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Pillai-Type Equations with Lucas Numbers and S-Unit Solutions

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ABSTRACT: In this paper, we investigate the exponential Diophantine equation $L_n - 5^x 7^y = c$, where L_n denotes the n-th Lucas number. The Lucas sequence is defined by the initial values $L_0 = 2$, $L_1 = 1$, and the recurrence relation $L_{n+2} = L_{n+1} + L_n$ for all $n \geq 0$. We show that when $c = 0$, the equation admits exactly two distinct solutions. Moreover, for any $c \in \mathbb{N}$, we prove that there is no integer c for which the equation has at least three distinct solutions $(n, x, y) \in \mathbb{Z}^3_{\geq 0}$.

Keywords: Lucas numbers, S-units, linear form in logarithms, p-adic numbers, Pillai's problem.

Contents

1. Introduction

1.1. Background

The Lucas sequence $\{L_n\}_{n\geq 0}$ is a second-order linear recurrence defined by

$$L_{n+2} = L_{n+1} + L_n, \quad \text{for every } n \geq 0,$$

with initial conditions $L_0 = 2$ and $L_1 = 1$. Its first few terms are

$$(2, 1, 3, 4, 7, 11, 18, 29, 47, 76, 123, 199, 322, 521, 843, 1364, \dots).$$

A well-known Diophantine problem related to exponential equations is the one introduced by Pillai, which asks for integer solutions of

$$a^x - b^y = c, \tag{1.1}$$

where $a, b > 1$ and c are fixed integers. In his pioneering work [1], Pillai proved that when a and b are coprime positive integers and the absolute value of c exceeds a certain bound $c_0(a, b)$, equation (1.1) has at most one solution (x, y) in integers.

Over time, several generalizations of this problem have been considered. In particular, many authors replaced one of the exponential sequences (a^x) or (b^y) with sequences having similar exponential growth, such as Fibonacci, Tribonacci, Pell, or Lucas numbers, or even with generalized Fibonacci sequences.

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These extensions (see for instance [2], [3], [4], [9], [5], [18], [19]) show that the finiteness property established by Pillai generally persists in these broader contexts.

From an algebraic point of view, if K is a number field with ring of integers R, an element $x \in K$ is called an S-unit when the ideal it generates factors only into primes belonging to a finite set S. In the rational case, this means that the numerator and denominator of x have no prime factors outside S. Equations similar to (1.1) involving S-units have been explored in works such as [6], [7] and [8]. For example, in [7], the powers of a were replaced by Fibonacci numbers, while in [6], Lucas numbers played the same role.

In the present paper, we investigate the case where the sequence $\{L_n\}$ of Lucas numbers interacts with prime powers. Specifically, we study the exponential Diophantine equation

$$L_n - 5^x 7^y = c, \tag{1.2}$$

where $n, x, y \in \mathbb{Z}_{\geq 0}$.

1.2. Main Results

Our main findings can be summarized as follows.

Theorem 1.1 *When $c = 0$, the exponential Diophantine equation (1.2) admits precisely two non-negative integer solutions, given by*

$$(n, x, y) = (1, 0, 0) \quad \text{and} \quad (n, x, y) = (4, 0, 1).$$

Theorem 1.2 *There exists no integer $c \in \mathbb{N}$ for which equation (1.2) possesses three or more distinct triples $(n, x, y) \in \mathbb{Z}^3_{\geq 0}$ satisfying it.*

2. Methods

2.1. Preliminaries

We begin by recalling the classical closed-form expression, often referred to as the Binet formula, for the Lucas sequence. It can be written as

$$L_n = \alpha^n + \beta^n, \quad \text{for every } n \geq 0, \tag{2.1}$$

where

$$\alpha = \frac{1+\sqrt{5}}{2} \quad \text{and} \quad \beta = \frac{1-\sqrt{5}}{2}.$$

It is worth observing that these two constants satisfy the simple relation $\beta = -\alpha^{-1}$. Then

$$L_n = \alpha^n(1 + (-1)^n \alpha^{-2n}). \tag{2.2}$$

The characteristic polynomial associated with the Lucas sequence $(L_n)_{n\geq 0}$ is given by $\psi(X) = X^2 - X - 1$. The polynomial $\psi(X)$ is irreducible over $\mathbb{Q}[X]$ and has roots α and β. Numerical approximations yield the following estimates:

$$1.61 < \alpha < 1.62,$$
$$0.61 < |\beta| < 0.62.$$

Assuming that $n \geq 10$, then

$$0.999\alpha^n < \alpha^n(1 - \alpha^{-20}) \leq L_n \leq \alpha^n(1 + \alpha^{-20}) < 1.001\alpha^n. \tag{2.3}$$

In the situation where $c > 0$, it follows that

$$5^x 7^y = L_n - c \leq L_n - 1 \leq \alpha^n \quad \text{by (2.2)}. \tag{2.4}$$

This implies that $x \log 5 + y \log 7 < n \log \alpha$, so that

$$x < n\frac{\log \alpha}{\log 5} < 0.3n \quad \text{and} \quad y < n\frac{\log \alpha}{\log 7} < 0.25n. \tag{2.5}$$

We also note a straightforward result from calculus, as presented in [7, Lemma 1].

Lemma 2.1 *[7, Lemma 1] Let x be a real number such that $|x| < \frac{1}{2}$. Then $|\log(1+x)| < \frac{3}{2}|x|$.*

We conclude this section by presenting an analytic argument, as outlined in [20, Lemma 7].

Lemma 2.2 *Let $m \geq 1$. For any integer T such that $T > (4m^2)^m$ and $T > \dfrac{z}{(\log z)^m}$, we have*

$$z < 2^m T (\log T)^m.$$

2.2. Linear Forms in Logarithms

In our work, we often rely on lower bounds of Baker-type for nonzero linear forms in two or three logarithms of algebraic numbers. Such bounds have been extensively studied in the literature, notably by Baker and Wüstholz [10] and Matveev [11].

Before stating these inequalities, we recall the definition of the height of an algebraic number.

Definition 2.1 Let λ be an algebraic number of degree d with minimal polynomial over the integers

$$a_0 x^d + a_1 x^{d-1} + \cdots + a_d = a_0 \prod_{i=1}^{d} (x - \lambda^{(i)}),$$

where the leading coefficient a_0 is positive. The logarithmic height of λ is defined by

$$h(\lambda) := \frac{1}{d}\left(\log a_0 + \sum_{i=1}^{d} \log \max\{|\lambda^{(i)}|, 1\}\right).$$

The subsequent properties of the logarithmic height function $h(\cdot)$ will be taken as given throughout the paper, without additional references:

$$\begin{aligned} h(\zeta \pm \eta) &\leq h(\zeta) + h(\eta) + \log(2), \\ h(\zeta\eta^{\pm 1}) &\leq h(\zeta) + h(\eta), \\ h(\zeta^s) &= |s| h(\zeta) \quad (s \in \mathbb{Z}). \end{aligned} \tag{2.6}$$

A linear form in logarithms is defined as an expression of the type

$$\Lambda = b_1 \log \delta_1 + \cdots + b_s \log \delta_s, \tag{2.7}$$

where $\delta_1, \ldots, \delta_s$ are positive real algebraic numbers, and $b_1, \ldots, b_s$ are nonzero integers. Let $\mathbb{L} := \mathbb{Q}(\delta_1, \ldots, \delta_s)$ be the field generated by the δ_i's, and let D represent the degree of $\mathbb{L}$. Put $\Gamma = e^{\Lambda} - 1$. With this notation, we begin by stating the main result of Matveev [11], which leads to the following estimate.

Theorem 2.1 (Matveev, [11]) *Let $\mathbb{Q}(\delta_1, \ldots, \delta_s)$ be a number field of degree D over $\mathbb{Q}$. Assume that $\Gamma \neq 0$. Then we have*

$$\log |\Gamma| > -1.4 \times 30^{s+3} \times s^{4.5} \times D^2 (1 + \log D)(1 + \log B) A_1 \cdots A_s,$$

where $B \geq \max\{|b_1|, \ldots, |b_s|\}$, and for each $i = 1, \ldots, s$, $A_i \geq \max\{D h(\delta_i), |\log \delta_i|, 0.16\}$.

This result is the version of Bugeaud, Mignotte, and Siksek ([17, Theorem 9.4]).

In addition, we utilize a p-adic version of Laurent's result, as developed by Bugeaud and Laurent in [21, Corollary 1]. Prior to presenting their result, we first define the requisite concepts.

Definition 2.2 Let p be a prime number. The p-adic valuation of an integer x, denoted $v_p(x)$, is given by

$$v_p(x) := \begin{cases} \max\{k \in \mathbb{N} : p^k \mid x\}, & \text{if } x \neq 0, \\ \infty, & \text{if } x = 0. \end{cases}$$

Moreover, for a rational number $x = \dfrac{a}{b}$, where a and b are integers, the p-adic valuation is defined as

$$v_p(x) = v_p(a) - v_p(b).$$

The expression for $v_p(x)$ in the case of rational numbers, as given in Definition 2.2, does not depend on the particular representation of x as a fraction of integers. From this definition, it follows that for any rational number x,

$$v_p(x) = \text{ord}_p(x),$$

where $\text{ord}_p(x)$ denotes the exponent of p in the prime factorization of x. For instance, we have $v_5\left(\frac{9}{25}\right) = -2$.

Next, for an algebraic number λ, we define its p-adic valuation as

$$v_p(\lambda) := \frac{v_p(a_d/a_0)}{d},$$

where a_0 and a_d are integers associated with λ as in Definition 2.1, and d is the degree of λ. For instance, if $x = \frac{a_d}{a_0}$ is a rational number in lowest terms with $a_0 \geq 1$, then its minimal polynomial is $f(X) = a_0 X - a_d$, which has degree 1. In this case, $v_p(x) = v_p\left(\frac{a_d}{a_0}\right)$, which is consistent with Definition 2.2. The p-adic valuation gives rise to a corresponding absolute value.

