



T_D Reflexive Spaces

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ABSTRACT: In this paper, we characterize strongly open subsets of a reflexive space and show that there is a bijection from the set \mathbf{M}_B of all strongly open subsets of a reflexive space B onto the set $\mathbf{Eq}(B)$ of all equivalence relations on B . Moreover, we characterize each of T_D , T_1 and T_2 reflexive spaces and compare them.

Keywords: Topological category, T_D space, strongly open, reflexive space.

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1. Introduction

In 1969, Aull and Thron [3] introduced the T_D topological spaces, which are important in the theory of non-Hausdorff spaces. In 1979, Hoffmann [13] has defined T_D space in terms of irreducible filters. In 2025, sub-Hausdorff spaces were studied by Denniston and others [11] and in 2024, Abulghalwa [1] extended T_D space to the topological category.

The notion of strong closedness in topological categories was defined in [9] by using initial, final and discrete structures which are available in a topological category and used it to generalize each of compact, sober, $T_i, i = 1, 2, 3, 4$ objects in topological categories [4,7].

The aim of this paper is:

1. to characterize strongly closed and open subsets of a reflexive space and show that there is a bijection from the set \mathbf{M}_B of all strongly open subsets of a reflexive space B onto the set $\mathbf{Eq}(B)$ of all equivalence relations on B .
2. to examine the relationships between T_D reflexive space and each of T_0 , $PreT_2$, T_1 and T_2 reflexive spaces as well as compare them.

2. Preliminaries

Let \mathcal{B} be a category with finite products and cokernel pairs (i.e., given any morphism $f : A \rightarrow B$ in \mathcal{B} , the pushout of f with itself) and $B \in Ob(\mathcal{B})$. We denote by B^n , the product of B with itself n times and by $\pi_j : B^n \rightarrow B$ the j th projection morphism, $j = 1, 2, \dots, n$. If $A \in Ob(\mathcal{B})$ and $f_i : A \rightarrow B$ are morphisms in \mathcal{B} , then there exists a unique morphism $f = (f_1, f_2, \dots, f_n) : A \rightarrow B^n$ such that $\pi_i f = f_i$ for $i = 1, 2, \dots, n$. Let $1_B : B \rightarrow B$ be the identity morphism and $f : B \rightarrow B$ be the constant morphism,

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2020 *Mathematics Subject Classification*: 54B30, 54D10, 54A05, 54A20.

Submitted January 06, 2026. Published February 14, 2026

i.e, there exists a morphism $x : 1 \rightarrow B$ such that $x \circ h = f$, where $h : B \rightarrow 1$ is the unique morphism from B to the terminal object 1 . Define $p_1, \nabla_x : B \vee_x B \rightarrow B$ to be $f + 1_B$ and $1_B + 1_B$ respectively, where $B \vee_x B$ denotes the cokernel of $x : 1 \rightarrow B$ along itself, i.e., if i_1 and $i_2 : B \rightarrow B \vee_x B$ denote the inclusion of B as the first and second factor, respectively, then $i_1 \circ x = i_2 \circ x$ is a pushout diagram.

$$\begin{array}{ccc} 1 & \xrightarrow{x} & B \\ x \downarrow & & \downarrow i_2 \\ B & \xrightarrow{i_1} & B \vee_x B \end{array}$$

Note that for morphisms $(f, 1_B) : B \rightarrow B^2$ and $(1_B, 1_B) : B^2 \rightarrow B^2$, one has $(f, 1_B) \circ x = (x, x) = (1_B, 1_B) \circ x$ and consequently $S_x = (p_1, \nabla_x) : B \vee_x B \rightarrow B^2$ is the unique morphism called the skewed x -axis morphism for which $S_x \circ i_2 = (f, 1_B)$ and $S_x \circ i_1 = (1_B, 1_B)$, i.e.,

$$\begin{array}{ccc} 1 & \xrightarrow{x} & B \\ x \downarrow & & \downarrow i_2 \\ B & \xrightarrow{i_1} & B \vee_x B \end{array} \begin{array}{l} \xrightarrow{(1_B, 1_B)} \\ \xrightarrow{S_x = (p_1, \nabla_x)} \\ \xrightarrow{(f, 1_B)} \end{array} B^2$$

Figure 1: Pushout Diagram

Note that $\pi_1 \circ S_x = p_1$ and $\pi_2 \circ S_x = \nabla_x$ [12].

Similarly, $1_B \circ x = x$ and consequently, there exists a unique morphism $\nabla_x : B \vee_x B \rightarrow B$ called the fold morphism at x and $\nabla_x i_k = 1_B$ for $k = 1, 2$ [12].

Let B be a set, $x \in B$, and the wedge $B \vee_x B$ be taking two distinct copies of B and identifying them at the point x . A point a in $B \vee_x B$ will be denoted by a_1 (resp. a_2) if a is in the first (second) component of $B \vee_x B$. The skewed x -axis map $S_x : B \vee_x B \rightarrow B^2$ is given by $S_x(a_1) = (a, a)$ and $S_x(a_2) = (x, a)$. The fold map at x , $\nabla_x : B \vee_x B \rightarrow B$ is given by $\nabla_x(a_i) = a$ for $i = 1, 2$ [4].

