT-TL-G Family of Distributions

In this section, we present the sub-model of the TI-HT-TL-G family of distributions when the baseline CDF $G(x, \Delta)$ is specified. We present the special case of Rayleigh distribution given by [29], CDF, and PDF are given as follows:

$$T(x) = 1 - e^{-\beta x^2} \tag{3.1}$$

and

$$t(x) = 2\beta x e^{-\beta x^2} \tag{3.2}$$

respectively.

Substituting Eq. (3.1) into Eq. (3.2) gives the CDF of the TI-HT-TLR distribution as follows:

$$G(x; \theta, \beta) = 1 - \left(\frac{e^{-2\beta x^2}}{1-(1-\theta)\left(1-e^{-2\beta x^2}\right)}\right)^{\theta} \tag{3.3}$$

also substituting Eq. (3.2) and Eq. (3.1) into Eq. (2.6) gives the pdf of the TI-HT-TLR distribution.

$$g(x; \theta, \beta) = \frac{4\theta^2 \beta x e^{-2\theta\beta x^2}}{\left[1-(1-\theta)\left(1-e^{-2\beta x^2}\right)\right]^{\theta+1}} \tag{3.4}$$

for $\theta, \beta > 0$.

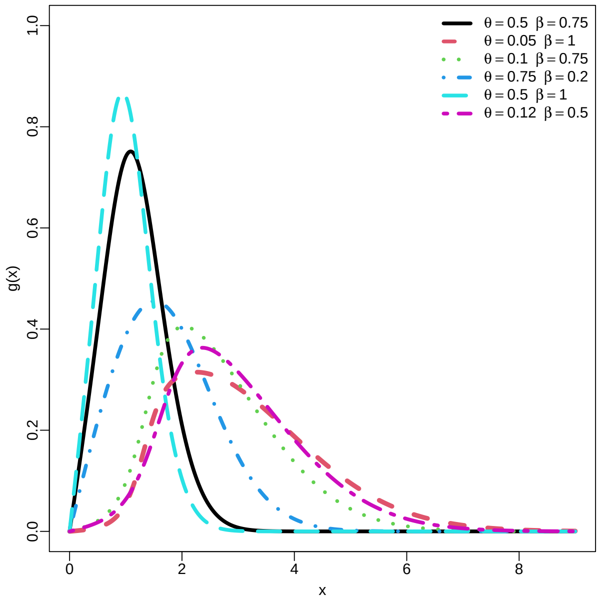


Figure 1: pdf Plots of TI-HT-TLR Distribution

3.1. Expansion of density function of TI-HT-TLR

Given the pdf of TI-HT-TLR in Eq. (3.4)

$$g(x;\theta,\beta) = \frac{4\theta^2\beta x e^{-2\theta\beta x^2}}{\left(1-(1-\theta)\left(1-e^{-2\beta x^2}\right)\right)^{\theta+1}}$$

$$g(x;\theta,\beta) = 4\theta^2\beta x e^{-2\theta\beta x^2} \times \left(1-(1-\theta)\left(1-e^{-2\beta x^2}\right)\right)^{-(\theta+1)}$$

$$\begin{aligned} g(x;\theta,\beta) &= 4\theta^2\beta x e^{-2\theta\beta x^2} \times \sum_{i=0}^{\infty}\binom{\theta+1}{i}(1-\theta)^i\left(1-e^{-2\beta x^2}\right)^i \\ &= 4\theta^2\beta x e^{-2\theta\beta x^2} \times \sum_{i=0}^{\infty}\binom{\theta+1}{i}(1-\theta)^i\sum_{k=0}^{\infty}(-1)^k\binom{i}{k}e^{-2k\beta x^2} \\ &= 4\theta^2\beta x e^{-2\theta\beta x^2} \times \sum_{i=0}^{\infty}\sum_{k=0}^{\infty}\binom{\theta+1}{i}(1-\theta)^i(-1)^k\binom{i}{k}e^{-2k\beta x^2} \end{aligned}$$

$$g(x;\theta,\beta) = 4\theta^2\beta \times \sum_{i=0}^{\infty}\sum_{k=0}^{\infty}\binom{\theta+1}{i}(1-\theta)^i(-1)^k\binom{i}{k}\,x\,e^{-2\beta x^2(\theta+k)} = Q_{ik} \tag{3.5}$$

3.2. Survival Function

The survival function of a random variable X ∼TI-HT-TLR (θ,β) is written as,

$$S(x,\theta,\beta) = 1-G(x,\theta,\beta) = \left(\frac{e^{-2\beta x^2}}{1-(1-\theta)(1-e^{-2\beta x^2})}\right)^{\theta}$$

3.3. Hazard Function

The hazard rate of a random variable X ∼TI-HT-TLR(θ,β) is written as,

$$h(x) = \frac{g(x,\theta,\beta)}{1-G(x,\theta,\beta)} = \frac{g(x,\theta,\beta)}{S(x,\theta,\beta)} = \frac{4\theta^2\beta x}{1-(1-\theta)(1-e^{-2\beta x^2})}$$

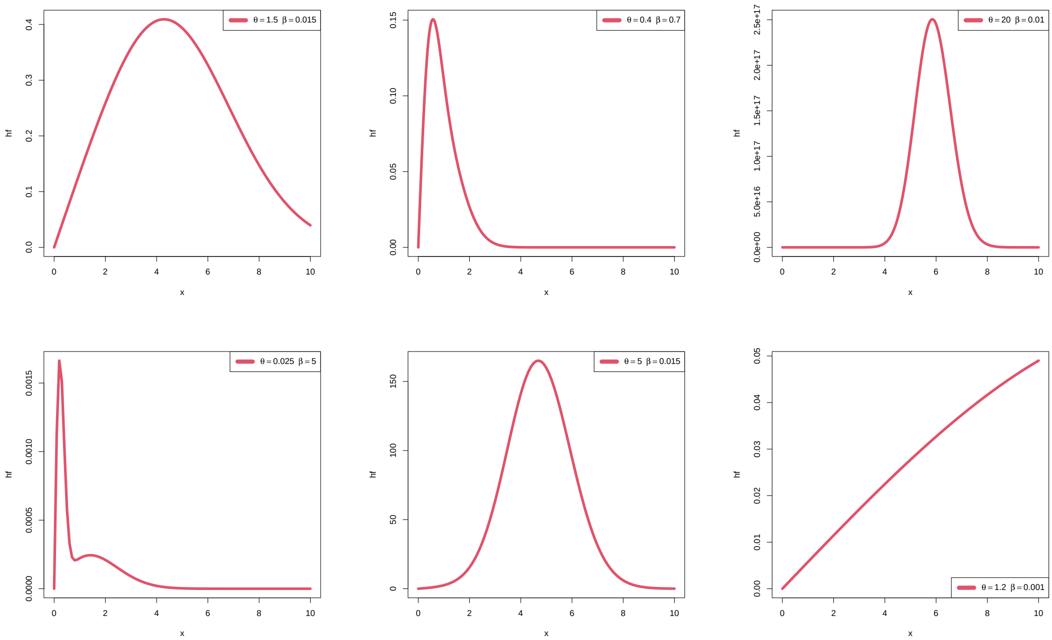


Figure 2: hazard function Plots of TI-HT-TLR Distribution

The shape of the hazard function plots in Figure (2) displays a variety of typical shapes. They are made up of bimodal-shaped curves, right-skewed, and left-skewed patterns, bell-shaped and symmetric shapes. There is even one with a predominantly straight-line or linear pattern. These shapes reflect the flexibility of the TI-HT-TLR distribution to model a wide range of failure behaviors and hazard dynamics under various parameter settings.

3.4. Quantile Function of TI-HT-TLR Distribution

Let $X \sim$ TI-HT-TLR(θ, β) and the cumulative distribution function (CDF) be represented by Eq. (3.3). The quantile function, which provides the inverse relationship between the distribution function and the corresponding quantiles, is obtained by solving the equation:

$$p = 1 - \left(\frac{e^{-2\beta x^2}}{1-(1-\theta)(1-e^{-2\beta x^2})} \right)^{\theta} \Rightarrow 1 - p = \left(\frac{e^{-2\beta x^2}}{1-(1-\theta)(1-e^{-2\beta x^2})} \right)^{\theta}$$

$$\Rightarrow (1-p)^{1/\theta} = \frac{e^{-2\beta x^2}}{1-(1-\theta)(1-e^{-2\beta x^2})}$$

simplifying gives the quantile function;

$$Q(p) = \left[-\frac{1}{2\beta} \ln \left(\frac{\theta(1-p)^{1/\theta}}{1-(1-p)^{1/\theta}(1-\theta)} \right) \right]^{1/2} \tag{3.6}$$

The quantile function derived in Eq. (3.6) plays a crucial role in simulating random samples from the TI-HT-TLR distribution, making it especially valuable for simulation purposes. When the probability level $p = 1/2$, the function yields the median of the distribution, representing the typical lifetime in reliability contexts. Additionally, the quantile function can be employed to derive measures such as skewness and kurtosis, which provide insights into the shape and tail behavior of the distribution.

Using iterative methods by making use of R software, we present quantiles for the TI-HT-TLR distribution for some selected values of parameters. The results are shown in Table 1

Table 1: Quantiles for the TI-HT-TLR Distribution

q	(0.9,1.5)	(0.5,1.0)	(1.0,0.5)	(1.5,1.5)	(0.5,0.1)
0.1	0.2076	0.4386	0.3246	0.1257	1.3869
0.2	0.3011	0.6139	0.4724	0.1840	1.9414
0.3	0.3795	0.7502	0.5972	0.2343	2.3722
0.4	0.4526	0.8708	0.7147	0.2825	2.7535
0.5	0.5254	0.9864	0.8326	0.3319	3.1192
0.6	0.6019	1.1051	0.957	0.3854	3.4945
0.7	0.6871	1.2359	1.0973	0.4470	3.9084
0.8	0.7909	1.3950	1.2686	0.5245	4.4112
0.9	0.9409	1.6269	1.5174	0.6408	5.1446

3.5. Moment

The j^{th} moment of the TI-HT-TLR distribution is derived as

$$\mu_j' = \int_0^\infty x^j g(x,\theta,\beta)dx$$

$$\mu_j' = \int_0^\infty x^j \frac{4\theta^2\beta x e^{-2\theta\beta x^2}}{1-(1-\theta)(1-e^{-2\beta x^2})^{\theta+1}}dx$$

$$\mu_j' = 4\theta^2\beta \int_0^\infty x^{j+1} e^{-2\theta\beta x^2}(1-(1-\theta)(1-e^{-2\beta x^2}))^{-(\theta+1)}dx$$

Applying general binomial expansion in Eq. (3.5), we get

$$\mu_j' = 4\theta^2\beta \sum_{i=0}^{\infty}\sum_{k=0}^{\infty}\binom{\theta+1}{i}(1-\theta)^i(-1)^k\binom{i}{k}\int_0^\infty x^{j+1}e^{-(\theta+k)2\beta x^2}dx$$

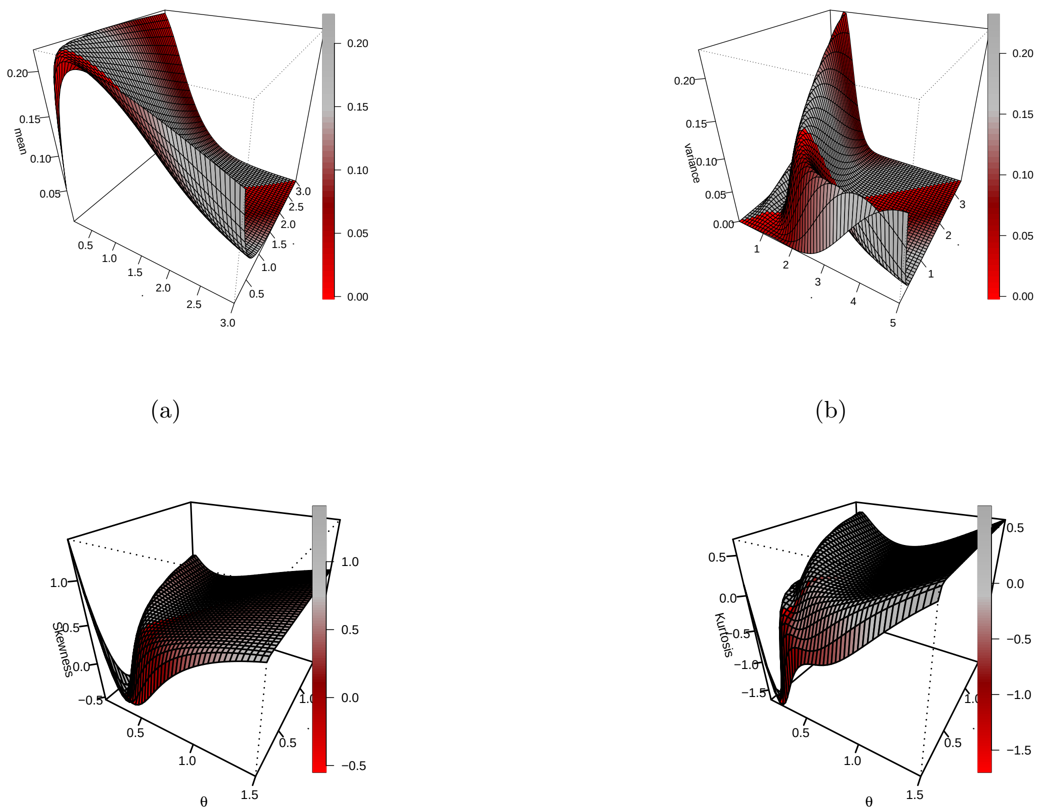
The j^{th} moment is given as

$$\mu_j' = \theta^2\,\Gamma\left(\frac{j}{2}+1\right)\sum_{i=0}^{\infty}\sum_{k=0}^{i}\binom{\theta+1}{i}(1-\theta)^i(-1)^k\binom{i}{k}\left(\frac{1}{2\beta(\theta+k)}\right)^{\frac{j}{2}+1} \tag{3.7}$$