In a manner analogous to the previous context, let λ_1 and λ_2 be algebraic numbers over $\mathbb{Q}$, regarded as elements of the field $\mathbb{K}_p := \mathbb{Q}_p(\lambda_1, \lambda_2)$, where $D := [\mathbb{Q}_p(\lambda_1, \lambda_2) : \mathbb{Q}_p]$. As in Theorem 2.1, we employ a modified height function. In particular, the adjusted height of λ_i is defined as

$$h'(\lambda_i) \geq \max\left\{h(\lambda_i), \frac{\log(p)}{D}\right\}, \quad \text{for} \quad i = 1, 2.$$

Lemma 2.3 (Bugeaud and Laurent, [21]) *Let b_1, b_2 be positive integers, and let λ_1 and λ_2 be multiplicatively independent algebraic numbers such that $v_p(\lambda_1) = v_p(\lambda_2) = 0$. Define*

$$E := \frac{b_1}{h'(\lambda_2)} + \frac{b_2}{h'(\lambda_1)} \quad \textit{and} \quad F := \max\left\{\log E + \log\log p + 0.4,\ 10,\ 10\log p\right\}.$$

Then, the p-adic valuation of $\lambda_1^{b_1}\lambda_2^{b_2} - 1$ satisfies

$$v_p\left(\lambda_1^{b_1}\lambda_2^{b_2} - 1\right) \leq \frac{24pg}{(p-1)(\log p)^4} F^2 D^4\, h'(\lambda_1)h'(\lambda_2),$$

where $g > 0$ denotes the smallest integer such that $v_p(\lambda_i^g - 1) > 0$ for $i = 1, 2$.

To employ Lemma 2.3, it is necessary to verify that λ_1 and λ_2 are multiplicatively independent. In our setting, this amounts to ensuring that α and

$$\tau(t) = \frac{\alpha^t - 1}{\beta^t - 1} \tag{2.8}$$

are multiplicatively independent. The following result, taken from [7, Lemma 5], is useful in this context.

Lemma 2.4 *[7, Lemma 5] Let $t \geq 1$ be an integer. The algebraic numbers α and $\tau(t)$ are multiplicatively dependent if and only if $t = 1$, $t = 3$, or t is even. In these cases, we have*

$$\tau(1) = -\alpha^{-2}, \quad \tau(3) = -\alpha^2, \quad \textit{and} \quad \tau(2t) = -\alpha^{2t}.$$

Lemma 2.4 shows that in infinitely many instances (t even or $t = 1, 3$), Lemma 2.3 does not apply, and one must instead estimate $v_p(\alpha^x \pm 1)$. This can be achieved using the p-adic logarithm $\log_p$ (see [22], Sect. II.2.4 for details).

For an algebraic number x, define $|x|_p := p^{-v_p(x)}$. Let $\mathbb{C}_p$ denote the complex p-adic field, which is complete with respect to $|\cdot|_p$ and algebraically closed. The p-adic logarithm $\log_p x$ is defined on the ball

$$D\Big(0, p^{-\frac{1}{p-1}}\Big) := \{\xi \in \mathbb{Q}_p : |\xi - 1|_p < p^{-\frac{1}{p-1}}\},$$

by the convergent series

$$\log_p \xi := -\sum_{i=1}^{\infty} \frac{(1-\xi)^i}{i}.$$

It satisfies the usual property

$$\log_p(xy) = \log_p x + \log_p y, \quad \text{for } x, y \in D\Big(0, p^{-\frac{1}{p-1}}\Big),$$

and moreover,

$$|\log_p \xi|_p = |\xi - 1|_p, \quad \text{and} \quad v_p(\log_p \xi) = v_p(\xi - 1), \tag{2.9}$$

for all $\xi \in D\Big(0, p^{-\frac{1}{p-1}}\Big)$.

In practice, these calculations often produce upper bounds on the variables that are too large, so reduction techniques are required. In this paper, we make use of the following result related to continued fractions (see [23, Theorem 8.2.4]).

Lemma 2.5 *(Legendre). Let μ be an irrational number, and let its continued fraction expansion be given by $[a_0, a_1, a_2, \ldots]$. Define the convergents of the continued fraction of μ as*

$$\frac{p_i}{q_i} = [a_0, a_1, a_2, \ldots, a_i], \quad \textit{for all } i \geq 0,$$

and let M be a positive integer. Let N be the smallest integer such that $q_N > M$. Then, for the quantity

$$a(M) := \max\{a_i : i = 0, 1, 2, \ldots, N\},$$

the following inequality holds for all pairs (r, s) of positive integers with $0 < s < M$:

$$\left|\mu - \frac{r}{s}\right| > \frac{1}{(a(M)+2)s^2}.$$

We will need the following result, taken from [24, Lemma 2.6].

Lemma 2.6 *(Adapted from [24, Lemma 2.6]) The equation*

$$\frac{1-\gamma^w}{1-\gamma^h} = \frac{1-\alpha^w}{1-\alpha^h} \tag{2.9}$$

has no integer solutions h, w satisfying $h > w \geq 1$.

Lemma 2.7 *For any positive integer x and $z \in \{\alpha, \beta\}$, we have:*

1.

$$\nu_5(z^x - 1) = \begin{cases} \nu_5(x) + \frac{1}{2}, & \textit{if } x \equiv 0 \pmod 4, \\ 0, & \textit{if } x \not\equiv 0 \pmod 4. \end{cases}$$

$$\nu_5(z^x + 1) = \begin{cases} \nu_5(x) + \frac{1}{2}, & \textit{if } x \equiv 2 \pmod 4, \\ 0, & \textit{if } x \not\equiv 2 \pmod 4. \end{cases}$$

2.

$$\nu_7(z^x-1)=\begin{cases}\nu_7(x)+1, & \textit{if } x\equiv 0 \pmod{16},\\ 0, & \textit{if } x\not\equiv 0 \pmod{16}.\end{cases}$$

$$\nu_7(z^x+1)=\begin{cases}\nu_7(x)+1, & \textit{if } x\equiv 8 \pmod{16},\\ 0, & \textit{if } x\not\equiv 8 \pmod{16}.\end{cases}$$

Proof: We provide the proof in the case $z=\alpha$; the case $z=\beta$ follows by the same argument.
For $v_5(\alpha^x-1)$:

- If $x\equiv 0 \pmod 4$. Since $\alpha^4\equiv 1 \pmod 5$, we expect a non-zero valuation. Using a generalized Lifting The Exponent (LTE) property, we have $v_5(\alpha^n-1)=v_5(\alpha^4-1)+v_5(\frac{n}{4})$. We need to compute $v_5(\alpha^4-1)$. Since $\alpha^2-1=\alpha$ and $(\alpha^2+1)=\frac{5+\sqrt{5}}{2}$. it follows that $\alpha^4-1=\frac{5+\sqrt{5}}{2}\cdot\alpha$. To calculate $v_5(\alpha^4-1)$, we use the valuation on $\mathbb{Q}_5(\sqrt{5})$, where $v_5(\sqrt{5})=\frac{1}{2}$ and
$$\begin{aligned} v_5(\tfrac{5+\sqrt{5}}{2}\cdot\alpha) &= v_5(5+\sqrt{5})-v_5(2)+v_5(\alpha)\\ &= v_5(\sqrt{5}(\sqrt{5}+1))\\ &= v_5(\sqrt{5})+v_5(1+\sqrt{5})\\ &= \tfrac{1}{2}.\end{aligned}$$
Since $\sqrt{5}+1$ is a 5-adic unit (it is norm is - 4, which is not divisible by 5).
So, if $x\equiv 0 \pmod 4, v_5(\alpha^x-1)=v_5(\frac{x}{4})+\frac{1}{2}=v_5(x)+\frac{1}{2}$.

- If $x\not\equiv 0 \pmod 4$. Then $\alpha^x\not\equiv 1 \pmod 5$, so $v_5(\alpha^x-1)=0$.

For $v_5(\alpha^x+1)$:

- If $x\equiv 2 \pmod 4$. Then $\alpha^x\equiv\alpha^2\equiv 4\equiv -1 \pmod 5$. $v_5(\alpha^x+1)=v_5(\alpha^2+1)+v_5(\frac{x}{2})$, and $v_5(\alpha^2+1)=v_5(\frac{5+\sqrt{5}}{2})=\frac{1}{2}$. So, if $x\equiv 2 \pmod 4, v_5(\alpha^x+1)=v_5(\frac{x}{2})+\frac{1}{2}=v_5(x)+\frac{1}{2}$.

- If $x\not\equiv 2 \pmod 4$. Then $\alpha^x\not\equiv -1 \pmod 5$, so $v_5(\alpha^x+1)=0$.

For $v_7(\alpha^x-1)$:

- If $x\equiv 0 \mod 16$: Since $\alpha^{16}\equiv 1 \pmod 7$, we expect a non-zero valuation. $v_7(\alpha^x-1)=v_7(\alpha^{16}-1)+v_7(x/16)=v_7(\alpha^{16}-1)+v_7(x)$. We need to compute $v_7(\alpha^{16}-1)$. Since 7 is inert, we are working in $\mathbb{Z}_7[\alpha]$. It is common that for inert primes and minimal polynomial, $v_p(\alpha^{\text{order}}-1)=1$. This is analogues to $v_3(\alpha^8-1)=1$ in [18]. So $v_7(\alpha^x-1)=v_7(x)+1$.

- If $x\not\equiv 8 \mod 16$. Then $\alpha^x\not\equiv 1 \mod 7$, so $v_7(\alpha^x-1)$.

For $v_7(\alpha^x+1)$:

- If $x\equiv 8 \pmod{16}$. We have $\alpha^x\equiv\alpha^8\equiv -1 \pmod 7$. Then $v_7(\alpha^x+1)=v_7(\alpha^8+1)+v_7(x/8)=v_7(\alpha^8+1)+v_7(x)$. We need to compute $v_7(\alpha^8+1)$. Since $\alpha^8\equiv -1 \pmod 7, v_7(\alpha^8+1)\geq 1$. This is analogues to $v_3(\alpha^4+1)=1$ in [18]. So, if $x\equiv 8 \mod 16,\ v_7(\alpha^x+1)=v_7(x)+1$.

- If $x\not\equiv 8 \pmod{16}$. Then $\alpha^x\not\equiv -1 \pmod 7$, so $v_7(\alpha^x+1)=0$.

□

However, as current continued fraction techniques do not yield lower bounds for linear forms in more than two variables with bounded integer coefficients, we instead employ a method based on the LLL algorithm, described below.

2.3. Reduced Bases for Lattices and LLL-Reduction Methods

Let k be a positive integer. A subset $\mathfrak{L}$ of the real vector space $\mathbb{R}^k$ is called a *lattice* if there exist vectors $b_1, b_2, \ldots, b_k \in \mathbb{R}^k$ such that

$$\mathfrak{L} = \sum_{i=1}^{k} \mathbb{Z} b_i = \left\{ \sum_{i=1}^{k} r_i b_i \;\middle|\; r_i \in \mathbb{Z} \right\}.$$

The vectors $b_1, b_2, \ldots, b_k$ are said to form a *basis* of $\mathfrak{L}$, and they generate the lattice. The integer k is referred to as the *rank* of $\mathfrak{L}$. The *determinant* of the lattice $\mathfrak{L}$, denoted by $\det(\mathfrak{L})$, is defined as

$$\det(\mathfrak{L}) = \big|\det(b_1, b_2, \ldots, b_k)\big|.$$

where the b_i's are expressed as column vectors. This determinant is a positive real number and remains invariant under the choice of basis (refer to [14], Section 1.2).

Consider $b_1, b_2, \ldots, b_k$, a set of linearly independent vectors in $\mathbb{R}^k$. The Gram–Schmidt orthogonalization process provides a method to iteratively construct a set of orthogonal vectors b_i^* (for $1 \leq i \leq k$) along with corresponding coefficients $\mu_{i,j}$ (where $1 \leq j \leq i \leq k$). These are defined as follows:

$$b_i^* = b_i - \sum_{j=1}^{i-1} \mu_{i,j} b_j^*, \quad \mu_{i,j} = \frac{\langle b_i, b_j^* \rangle}{\langle b_j^*, b_j^* \rangle},$$

where $\langle \cdot, \cdot \rangle$ denotes the standard inner product in $\mathbb{R}^k$. The vector b_i^* represents the orthogonal projection of b_i onto the orthogonal complement of the subspace spanned by $b_1, \ldots, b_{i-1}$. As a result, b_i^* is orthogonal to $b_1^*, \ldots, b_{i-1}^*$ for all $1 \leq i \leq k$.

Hence, the sequence $b_1^*, b_2^*, \ldots, b_k^*$ constitutes an orthogonal basis of $\mathbb{R}^k$.

Definition 2.3 A basis $b_1, b_2, \ldots, b_n$ of a lattice $\mathfrak{L}$ is said to be *reduced* if the following conditions hold:

$$|\mu_{i,j}| \leq \frac{1}{2}, \quad \text{for all } 1 \leq j < i \leq n, \quad \text{and} \quad \|b_i^* + \mu_{i,i-1} b_{i-1}^*\|^2 \geq \frac{2}{3}\|b_{i-1}^*\|^2, \quad \text{for all } 1 < i \leq n,$$

where $\|\cdot\|$ denotes the Euclidean norm.

The constant $\frac{2}{3}$ in the second inequality is not fixed; it may be replaced by any real number in the interval $\left[\frac{1}{4}, 1\right]$, (see [12, Section 1]).

Let $\mathfrak{L} \subseteq \mathbb{R}^k$ be a k-dimensional lattice with a reduced basis $b_1, \ldots, b_k$, and let B be the matrix whose columns are $b_1, \ldots, b_k$. We define the function $\ell(\mathfrak{L}, v)$ as follows:

$$\ell(\mathfrak{L}, v) = \begin{cases} \min_{u \in \mathfrak{L}} \|u - v\|, & \text{if } v \notin \mathfrak{L}, \\ \min_{u \in \mathfrak{L} \setminus \{0\}} \|u\|, & \text{if } v \in \mathfrak{L}, \end{cases}$$

where $\|\cdot\|$ denotes the Euclidean norm in $\mathbb{R}^k$. It is a well-known result that the LLL algorithm can be used to compute a polynomial-time lower bound for $\ell(\mathfrak{L}, v)$. Specifically, there exists a positive constant c_1 such that $\ell(\mathfrak{L}, v) \geq c_1$ (see [22], Section V.4).

Lemma 2.8 *Let $v \in \mathbb{R}^k$ and $z = B^{-1} v$, where $z = (z_1, \ldots, z_k)^T$. Define the following:*

(i) If $v \notin \mathfrak{L}$, let i_0 be the largest index such that $z_{i_0} \neq 0$, and put $\sigma := \{z_{i_0}\}$, where $\{\cdot\}$ denotes the fractional part or the distance to the nearest integer.

(ii) If $v \in \mathfrak{L}$, put $\sigma := 1$.

Additionally, let

$$c_2 := \max_{1 \leq j \leq k} \left\{ \frac{\|b_1\|^2}{\|b_j^*\|^2} \right\}.$$

Then we have

$$c_1^2 := c_2^{-1} \sigma^2 \|b_1\|^2$$

In the context of our application, we consider real numbers $\eta_0, \eta_1, \ldots, \eta_k$ that are linearly independent over $\mathbb{Q}$. Moreover, we assume the existence of two positive constants c_3 and c_4 such that

$$|\eta_0 + a_1\eta_1 + \cdots + a_k\eta_k| \leq c_3 \exp(-c_4 H), \tag{2.10}$$

where the integers a_i are subject to the bounds $|a_i| \leq A_i$, for some given constants A_i, where $1 \leq i \leq k$. For simplicity, we define $A_0 := \max_{1\leq i\leq k}\{A_i\}$. The primary approach, based on the work of [13], involves approximating the linear form in inequality(2.10) through a lattice construction. Specifically, we consider the lattice $\mathfrak{L}$ generated by the columns of the matrix

$$A = \begin{pmatrix} 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \\ \lfloor M\eta_1 \rfloor & \lfloor M\eta_2 \rfloor & \cdots & \lfloor M\eta_{k-1} \rfloor & \lfloor M\eta_k \rfloor \end{pmatrix},$$

where M is a sufficiently large constant, typically chosen around the order of A_0^k. Suppose that an LLL-reduced basis $b_1, \ldots, b_k$ for $\mathfrak{L}$ has been computed, and that a lower bound $\ell(\mathfrak{L}, y) \geq c_1$ holds, where $v := (0, 0, \ldots, -\lfloor M\eta_0 \rfloor)$. The value of c_1 can be determined using the results from Lemma 2.8. Under these assumptions, the following result is analogous to [22, Lemma VI.1].

Lemma 2.9 *(Adapted from [22, Lemma VI.1]) Define*

$$S := \sum_{i=1}^{k-1} A_i^2 \quad \text{and} \quad T := \frac{1 + \sum_{i=1}^{k} A_i}{2}.$$

If $c_1^2 \geq T^2 + S$, then the inequality (2.10) implies that we must have either

$$a_1 = a_2 = \cdots = a_{k-1} = 0, \quad \text{and} \quad a_k = -\frac{\lfloor M\eta_0 \rfloor}{\lfloor M\eta_k \rfloor},$$

or

$$H \leq \frac{1}{c_4}\left(\log(Mc_3) - \log\left(\sqrt{c_1^2 - S} - T\right)\right).$$

2.4. Bounds for Solutions to S-unit Equations

The aim of this subsection is to derive a result from the following proposition.

Proposition 2.1 . *Let $\Delta > 10^{80}$ be a fixed integer. Suppose that*

$$5^x 7^y - 5^{x_1} 7^{y_1} = \Delta. \tag{2.11}$$

Then, we have

$$5^x 7^y < \Delta (\log \Delta)^{60 \log\log \Delta}.$$

Proof: Let $u := 5^x 7^y$ and $v := 5^{x_1} 7^{y_1}$, then $u - v = \Delta$. We note that

$$\max\{|x - x_1|, |y - y_1|\} \log 5 \leq x \log 5 + y \log 7 = \log u.$$

Let us divide equation (2.11) through u, then we obtain

$$|5^{x_1 - x} 7^{y_1 - y} - 1| < \frac{\Delta}{u}. \tag{2.12}$$

In view Proposition 2.1 we may assume that $u > 2\Delta$, and by Lemma 2.1 we obtain

$$|\Lambda| = |(x - x_1)\log 5 + (y - y_1)\log 7| < \frac{3}{2} \cdot \frac{\Delta}{u}. \tag{2.13}$$

Since $\log 5$ and $\log 7$ are linearly independent over $\mathbb{Q}$, the linear form Λ can vanish only in the case where $x = x_1$ and $y = y_1$, which contradicts the assumption that $\Delta \neq 0$. Therefore, by applying [7, Lemma 3] with $D = 1$, $\log A_1 = \log 5$ and $\log A_2 = \log 7$, we derive

$$b' \leq 2\max\{|x - x_1|, |y - y_1|\} \leq \frac{2}{\log 5}\log u.$$

Thut is

$$\max\{\log b' + 0.38, 30\} \leq \log(\frac{2}{\log 5}\log u) + 0.38 \leq \log(1.82\log u).$$

Provided that $\log(1.82\log u) \geq 30$. Laurent's lower bound for linear forms in two logarithms yield's

$$17.9(\log(1.82\log u))^2\log 5\log 7 \geq -\log\frac{3}{2} - \log\Delta + \log u.$$

If we substitue $u = \Delta(\log\Delta)^{60\log\log\Delta}$ into this inequality we obtain

$$\begin{aligned} 17.9\cdot(\log 1.82 + \log\log(\Delta(\log\Delta)^{60\log\log\Delta}))^2\log 5\log 7 &= 17.9\cdot\log(1.82\log\Delta + 1.82\cdot 60 \\ &\quad \cdot(\log\log\Delta)^2)^2\log 5\log 7 \\ &\geq -\log\frac{3}{2} - \log\Delta + \log\Delta + 60(\log\log\Delta)^2 \\ 56.1\log(1.82\log\Delta + 109.3(\log\log\Delta)^2)^2 &\geq -0.406 + 60(\log\log\Delta)^2. \end{aligned}$$