The first four moments are obtained by replacing $j=1, j=2, j=3$ and $j=4$ in Eq. (3.7). Some of the most significant are the mean (μ), variance and coefficient of variation (CV) of the random variable X, where the first moment μ_1' is the mean (μ), and the variance represented by $\mu_2 = \mu_2' - \mu_2'^2$ and the $CV = \sqrt{\text{Var}(X)}/\mu$. The first five moments of TI-HT-TLR and the standard deviation (SD), coefficient of variation (CV), coefficient of skewness and coefficient of kurtosis with their different parameter values are presented in Table 2

Table 2: Moments of TI-HT-TLR Distribution

	$E(X)$	$E(X^2)$	$E(X^3)$	$E(X^4)$	$E(X^5)$	SD	CV	CS	CK
(0.9,0.5)	0.5554	0.3892	0.3193	0.2952	0.3005	0.2841	0.5116	0.5874	3.1785
(0.5,1.0)	1.0172	1.2465	1.7420	2.7017	4.5728	0.4603	0.4525	0.4427	3.1323
(1.0,0.5)	0.8862	1.0000	1.3293	2.0000	3.3234	0.4633	0.5227	0.6311	3.2451
(1.5,1.5)	0.3626	0.1728	0.0995	0.0661	0.0493	0.2033	0.5606	0.8202	3.6832
(0.5,0.1)	3.2165	12.4645	55.0861	270.1719	1446.0615	1.4555	0.4525	0.4427	3.1323



(a) (b) (c) (d)

Figure 3: Mean, Variance, Skewness, and Kurtosis of TI-HT-TL-R Distribution

Figure (3) displays the mean, variance, skewness, and kurtosis. The plot in Figure (3) illustrates the behavior of the key moments of the TIHTTLR distribution as a function of its parameters θ and β case (a) illustrates that the mean of the TIHTTLR distribution is nonlinear with regard to θ and β. It is a maximum when θ takes moderate values and β takes small values and decreases as both parameters increase. This indicates that larger θ and *beta* are heavy tailed around small values, producing a dome-shaped mean surface that reflects the joint impact of the parameters on tail and scale behavior. The variance as shown in case (b) has a non-monotonic and nonlinear relationship with θ and β. It increases with both parameters at first, reaching a definite maximum before declining slowly. This is an indication of extreme sensitivity of the dispersion of the distribution to the interaction of θ and β, i.e., intermediate parameter values produce the most variation in results. Similarly, in case (c) the skewness is mostly positive, indicating a right-skewed shape with a heavier right tail. The extent of asymmetry is highest at small values of θ and β and diminishes progressively as the two parameters get larger, showing a tendency towards symmetry for large values of the parameters. Also in case (d), the kurtosis shows that the TIHTTLR distribution is largely leptokurtic with heavier tails and a more peaked shape than the normal distribution. Kurtosis decreases slightly with increasing θ and β, meaning that the distribution becomes less heavy-tailed and less peaked for larger parameter values.

3.6. Moment Generating Function (MGF)

The moment generating function for the TI-HT-TLR distributed random variable X is given as

$$M_x(t) = E[e^{tx}] = \int_0^\infty e^{tx} g(x) dx$$

$$M_x(t) = \int_0^\infty e^{tx} \frac{4\theta^2 \beta x e^{-2\theta\beta x^2}}{1-(1-\theta)(1-e^{-2\beta x^2})^{\theta+1}} dx$$

Using the series expansion for $g(x)$ in Eq. (3.5) we have

$$M_x(t) = 4\theta^2\beta \sum_{i,k=0}^{\infty} \binom{\theta+1}{i} (1-\theta)^i (-1)^k \int_0^\infty x e^{tx} e^{-(\theta+k)2\beta x^2} dx$$

$$M_x(t) = \theta^2 \sum_{i,k=0}^{\infty} \binom{\theta+1}{i} (1-\theta)^i \binom{i}{k} (-1)^k (\theta+k) e^{\frac{t^2}{8\beta(\theta+k)}}$$

3.7. Reliability Function under the TI-HT-TLR Model

Let $X \sim$ TI-HT-TLR(θ_1, β_1) be the component strength, and $Z \sim$ TI-HT-TLR(θ_2, β_2), be the stress imposed on the same system. The reliability function, which provides the probability that the component strength is higher than the induced stress, is expressed as, $R = P(Z < X)$. "This reliability measure can be calculated by integrating the joint distribution of the random variable for stress and the random variable for strength over the appropriate domain. In case the two random variables are independent, then the reliability function" simplifies to:

$$\begin{aligned} R &= \int_0^\infty \left[\int_0^x g(x)G(z)\, dz\right] dx \\ &= \int_0^\infty \frac{4\theta_1^2\beta_1 x e^{-2\theta_1\beta_1 x^2}}{\left(1-(1-\theta_1)(1-e^{-2\beta_1 x^2})\right)^{\theta_1+1}} \\ &\quad \times \left[1 - \left(\frac{e^{-2\beta_2 x^2}}{1-(1-\theta_2)(1-e^{-2\beta_2 x^2})}\right)^{\theta_2}\right] dx \end{aligned}$$

$$\begin{aligned} R = 4\theta_1^2\beta_1 &\int_0^\infty x e^{-2\theta_1\beta_1 x^2} \left[1-(1-\theta_1)\left(1-e^{-2\beta_1 x^2}\right)\right]^{-(\theta_1+1)} \\ &\times \left\{1 - e^{-2\theta_2\beta_2 x^2}\left[1-(1-\theta_2)\left(1-e^{-2\beta_2 x^2}\right)\right]^{-\theta_2}\right\} dx \end{aligned}$$

$$\begin{aligned} R = P(z<x) = 4\theta_1^2\beta_1 &\sum_{i=0}^{\infty}\sum_{k=0}^{\infty} \binom{\theta_1+1}{i}(1-\theta_1)^i \binom{i}{k}(-1)^k \\ &\times \int_0^\infty x e^{-2(\theta_1+k)\beta_1 x^2} \left\{1 - e^{-2\theta_2\beta_2 x^2}\left[1-(1-\theta_2)\left(1-e^{-2\beta_2 x^2}\right)\right]^{-\theta_2}\right\} dx \end{aligned}$$

3.8. Order Statistics

Order statistics play a vital role in solving intricate problems in statistical analysis. Consider a random sample $X_1, X_2, ...X_n$ from a population. The r^{th} order statistic, $X(r)$ for $r = 1, 2, \ldots, n$, is the r^{th} smallest value after arranging the sample in ascending order. Order statistics are particularly useful in analyzing data that obeys the TIHTTLR distribution.

$$g_{r:n}(x, \theta, \beta) = \frac{n!}{(r-1)!(n-r)!} g(x) \left[G(x)\right]^{r-1} \left[1-G(x)\right]^{n-r}$$

$$g_{(r)}(x;\theta,\beta) = \frac{n!}{(r-1)!\,(n-r)!}\left[1-\left(\frac{e^{-2\beta x^2}}{1-(1-\theta)\left(1-e^{-2\beta x^2}\right)}\right)^{\theta}\right]^{r-1}$$
$$\times\left[\left(\frac{e^{-2\beta x^2}}{1-(1-\theta)\left(1-e^{-2\beta x^2}\right)}\right)^{\theta}\right]^{n-r}\times\left(\frac{4\theta^2\beta x e^{-2\theta\beta x^2}}{\left[1-(1-\theta)\left(1-e^{-2\beta x^2}\right)\right]^{\theta+1}}\right).$$

The pdf for the maximum order statistics is derived by setting $r = n$, yielding

$$g_{(n)}(x) = n\left[1-\left(\frac{e^{-2\beta x^2}}{1-(1-\theta)(1-e^{-2\beta x^2})}\right)^{\theta}\right]^{n-1}\left(\frac{4\theta^2\beta x e^{-2\theta\beta x^2}}{\left[1-(1-\theta)(1-e^{-2\beta x^2})\right]^{\theta+1}}\right)$$

The pdf of the first-order statistics is derived by setting $r = 1$

$$g_{(1)}(x) = \frac{4n\theta^2\beta x e^{-2\beta x^2(1+\theta(n-1))}}{\left[1-(1-\theta)(1-e^{-2\beta x^2})\right]^{\theta n+1}}$$

3.9. Measure of uncertainty

In the context of statistical inference, entropy is a quantification of unpredictability or uncertainty about a random variable. One among a number of entropy definitions, which is used extensively, is Rényi entropy, introduced by [15], which generalizes Shannon entropy by the introduction of a parameter that emphasizes different aspects of the distribution. For TI-HT-TLR, it is derived thus:

$$R_E = \frac{1}{1-v}\log\int_0^\infty g^v(x)dx = \frac{1}{1-v}\log\int_0^\infty 4\theta^2\beta^v x^v e^{-2v\theta\beta x^2}(1-(1-\theta)(1-e^{-2\beta x^2}))^{-v(\theta+1)}dx$$
$$= \frac{1}{1-v}\log\int_0^\infty 4\theta^2\beta^v x^v e^{-v(\theta+k)2\beta x^2}\sum_{i,k=0}^{\infty}\binom{v\theta+i+i-1}{i}(1-\theta)^i\binom{i}{k}(-1)^k dx$$

$$R_E = \frac{1}{1-v}\log\left(4\theta^2\beta^v\frac{1}{2}\Gamma\left(\frac{v+1}{2}\right)\sum_{i,k=0}^{\infty}\binom{v\theta+i+i-1}{i}(1-\theta)^i\binom{i}{k}(-1)^k(2(v\theta+k)\beta)^{-\frac{v+1}{2}}\right)$$

Where $v \neq 1$ but $v > 0$.

4. Estimation of Parameters

In this, various parameter estimation techniques were employed to effectively model and describe the data. They aim to determine the values of the unknown parameters that best describe the behavior of the observed data set. The approaches considered are maximum likelihood estimate (MLE), maximum product spacing (MPS), least squares (LS), weighted least squares (WLS), Cramér-von Mises (CVM), Anderson-Darling (AD), and the right-tailed Anderson-Darling (RTAD) approach. Moreover, Bayesian estimation was used in selected samples under different loss functions, namely squared error loss (SEL), linear exponential loss (LINEX), and generalized entropy loss (GEL). Performance of every estimation was compared via comparative examination based on bias and root mean square error (RMSE).

4.1. Maximum likelihood estimation (MLE)

The parameters of the TI-HT-TLR distribution can be estimated using the maximum likelihood estimation method. If $x_1, x_2, \ldots, x_n$ are random samples and given the expression in Eq. (3.5), the likelihood function is:

$$L(\theta,\beta) = \prod_{i=1}^{n} g(x_i,\theta,\beta) = \prod_{i=1}^{n}\frac{4\theta^2\beta x_i e^{-2\theta\beta x_i^2}}{[1-(1-\theta)(1-e^{-2\beta x_i^2})]^{\theta+1}}$$

While the log-likelihood function is expressed as:

$$\ell(\theta,\beta) = n\ln(4) + 2n\ln(\theta) + n\ln(\beta) + \sum_{i=1}^{n}\ln(x_i) - (\theta+1)\sum_{i=1}^{n}\ln\left(1-(1-\theta)(1-e^{-2\beta x_i^2})\right) - 2\theta\beta\sum_{i=1}^{n}x_i^2$$

Taking partial derivative with respect to θ.

$$\frac{\partial\ell}{\partial\theta} = \frac{2n}{\theta} - 2\beta\sum_{i=1}^{n}x_i^2 + (\theta+1)\sum_{i=1}^{n}\frac{(1-e^{-2\beta x_i^2})}{1-(1-\theta)(1-e^{-2\beta x_i^2})} - \sum_{i=1}^{n}\ln\left(1-(1-\theta)(1-e^{-2\beta x_i^2})\right) \tag{4.1}$$

Similarly, taking the partial derivative with respect to β

$$\frac{\partial\ell}{\partial\beta} = \frac{n}{\beta} - 2\theta\sum_{i=1}^{n}x_i^2 + (\theta+1)\sum_{i=1}^{n}\frac{2(1-\theta)x_i^2e^{-2\beta x_i^2}}{1-(1-\theta)(1-e^{-2\beta x_i^2})} \tag{4.2}$$

The MLEs $(\hat{\theta},\hat{\beta})$ for (θ,β) are obtained by equating the score function that is, the partial derivatives of the log-likelihood function with respect to each parameters, given in Eq. (4.1) and Eq. (4.2) to zero and solving the resulting system of equations simultaneously. Since these equations are non-linear, they are solved iteratively using the Newton-Raphson method with the help of R software to obtain the MLEs of the unknown parameters.

Newton Raphson Algorithm for TIHTTLR distribution

In this study, the Newton–Raphson iterative method is employed to approximate the estimates.