Since we assume that $\Delta > 10^{80}$ we have $109.3(\log\log\Delta)^2 > 15.98\log\Delta$, and also $0.406 < 0.015(\log\log\Delta)^2$. Thus we get

$$\begin{aligned} 56.1\log(1.82\log\Delta + 15.98\log\Delta)^2 &\geq 60(\log\log\Delta)^2 - 0.015(\log\log\Delta)^2 \\ 56.1\log(17.80\log\Delta)^2 &\geq 59.985(\log\log\Delta)^2, \end{aligned}$$

which does not hold if $\Delta > 10^{80}$. Thus proposition 2.1 holds under the assumption that $\log(1.82\log u) \geq 30$, i.e that $\log u > 5.87\cdot 10^{12}$. Let us assume that $\log u \leq 5.87\cdot 10^{12}$, which implies that

$$x\log 5 \leq 5.87\cdot 10^{12} \quad \text{and} \quad y\log 7 \leq 5.87\cdot 10^{12},$$

$$x \leq \frac{5.87}{\log 5}\cdot 10^{12} < 3.65\cdot 10^{12} \quad \text{and} \quad y \leq \frac{5.87}{\log 7}\cdot 10^{12} < 3.02\cdot 10^{12}.$$

Then

$$x, y < 3.65\cdot 10^{12}.$$

Suppose, for the sake of argument, that $u \geq \Delta(\log\Delta)^{60\log\log\Delta}$. Under this assumption, inequality (2.13) becomes

$$|(x - x_1) + (y - y_1)\frac{\log 7}{\log 5}| < \frac{3}{2\log 5(\log\Delta)^{60\log\log 10^{80}}} < 1.03\cdot 10^{-709}.$$

Observe that the 25-th convergent $\frac{p_{25}}{q_{25}}$ to $\frac{\log 7}{\log 5}$ satisfies $p_{25}, q_{25} > 3.65\cdot 10^{12}$, and we get

$$|(x - x_1) + (y - y_1)\frac{\log 7}{\log 5}| < 1.03\cdot 10^{-709} < |p_{25} + q_{25}\frac{\log 7}{\log 5}|.$$

This, however, contradicts the optimal approximation property of continued fractions (see, for instance, [16, Theorem 182]). Therefore, Proposition (2.1) also holds when $\log u \leq 5.87\times 10^{12}$. □

In preparation for the proofs of our main theorems, we establish the following corollary, which will be of practical use later. For convenience, let us denote $X = x\log 5 + y\log 7$ and $X_1 = x_1\log 5 + y_1\log 7$.

Corollary 2.1 *Suppose that the tuple (n, n_1, x, x_1, y, y_1) is a solution of*

$$L_n - 5^x 7^y = L_{n_1} - 5^{x_1}7^{y_1},$$

with $n > 385$ and $n > n_1$. Then

$$0.38\alpha^n < \exp(X) < 2\alpha^n(n\log\alpha)^{60\log(n\log\alpha)},$$

Proof: If $n > 385$, then

$$\Delta = 5^x 7^y - 5^{x_1} 7^{y_1} = L_n - L_{n_1} \geq L_{n-2} > L_{383} > 10^{80}.$$

So, we apply Proposition 2.1 with $\Delta = L_n - L_{n_1} < L_n \leq 1.001\,\alpha^n$ by (2.3). This yields

$$\begin{aligned} \exp(X) &< \Delta(\log(\Delta))^{60 \log\log \Delta} < 1.001\alpha^n(\log(1.001\alpha^n))^{60\log\log(1.001\alpha^n)} \\ &< 2\alpha^n(n\log\alpha)^{60\log(n\log\alpha)}. \end{aligned} \tag{2.14}$$

To explain the preceding computation, observe that

$$\begin{aligned} 1.001(\log(1.001\alpha^n))^{60\log\log(1.001\alpha^n)} &= 1.001(n\log\alpha + \log 1.001)^{60\log\log(1.001\alpha^n)} \\ &< 1.001(n\log\alpha)^{60\log(n\log\alpha+\log 1.001)}\left(1+\frac{\log(1.001)}{n\log\alpha}\right)^{60\log(n\log\alpha+\log(1.001))} \\ &< 1.001(n\log\alpha)^{60\log(n\log\alpha)(1+\log(1.001))/(n\log\alpha)} \\ &\quad\cdot\exp\left(\frac{60\log(n\log\alpha+\log(1.001))\cdot\log(1.001)}{n\log\alpha}\right) \\ &< 1.001(n\log\alpha)^{60\log(n\log\alpha)}(n\log\alpha)^{60\log(n\log\alpha)(\log(1.001)/(n\log\alpha)} \\ &\quad\cdot\exp\left(\frac{60\log(n\log\alpha+\log(1.001))\log(1.001)}{n\log\alpha}\right) \\ &< 1.001(n\log\alpha)^{60\log(n\log\alpha)}\cdot\exp\left(\frac{60\log(n\log\alpha)\cdot\log(n\log\alpha)\log(1.001))}{n\log\alpha}\right) \\ &\quad\cdot\exp\left(\frac{60\log(n\log\alpha+\log(1.001))\log(1.001)}{n\log\alpha}\right). \end{aligned}$$

In the argument above, we only relied on the inequality $\log(1+y) < y$, which holds for all positive real numbers y. Within the last two exponential terms, when $n > 500$, the first term does not exceed 0.008 and the second is bounded by 0.002. Hence, together with the factor 1.001, these contributions sum to at most

$$1.001 \cdot \exp(0.008) \cdot \exp(0.002) < 2,$$

which establishes (2.14). Conversely,

$$0.38\alpha^n < 0.999\alpha^n - 1.001\alpha^{n-1} < L_n - L_{n-1} < 5^x7^y - 5^{x_1}7^{y_1} < 5^x7^y = \exp(X).$$

Which gives

$$0.38\alpha^n < \exp(X). \tag{2.15}$$

Combining (2.14) and (2.15), we get

$$0.38\alpha^n < \exp(X) < 2\alpha^n(n\log\alpha)^{60\log(n\log\alpha)},$$

and taking logarithms both sides gives

$$n\log\alpha + \log 0.38 < X < \log 2 + n\log\alpha + 60(\log(n\log\alpha))^2 < 1 + n\log\alpha + 60(\log(n\log\alpha))^2.$$

□

3. Proof of Theorem 1.1

This section is devoted to proving Theorem 1.2 in the case $c = 0$. Under this assumption, equation (1.2) simplifies to the Diophantine equation

$$L_n = 5^x 7^y. \tag{3.1}$$

To handle this case, recall that for every integer $n > 12$, the Lucas number L_n admits at least one primitive prime factor, that is, a prime dividing L_n but not dividing any earlier term L_m with $m < n$. This follows from Carmichael's Primitive Divisor Theorem and its extensions. Therefore, L_n cannot be divisible by 7 for $n > 12$, since 7 already divides L_4, and thus would not be a primitive divisor. Moreover, a Lucas number cannot be a pure power of 5. Hence, equation (3.1) has no solutions for $n > 12$. Thus, we only need to check the values of L_n for $0 \leq n \leq 12$, and determine for which values L_n is a product of powers of 5 and 7. A direct computation shows that the only such values of n are those listed in Theorem 1.1.

4. Poof of theorem 1.2

In this section, we prove Theorem 1.2. Note that when $n \leq 2000$ in equation (1.2), we have $x \leq 600$ and $y \leq 500$ according to inequality (2.5). An exhaustive computer search using `Maple` explored all triples (n, x, y) with $0 \leq n \leq 2000$, $0 \leq x \leq 600$ and $0 \leq y \leq 500$, and searched for all values of c that admit at least three representations of the form $L_n - 5^x 7^y$. This search returned no solution, as stated in Theorem 1.2.

4.1. An absolute upper bound for n in the case that $c \in \mathbb{N}$

From this point onward, we assume that $n > 2000$ and focus on deriving an upper bound for n. Let (n, x, y), (n_1, x_1, y_1) and (n_2, x_2, y_2) be non-negative integers satisfying

$$L_n - 5^x 7^y = L_{n_1} - 5^{x_1} 7^{y_1} = L_{n_2} - 5^{x_2} 7^{y_2}.$$

It is important to note that $n = n_1$ is not possible, at this would imply $x = x_1$ and $y = y_1$, leading to the same representation of c. Therefore, without loss of generality, we may assume $n > n_1 > n_2$.

Lemma 4.1 *Let $c \geq 1$ such that the Diophantine equation (1.2) admits at least two distinct representations given by*

$$c = L_n - 5^x 7^y = L_{n_1} - 5^{x_1} 7^{y_1}.$$

Then

$$n - n_1 < 1.9 \cdot 10^{13} \log n. \tag{4.1}$$

Proof: Since $c > 0$, we have $L_n > 5^x 7^y$ and $L_{n_1} > 5^{x_1} 7^{y_1}$. We now return to equation (1.2) and rewrite it as

$$\begin{aligned} L_n - 5^x 7^y &= L_{n_1} - 5^{x_1} 7^{y_1} \\ 0 \leq c - 1 < c - \beta^n = \alpha^n - 5^x 7^y &= \alpha^{n_1} - 5^{x_1} 7^{y_1} + \beta^{n_1} - \beta^n \\ &\leq \alpha^{n_1} - 5^{x_1} 7^{y_1} + |\beta^{n_1} - \beta^n| \leq \alpha^{n_1}, \end{aligned}$$

where we have used the fact that $|-\beta^n + \beta^{n_1}| \leq -\beta + \beta^2 = 1$ for all $n > 2500$. So we conclude that

$$\left|5^x 7^y \alpha^{-n} - 1\right| < \alpha^{-(n - n_1)}. \tag{4.2}$$

We now apply Theorem 2.1 to the left-hand side of (4.2). Define

$$\Gamma_0 := 5^x 7^y \alpha^{-n} - 1.$$

Notice that $\Gamma_0 \neq 0$; otherwise, we would have $\alpha^n = 5^x 7^y \in \mathbb{Z}$. Applying any automorphism that maps α to β, we obtain $\beta^n = 5^x 7^y$. This leads to a contradiction, since $|\beta^n| < 1$, whereas $5^x 7^y \geq 1$ for all $x, y \geq 0$. Next, we use the field $K := \mathbb{Q}(\sqrt{5})$ of degree $D = 2$. Here $t := 3$, and put

$$\gamma_1 := 5, \quad \gamma_2 := 7, \quad \gamma_3 := \alpha,$$

and

$$b_1 := x, \quad b_2 := y, \quad b_3 := -n.$$

Next, we have

$$\max\{|b_1|, |b_2|, |b_3|\} = \max\{x, y, n\} = n.$$

We may therefore take $B := n$. Also, $A_i \geq \max\{Dh(\gamma_i), |\log \gamma_i|, 0.16\}$ for all $i = 1, 2, 3$. So

$$A_1 := Dh(\gamma_1) = 2 \log 5, \quad A_2 := Dh(\gamma_2) = 2 \log 7, \quad A_3 := Dh(\gamma_3) = \log \alpha.$$

Then by Theorem 2.1, we get

$$\begin{aligned} \log|\Gamma| &> -1.4 \cdot 30^6 \cdot 3^{4.5} \cdot 2^2 (1 + \log 2)(1 + \log n)(2 \log 5)(2 \log 7)(\log \alpha) \\ &> -5.85 \cdot 10^{12} (1 + \log n). \end{aligned} \tag{4.3}$$

Simplifying this gives

$$\log|\Gamma| > -8.77\cdot 10^{12}\cdot\log n. \tag{4.4}$$

Where the last inequality holds for $n > 2000$. Comparing (4.2) and (4.4), we get

$$n - n_1 < 1.9\cdot 10^{13}\log n.$$

This proves Lemma 4.1 □

We now present and prove the following result.

Lemma 4.2 *Let $c \geq 1$, $X := x\log 5 + y\log 7$ and $X_1 := x_1\log 5 + y_1\log 7$. Then*

$$X - X_1 < 9.7\cdot 10^{27}(\log n)^2.$$

Proof: We return to equation (1.2) and rewrite it in the form

$$\alpha^n - \alpha^{n_1} - 5^x7^y = -5^{x_1}7^{y_1} + \beta^{n_1} - \beta^n.$$

By factoring the terms, we obtain

$$\frac{\alpha^{n_1}\left(\alpha^{n-n_1}-1\right)}{5^x7^y} - 1 = \frac{-5^{x_1}7^{y_1}}{5^x7^y} + \frac{\beta^{n_1}-\beta^n}{5^x7^y} = \frac{-1}{\exp(X-X_1)} + \frac{\beta^{n_1}-\beta^n}{\exp(X)},$$

and taking absolute values, we get

$$\left|\frac{\alpha^{n_1}(\alpha^{n-n_1}-1)}{5^x7^y} - 1\right| \leq \frac{1}{\exp(X)} + \frac{1}{\exp(X-X_1)} \leq 2\exp(-(X-X_1)), \tag{4.5}$$

where we have used the fact that $|-\beta^n + \beta^{n_1}| \leq -\beta + \beta^2 = 1$ for all $n > 2000$. Moreover, if $X - X_1 > 1.4$, then $2\exp(-(X-X_1)) < \dfrac{1}{2}$. Let $\Gamma_1 := \alpha^{n_1}(\alpha^{n-n_1}-1)5^{-x}7^{-y} - 1$. Then

$$|\Gamma_1| \leq 2\exp(-(X-X_1)). \tag{4.6}$$

Observe that $\Gamma_1 \neq 0$; otherwise, we would have $\frac{\alpha^n-\alpha^{n_1}}{5^x7^y} = 1$. Taking the algebraic conjugates, we obtain $1 = \frac{\beta^n-\beta^{n_1}}{5^x7^y} < 1$, a contradiction. Therefore, $\Gamma_1 \neq 0$. As before, we work in the field $\mathbb{Q}(\sqrt{5})$, which has degree $D = 2$.
Here, $t := 4$,

$$\gamma_1 := 5, \quad \gamma_2 := 7, \quad \gamma_3 := \alpha, \quad \gamma_4 := \alpha^{n-n_1} - 1,$$
$$b_1 := -x, \quad b_2 := -y, \quad b_3 := n_1, \quad b_4 := 1.$$

Next, $\max\{|b_1|,|b_2|,|b_3|,|b_4|\} = \max\{x,y,1,n_1\} < n$, so we can take $B := n$. As before, we can still take $A_1 := 2\log 5$, $A_2 := 2\log 7$ and $A_3 := \log\alpha$, as before and

$$\begin{aligned}2h(\gamma_4) &= 2h((\alpha^{n-n_1}-1))\\ &\leq 2(n-n_1)h(\alpha) + 2\log 2\\ &< 2\cdot 10^{13}\log\alpha\log n\\ &< 10^{13}\log n,\end{aligned}$$

by (4.1). Therefore, we take $A_4 = 10^{13}\log n$. Then, by Theorem 2.1,

$$\begin{aligned}\log|\Gamma_1| &> -1.4\cdot 30^7\cdot 4^{4.5}\cdot 2^2(1+\log 2)(1+\log n)(2\log 5)(2\log 7)(\log\alpha)(10^{13}\log n)\\ &> -9.61\cdot 10^{27}(\log n)^2.\end{aligned} \tag{4.7}$$

Comparing (4.5) and (4.7), we get

$$X - X_1 < 9.7\cdot 10^{27}(\log n)^2. \tag{4.8}$$

This completes the proof of Lemma 4.2. □

Next, define

$$x_{\min} := \min\{x, x_1\}, \quad y_{\min} := \min\{y, y_1\}.$$

We now present and prove the following result.

Lemma 4.3 *Assume that $c \geq 1$. Then, either*

$$x_{\min}, y_{\min} < 6.2 \cdot 10^{13} (\log n)^3,$$

or

$$n < 37000.$$

Proof: Once again, we consider equation (1.2) and assume that it admits two distinct solutions, namely (n, x, y) and (n_1, x_1, y_1). We then rewrite it as

$$\alpha^n - \alpha^{n_1} + \beta^n - \beta^{n_1} = 5^x 7^y - 5^{x_1} 7^{y_1}.$$

By factoring the terms, we obtain

$$\frac{\alpha^{n_1}\left(\alpha^{n-n_1}-1\right)}{\beta^{n_1}\left(\beta^{n-n_1}-1\right)} + 1 = \frac{5^{x_{min}} 7^{y_{min}} \left(5^{x-x_{min}} 7^{y-y_{min}} - 5^{x_1 - x_{min}} 7^{y_1 - y_{min}}\right)}{\beta^{n_1}(\beta^{n-n_1}-1)}. \tag{4.9}$$

Let us denote $A := 5^{x-x_{min}} 7^{y-y_{min}} - 5^{x_1-x_{min}} 7^{y_1-y_{min}}$. Since $v_p(\beta) = 0$ for $p = 5, 7$, we have

$$v_5\left(\left(\frac{\alpha}{\beta}\right)^{n_1} \frac{(\alpha^{n-n_1}-1)}{(\beta^{n-n_1}-1)} + 1\right) = x_{min} - v_5(\beta^{n-n_1}-1) + v_5(A),$$

or equivalently

$$x_{min} = v_5\left(\left(\frac{\alpha}{\beta}\right)^{n_1} \frac{\alpha^{n-n_1}-1}{\beta^{n-n_1}-1} + 1\right) + v_5(\beta^{n-n_1}-1) - v_5(A).$$

Then

$$x_{\min} \leq v_5\left(\left(\frac{\alpha}{\beta}\right)^{n_1} \frac{\alpha^{n-n_1}-1}{\beta^{n-n_1}-1} + 1\right) + v_5(\beta^{n-n_1}-1), \tag{4.10}$$

and similarly

$$y_{\min} \leq v_7\left(\left(\frac{\alpha}{\beta}\right)^{n_1} \frac{\alpha^{n-n_1}-1}{\beta^{n-n_1}-1} + 1\right) + v_7(\beta^{n-n_1}-1). \tag{4.11}$$

Next, we estimate $v_p(\beta^{n-n_1}-1)$ for $p = 5, 7$. By Lemma 2.7,

$$\begin{aligned} v_p(\beta^{n-n_1}-1) &\leq 1 + v_p(n-n_1) \leq 1 + \frac{\log(n-n_1)}{\log p} \\ &< 1 + \frac{\log(1.9 \cdot 10^{13} \log n)}{\log p} \\ &< 1 + \frac{6 \log n}{\log p}. \end{aligned}$$

Under the assumption that $n > 2000$, we proceed to estimate the first terms on the right hand side of (4.10) and (4.11), respectively.
Assuming $n - n_1$ is even, we have by Lemma 2.7 and 2.4

$$\begin{aligned} v_p\left(\left(\frac{\alpha}{\beta}\right)^{n_1} \frac{\alpha^{n-n_1}-1}{\beta^{n-n_1}-1} + 1\right) &= v_p\left(-\left(\frac{\alpha}{\beta}\right)^{n_1} \alpha^{n-n_1} + 1\right) \\ &= v_p(-\alpha^{n_1+n-n_1}(-\alpha^{-1})^{-n_1} + 1) \\ &= v_p(-(-1)^{n_1}\alpha^{n+n_1} + 1) \\ &= v_p(\alpha^{n+n_1} \pm 1) \\ &< 1 + \frac{\log(n+n_1)}{\log p} \\ &< 1 + \frac{\log 2n}{\log p}. \end{aligned}$$

Therefore, we get inequalities

$$x_{min} < 1 + \frac{\log 2n}{\log 5} + 1 + \frac{6 \log n}{\log 5} < 3 + \frac{7 \log n}{\log 5} < 9 \log n \tag{4.12}$$

and

$$y_{min} < 1 + \frac{\log 2n}{\log 7} + 1 + \frac{6 \log n}{\log 7} < 3 + \frac{7 \log n}{\log 7} < 8 \log n. \tag{4.13}$$

If $n-n_1=1$, then by Lemma 2.7 and 2.4, we get

$$\begin{aligned} v_p\left(\left(\frac{\alpha}{\beta}\right)^{n_1}\left(\frac{\alpha^{n-n_1}-1}{\beta^{n-n_1}-1}\right)+1\right) &= v_p\left(\left(\frac{\alpha}{-\alpha^{-1}}\right)^{n_1}\left(-\alpha^{-2}\right)+1\right)\\ &= v_p((-\alpha)^{n_1}\alpha^{n_1}(-\alpha)^{-1}+1)\\ &= v_p((-1)^{n_1+1}\alpha^{n_1-2}+1)\\ &= v_p(\alpha^{n_1-2}\pm 1)\\ &< 1+v_p(n_1-2)\\ &< 1+\frac{\log(n_1-2)}{\log p}\\ &< 1+\frac{\log n}{\log p}. \end{aligned}$$

An identical inequality arises in the case when $n-n_1=3$.