The Hessian matrix of second-order partial derivatives is defined as

$$H(\boldsymbol{w}) = \begin{pmatrix} \frac{\partial^2\ell}{\partial\theta^2} & \frac{\partial^2\ell}{\partial\theta\partial\beta} \\ \frac{\partial^2\ell}{\partial\beta\partial\theta} & \frac{\partial^2\ell}{\partial\beta^2} \end{pmatrix} \tag{4.3}$$

where the individual elements are given as follows:

$$\frac{\partial^2\ell}{\partial\theta^2} = \frac{2n}{\theta} - \sum_{i=1}^{n}\frac{1-e^{-2\beta x_i^2}}{\theta+(1-\theta)e^{-2\beta x_i^2}} + (\theta+1)\sum_{i=1}^{n}\frac{(1-e^{-2\beta x_i^2})^2}{\theta+(1-\theta)e^{-2\beta x_i^2}} \tag{4.4}$$

$$\frac{\partial^2\ell}{\partial\beta^2} = \frac{n}{\beta^2} - 2(\theta+1)(1-\theta)\sum_{i=1}^{n}\frac{2x_i^4e^{-2\beta x_i^2}\left(\theta+(1-\theta)e^{-2\beta x_i^2}\right)+2(1-\theta)x_i^4\left(e^{-2\beta x_i^2}\right)^2}{\theta+(1-\theta)e^{-2\beta x_i^2}} \tag{4.5}$$

$$\begin{aligned}\frac{\partial^2\ell}{\partial\theta\partial\beta} = \frac{\partial^2\ell}{\partial\beta\partial\theta} &= -2\sum_{i=1}^{n}x_i^2 - 2\theta\sum_{i=1}^{n}\frac{x_i^2e^{-2\beta x_i^2}}{\theta+(1-\theta)e^{-2\beta x_i^2}} \\ &+ (\theta+1)(1-\theta)\sum_{i=1}^{n}\frac{x_i^2e^{-2\beta x_i^2}\left(1-e^{-2\beta x_i^2}\right)}{\left(\theta+(1-\theta)e^{-2\beta x_i^2}\right)^2}\end{aligned} \tag{4.6}$$

The Newton-Raphson Technique is then applied to iteratively solve the system on nonlinear score equations and obtain the maximum likelihood estimate of the model parameters.

Algorithm 1: Newton-Raphson procedure for TIHTTLR MLE

Require: Observed sample $y = (y_1,\ldots,y_n)$, initial parameter vector $\boldsymbol{\theta}^{(0)} = (\theta^{(0)},\beta^{(0)})$, convergence tolerance ξ, and maximum number of iterations N.

Ensure: Maximum likelihood estimates $\hat{\boldsymbol{\theta}} = (\hat{\theta},\hat{\beta})$ and asymptotic variance–covariance matrix U.

- Set $k \leftarrow 0$
- Set $\boldsymbol{\theta} \leftarrow \boldsymbol{\theta}^{(0)}$
- Repeat

- Evaluate the score vector at $\boldsymbol{\theta}^{(k)}$ $S^{(k)} = \Delta\ell(\boldsymbol{\theta}^{(k)}) = \left(\frac{\partial \ell}{\partial \theta}, \frac{\partial \ell}{\partial \beta}\right)^{\top}$
- Compute the Hessian matrix at $\boldsymbol{\theta}^{(k)}$
- $H^{(k)} = \begin{pmatrix} \frac{\partial^2 \ell}{\partial \theta^2} & \frac{\partial^2 \ell}{\partial \theta \partial \beta} \\ \frac{\partial^2 \ell}{\partial \beta \partial \theta} & \frac{\partial^2 \ell}{\partial \beta^2} \end{pmatrix}_{\boldsymbol{\theta}^{(k)}}$
- Determine the Newton step direction $\boldsymbol{\gamma}^{(k)}$
- Solve the linear system: $H^{(k)}\boldsymbol{\gamma}^{(k)} = -S^{(k)}$
- Update the parameter vector:
  $\boldsymbol{\theta}^{(k+1)} = \boldsymbol{\theta}^{(k)} + \boldsymbol{\gamma}^{(k)}$
- Check convergence criterion
  If$\|\boldsymbol{\gamma}^{(k)}\| < \xi$ or $k \geq N$, terminate the iteration
- Otherwise, set $k \leftarrow k+1$, and repeat the process
- Untilconvergence Set $\hat{\boldsymbol{\theta}} = \boldsymbol{\theta}^{(k+1)}$ as MLE Estimates
- Compute the observed Fisher information matrix: $I_0 = -H(\hat{\boldsymbol{\theta}})$
- Obtain the asymptotic variance–covariance matrix: $U = I_0^{-1}$

4.2. Least Square Estimation (LSE) Method

Least squares estimation is a technique most commonly used to estimate the parameters of distributions by minimizing the squared empirical-theoretical distribution function differences. Initially proposed by [16] as a parameter estimation method for the Beta distribution, the method has since been adapted for application with other distributions, we write $E\left[Gx_{i:n} \mid \theta, \beta\right] = \frac{i}{n+1}$ and$Z\left[G_{X_{(i:n)}} \mid \theta, \beta\right] = \frac{i(n-i+1)}{(n+1)^2(n+2)}$. The least square estimates $\hat{\theta}_{\text{LSE}}$, and $\hat{\beta}_{LSE}$ of the parameter θ, and β are obtained by minimizing the function $L(\theta, \beta)$ with respect to (w. r. t) θ, and β

$$L(\theta, \beta) = \sum_{i=1}^{n}\left[G\left(x_{(i:n)} \mid \theta, \beta\right) - \frac{i}{n+1}\right]^2$$

The estimate are derived by solving set of non-linear equations.

$$\sum_{i=1}^{n}\left[G\left(x_{(i:n)} \mid \theta, \beta\right) - \frac{i}{n+1}\right]\Delta_1\left(x_{(i:n)} \mid \theta, \beta\right) = 0 \tag{4.7}$$

$$\sum_{i=1}^{n}\left[G\left(x_{(i:n)} \mid \theta, \beta\right) - \frac{i}{n+1}\right]\Delta_2\left(x_{(i:n)} \mid \theta, \beta\right) = 0 \tag{4.8}$$

Where

$$\Delta_1\left(x_{(i:n)} \mid \theta, \beta\right) = \left(\frac{e^{-2\beta x^2}}{1-(1-\theta)\left(1-e^{-2\beta x^2}\right)}\right)^{\theta}\left[\ln\left(\frac{e^{-2\beta x^2}}{1-(1-\theta)\left(1-e^{-2\beta x^2}\right)}\right) - \frac{\theta\left(1-e^{-2\beta x^2}\right)}{1-(1-\theta)\left(1-e^{-2\beta x^2}\right)}\right] \tag{4.9}$$

$$\Delta_2\left(x_{(i:n)} \mid \theta, \beta\right) = 2\theta x^2\left(\frac{e^{-2\beta x^2}}{1-(1-\theta)\left(1-e^{-2\beta x^2}\right)}\right)^{\theta}\left(\frac{\theta+(1-\theta)-e^{-2\beta x^2}}{1-(1-\theta)\left(1-e^{-2\beta x^2}\right)}\right) \tag{4.10}$$

Eq. (4.9), and Eq. (4.10) are obtained by partially differentiating the CDF of TI-HT-TLR contained in Eq. (3.3) with respect to θ, and β respectively.

4.3. Weighted-Least Squares Estimate (WLSE)

The WLSE $\hat{\theta}_{WLSE}$, and $\hat{\beta}_{WLSE}$ of TI-HT-TLR distribution with the parameter θ, and β are obtained by minimizing the function $W(\theta, \beta)$ with respect to θ, and β

$$W(\theta, \beta) = \sum_{i=1}^{n} \frac{(n+1)^2(n+2)}{i(n-i+1)} \left[G\left(x_{(i:n)} \mid \theta, \beta\right) - \frac{i}{n+1} \right]^2$$

Deriving the following non-linear equation yields the estimate.

$$\sum_{i=1}^{n} \frac{(n+1)^2(n+2)}{i(n-i+1)} \left[G\left(x_{(i:n)} \mid \theta, \beta\right) - \frac{i}{n+1} \right] \Delta_1\left(x_{(i:n)} \mid \theta, \beta\right) = 0 \tag{4.11}$$

$$\sum_{i=1}^{n} \frac{(n+1)^2(n+2)}{i(n-i+1)} \left[G\left(x_{(i:n)} \mid \theta, \beta\right) - \frac{i}{n+1} \right] \Delta_2\left(x_{(i:n)} \mid \theta, \beta\right) = 0 \tag{4.12}$$

Where $\Delta_1\left(x_{(i:n)} \mid \theta, \beta\right)$, and $\Delta_2\left(x_{(i:n)} \mid \theta, \beta\right)$ are defined in Eq. (4.9), Eq (4.10) respectively.

4.4. Maximum Product Spacing Estimation (MPSE) Method

MPSE provides an alternative to the maximum likelihood method, especially in cases where likelihood functions are either complex or contain flat regions. First introduced by [17], the method focuses on maximizing the geometric mean of spacings of the distribution function. then, the MPSE for the TI-HT-TLR is given as follows:

$$F_S(\theta, \beta \mid \text{data}) = \left[\prod_{i=1}^{n+1} A_i(x_i, \theta, \beta) \right]^{\frac{1}{n+1}}$$

Where $A_i(x_i, \theta, \beta) = G(x_i, \theta, \beta) - G(x_{i-1}, \theta, \beta)$, $i = 1, 2, \ldots, n$. likewise, we can also maximize the function

$$M(\theta, \beta) = \frac{1}{n+1} \sum_{i=1}^{n+1} \ln A_i(\theta, \beta) \tag{4.13}$$

Taking the first derivative of $M(\theta, \beta)$ w. r. t θ, and β, and solving the resulting non-linear equation, at $\frac{\partial M(\theta,\beta)}{\partial \theta} = 0$, and $\frac{\partial M(\theta,\beta)}{\partial \beta} = 0$, the parameter estimates value can be obtained.

4.5. Cramer-von-Mises Estimation (CVME)

The Cramér–von Mises estimation method is based on minimizing squared differences between empirical and theoretical cumulative distribution functions over the data range. The approach is distinct from other estimation techniques and is known to be robust for goodness-of-fit assessment. For TI-HT-TLR distribution, parameter estimates $\hat{\theta}_{CVME}$, and $\hat{\beta}_{CVME}$ are obtained by minimizing the following function,

$$C(\theta, \beta) = \arg\min \left[\frac{1}{12n} + \sum_{i=1}^{n} \left(G(x_{(i:n)} \mid \theta, \beta) - \frac{2i-1}{2n} \right)^2 \right].$$

The estimates are obtained by solving the non-linear equations,

$$\sum_{i=1}^{n} \left(G(x_{(i:n)} \mid \theta, \beta) - \frac{2i-1}{2n} \right) \Delta_1\left(x_{(i:n)} \mid \theta, \beta\right) = 0, \tag{4.14}$$

$$\sum_{i=1}^{n} \left(G(x_{(i:n)} \mid \theta, \beta) - \frac{2i-1}{2n} \right) \Delta_2\left(x_{(i:n)} \mid \theta, \beta\right) = 0, \tag{4.15}$$

Where $\Delta_1\left(x_{(i:n)} \mid \theta, \beta\right)$, and $\Delta_2\left(x_{(i:n)} \mid \theta, \beta\right)$, is as defined in Eq. (4.9), and Eq. (4.10) respectively.

4.6. Anderson Darling Estimation (ADE)

The Anderson Darling estimate $\hat{\theta}_{ADE}$, and $\hat{\beta}_{ADE}$ of the parameters θ, and β of the TI-HT-TLR distribution are the solution to the problem of minimizing the function $A(\theta,\beta)$ in terms of both θ, and β.

$$A(\theta,\beta) = \arg\min \sum_{i=1}^{n}(2i-1)\left[\ln G\left(x_{(i:n)} \mid \theta,\beta\right) + \ln\left(1 - G\left(x_{(n+1-i:n)} \mid \theta,\beta\right)\right)\right]$$

The estimates are obtained by solving the following sets of non-linear equations,

$$\sum_{i=1}^{n}(2i-1)\left\{\frac{\Delta_1\left(x_{(i:n)} \mid \theta,\beta\right)}{G\left(x_{(i:n)} \mid \theta,\beta\right)} - \frac{\Delta_1 G\left(x_{(n+1-i:n)} \mid \theta,\beta\right)}{1-G\left(x_{(n+1-i:n)} \mid \theta,\beta\right)}\right\} = 0, \tag{4.16}$$

$$\sum_{i=1}^{n}(2i-1)\left\{\frac{\Delta_2\left(x_{(i:n)} \mid \theta,\beta\right)}{G\left(x_{(i:n)} \mid \theta,\beta\right)} - \frac{\Delta_2 G\left(x_{(n+1-i:n)} \mid \theta,\beta\right)}{1-G\left(x_{(n+1-i:n)} \mid \theta,\beta\right)}\right\} = 0, \tag{4.17}$$

Where $\Delta_1\left(x_{(i:n)} \mid \theta,\beta\right)$, and $\Delta_2\left(x_{(i:n)} \mid \theta,\beta\right)$ is as defined in Eq. (4.9), and Eq. (4.10) respectively.

4.7. Right-Tailed Anderson-Darling Estimation (RTADE)

The Right-Tailed Anderson-Darling estimates denoted by $\hat{\theta}_{RTADE}$, and $\hat{\beta}_{RTADE}$ for the TI-HT-TLR distribution parameters θ, and β are derived by minimizing the function $R(\theta,\beta)$ with respect to (w. r. t) both θ and β

$$R(\theta,\beta) = \arg\min\left[\frac{n}{2} - 2\sum_{i=1}^{n} G\left(x_{(i:n)} \mid \theta,\beta\right) - \frac{1}{n}\sum_{i=1}^{n}(2i-1)\ln\left\{1 - G\left(x_{(n+1-i:n)} \mid \theta,\beta\right)\right\}\right]$$

The estimate can be obtained by solving the following set of non-linear equation

$$-2\sum_{i=1}^{n}\frac{\Delta_1\left(x_{(i:n)} \mid \theta,\beta\right)}{G\left(x_{(i:n)} \mid \theta,\beta\right)} + \frac{1}{n}\sum_{i=1}^{n}(2i-1)\frac{\Delta_1 G\left(x_{(n+1-i:n)} \mid \theta,\beta\right)}{1-G\left(x_{(n+1-i:n)} \mid \theta,\beta\right)} = 0 \tag{4.18}$$

$$-2\sum_{i=1}^{n}\frac{\Delta_2\left(x_{(i:n)} \mid \theta,\beta\right)}{G\left(x_{(i:n)} \mid \theta,\beta\right)} + \frac{1}{n}\sum_{i=1}^{n}(2i-1)\frac{\Delta_2 G\left(x_{(n+1-i:n)} \mid \theta,\beta\right)}{1-G\left(x_{(n+1-i:n)} \mid \theta,\beta\right)} = 0 \tag{4.19}$$

Where $\Delta_1\left(x_{(i:n)} \mid \theta,\beta\right)$, and $\Delta_2\left(x_{(i:n)} \mid \theta,\beta\right)$ is as defined Eq. (4.9), and Eq. (4.10) respectively. The estimates given in Eq. (4.1), (4.2), (4.7), (**??**), (4.11), (4.12), (4.13), (4.14), (4.15), (4.16), (4.17), (4.18), and (4.19) they are computed using the optim() function in R, which applies the Newton-Raphson iterative procedure.