Now suppose that $n-n_1\geq 5$ is odd. In this case, Lemma 2.3 can be applied to the first terms on the right-hand sides of (4.10) and (4.11), respectively. Here we note that $\gamma_1=\dfrac{\alpha}{\beta}$ and $\gamma_2=\dfrac{\alpha^{n-n_1}-1}{\beta^{n-n_1}-1}$ are multiplivatively independent by Lemma 2.4. Furthermore, $h(\gamma_1)=\log(\alpha)$. We choose $h'(\gamma_1)=\dfrac{\log 5}{2}$ in (4.10) and $h'(\gamma_1)=\dfrac{\log 7}{2}$ in (4.11). Moreover,

$$\begin{aligned} h(\gamma_2) &= h\left(\frac{\alpha^{n-n_1}-1}{\beta^{n-n_1}-1}\right)\leq 2h(\alpha^{n-n_1}-1)\\ &< 2((n-n_1)h(\alpha)+\log 2)\\ &= (n-n_1)\log\alpha+\log 4\\ &< 10^{13}\log n. \end{aligned}$$

Therefore,

$$E=\frac{b_1}{h'(\gamma_2)}+\frac{b_2}{h'(\gamma_1)}=\frac{n_1}{h'(\gamma_1)}+\frac{1}{h'(\gamma_2)}\leq\frac{n}{h'(\gamma_1)}<n^2.$$

Assuming that $n\geq 37000$, then

$$\begin{aligned} F=\max\{\log E+\log\log p+0.4, 10, 10\log p\} &< 2\log n+\log\log p+0.4\\ &< 2\log n+1.1, \end{aligned}$$

in both cases. Moreover, we may choose $g=3$ when $p=5$ and $g=4$ when $p=7$. It then follows from (4.10) that

$$\begin{aligned} x_{min} &\leq \frac{24pg}{(p-1)(\log p)^4}F^2D^4h'(\gamma_1)h'(\gamma_2)+v_5(\beta^{n-n_1}-1)\\ &\leq \frac{24\cdot 5\cdot 3}{(5-1)(\log 5)^4}(2\log n+1.1)^2\cdot 2^4\log\tfrac{\log 5}{2}\cdot 10^{13}\log n+1+\frac{6\log n}{\log 5}\\ &< 4.9\cdot 10^{16}\cdot(\log n)^3\left(2+\frac{1.1}{\log 37000}\right)^2+1+\frac{6\log n}{\log 5}\\ &< (4.9\cdot 10^{16}\cdot\left(2+\frac{1.1}{\log 37000}\right)^2+3)(\log n)^3\\ &< 6.2\cdot 10^{16}(\log n)^3. \end{aligned}$$

Similarly, (4.11) gives

$$\begin{aligned} y_{min} &\leq \frac{24\cdot 7\cdot 4}{(7-1)(\log 7)^4}(\log n)^2\left(2+\frac{1.3}{\log 37000}\right)^2\cdot 2^4\frac{\log 7}{2}\cdot 10^{13}\log n+1+\frac{6\log n}{\log 7}\\ &< 1.1\cdot 10^{16}(\log n)^3+1+\frac{6\log n}{\log 3}\\ &< 2.4\cdot 10^{16}(\log n)^3. \end{aligned}$$

This completes the proof of Lemma 4.3. □

Now, we consider a third solution (n_2,x_2,y_2), with $n>n_1>n_2$ and we find an absolute bound for n. We prove the following result.

Lemma 4.4 *If $c > 0$ and $n > 37000$, then*

$$n < 9.7 \cdot 10^{34}, \quad x < 3.10^{34}, \quad y < 2.5 \cdot 10^{34}$$

Proof: Lemma 4.3 states that, out of any two solutions, the minimal values of x and y are bounded by $6.2 \cdot 10^{16}(\log n)^3$. Therefore, among three solutions, at most one can have x exceeding $6.2 \cdot 10^{16}(\log n)^3$, and at most one can have y exceeding this bound. Hence, at least one of the solutions must have both x and y bounded by this quantity. In particular, this shows that the minimal solution satisfies

$$X_2 = x_2 \log 5 + y_2 \log 7 < (6.2 \log 5 + 6.2 \log 7) \cdot 10^{16}(\log n)^3 < 2.3 \cdot 10^{17}(\log n)^3.$$

From which Lemma 4.2 follows

$$\begin{aligned} n \log \alpha < X - \log 0.38 &< X + 1 = X_2 + (X_1 - X_2) + (X - X_1) + 1 \\ &< 2.3 \cdot 10^{17}(\log n)^3 + 2 \cdot 9.7 \cdot 10^{27}(\log n)^2 + 1 \quad \text{Which implies} \\ &< 2.10^{28}(\log n)^3. \end{aligned}$$

$$\frac{n}{(\log n)^3} < \frac{2.10^{28}}{\log \alpha} < 4.2 \cdot 10^{28}. \tag{4.14}$$

We apply Lemma 2.2 to inequality (4.14) above with $z = n, m = 3, T = 4.2 \cdot 10^{28}$. Since $T > (4.3^2)^3$ and by (4.14) we get $n < 2^m T(\log T)^m = 2^3 \cdot 4.2 \cdot 10^{28}(\log(4.2 \cdot 10^{28})^3 < 9.7 \cdot 10^{34}$.
Further, we have by Corollary 2.1

$$X < 1 + n \log \alpha + 60(\log(n \log \alpha))^2$$

or, equivalently

$$\begin{aligned} x \log 5 + y \log 7 &< 1 + 9.7 \cdot 10^{34} \log \alpha + 60(\log(9.7 \cdot 10^{34} \log \alpha))^2 \\ &< 4.7 \cdot 10^{34}. \end{aligned}$$

This gives

$$x < 3 \cdot 10^{34} \quad \text{and} \quad y < 2.5 \cdot 10^{34},$$

This completes the proof of Lemma 4.4. □

4.2. Reduction of the Upper Bound on n

In this subsection, we employ the LLL-reduction algorithm, the theory of continued fractions, and p-adic reduction techniques as introduced in [15] to derive a significantly smaller bound for n. Which will conclude the prof of Theorem 1.2.

To begin, we return to equation (4.2). Assuming that $n - n_1 \geq 2$, we can write

$$|\Lambda_0| = \left| x \log 5 + y \log 7 - n \log \alpha \right| < \frac{3}{2} \alpha^{-(n-n_1)},$$

where we applied Lemma 2.1 under the assumption $n - n_1 \geq 2$, since $\alpha^{-(n-n_1)} \leq \alpha^{-2} \leq \frac{1}{2}$. We consider the approximation lattice

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \lfloor M \log 5 \rfloor & \lfloor M \log 7 \rfloor & \lfloor M \log \alpha \rfloor \end{pmatrix},$$

with $M := 10^{105}$ and choose $v := (0, 0, 0)$. Now, by Lemma 2.8, we get

$$l(\mathfrak{L}, v) \geq c_1 = 3.13 \cdot 10^{35} \quad \text{and} \quad c_2 = 5.7 \cdot 10^{35}.$$

Moreover, by Lemma 4.4, we have

$$x < A_1 := 3 \cdot 10^{34}, \quad y < A_2 := 2.5 \cdot 10^{34}, \quad n < A_3 := 9.7 \cdot 10^{34}.$$

So, Lemma 2.9 gives $S = 1.525 \cdot 10^{69}$ and $T = 7.60 \cdot 10^{34}$. Since $c_2^2 \geq T^2 + S$, then choosing $c_3 := \frac{3}{2}$ and $c_4 := \log \alpha$, we get

$$H := n - n_1 \leq \frac{1}{\log \alpha}\left(\log\left(10^{105} \cdot \frac{3}{2}\right) - \log\left(\sqrt{(3.13 \times 10^{35})^2 - 1.525 \times 10^{69}} - 7.60 \times 10^{34}\right)\right).$$

Then

$$n - n_1 \leq 99.$$

Next, we revisit equation (4.5). Assume that $X - X_1 \geq 2$. We can then write

$$|\Lambda_1| = \left|n_1 \log \alpha + \log(\alpha^{n-n_1} - 1) - x \log 5 - y \log 7\right| < 3 \exp(-(X - X_1)),$$

where we applied Lemma 2.1 along with the fact that $2 \exp(-(X - X_1)) \leq 2 \exp(-2) < \frac{1}{2}$. Thus, we proceed by utilizing the same approximation lattice

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \lfloor M \log 5 \rfloor & \lfloor M \log 7 \rfloor & \lfloor M \log \alpha \rfloor \end{pmatrix}.$$

However, setting $M := 10^{106}$ and choosing $v := (0, 0, -\lfloor M \log(\alpha^{n-n_1} - 1) \rfloor)$, we observe that for all values $1 \leq n - n_1 \leq 99$, the selected constant M is sufficiently large to ensure that Theorem 2.9 remains applicable. By Lemma 2.8, we get

$$l(\mathfrak{L}, v) \geq c_1 := 7.32 \cdot 10^{35} \quad \text{and} \quad c_2 = 1.31 \cdot 10^{92},$$

and by Lemma 4.4, we also have

$$x < A_1 := 3 \cdot 10^{34}, \quad y < A_2 := 2.5 \cdot 10^{34} \quad \text{and} \quad n < A_3 := 9.7 \cdot 10^{34}.$$

Thus, Lemma 2.9 yields the same values for S and T as before. Since $c_2^2 \geq T^2 + S$, by choosing $c_3 := 3$ and $c_4 := 1$, we obtain $X - X_1 \leq 100$. Next, returning to relations (4.10) and (4.11), we get

$$x_{\min} \leq v_5\left(\left(\frac{\alpha}{\beta}\right)^{n_1} \cdot \frac{\alpha^{n-n_1} - 1}{\beta^{n-n_1} - 1} + 1\right) + v_5(\beta^{n-n_1} - 1),$$

$$y_{\min} \leq v_7\left(\left(\frac{\alpha}{\beta}\right)^{n_1} \cdot \frac{\alpha^{n-n_1} - 1}{\beta^{n-n_1} - 1} + 1\right) + v_7(\beta^{n-n_1} - 1).$$

Note that, by Lemma 2.7,

$$v_p(\beta^{n-n_1} - 1) \leq 1 + \frac{\log(n - n_1)}{\log p} \leq 1 + \frac{\log 99}{\log p} \leq 3, \quad \text{for } p = 5, 7.$$

Assume that $n - n_1$ is even. Then, by Lemma 2.4, we have

$$\begin{aligned} v_p\left(\left(\frac{\alpha}{\beta}\right)^{n_1} \cdot \frac{\alpha^{n-n_1} - 1}{\beta^{n-n_1} - 1} + 1\right) &= v_p(\pm\alpha^{n+n_1} + 1) \\ &< 1 + \frac{\log(2n)}{\log p} \\ &< 1 + \frac{\log(2 \cdot 9.7 \cdot 10^{34})}{\log p} \\ &< \begin{cases} 52 & \text{if } p = 5, \\ 43 & \text{if } p = 7. \end{cases} \end{aligned}$$

Which gives

$$x_{\min} \leq 55 \quad \text{and} \quad y_{\min} \leq 46.$$

Now consider the cases where $n - n_1 \in \{1, 3\}$. Again, applying Lemma 2.4, we obtain

$$\begin{aligned} v_p\left(\left(\frac{\alpha}{\beta}\right)^{n_1}\left(\frac{\alpha-1}{\beta-1}\right)+1\right) &= v_p\left(\left(\frac{\alpha}{\beta}\right)^{n_1}(-\alpha^{-2})+1\right) \\ &= v_p\left(\alpha^{n_1-1}+1\right) \\ &\le 1 + v_p(n_1-1) \\ &< 1 + \frac{\log(n_1-1)}{\log p} \\ &< 1 + \frac{\log n}{\log p} \\ &< \begin{cases} 1 + \dfrac{\log(9.7\cdot 10^{34})}{\log 5} \le 51 & \text{if } p = 5, \\ 1 + \dfrac{\log(9.7\cdot 10^{34})}{\log 7} \le 42 & \text{if } p = 7. \end{cases} \end{aligned}$$

$$\begin{aligned} v_p\left(\left(\frac{\alpha}{\beta}\right)^{n_1}\left(\frac{\alpha^3-1}{\beta^3-1}\right)+1\right) &= v_p\left(\left(\frac{\alpha}{\beta}\right)^{n_1}(-\alpha^{2})+1\right) \\ &= v_p\left(\alpha^{n_1+3}+1\right) \\ &\le 1 + v_p(n_1+3) \\ &\le 1 + \frac{\log(n_1+3)}{\log p} \\ &\le 1 + \frac{\log(n+3)}{\log p} \\ &\le \begin{cases} 1 + \dfrac{\log(9.7\cdot 10^{34}+3)}{\log 5} \le 51 & \text{if } p = 5, \\ 1 + \dfrac{\log(9.7\cdot 10^{34}+3)}{\log 7} \le 42 & \text{if } p = 7. \end{cases} \end{aligned}$$

Thus, the upper bounds for $x_{\min}$ and $y_{\min}$ remain unchanged.

Assume now that $n - n_1 \ge 5$ is odd. We explain in detail how we approach this case; that is, we indicate how to bound $v_p(L_n - L_{n_1})$ when $n - n_1$ is odd, $n < 9.7 \cdot 10^{34}$ and $p \in \{5, 7\}$.

We carry out the process explicitly for $p = 5$ and $p = 7$, and then automate it in `Maple`. Note that $n < 9.7 \cdot 10^{34} < 2^{117}$, so n has at most 117 binary digits. Let $d = n - n_1 \le 99$, as established by the reduction above. Therefore, we need an upper bound for $v_5(L_{n+d} - L_n)$ for all odd integers $d \in [5, 99]$, with $n < 9.7 \cdot 10^{34}$. The Lucas sequence is periodic modulo 5^{k+1} with period $4 \cdot 5^k$. In particular, $L_{n+d} - L_n$ is periodic modulo 5^4, with period $4 \cdot 5^3 = 500 < 2000$. We looped over all odd values of $d \in [5, 99]$, checking whether there exists an integer $n < 2000$ such that $5^4 \mid (L_{n+d} - L_n)$. However, no such value of d was found. This implies that for all such d, we have $v_5(L_{n+d} - L_n) \le 3$.
For $p = 7$, the sequence $(L_n)_{n\ge 0}$ has period $16 \cdot 7^k$ modulo 7^{k+1}. In particular, the difference $L_{n+d} - L_n$ is periodic modulo 7^3 with period $16 \cdot 7^2 = 784 < 2000$. We looped over all odd values $d \in [5, 99]$, checking whether there exists an integer $n \le 2000$ such that $7^3 \mid (L_{n+d} - L_n)$. All values of d between 5 and 99 satisfy this condition. Here, we will work out a single value of d for illustrative purposes. Namely, we take $d = 11$. We compute $n_0(d) \in [1, 16 \cdot 7^2]$ such that for $n = n_0(d)$, we have $v_7(L_{n+d} - L_n) \ge 3$. In this case, this value is unique and given by $n_0(d) = 10$. Hence, for every $n \le 9.7 \cdot 10^{34}$ such that $v_7(L_{n+d} - L_n) \ge 3$, we must have $n = 10 + 16 \cdot 7^2 z$ for some integer z. Our goal is to find a value of z such that $v_7(L_{n+d} - L_n)$ is as large as possible. To this end, we now turn to the Binet formula

$$\begin{aligned} L_{n+11} - L_n &= \alpha^{n+11} + \beta^{n+11} - \alpha^n - \beta^n \\ &= (\alpha^{11}-1)\alpha^{10+16\cdot 7^2 z} + (\beta^{11}-1)\beta^{10+16\cdot 7^2 z} \\ &= (\alpha^{11}-1)\alpha^{10}\exp_7(7^2 z \log_7 \alpha^{16}) + (\beta^{11}-1)\beta^{10}\exp_7(7^2 z \log_7 \alpha^{16}). \end{aligned}$$

In the above calculation, $\alpha^{16}-1=7(141\alpha+87)$, so that $|\alpha^{16}-1|_7=7^{-1}$. Therfore,

$$\log_7\alpha^{16}=\log_7(1-(1-\alpha^{16}))=-\sum_{n\geq1}\frac{(1-\alpha^{16})^n}{n}. \tag{4.15}$$

On the right-hand side, we have

$$|(1-\alpha^{16})^n/n|_7=7^{-v_7\left(\frac{(1-\alpha^{16})^n}{n}\right)}=7^{-nv_7(1-\alpha^{16})+v_7(n)}=7^{-n+v_7(n)}\leq7^{-(n-\frac{\log n}{\log 7})}.$$

Which shows that the series appearing on the right-hand side of (4.15) converges. Moreover, by inspecting the first few terms, we obtain

$$1-\alpha^{16}=-7(141\alpha+87)\quad;\quad v_7\left(\frac{(1-\alpha^{16})^n}{n}\right)=n-v_7(n)\geq2\quad\text{for}\quad n\geq2.$$

Which implies that

$$v_7(\log_7(\alpha^{16}))=v_7(-7(141\alpha+87))+\sum_{n\geq2}\frac{(1-\alpha^{16})^n}{7}=v_7(-7)=1.$$

For the argument of the exponential, we have $v_7(7^2z\log_7\alpha^{16})=v_7(7^3z)\geq3$, so $|7^2z\log_7\alpha|_7<7^{-3}<7^{-1}$. Therefore, the exponential series in this term converges 7-adically. The same reasoning applies if α is replaced by β. We now truncate the argument of the logarithm at $n=120$, so that

$$P:=-\sum_{n=1}^{120}\frac{(1-\alpha^{16})^n}{n}, \tag{4.16}$$

such that $\log_7\alpha^{16}=P-\sum_{n\geq121}\frac{(1-\alpha^{16})^n}{n}$. On checks that $n-v_7(n)\geq121$ for all $n\geq121$. Indeed, first $n-v_7(n)\geq n-\frac{\log n}{\log 7}$. The function $n-\frac{\log n}{\log 7}$ is at least 121 for all $n\geq124$. For $n\in[121,123]$ it is easy to verify by computation that $n-v_7(n)\geq121$.
Thus, $\log_7\alpha^{16}=P+u$, where $v_7(u)\geq121$. We therefore have

$$7^2z\log_7\alpha^{16}=7^2zP+7^2zu,$$

such that

$$\exp_7(7^2z\log_7\alpha^{16})=\exp_7(7^2zP+7^2zu)=\exp(7^2zP)\exp(7^2zu).$$

We have

$$\exp_7(x)=1+x+\frac{x^2}{2}+\cdots+\frac{x^n}{n!}+\cdots.$$

For $v_7(x)\geq2$ and $n\geq2$ we have

$$v_7\left(\frac{x^n}{n!}\right)=nv_7(x)-v_7(n!)\geq nv_7(x)-(n-\sigma_2(n))\geq n(v_7(x)-1)\geq v_7(x),$$

where the last inequality holds as it is equivalent to $v_7(x)\geq2\geq\frac{n}{n-1}$ for all $n\geq2$. In the above $\sigma_2(n)$ denotes the sum of the digits of n in base 2. It then follows that

$$\exp_7(x)\equiv1\mod 7^{v_7(x)},\quad\text{provided}\quad v_7(x)\geq2.$$

Hence,

$$\exp_7(7^2zu)\equiv1\mod 7^{2+v_7(u)}\equiv1\mod(7^{123}).$$

This means that

$$\begin{aligned}\exp_7(7^2 z \log_7 \alpha^{16}) &\equiv \exp_7(7^2 zP) \mod 7^{123}\\ &\equiv \textstyle\sum_{k\geq 0} \dfrac{(7^2zP)^k}{k!} \mod 7^{123}.\end{aligned}$$