4.8. Bayesian estimation

In this study, we discuss the Bayesian estimation (BE) of the unknown parameters of the TI-HT-TLR distribution. In the Bayesian approach, various loss functions such as the squared error loss, linear exponential (LINEX) loss, and generalized entropy loss can be utilized. We assume that the parameters θ and β have independent prior distributions and that they have gamma priors. The prior probability density functions are given by:

$$\mu_1(\theta) \propto \theta^{c_1-1}e^{-k_1\theta}, \quad \theta > 0,\ c_1 > 0,\ k_1 > 0$$
$$\mu_2(\beta) \propto \beta^{c_2-1}e^{-k_2\beta}, \quad \beta > 0,\ c_2 > 0,\ k_2 > 0$$

Where the hyper-parameters $c_i, q_i, i = 1,2$ were selected to show the prior information of the unknown parameters.

The joint prior for $\phi = (\theta, \beta)$ is given by

$$\mu(\phi) = \mu_1(\theta)\,\mu_2(\beta)$$
$$\propto \theta^{c_1-1}\beta^{c_2-1}e^{-(k_1\theta+k_2\beta)}$$

The posterior density given the observed data $X = (x_1, x_2, \ldots x_n)$ is given by: The likelihood is $l(\theta, \beta) = \prod_{i=1}^{n} g(x_i; \theta, \beta)$ Implying that the posterior density function is:

$$\mu(\phi|X) \propto L(\theta, \beta) \cdot \mu(\phi)$$

$$\mu(\phi \mid X) \propto \left[\prod_{i=1}^{n} g(x_i; \theta, \beta)\right] \theta^{c_1-1}\,\beta^{c_2-1}\,e^{-(k_1\theta+k_2\beta)}$$

Given any function, such as $l(\phi)$ under the squared error loss (SEL) function, the Bayes estimator is given as

$$\hat{\phi}_{\mathrm{BE_{SEL}}} = \mathbb{E}[l\phi \mid x] = \int_{\Phi} l\phi\,\mu(\phi \mid x)\,d\phi \tag{4.20}$$

The symmetric error loss (SEL) function inflicts an identical penalty for overestimation as well as underestimation. In the majority of real-world cases, one of the estimation errors may be more severe compared to the other. Due to this fact, the LINEX (Linear-Exponential) loss function is advised as an alternative. $(l(\phi), \hat{l}(\phi)) = e^{\hat{l}(\phi)-l(\phi)} - w(\hat{l}(\phi) - l(\phi)) - 1$ Where $w \neq 0$ is direction and magnitude controlling parameter. Overestimation is penalized more for a $w > 1$, and underestimation is penalized more when $w < 1$. For $w \to 0$, the LINEX loss converges to the SEL. For further detailed discussions on its use in Bayesian estimation, see [18] and [19].
The BE of $l(\phi)$ under this loss can be derived as

$$\hat{\phi}_{\mathrm{BE\_LINEX}} = E\left[e^{-wl(\phi)} \mid X\right] = -\frac{1}{w}\log\left[\int_{\phi^{\cdot}} e^{-wl(\phi)}\,\mu(\phi \mid x)\,d\phi\right] \tag{4.21}$$

A second alternative of the symmetric error loss function is the general entropy loss (GEL) function, which provides a more versatile framework for the imposition of estimation penalties. The GEL function was introduced by [20] and is in use particularly when different weights are needed to be assigned to overestimation and underestimation. The GEL function is given by: $(l(\phi), \hat{l}(\phi)) = \left(\frac{\hat{l}(\phi)}{l(\phi)}\right)^{\alpha} - \alpha\log\left(\frac{\hat{l}(\phi)}{l(\phi)}\right) - 1$ where the function form offers flexibility to code preferences for estimation errors in one direction rather than the other. The parameter $\alpha \neq 0$ represents the degree of asymmetry in the loss function. When $\alpha > 0$, overestimation is penalized more than underestimation, while $\alpha < 0$. Hence, underestimation is taken more seriously. Thus, the choice of reflects the decision maker's sensitivity to estimation errors of one type versus the other. The resulting Bayes estimator for this asymmetric loss is obtained by minimizing the posterior expected GEL, thereby incorporating both the asymmetry preferences and the information in the posterior distribution.

$$\hat{\phi}_{\mathrm{BE_{GEL}}} = \left(\mathbb{E}\left[(l(\phi))^{-\alpha} \mid X\right]\right)^{-\frac{1}{\alpha}} = \left[\int_{\phi} (l(\phi))^{-\alpha}\,\mu(\phi \mid x)\,d\phi\right]^{-\frac{1}{\alpha}} \tag{4.22}$$

As the posterior expressions derived in Eq. (4.20), (4.21), and (4.22) are complex, closed-form Bayes estimators cannot be obtained. To address this, we used the Markov Chain Monte Carlo (MCMC) method to estimate the desired estimates based on posterior samples. MCMC is a general-purpose computational device for sampling complex posterior distributions. Most often, an initial fraction of the sampled points, the burn-in, is omitted so that the chain may converge. The remaining samples are used to construct Bayes estimates with respect to the loss function used. For each posterior draw $\phi^{(j)} = (\theta^{(j)}, \beta^{(j)})$, the Bayes estimator under different loss functions, e.g. symmetric error loss (SEL), LINEX, and GEL, can be calculated as follows.

$$\hat{\phi}_{\mathrm{BE_{SEL}}} = \frac{1}{N - l_D}\sum_{j=l_D}^{N} \phi^{(j)}$$

$$\hat{\phi}_{\text{BE}_{\text{LINEX}}} = -\frac{1}{w}\log\left\{\frac{1}{N-l_D}\sum_{j=l_D}^{N}\exp\left(-w\phi^{(j)}\right)\right\}$$

$$\hat{\phi}_{\text{BE}_{\text{GEL}}} = \left[\frac{1}{N-l_D}\sum_{j=l_D}^{N}\left(\phi^{(j)}\right)^{\alpha}\right]^{-1/\alpha}$$

Where l_D represents the number of burn-in samples. For further details on MCMC see [22] and [21].

4.8.1. Credible Interval for Bayes Estimates. The credible intervals $100(1-\xi)\%$ for the parameters $\phi = (\theta, \beta)$ under the loss function discussed are

$$\hat{\phi}_{\text{BE}_{\text{SEL}}} \neq Z_{\xi/2}\sqrt{\text{Var}(\hat{\phi}_{\text{BE}_{\text{SEL}}})};\quad \hat{\phi}_{\text{BE}_{\text{LINEX}}} \neq Z_{\xi/2}\sqrt{\text{Var}(\hat{\phi}_{\text{BE}_{\text{LINEX}}})};\quad \hat{\phi}_{\text{BE}_{\text{GEL}}} \neq Z_{\xi/2}\sqrt{\text{Var}(\hat{\phi}_{\text{BE}_{\text{GEL}}})}$$

Here $Z_{\xi/2}$ is the standard normal percentile distribution with right-tailed probability.

5. Simulation

To compare and evaluate the performance of the various non-Bayesian estimators (MLE, MPS, LS, WLS, CvM, AD, RTAD) of the parameters of the TI-HT-TLR distribution, an extensive simulation study was conducted. For each of the estimation methods, the parameter estimates were calculated with 10,000 bootstrap replicates for four different sample sizes (n = 25,75,150,200). This allowed for a robust comparison of bias, root mean squared error (RMSE), and general estimation performance for conditions from small to large. Additionally, Bayesian estimation was conducted on the rest of the samples using the "Squared Error Loss (SEL), LINEX loss, and General Entropy Loss (GEL) loss functions. To compare and analyze the performance of all the estimation techniques, average bias and root mean square error (RMSE) were computed for each sample size for 10,000 iterations". The measures provided an overall evaluation of the precision and accuracy of each technique.

The study also considers different cases by varying the initial values of the parameters, which includes; Case I: $\theta = 1.15, \beta = 0.75$ Case II: $\theta = 1.15, \beta = 0.5$ Case III: $\theta = 1.75, \beta = 1.25$ Case IV: $\theta = 1.80, \beta = 0.80$

Table 3: Simulation for Case I

Type	Method	$n = 25$		$n = 75$		$n = 150$		$n = 200$	
		Bias	RMSE	Bias	RMSE	Bias	RMSE	Bias	RMSE
Non Bayesian	$\text{MLE}_{\hat{\theta}}$	1.42760	19.06019	0.43921	7.14895	0.13805	1.69984	0.09487	0.26125
	$\text{MLE}_{\hat{\beta}}$	0.07831	0.63679	0.05052	0.28129	0.02789	0.15398	0.00534	0.04485
	$\text{MPS}_{\hat{\theta}}$	0.19172	4.56238	0.01059	1.59865	0.05532	0.21763	0.02702	0.18053
	$\text{MPS}_{\hat{\beta}}$	0.68820	2.27352	0.39786	1.44508	0.13982	0.37672	0.07611	0.15039
	$\text{LS}_{\hat{\theta}}$	0.40311	4.16108	0.13228	1.36343	0.02416	0.27944	0.02911	0.20415
	$\text{LS}_{\hat{\beta}}$	0.36668	0.85695	0.15438	0.28472	0.07546	0.10012	0.04400	0.06316
	$\text{WLS}_{\hat{\theta}}$	0.54384	6.19866	0.19520	1.98285	0.05366	0.24281	0.04852	0.17295
	$\text{WLS}_{\hat{\beta}}$	0.32883	0.88197	0.10930	0.24137	0.04319	0.06791	0.02507	0.05159
	$\text{CvM}_{\hat{\theta}}$	0.91033	6.94581	0.28176	1.69857	0.09133	0.31145	0.07811	0.22022
	$\text{CvM}_{\hat{\beta}}$	0.14544	0.56370	0.08017	0.22814	0.04229	0.08673	0.02062	0.05630
	$\text{AD}_{\hat{\theta}}$	0.69238	6.43782	0.19885	1.50299	0.05553	0.23845	0.04987	0.17209
	$\text{AD}_{\hat{\beta}}$	0.24777	0.80680	0.08688	0.19602	0.04126	0.06607	0.02804	0.07473
	$\text{RTAD}_{\hat{\theta}}$	0.60290	3.48431	0.22152	1.01098	0.06787	0.24369	0.07275	0.22241
	$\text{RTAD}_{\hat{\beta}}$	0.23230	0.84579	0.07532	0.25331	0.03343	0.06139	0.01679	0.05499
Bayesian	$\text{SEL}_{\hat{\theta}}$	0.56179	0.47433	0.56969	0.38610	0.57711	0.36945	0.57630	0.36051
	$\text{SEL}_{\hat{\beta}}$	2.57397	13.5726	2.27918	11.0539	2.09419	9.34322	1.96090	7.77003
	$\text{LINEX1}_{\hat{\theta}}$	0.55836	0.47382	0.56743	0.38429	0.57546	0.36786	0.57481	0.35901
	$\text{LINEX1}_{\hat{\beta}}$	2.75215	16.4132	2.35021	12.6853	1.95163	7.70898	1.85701	6.37875
	$\text{LINEX2}_{\hat{\theta}}$	0.56517	0.47498	0.57193	0.38791	0.57876	0.37105	0.57777	0.36200
	$\text{LINEX2}_{\hat{\beta}}$	2.33840	10.5174	2.08068	8.29047	1.94286	7.08281	1.85029	6.15619
	$\text{GEL1}_{\hat{\theta}}$	0.56679	0.47905	0.57335	0.39003	0.57993	0.37265	0.57882	0.36335
	$\text{GEL1}_{\hat{\beta}}$	2.52546	13.0287	2.23792	10.5634	2.06207	8.94469	1.93620	7.49811
	$\text{GEL2}_{\hat{\theta}}$	0.57677	0.48869	0.58070	0.39803	0.58559	0.37913	0.58387	0.36911
	$\text{GEL2}_{\hat{\beta}}$	2.43224	12.0374	2.15924	9.68565	2.00099	8.24134	1.88863	7.00776

Table 4: Simulation for Case II

Type	Method	$n=25$		$n=75$		$n=150$		$n=200$	
		Bias	RMSE	Bias	RMSE	Bias	RMSE	Bias	RMSE
Non Bayesian	$\text{MLE}_{\hat\theta}$	1.09176	8.93171	0.41519	2.23717	0.20342	0.83982	0.10153	0.27743
	$\text{MLE}_{\hat\beta}$	0.05841	0.34231	0.01760	0.12430	0.00272	0.02476	0.00231	0.01788
	$\text{MPS}_{\hat\theta}$	0.20914	3.42801	0.12577	1.49973	0.06499	0.59076	0.00923	0.23500
	$\text{MPS}_{\hat\beta}$	0.52296	1.67825	0.15030	0.44775	0.04319	0.07079	0.03289	0.02203
	$\text{LS}_{\hat\theta}$	0.31391	2.42226	0.18790	1.14788	0.10383	0.47787	0.05386	0.20500
	$\text{LS}_{\hat\beta}$	0.23917	0.49416	0.06210	0.09630	0.02140	0.02927	0.01581	0.01966
	$\text{WLS}_{\hat\theta}$	0.46177	3.71309	0.24530	1.54239	0.11776	0.41106	0.06698	0.18255
	$\text{WLS}_{\hat\beta}$	0.19601	0.45178	0.03831	0.07043	0.00970	0.02433	0.00887	0.01760
	$\text{CvM}_{\hat\theta}$	0.67963	3.42599	0.31945	1.29739	0.15644	0.46439	0.10338	0.28275
	$\text{CvM}_{\hat\beta}$	0.07202	0.26618	0.01823	0.07345	0.00242	0.02681	0.00143	0.01870
	$\text{ADE}_{\hat\theta}$	0.55111	3.28831	0.25348	1.16869	0.12340	0.39893	0.07367	0.20303
	$\text{ADE}_{\hat\beta}$	0.12716	0.37050	0.03157	0.07459	0.00773	0.02448	0.00750	0.01768
	$\text{RTADE}_{\hat\theta}$	0.42585	1.85879	0.24621	0.86917	0.14533	0.45735	0.09652	0.23814
	$\text{RTADE}_{\hat\beta}$	0.18621	0.66079	0.03511	0.10016	0.00902	0.02893	0.00648	0.02161
Bayesian	$\text{SEL}_{\hat\theta}$	0.59456	0.51342	0.60191	0.42458	0.60516	0.40364	0.60226	0.39181
	$\text{SEL}_{\hat\beta}$	1.96091	7.94905	1.66740	6.01388	1.45168	4.36036	1.32549	3.18312
	$\text{LINEX1}_{\hat\theta}$	0.59158	0.51312	0.59994	0.42294	0.60367	0.40215	0.60090	0.39039
	$\text{LINEX1}_{\hat\beta}$	1.96660	8.12995	1.62280	5.73535	1.33348	3.17626	1.23963	2.25344
	$\text{LINEX2}_{\hat\theta}$	0.59750	0.51385	0.60387	0.42622	0.60663	0.40512	0.60361	0.39324
	$\text{LINEX2}_{\hat\beta}$	1.91553	7.46660	1.64179	5.63483	1.45058	4.24270	1.33723	3.31062
	$\text{GEL1}_{\hat\theta}$	0.59885	0.51738	0.60514	0.42809	0.60773	0.40661	0.60463	0.39455
	$\text{GEL1}_{\hat\beta}$	1.96050	8.01692	1.65080	5.88648	1.43300	4.20110	1.31163	3.09623
	$\text{GEL2}_{\hat\theta}$	0.60737	0.52541	0.61159	0.43517	0.61288	0.41261	0.60936	0.40007
	$\text{GEL2}_{\hat\beta}$	1.90748	7.59388	1.64999	5.98756	1.42092	4.16321	1.28456	2.93545

Table 5: Simulation for Case III

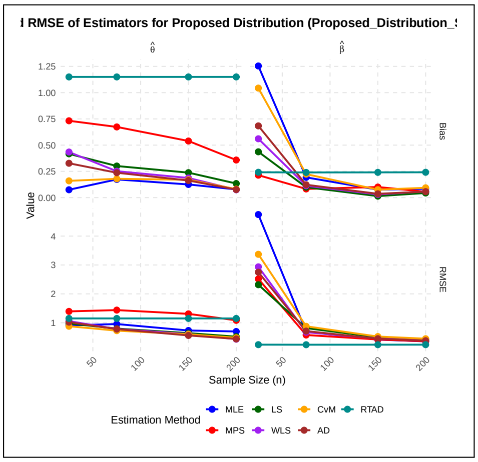
Type	Method	$n=25$		$n=75$		$n=150$		$n=200$	
		Bias	RMSE	Bias	RMSE	Bias	RMSE	Bias	RMSE
Non Bayesian	$\text{MLE}_{\hat\theta}$	3.57643	142.68064	0.68623	10.63149	0.27739	1.26300	0.17233	0.90456
	$\text{MLE}_{\hat\beta}$	0.04408	0.86237	0.13388	0.80247	0.14477	0.72562	0.16127	0.79355
	$\text{MPS}_{\hat\theta}$	0.70259	36.70231	0.13697	2.47470	0.18776	1.16823	0.20875	0.94423
	$\text{MPS}_{\hat\beta}$	0.63182	1.81510	0.78046	2.41731	0.59736	1.97875	0.55005	1.84758
	$\text{LS}_{\hat\theta}$	1.44346	31.73346	0.41135	4.30184	0.21837	1.88124	0.07162	1.34390
	$\text{LS}_{\hat\beta}$	0.32885	0.96892	0.27918	0.69413	0.20828	0.49301	0.21695	0.42446
	$\text{WLS}_{\hat\theta}$	1.74880	56.47062	0.44695	4.05775	0.22887	1.58253	0.11244	1.10830
	$\text{WLS}_{\hat\beta}$	0.32100	1.07959	0.25747	0.77219	0.17153	0.47675	0.16357	0.39940
	$\text{CvM}_{\hat\theta}$	2.82976	61.64289	0.74582	5.27874	0.38516	2.14853	0.20086	1.45300
	$\text{CvM}_{\hat\beta}$	0.05946	0.76913	0.14807	0.60891	0.13458	0.43835	0.14621	0.35186
	$\text{ADE}_{\hat\theta}$	1.82510	40.37347	0.47069	3.71552	0.21812	1.47600	0.10503	1.04490
	$\text{ADE}_{\hat\beta}$	0.22653	1.11757	0.22256	0.72944	0.16940	0.48388	0.15624	0.37100
	$\text{RTADE}_{\hat\theta}$	1.64869	22.37155	0.45145	3.69498	0.17805	1.29965	0.09842	0.96414
	$\text{RTADE}_{\hat\beta}$	0.24521	1.23309	0.24416	0.83292	0.18496	0.54561	0.15759	0.40443
Bayesian	$\text{SEL}_{\hat\theta}$	0.84611	0.96648	0.84187	0.78589	0.86672	0.79440	0.88173	0.81063
	$\text{SEL}_{\hat\beta}$	4.22477	30.43351	3.79259	22.66490	3.66552	17.87298	3.65605	16.20723
	$\text{LINEX1}_{\hat\theta}$	0.83695	0.95771	0.83568	0.77710	0.86247	0.78791	0.87828	0.80513
	$\text{LINEX1}_{\hat\beta}$	5.02161	48.50792	3.93942	24.98880	3.72901	17.28580	3.84414	18.38112
	$\text{LINEX2}_{\hat\theta}$	0.85510	0.97556	0.84800	0.79475	0.87093	0.80093	0.88516	0.81614
	$\text{LINEX2}_{\hat\beta}$	3.47701	18.36733	3.27082	14.33560	3.34717	13.54053	3.41697	13.45211
	$\text{GEL1}_{\hat\theta}$	0.85698	0.98624	0.84885	0.79786	0.87148	0.80253	0.88562	0.81733
	$\text{GEL1}_{\hat\beta}$	4.09214	28.27276	3.69036	20.99139	3.59963	16.94424	3.60628	15.61972
	$\text{GEL2}_{\hat\theta}$	0.87913	1.02798	0.86313	0.82309	0.88115	0.81940	0.89348	0.83105
	$\text{GEL2}_{\hat\beta}$	3.84198	24.51739	3.50138	18.22240	3.47721	15.43256	3.51143	14.61930

Table 6: Simulation for Case IV

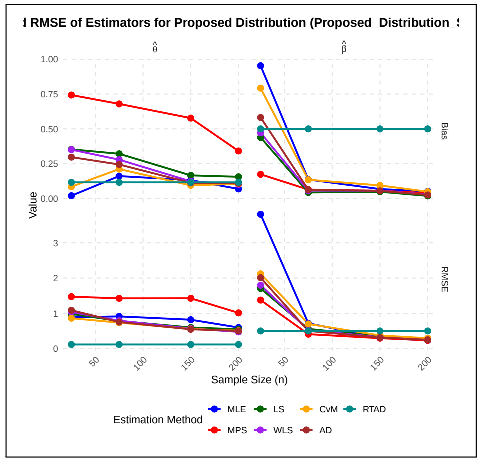
Type	Method	$n=25$ Bias	$n=25$ RMSE	$n=75$ Bias	$n=75$ RMSE	$n=150$ Bias	$n=150$ RMSE	$n=200$ Bias	$n=200$ RMSE
Non Bayesian	$\text{MLE}_{\hat\theta}$	2.81509	103.45752	0.38913	4.63158	0.17747	0.77338	0.09110	0.40589
	$\text{MLE}_{\hat\beta}$	0.06064	0.59141	0.12667	0.49361	0.04207	0.23406	0.02443	0.06617
	$\text{MPS}_{\hat\theta}$	0.43175	15.88123	0.14842	2.63876	0.09129	0.63251	0.10662	0.39683
	$\text{MPS}_{\hat\beta}$	0.63606	1.90172	0.49528	1.55922	0.23962	0.91984	0.13157	0.33880
	$\text{LS}_{\hat\theta}$	0.70819	12.39191	0.05981	2.28219	0.07041	0.78395	0.01577	0.51863
	$\text{LS}_{\hat\beta}$	0.36305	0.83978	0.22569	0.37974	0.07787	0.12390	0.06801	0.08706
	$\text{WLS}_{\hat\theta}$	0.93681	17.08734	0.13462	2.63507	0.10921	0.67820	0.05555	0.43663
	$\text{WLS}_{\hat\beta}$	0.32150	0.87665	0.17593	0.32424	0.04369	0.08088	0.03885	0.05806
	$\text{CvM}_{\hat\theta}$	1.57334	19.03110	0.30320	3.04730	0.17935	0.86779	0.09613	0.55098
	$\text{CvM}_{\hat\beta}$	0.13016	0.55382	0.14250	0.31458	0.04147	0.10602	0.03941	0.07106
	$\text{ADE}_{\hat\theta}$	1.37004	24.55519	0.16628	2.61937	0.11271	0.67030	0.05536	0.43448
	$\text{ADE}_{\hat\beta}$	0.21526	0.73963	0.15450	0.28938	0.04110	0.07673	0.03875	0.05827
	$\text{RTADE}_{\hat\theta}$	0.99996	11.60004	0.29772	5.01241	0.15888	0.77112	0.06808	0.44815
	$\text{RTADE}_{\hat\beta}$	0.27208	1.11943	0.15301	0.37765	0.02514	0.06849	0.03596	0.06399
Bayesian	$\text{SEL}_{\hat\theta}$	0.97597	1.20469	0.97640	1.03115	0.99095	1.02510	0.99922	1.03117
	$\text{SEL}_{\hat\beta}$	3.65132	22.82501	3.16252	15.46627	2.98889	11.98799	2.94224	10.58738
	$\text{LINEX1}_{\hat\theta}$	0.96919	1.19823	0.97187	1.02366	0.98773	1.01937	0.99649	1.02616
	$\text{LINEX1}_{\hat\beta}$	3.72778	24.62533	3.04225	13.01998	2.97698	10.94940	2.96881	10.18530
	$\text{LINEX2}_{\hat\theta}$	0.98263	1.21147	0.98089	1.03865	0.99415	1.03082	1.00194	1.03617
	$\text{LINEX2}_{\hat\beta}$	3.23739	16.56195	2.91975	12.04130	2.84221	10.16726	2.82399	9.39284
	$\text{GEL1}_{\hat\theta}$	0.98424	1.22098	0.98191	1.04191	0.99486	1.03272	1.00255	1.03768
	$\text{GEL1}_{\hat\beta}$	3.57475	21.76880	3.12151	15.00623	2.98320	12.13090	2.93461	10.72454
	$\text{GEL2}_{\hat\theta}$	1.00091	1.25462	0.99307	1.06412	1.00275	1.04830	1.00922	1.05088
	$\text{GEL2}_{\hat\beta}$	3.46205	20.46404	3.02053	13.82881	2.91222	11.38902	2.86871	10.08077

Tables(3–6) indicates that the results for the non-Bayesian estimators show improved performance (smaller RMSE and bias) as sample size increases. Among non-Bayesian methods, RTADE, WLS, and CvM consistently show higher accuracy for both parameters, especially for large n, the MLE fails in estimating θ in the case of small samples owing to large variance, but this improves considerably with large n, and the MPS performs well for θ but comparatively higher bias for β.

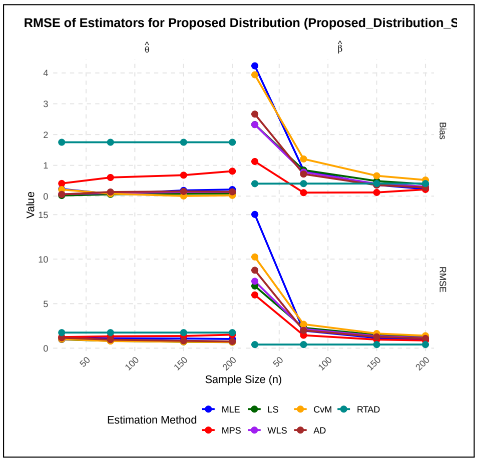
The Bayesian estimators have low and stable bias for β for all sample sizes. Estimation of β under Bayesian estimators is higher in bias and RMSE, especially under SEL and LINEX1. Among Bayesian approaches, LINEX2 and GEL2 perform relatively better in estimation accuracy for β particularly as n becomes large.