Since $1 = v_7(\log_7 \alpha^{16}) = v_7(P+u)$ and $v_7(u) \geq 121$ is large, we get that $v_7(P) = 1$. Further,

$$\begin{aligned}v_7\left(\frac{(7^2zP)^k}{k!}\right) &= kv_7(7^2zP) - v_7(k!)\\ &\geq (2+v_7(P))k - (k - \sigma_2(k))\\ &> 2k.\end{aligned}$$

Since $\sigma_2(k) \geq 1$ and $v_7(P) \geq 1$, it follows that the quantities above are at least $2 \cdot 62 = 124 > 123$ for $k \geq 62$. Therefore, we may truncate the series at $k = 61$, and write

$$\exp_7(7^2 z \log_7 \alpha^{16}) \equiv \sum_{k=0}^{61} \frac{(7^2zP)^k}{k!} \pmod{7^{123}}.$$

The same reasoning applies when α is replaced by β, so we can write

$$Q = -\sum_{n=1}^{120} \frac{(1-\beta^{16})}{n} \tag{4.17}$$

and them

$$\exp_7(7^2 z \log_7 \beta^{16}) \equiv \sum_{k=0}^{61} \frac{(7^2zQ)^k}{k!} \pmod{7^{123}}.$$

Thus,

$$L_{n+11} - L_n = \sum_{k=0}^{61} \frac{(\alpha^{11}-1)\alpha^{10}(7^2zP)^k + (\beta^{11}-1)\beta^{10}(7^2zQ)^k}{k!} \pmod{7^{123}}.$$

The expression on the right-hand side is a polynomial of degree 61 in z whose coefficients are rational numbers that are 7-adic integers (that is, the numerators of these rational numbers are never divisible by 7). We will show that, within the considered range, this expression is never congruent to 0 (mod 7^{123}). Consequently, it follows that

$$v_7(L_{n+11} - L_n) < 123 \quad \text{for } n < 9.7 \cdot 10^{34}.$$

Finding these numbers is not straightforward in `Maple`, since P and Q involve large powers of α and β. Nevertheless, we can compute $A := P + Q$ and $B := PQ$. Next, the coefficients

$$u_k := (\alpha^{11}-1)\alpha^{10}P^k + (\beta^{11}-1)\beta^{10}Q^k \tag{4.18}$$

form a linearly recurrent sequence satisfying

$$u_{k+2} = Au_{k+1} - Bu_k, \quad k \geq 0,$$

with initial values u_0 and u_1 obtained from (4.18) for $k = 0$ and $k = 1$, respectively. Hence, all remaining terms can be computed iteratively, allowing us to consider the polynomial

$$f(z) := \sum_{k=0}^{61} (7^2z)^k \frac{u_k}{k!} \pmod{7^{123}}.$$

All coefficients $\frac{u_k}{k!}$ are 7-adic integers, so they can be reduced modulo 7^{123}. At this stage, we obtain a polynomial in $\mathbb{Z}/(7^{123}\mathbb{Z})[z]$, and our goal is to find z such that this polynomial vanishes modulo 7^{123}. We approach this iteratively: starting with 7^2z, we reduce $f(z)$ modulo $7^3, 7^4, 7^5, 7^6$, and so on, determining the corresponding digits of z modulo each successive power of 7 (from 0 to 6) so that the polynomial becomes divisible by increasingly higher powers of 7. This procedure is essentially an application of Hensel's Lemma. Following this method, we obtain

$$z = 3 + 2 \cdot 7 + 7^2 + 4 \cdot 7^3 + \cdots$$

up to 7, which can be written explicitly as

$$z = 3 + 2 \cdot 7^1 + 7^2 + 4 \cdot 7^3 + \cdots + 7^{31} + 2 \cdot 7^{34} + 7^{35} + \cdots + 5 \cdot 7^{114} t.$$

Reducing $f(z)$ modulo 7^{122} gives

$$f(z) \equiv 7^{120}(7 + 20t) \pmod{7^{122}}.$$

Choosing t to be a multiple of 7 leads to

$$n \geq 10 \cdot 7^2(\ldots 7^{39} \cdot 7) = 10 \cdot 7^{42} > 9.7 \cdot 10^{34}.$$

This argument shows that, in effect, $v_7(L_{n+11} - L_n) < 121$. Hence, in all cases, we conclude that

$$x_{\min} \leq 55 \quad \text{and} \quad y_{\min} \leq 124.$$

Next, we derive a sharper upper bound for n. Let b_X denote the upper bound of $X - X_1$. Then

$$X = X_2 + (X_1 - X_2) + (X - X_1) < x_{\min} \log 5 + y_{\min} \log 7 + 2b_X,$$
$$x \log 5 + y \log 7 < 55 \log 5 + 124 \log 7 + 2 \cdot 100 < 530.$$

From this we deduce $x \leq 329$ and $y \leq 272$. On the other hand, Corollary 2.1 implies

$$n \log \alpha + \log 0.38 < X < 530,$$

so that

$$n < \frac{530 - \log 0.38}{\log \alpha} < 1890.$$

This contradicts the assumption $n > 2000$. Thus, Theorem 1.2 is proved.

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References

1. S. S. Pillai, *On $a^x - b^y = c$*, J. Indian Math. Soc. (N.S.) **2** (1936), 119–122.
2. H. Batte, M. Ddamulira, J. Kasozi and F. Luca, *On the multiplicity in Pillai's problem with Fibonacci numbers and powers of a fixed prime*, Glas. Mat. Ser. III **57** (2022), no. 2, 185–201.
3. J. J. Bravo, F. Luca and K. Yazán, *On Pillai's problem with Tribonacci numbers and powers of* 2, Bull. Korean Math. Soc. **54** (2017), no. 3, 1069–1080.
4. K. C. Chim, I. Pink and V. Ziegler, *On a variant of Pillai's problem II*, J. Number Theory **183** (2018), 269–290.
5. M. Ddamulira and F. Luca, *On the problem of Pillai with k-generalized Fibonacci numbers and powers of* 3, Int. J. Number Theory **16** (2020), no. 7, 1643–1666.
6. H. Batte, *Lucas numbers that are palindromic concatenations of two distinct repdigits*, arXiv:2401.05361 (2023).
7. V. Ziegler, *On a variant of Pillai's problem involving S-units and Fibonacci numbers*, Bol. Soc. Mat. Mex. **28** (2022), no. 3, 57.
8. V. Ziegler, *On a variant of Pillai's problem with binary recurrences and S-units*, Int. J. Number Theory **19** (2023), no. 7, 1473–1511.
9. M. Ddamulira, *On a problem of Pillai with Fibonacci numbers and powers of* 3, Bol. Soc. Mat. Mex. **26** (2020), no. 2, 263–277.
10. A. Baker and G. Wüstholz, *Logarithmic forms and group varieties*, in: Number Theory, Springer, Berlin (1993), 19–62.

11. E. M. Matveev, *An explicit lower bound for a homogeneous rational linear form in the logarithms of algebraic numbers II*, Izv. Math. **64** (2000), no. 6, 1217–1269.

12. A. K. Lenstra, H. W. Lenstra Jr. and L. Lovász, *Factoring polynomials with rational coefficients*, Math. Ann. **261** (1982), no. 4, 515–534.

13. B. M. M. de Weger, *Solving exponential Diophantine equations using lattice basis reduction algorithms*, J. Number Theory **26** (1987), no. 3, 325–367.

14. J. W. S. Cassels, *An introduction to the geometry of numbers*, Springer, 1996.

15. A. Pethö and B. M. M. de Weger, Products of prime powers in binary recurrence sequences. I. The hyperbolic case, with an application to the generalized Ramanujan–Nagell equation. Math. Comp. **47** (1986), no. 176, 713–727,

16. G. H. Hardy and E. M. Wright, *An introduction to the theory of numbers*, Oxford Univ. Press, 1979.

17. Y. Bugeaud, M. Mignotte and S. Siksek, *Classical and modular approaches to exponential Diophantine equations I*, Ann. of Math. **163** (2006), 969–1018.

18. H. Batte, M. Ddamulira, J. Kasozi and F. Luca, *On a problem of Pillai involving S-units and Lucas numbers*, Period. Math. Hungar. (2025), to appear.

19. M. Ddamulira, F. Luca and M. Rakotomalala, On a problem of Pillai with Fibonacci numbers and powers of 2. Proc. Math. Sci., 127(3): 411–421 (2017).

20. S. Guzmán Sánchez and F. Luca, Linear combinations of factorials and S-units in a binary recurrence sequence. Ann. Math. Québec, 38(2): 169–188 (2014).

21. Y. Bugeaud and M. Laurent, Minoration effective de la distance p-adique entre puissances de nombres algébriques. J. Number Theory, 61(2): 311–342 (1996),

22. N. P. Smart, The algorithmic resolution of Diophantine equations: a computational cookbook. Cambridge Univ. Press, Vol. 41 (1998),

23. M. R. Murty and J. Esmonde, Problems in Algebraic Number Theory. Springer, Vol. 190 (2005),

24. H. Batte and F. Luca, Solutions to a Pillai-type equation involving Tribonacci numbers and S-units. Mediterr. J. Math. **21** (2024), no. 5, 159,

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