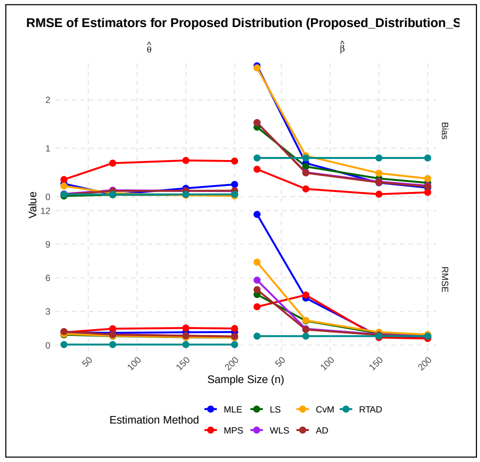
(a)



(b)



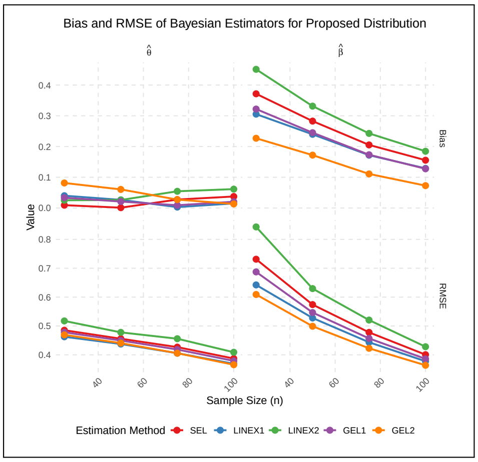
(c)



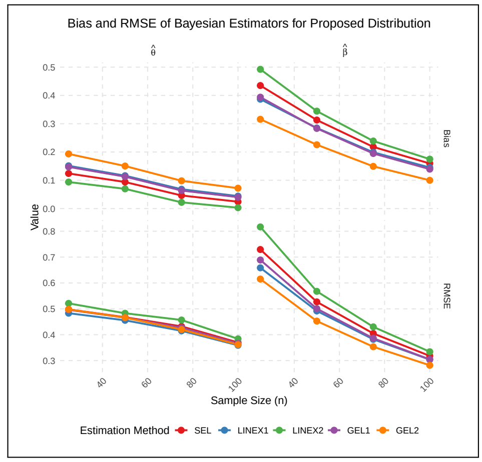
(d)

Figure 4: plot of the non-Bayesian simulation result of (a) Case I, (b) II, (c) III, (d) IV

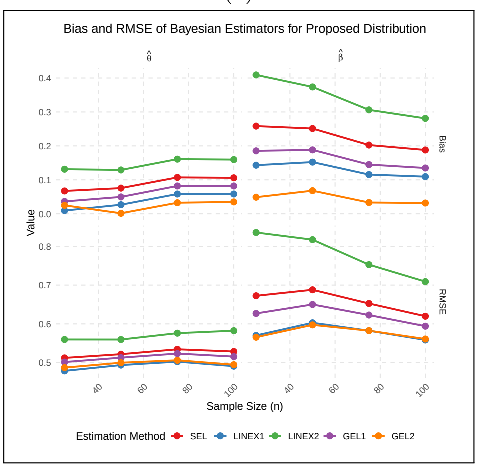
Figure 4, compares Bias and RMSE for $\hat{\theta}$ and $\hat{\beta}$ across sample sizes ($n = 25, 75, 150, 200$) using seven estimation methods. Bias and RMSE decline as n increases, confirming estimator consistency. For $\hat{\theta}$, MLE maintains minimal bias and low RMSE at all n, while MPS improves steadily; CvM, AD, LS and WLS perform moderately well, but RTAD shows persistently high bias. For $\hat{\beta}$, MLE and MPS start with higher bias at $n = 25$ but converge rapidly to near unbiased, achieving the lowest RMSE at large n.



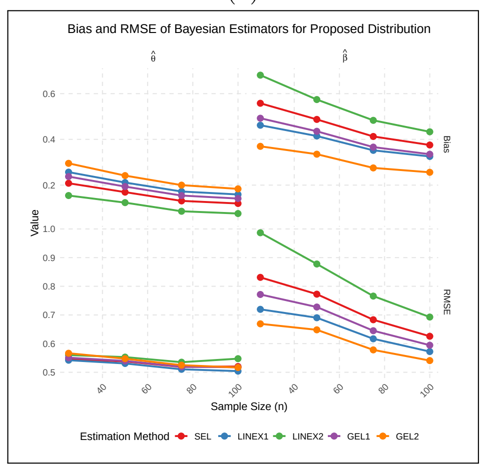
(a)



(b)



(c)



(d)

Figure 5: Plot of the Bayesian simulation result of (a) Case I, (b) II, (c) III, (d) IV

Figure 5 shows the plots of the bias and root mean square error (RMSE) of the Bayesian estimators for the symmetric (SEL) and asymmetric loss functions (LINEX1, LINEX2, GEL1, GEL2) of TI-HT-TLR distribution. The results are presented for different sample sizes to compare the small and large sample behavior of the estimators. Overall, findings confirm that bias and RMSE diminish with higher sample sizes as would happen in the asymptotic behavior of Bayesian estimators. Variation between methods is more prominent when sample sizes are lower ($n = 25$) and ($n = 75$), with some estimators exhibiting more bias and variability. With a higher sample size ($n = 150$), however, all estimators converge to stable and effectively unbiased estimators, exhibiting consistency and higher efficiency. A further observation shows that performance of the estimators varies from parameter to parameter. On the shape parameter θ, SEL, LINEX1, and GEL1 are better consistently on bias and RMSE, especially for moderate to large sample sizes. However, for the scale parameter β, GEL2 and LINEX2 have the lowest estimation error for nearly all sample sizes. These observations highlight the importance the chosen loss function has in identifying estimator efficiency, particularly in finite samples. Collectively, while all the Bayesian estimators are consistent, they are efficient depending on both the sample size as well as parameter of interest. SEL, LINEX1, and GEL1 are recommended in application for estimation of θ, but GEL2 and LINEX2 are preferable for β. The above thus suggests that optimal choice of loss function can greatly improve inference under the TI-HT-TLR distribution.

6. Real Data Applications

In this section, we apply real data to the TIHTTLR distribution to illustrate the modeling performance. The first data (Data I) is the daily ratio of newly reported Covid-19 mortality to new cases in Italy for 111 consecutive days between 1 April and 20 July 2020. Data-I was retrieved from https://covid19.who.int/ and presented in Table (7). The second data (Data II) is on the mortality rate due to COVID-19 in Mexico. Data-II was also obtained from https://covid19.wh and presented in Table (8). The third data (Data III) is the number of persons who tested positive for malaria using the Rapid Diagnostic Test (RTD) between 2022 and 2024 in Anambra State, Nigeria. Data-III were obtained from the Anambra State Ministry of Health, Nigeria and are presented in Table (9)

Table 7: Covid-19 daily mortality cases recoreded in Italy

0.2070	0.1520	0.1628	0.1666	0.1417	0.1221	0.1767	0.1987
0.1408	0.1456	0.1443	0.1319	0.1053	0.1789	0.2032	0.2167
0.1387	0.1646	0.1375	0.1421	0.2012	0.1957	0.1297	0.1754
0.1390	0.1761	0.1119	0.1915	0.1827	0.1548	0.1522	0.1369
0.2495	0.1253	0.1597	0.2195	0.2555	0.1956	0.1831	0.1791
0.2057	0.2406	0.1227	0.2196	0.2641	0.3067	0.1749	0.2148
0.2195	0.1993	0.2421	0.2430	0.1994	0.1779	0.0942	0.3067
0.1965	0.2003	0.1180	0.1686	0.2668	0.2113	0.3371	0.1730
0.2212	0.4972	0.1641	0.2667	0.2690	0.2321	0.2792	0.3515
0.1398	0.3436	0.2254	0.1302	0.0864	0.1619	0.1311	0.1994
0.3176	0.1856	0.1071	0.1041	0.1593	0.0537	0.1149	0.1176
0.0457	0.1264	0.0476	0.1620	0.1154	0.1493	0.0673	0.0894
0.0365	0.0385	0.2190	0.0777	0.0561	0.0435	0.0372	0.0385
0.0769	0.1491	0.0802	0.0870	0.0476	0.0562	0.0138	

Table 8: COVID-19 mortality rate in Mexico

4.4130	3.0525	4.6955	7.4810	5.1915	3.6335	6.6100	8.2490
5.8325	3.0075	5.4275	3.0610	3.3280	1.7200	2.9270	5.3425
5.0175	2.6210	2.1720	2.5715	3.8150	7.3020	3.9515	3.1850
1.7685	3.1635	2.3650	1.6075	4.6420	6.4390	4.4065	5.0215
3.6300	2.9925	3.2060	1.6975	2.2120	4.9675	3.9200	4.7750
1.7495	1.8755	3.4840	1.6430	5.0790	4.0540	3.3485	3.5755
3.2800	1.0385	1.8890	1.4940	1.6680	3.4070	4.1625	3.9270
4.2755	1.6140	3.7430	3.3125	3.0700	2.4545	2.3305	2.6960
6.0210	4.3480	0.9075	1.6635	2.7030	3.0910	0.5205	0.9000
2.4745	2.0445	1.6795	1.0350	1.6490	2.6585	2.7210	2.2785
2.1460	1.2500	3.2675	2.3240	2.3485	2.7295	2.0600	1.9610
1.6095	0.7010	1.2190	1.6285	1.8160	1.6165	1.5135	1.1760
0.6025	1.6090	1.4630	1.3005	1.0325	1.5145	1.0290	1.1630
1.2530	0.9615						

Table 9: Persons Testing Positive for Malaria by RDT in Anambra State, Nigeria (2022-2024)

0.8076	0.2421	0.2332	0.1850	0.5503	0.5395	0.6270	0.4925
0.5869	0.6127	0.8375	0.8743	0.6488	0.2996	0.2670	0.2891
0.6910	0.4482	0.4791	0.7007	0.6245	0.7172	0.4870	0.5532
0.5050	0.6602	0.6593	0.7270	0.3709	0.2879	0.3938	0.3884
0.5151	0.5917	0.9339	0.6627	0.6127	0.4488	0.4716	0.4729
0.4966	0.4335	0.5370	0.6749	0.8185	0.6653	0.2575	0.3296
0.3011	0.7877	0.8287	0.8518	0.4058	0.3855		

Table 10: Descriptive Statistics for the three Datasets

Datasets	n	Mean	SD	Med	Minimum	Maximum	R	S	K	Se
Data I	111	0.17	0.08	0.07	0.014	0.5	0.48	0.77	4.90	0.007
Data II	106	2.91	1.62	2.63	0.52	8.25	7.73	0.97	3.67	0.16
Data III	54	0.542	0.189	0.538	0.185	0.934	0.749	0.066	2.174	0.026

where SD is the standard deviation, Med is the median, R is the range, S is the skewness, and K is the kurtosis. Most Important Observations regarding descriptive Statistics in Table (10). The analysis of the three data sets reveals various characteristics in magnitude, dispersion, and shape:

Data I ($n = 111$) has the lowest range and mean of the three sets. It is right skewed but is most typical of high kurtosis, indicating a distribution much more leptokurtic (heavier tailed and more peaked) than Data II or Data III.

Data II ($n = 106$) is the most extensive and most dispersed set. It possesses the largest mean, standard deviation, and range, meaning the widest total spread. It is also most skewed to the right, meaning its mass lies on the lower values with a long right tail leading towards the maximum value.

Data III ($n = 54$) has the smallest sample size. It is the most symmetric, as attested to by its mean being almost equal to its median. Moreover, it has the least kurtosis, which suggests its tails are the lightest of the three datasets.

Table 11: Model comparison and goodness of fit tests for the three Datasets

Datasets	Distributions	LL	AIC	CAIC	BIC	HQIC	W	A	K-S	p-value
Data I	TIHTTLR	129.63	-255.27	-255.1589	-249.8509	-253.0717	0.0712	0.4747	0.0592	0.8317
	TIHTR	127.47	-250.9604	-250.8493	-245.5413	-248.762	0.1247	0.7677	0.1081	0.1494
	TLR	235.23	-466.4558	-466.3447	-461.0367	-464.2574	0.1410	0.8671	0.2952	7.96×10^{-9}
	Rayleigh	127.54	-253.0873	-253.0506	-250.3777	-251.9881	0.1236	0.7611	0.1058	0.1661
	Gumbel	127.1	-250.2047	-250.0936	-244.7856	-248.0063	0.1745	1.1019	0.0816	0.4506
	Log Normal	117.7	-231.3927	-231.2816	-225.9737	-229.1944	0.5409	3.094	0.1327	0.0401
Data II	TIHTTLR	-190.07	384.1364	384.2529	389.4632	386.2954	0.077	0.4829	0.05480	0.9079
	TIHTR	-190.01	384.1064	384.229	389.4532	386.2544	0.0774	0.4826	0.0548	0.901
	TLR	-97.21	198.4292	198.5457	203.7561	200.5882	0.0961	0.6167	0.2553	1.99×10^{-6}
	Rayleigh	-190.98	383.9593	383.9978	386.6227	385.0388	0.1082	0.6978	0.0844	0.4368
	Gumbel	-190.2	384.402	384.5185	389.7289	386.5611	0.0824	0.4961	0.0850	0.4272
Data III	TIHTTLR	12.18	-20.369	-20.134	-16.391	-18.835	0.057	0.424	0.087	0.819
	TIHTR	12.12	-20.54	-20.133	-16.38	-18.74	0.0572	0.423	0.086	0.810
	TLR	89.24	174.253	-174.253	-170.510	-172.954	0.0907	0.603	0.385	2.23×10^{-7}
	Rayleigh	6.73	-11.458	-11.381	-9.469	-10.691	0.040	0.304	0.161	0.123
	Gumbel	11.95	-19.904	-19.670	-15.926	-18.370	0.100	0.671	0.088	0.803
	log-Normal	11.65	-19.299	-19.063	-15.321	-17.765	0.121	0.782	0.097	0.688

Table (11 presents the comparison of the fit of six different probability distributions, the TIHTR by [34], TLR by [31], Rayleigh given by [29], Gumbel introduced by [32] and Log-Normal on three datasets. In each of the three datasets, the TIHTTLR distribution provides the best fit every time. It provides the highest Log-Likelihood (LL) and lowest values for the AIC, BIC, and HQIC information criteria, which penalize complexity. For all the data sets, TIHTTLR models have Kolmogorov-Smirnov (K-S) p-values $(0.8317, 0.9079, 0.819)$ and very small W and A statistics. This justifies that the TIHTTLR model accurately model the empirical distribution of the data,

Table 12: MLEs and standard errors for the parameters of the three Datasets

Datasets	Distributions	$\hat{\theta}_{\text{MLE}}$	$\hat{\beta}_{\text{MLE}}$
Data I	TIHTTLR	0.4824(0.1622)	38.2974(17.7705)
	TIHTR	64.0543(13.3334)	0.0073 (0.0020)
	TLR	46.6692(3.9551)	2.8643(0.3486)
	Rayleigh	29.4438(2.7947)	-
	Gumbel	0.1299(0.0068)	0.0680(0.0049)
	Log Normal	-1.9264(0.0546)	0.5754(0.0386)
Data II	TIHTTLR	1.7107 (0.763)	0.01849 (0.0152)
	TIHTR	1.7084 (0.9045)	0.0371 (0.0360)
	TLR	0.1399 (0.0134)	2.4479 (0.3231)
	Rayleigh	0.0902 (0.0088)	-
	Gumbel	2.1817 (0.1238)	1.2129 (0.0954)
Data III	TIHTTLR	0.206 (0.158)	9.353 (6.879)
	TIHTR	0.205 (0.148)	18.710 (13.766)
	TLR	7.287 (0.773)	9.509 (2.037)
	Rayleigh	3.041 (0.414)	-
	Gumbel	0.449 (0.0247)	0.172 (0.018)
	Log-Normal	-0.681 (0.052)	0.385 (0.0371)

Table (12) presents the Maximum Likelihood Estimates (MLEs) and their corresponding Standard Errors (SEs) for the parameters of six distributions fitted to three datasets.

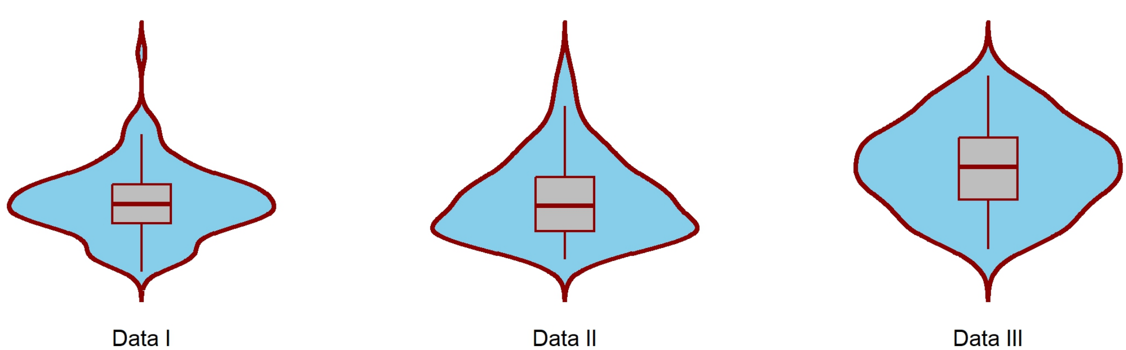


Figure 6: Boxplot superimposed on Violin plot for the datasets

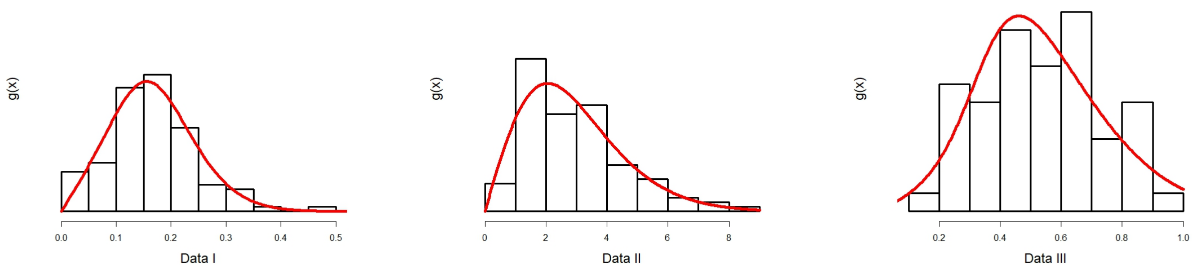


Figure 7: Density plot for the datasets

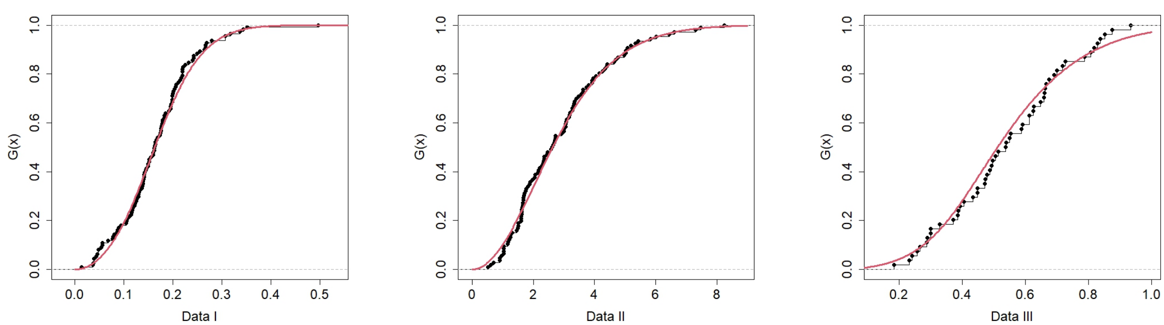


Figure 8: Empirical versus Theoretical CDF plot for the Datasets

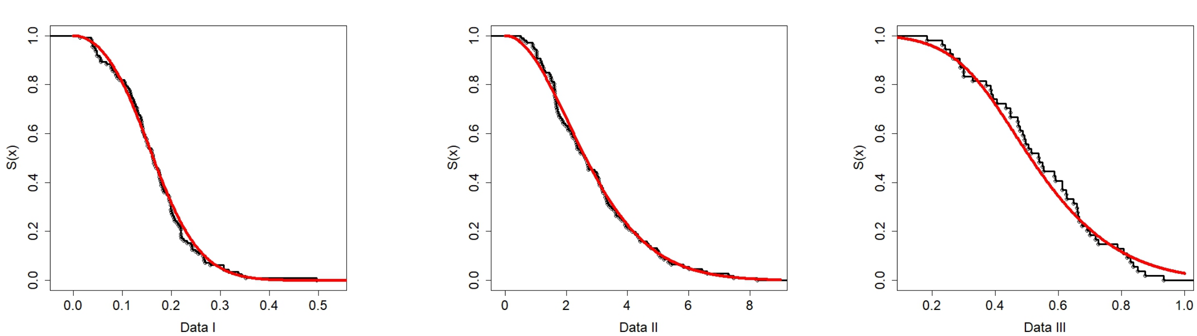


Figure 9: Empirical versus Theoretical survival function for the Datasets

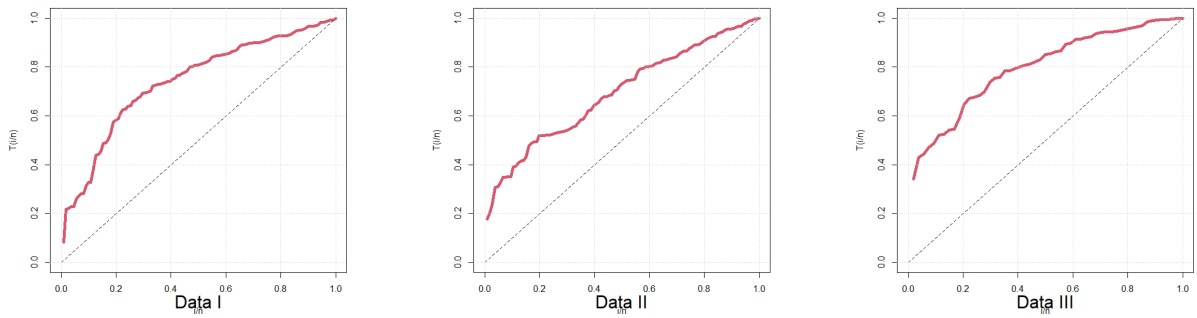


Figure 10: TTT plots for the Datasets

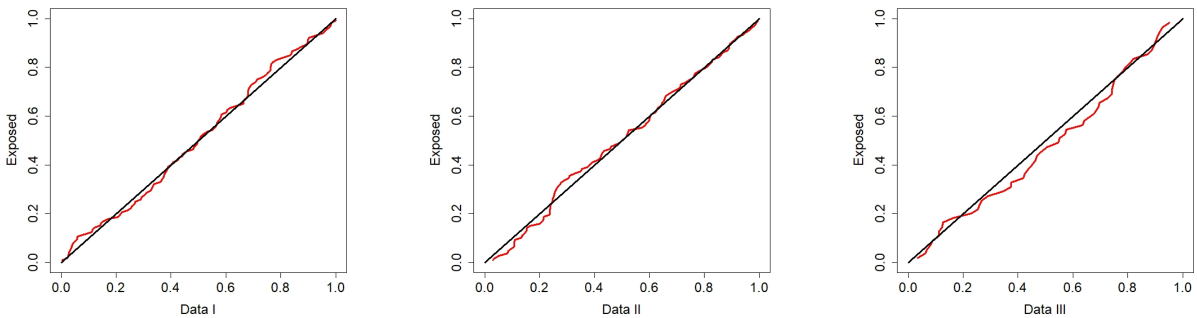


Figure 11: P-P plot for the Datasets

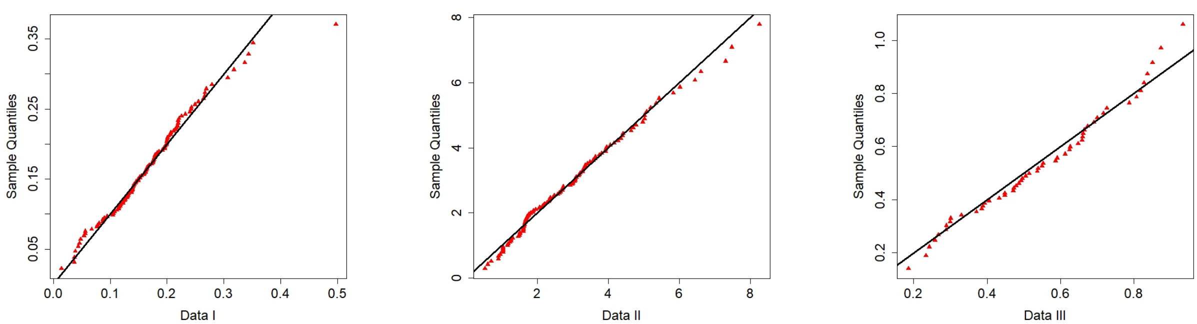


Figure 12: Q-Q plot for the Datasets

The following inference are drawn from figure (6-12)

i Figure (6) (Boxplot and Violin Plot): Data I's distribution is highly leptokurtic, with its shape being very narrow and peaked with a sharp peak, graphically validating the high kurtosis value. There is a little asymmetry, with the distribution decreasing more slowly towards higher values, as one would anticipate with the positive skewness. The Data III plot is the most symmetric of the three. The violin shape is nearly equal on both sides of the middle axis. Besides, the overall shape is wider and less pointed compared to Data I, reflecting lowest kurtosis ii Figure (7) (Histograms, superimposed with the fit probability density function (PDF, red line)), Data I, The histogram is clearly right-skewed, validating the initial descriptive statistics. Data II histogram has a wider spread, and this reflects its high dispersion. The distribution is highly right-skewed. The Data III histogram illustrates a relatively narrow and good balance spread, the distribution is very much more bell-shaped than the others.

iii Figure (8) (ECDF vs Theoretical CDF plot): The CDF plots graphical confirmation of the goodness-of-fit statistics, the overall trend of the observed data is well in line with that of the theoretical TIHTTLR model. This shows that the proposed TIHTTLR is an exceedingly good model for Data I, II, III

iv Figure (9) (Empirical vs. Theoretical Survival Function Plot): In all three cases, the survival function plots confirm conclusions from CDF and goodness-of-fit test: the theoretical survival function provides a very good and exact approximation for the empirical survival probability for Data I, Data II, and Data III.

v Figure (10) (TTT Plots): In each of the three plots, the red curve is above the dashed diagonal line. This implies that the theoretical TIHTTLR distribution overestimates data points in the lower and middle quantiles.

vi Figure (11) (P-P Plots): The tight tracking of the empirical probabilities against the theoretical diagonal line guarantees a good-fitting of TIHTTLR distribution of Data I, Data II, and Data III.

vii Figure (12) (Q-Q Plots): The nearly exact conformity across all three datasets guarantees that the theoretical TIHTTLR distribution closely replicates the empirical quantile behavior of Data I, Data II, and Data III.

7. Conclusion

In this research, we successfully developed the Type I Heavy-Tailed Topp-Leone G (TI-HT-TL-G) family of distributions. The new family has been specifically designed to overcome the limitations of conventional models when dealing with complex real-world data that are both skewed and heavy-tailed. By combining the Type I Heavy-Tailed and Topp-Leone G distributions with the Rayleigh distribution as the sub-model, the new distribution (TI-HT-TLR) possesses greater flexibility and a superior ability to handle extreme values. These extreme values are of significant importance in applications such as financial risk, insurance, engineering, and biomedical science. The paper provides a theoretical foundation for the new distribution by computing its major statistical characteristics, including moments, quantile functions, and stochastic orderings. The model's applicability was verified by fitting the distribution to three diverse datasets and conducting simulation experiments using both Bayesian and conventional methods of parameter estimation. The findings validate the superior ability of the TI-HT-TLR distribution to yield a more general and realistic statistical modeling process for a wide range of scientific and biomedical problems.

Competing Interest

There is no known conflict of interest.

References

1. Nkomo, W., Oluyede, B., and Chipepa, F. (2025). Type I heavy-tailed family of generalized Burr III distributions: properties, actuarial measures, regression and applications. *Statistics in Transition new series*, 26(1), 93–115.

2. Beirlant, J., Matthys, G., and Dierckx, G. (2001). Heavy-tailed distributions and rating. *ASTIN Bulletin: The Journal of the IAA*, 31(1), 37–58.

3. Zhao, W., Khosa, S. K., Ahmad, Z., Aslam, M., and Afify, A. Z. (2020). Type-I heavy tailed family with applications in medicine, engineering and insurance. *PloS one*, 15(8), e0237462.

4. Prakash, A., Maurya, R. K., Alsadat, N., and Obulezi, O. J. (2024). Parameter estimation for reduced Type-I Heavy-Tailed Weibull distribution under progressive Type-II censoring scheme. *Alexandria Engineering Journal*, 109, 935–949.

5. Afify, A. Z., Gemeay, A. M., and Ibrahim, N. A. (2020). The heavy-tailed exponential distribution: risk measures, estimation, and application to actuarial data. *Mathematics*, 8(8), 1276.

6. Alyami, S. A., Babu, M. G., Elbatal, I., Alotaibi, N., and Elgarhy, M. (2022). Type II half-logistic odd Fréchet class of distributions: Statistical theory and applications. *Symmetry*, 14(6), 1222.

7. Benkhelifa, L. (2022). Alpha power Topp-Leone Weibull distribution: Properties, characterizations, regression modeling and applications. *Journal of Statistics and Management Systems*, 25(8), 1945–1970.

8. Ekemezie, D. N., Anyiam, K. E., Kayid, M., Balogun, O. S., and Obulezi, O. J. (2024). DUS Topp–Leone-G Family of Distributions: Baseline Extension, Properties, Estimation, Simulation and Useful Applications. *Entropy*, 26(11), 973.

9. Oluyede, B., Dingalo, N., and Chipepa, F. (2023). The Topp-Leone-Harris-G family of distributions with applications. *International Journal of Mathematics in Operational Research*, 24(4), 554–582.

10. Oluyede, B., Gabanakgosi, M., and Warahena-Liyanage, G. (2024). The Marshall-Olkin-Topp-Leone-Gompertz-G Family of Distributions with Applications. *Statistics, Optimization & Information Computing*, 12(4), 882–906.

11. Teamah, A. A., Elbanna, A. A., and Gemeay, A. M. (2021). Heavy-tailed log-logistic distribution: properties, risk measures and applications. *Statistics, Optimization & Information Computing*, 9(4), 910–941.

12. Al-Shomrani, A., Arif, O., Shawky, A., Hanif, S., and Shahbaz, M. Q. (2016). Topp–Leone Family of Distributions: Some Properties and Application. *Pakistan Journal of Statistics and Operation Research*, 443–451.

13. Shaked, M., and Shanthikumar, J. G. (1994). *Stochastic orders and their applications.*

14. Szekli, R. (2012). *Stochastic ordering and dependence in applied probability* (Vol. 97). Springer Science & Business Media.

15. Rényi, A. (1961). On measures of entropy and information. In *Proceedings of the fourth Berkeley symposium on mathematical statistics and probability* (Vol. 4, pp. 547–562). University of California Press.

16. Swain, J. J., Venkatraman, S., and Wilson, J. R. (1988). Least-squares estimation of distribution functions in Johnson's translation system. *Journal of Statistical Computation and Simulation*, 29(4), 271–297.

17. Cheng, R., and Amin, N. (1979). Maximum product-of-spacings estimation with applications to the lognormal distribution. *Math report*, 791. Department of Mathematics, UWIST, Cardiff.

18. Varian, H. R. (1975). A Bayesian approach to real estate assessment. *Studies in Bayesian econometrics and statistics in honor of Leonard J. Savage*. North-Holland.

19. Doostparast, M., Akbari, M. G., and Balakrishna, N. (2011). Bayesian analysis for the two-parameter Pareto distribution based on record values and times. *Journal of Statistical Computation and Simulation*, 81(11), 1393–1403.

20. Calabria, R., and Pulcini, G. (1996). Point estimation under asymmetric loss functions for left-truncated exponential samples. *Communications in Statistics-Theory and Methods*, 25(3), 585–600.

21. Brooks, S. (1998). Markov chain Monte Carlo method and its application. *Journal of the royal statistical society: series D (the Statistician)*, 47(1), 69–100.

22. Van Ravenzwaaij, D., Cassey, P., and Brown, S. D. (2018). A simple introduction to Markov Chain Monte–Carlo sampling. *Psychonomic bulletin & review*, 25(1), 143–154.

23. Obulezi, O. J., Obiora-Ilouno, H. O., Osuji, G. A., Kayid, M., and Balogun, O. S. (2025). A new family of generalized distributions based on logistic-x transformation: sub-model, properties and useful applications. *Research in Statistics*, 3(1), 2477232.

24. Adepoju, A. A., Abdulkadir, S. S., and Jibasen, D. (2023). The Type I Half Logistics-Topp-Leone-G Distribution Family: Model, its Properties and Applications. *UMYU Scientifica*, 2(4), 9–22.

25. Hassan, A. S., Al-Omari, A. I., Hassan, R. R., and Alomani, G. A. (2022). The odd inverted Topp Leone–H family of distributions: Estimation and applications. *Journal of Radiation Research and Applied Sciences*, 15(3), 365–379.

26. Rasheed, N. (2020). A New Generalized-G class of distributions and its applications with Dagum distribution. *Research Journal of Mathematical and Statistical Sciences*, 2320, 6047.

27. Alzaatreh, A., Lee, C., and Famoye, F. (2013). A new method for generating families of continuous distributions. *Metron*, 71(1), 63–79.

28. El-Saeed, A. R., Obulezi, O. J., and Abd El-Raouf, M. (2025). Type II heavy tailed family with applications to engineering, radiation biology and aviation data. *Journal of Radiation Research and Applied Sciences*, 18(3), 101547.

29. Elshahhat, A., Ahmad, H. H., Rabaiah, A., and Abo-Kasem, O. E. (2024). Analysis of a new jointly hybrid censored Rayleigh populations. *AIMS Mathematics*, 9(2), 3740–3762.

30. Nwankwo, M. P., Alsadat, N., Kumar, A., Bahloul, M. M., and Obulezi, O. J. (2024). Group acceptance sampling plan based on truncated life tests for Type-I heavy-tailed Rayleigh distribution. *Heliyon*, 10(19).

31. Olayode, F. (2019). The Topp-Leone Rayleigh distribution with application. *Amer. J. Math. Stat*, 9(6), 215–220.

32. Gumbel, E. J. (1941). The return period of flood flows. *The annals of mathematical statistics*, 12(2), 163–190.

33. Oramulu, D. O., Alsadat, N., Kumar, A., Bahloul, M. M., & Obulezi, O. J. (2024). Sine generalized family of distributions: Properties, estimation, simulations and applications. *Alexandria Engineering Journal*, 109, 532–552.

34. Nwankwo, B. C., Obiora-Ilouno, H. O., Almulhim, F. A., SidAhmed Mustafa, M., & Obulezi, O. J. (2024). Group acceptance sampling plans for type-I heavy-tailed exponential distribution based on truncated life tests. *AIP Advances*, 14(3).

35. Obulezi, O. J. (2025). Obulezi distribution: a novel one-parameter distribution for lifetime data modeling. *Mod. J. Stat.*, 2(1), 32–74.

36. Orji, G. O., Etaga, H. O., Almetwally, E. M., Igbokwe, C. P., Aguwa, O. C., & Obulezi, O. J. (2025). A new odd reparameterized exponential transformed-x family of distributions with applications to public health data. *Innovation in Statistics and Probability*, 1(1), 88–118.

37. Chinedu, E. Q., Chukwudum, Q. C., Alsadat, N., Obulezi, O. J., Almetwally, E. M., & Tolba, A. H. (2023). New lifetime distribution with applications to single acceptance sampling plan and scenarios of increasing hazard rates. *Symmetry*, 15(10), 1881.

38. Tolba, A. H., Onyekwere, C. K., El-Saeed, A. R., Alsadat, N., Alohali, H., & Obulezi, O. J. (2023). A New Distribution for Modeling Data with Increasing Hazard Rate: A Case of COVID-19 Pandemic and Vinyl Chloride Data. *Sustainability*, 15(17), 12782.

39. Onyekwere, C. K., Aguwa, O. C., & Obulezi, O. J. (2025). An updated lindley distribution: Properties, estimation, acceptance sampling, actuarial risk assessment and applications. *Innovation in Statistics and Probability*, 1(1), 1–27.

40. Obulezi, O. J., Semary, H. E., Nadir, S., Igbokwe, C. P., Orji, G. O., Al-Moisheer, A. S., & Elgarhy, M. (2025). Type-I Heavy-Tailed Burr XII Distribution with Applications to Quality Control, Skewed Reliability Engineering Systems and Lifetime Data. *Comput. Model. Eng. Sci*, 144, 2991.

41. Ekemezie, D. N., Alghamdi, F. M., Aljohani, H. M., Riad, F. H., Abd El-Raouf, M. M., & Obulezi, O. J. (2024). A more flexible Lomax distribution: characterization, estimation, group acceptance sampling plan and applications. *Alexandria Engineering Journal*, 109, 520–531.

42. Elgarhy, M., Abdalla, G. S. S., Obulezi, O. J., & Almetwally, E. M. (2025). A tangent DUS family of distributions with applications to the Weibull model. *Scientific African*, e02843.

43. Obulezi, O. J., Obiora-Ilouno, H. O., Osuji, G. A., Kayid, M., & Balogun, O. S. (2025). Weibull Sine Generalized Distribution Family: Fundamental Properties, Sub-model, Simulations, with Biomedical Applications. *Electronic Journal of Applied Statistical Analysis*, 18(01), 183–212.

44. Obulezi, O. J., Oguadimma, E. E., Igbokwe, C. P., Chikwelu, C. U., & Salem, G. S. (2025). A New Unit Family of Distributions with Applications to Public Health, Environmental, and Financial Data. *Appl. Math*, 19(6), 1437–1461.

45. Anyiam, K. E., Alghamdi, F. M., Nwaigwe, C. C., Aljohani, H. M., & Obulezi, O. J. (2024). A new extension of Burr-Hatke exponential distribution with engineering and biomedical applications. *Heliyon*, 10(19).

46. Aforka, K. F., Semary, H. E., Onyeagu, S. I., Etaga, H. O., Obulezi, O. J., & Al-Moisheer, A. S. (2025). A New Exponentiated Power Distribution for Modeling Censored Data with Applications to Clinical and Reliability Studies. *Symmetry*, 17(7), 1153.

47. Obulezi, O. J., Nadir, S., Orji, G. O., Igbokwe, C. P., Abdalla, G. S. S., Faal, A., & Elgarhy, M. (2026). Burr III scaled inverse odds ratio-Rayleigh distribution for modeling asymmetric engineering, disease surveillance and epidemiological data. *Appl. Math*, 20(1), 69–93.

48. Obisue, E. I., Okoro, C. N., Obulezi, O. J., Abdalla, G. S. S., Almetwally, E. M., Faal, A., & Elgarhy, M. (2025). Kavya-Manoharan DUS Family of Distributions with Diverse Applications. *Journal of Statistics Applications & Probability*, 14(6).

49. Nadir, S., Aslam, M., Anyiam, K. E., Alshawarbeh, E., & Obulezi, O. J. (2025). Group acceptance sampling plan based on truncated life tests for the Kumaraswamy Bell–Rayleigh distribution. *Scientific African*, 27, e02537.

50. Anyiam, K. E., Elbarkawy, M. A., Almetwally, E. M., Obulezi, O. J., & Elgarhy, M. (2025). Arcsine Ratio Sine Generalized Distributions with Applications to Biomedical and Engineering Data. *Mesopotamian Journal of CyberSecurity*, 5(3), 1218–1271.

51. Chukwuma, P. O., Hamdani, H., Nejib Ouertani, M., Obiora-Ilouno, H. O., Dike, J. C., Obulezi, O. J., Jobarteh, M., & Elgarhy, M. (2025). Statistical Inference for the Alpha Logarithmic Power Transformed Exponential Distribution With Applications. *Engineering Reports*, 7(11), e70443.

52. Nadir, S., Ali, E. I., Semary, H. E., Aslam, M., Obulezi, O. J., Nasiru, S., & Elgarhy, M. (2025). Burr III Scaled Inverse Odds Ratio-Weibull Distribution for Modeling Asymmetric Medical and Engineering Data. *Engineering Reports*, 7(11), e70482.

53. Prakash, Aman, Maurya, Raj Kamal, & Obulezi, O. J. (2025). Parameter Estimation and Optimal Plan Under Progressive Censoring for New Exponentiated Transformed Weibull Distribution. *Iranian Journal of Science*, 1–20.

54. Udobi, J. I., Obiora-Ilouno, H. O., Obulezi, O. J., Ahmed, D. M., Almetwally, E. M., & Elgarhy, M. (2025). Bilal-G Family of Distributions with Applications to Biomedical and Reliability Engineering Data. *International Journal of Statistics in Medical Research*, 14, 697–733.

Julian I. Mbegbu,
Department of Statistics,
Faculty of Physical Sciences,
University of Benin, Benin City,
Nigeria.
E-mail address: julian.mbegbu@uniben.edu

and

Chinelo U. Chikwelu,
Department of Statistics,
Faculty of Physical Sciences,
Nnamdi Azikiwe University, P.O. Box 5025, Awka
Nigeria.
E-mail address: uc.anyadiegwu@unizik.edu.ng

and

Friday Ewere,
Department of Statistics,
Faculty of Physical Sciences,
University of Benin, Benin City,
Nigeria.
E-mail address: friday.ewere@uniben.edu

and

Nzei C. Lawrence,
Department of Statistics,
Faculty of Physical Sciences,
University of Benin, Benin City,
Nigeria.
E-mail address: lawrencedumebi@gmail.com