A functor $U : \mathcal{E} \rightarrow \mathcal{B}$ is said to be topological or \mathcal{E} is a topological category over \mathcal{B} if and only if the following conditions hold:

1. U is concrete, i.e., faithful (U is mono on hom sets) and amnestic (if $U(f) = id$ and f is an isomorphism, then $f = id$) [15].
2. U has small fibers, i.e., $U^{-1}(b)$ is a set for all b in \mathcal{B} .

3. For every U -source, i.e., family $g_i : b \rightarrow U(X_i)$ of maps in **Set**, the category of sets and functions, there exists a family $f_i : X \rightarrow X_i$ in \mathcal{E} such that $U(f_i) = g_i$ and if $U(h_i : Y \rightarrow X_i) = g_{i_k} : UY \rightarrow b \rightarrow U(X_i)$, then there exists a lift $\bar{k} : Y \rightarrow X$ of $k : UY \rightarrow UX$, i.e., $U(\bar{k}) = k$. This latter condition means that every U -source has an initial lift. It is well known that the existence of initial lifts of arbitrary U -source is equivalent to the existence of final lifts (the dual of the initial lifts) for arbitrary U -sink [2].

A topological functor $U : \mathcal{E} \rightarrow \mathcal{B}$ is said to be normalized if the constant objects, i.e., subterminals, have a unique structure [9]. Z is called a subspace of X if there exists monomorphism $i : Z \rightarrow X$ that is an initial lift (i.e., an embedding) and we denote it by $Z \subset X$.

3. Strongly Open Subsets Of Reflexive Spaces

In this section, we characterize strongly closed and open subsets of a reflexive space and show that there is a bijection from the set \mathbf{M}_B of all strongly open subsets of a reflexive space B onto the set $\mathbf{Eq}(B)$ of all equivalence relations on B .

Let \mathcal{B} be a category with finite products, a terminal object, and pushout. Let $U : \mathcal{E} \rightarrow \mathcal{B}$ be a normalized topological functor, x be the subterminal, $X \in \text{Ob}(\mathcal{E})$ with $U(X) = B$ and $Z \subset X$.

Definition 3.1 1. If the initial lift of the U -source $S_x : B \vee_x B \rightarrow U(X^2)$ and $\nabla_x : B \vee_x B \rightarrow UD(B)$ is discrete, then X is called T_1 at x [9].

2. If X/Z is T_1 at 1, then Z is called a strongly closed subobject of X , where X/Z is the pushout of $i : Z \rightarrow X$ with $1 : Z \rightarrow 1$ [9].

Remark 3.1 1. For $\mathcal{E} = \mathbf{Top}$, the category of topological spaces and continuous functions, and $\mathcal{B} = \mathbf{Set}$ if Z^C the complement of Z is strongly closed, then Z is said to be strongly open [4]. Note that $X \setminus Z$ is the final lift of the epi U -sink $\mathcal{Q} : U(X) = B \rightarrow B \setminus Z = (B \setminus Z) \cup 1$ identifying Z with 1 for $Z \neq \emptyset$, the initial object in **Set**. If $Z = \emptyset$, then $B \setminus Z = B \amalg 1$ [9].

2. A topological space is T_1 at x if and only if for any distinct point y from x , there is a neighborhood of each missing the other [4,5]. If a topological space is T_1 , then the notion strong openness (resp. closedness) coincides with the usual openness (resp. closedness) [4,6].
3. Note that a notion of closedness at the level of set-based topological categories was defined in [5]. In **Top**, if a space is T_1 , then the notions of closedness and strong closedness coincide. The notions of closedness and strong closedness may coincide [12]. By Theorem 3.1, 3.2, 3.9, and 3.10 of [5], these notions are independent of each other in a topological category, in general.

The category **RRel** of reflexive relation spaces has as objects (A_1, R) , where R is a reflexive relation on a set A_1 and as morphisms $f : (A_1, R) \rightarrow (B_1, S)$ are relation preserving functions, i.e, if sRt , then $f(s)Sf(t)$ for all $s, t \in A_1$. **RRel** is a topological category [17,10].

Proposition 3.1 1. An epimorphism $f : (A_1, R) \rightarrow (B_1, S)$ is final in **RRel** if and only if for all $s, t \in B_1$, sSt holds in B_1 precisely when there exist $u, v \in A_1$, such that uRv and $f(u) = s$ and $f(v) = t$ [17].

2. A source $f_i : (A_1, R) \rightarrow (B_i, R_i), i \in I$ is initial in **RRel** if and only if for all $u, v \in A_1$, uRv iff $f_i(u)R_i f_i(v)$ for all $i \in I$ [17].

Theorem 3.1 Let (A, R) be a reflexive space, $x \in A$, and Z be a nonempty subset of A .

1. (A, R) is T_1 at x if and only if for each $s \in A$ if sRx or xRs , then $s = x$.
2. The following are equivalent:

- i. Z is strongly closed.
- ii. For each $s \in A$ if there exist $t \in Z$ such that sRt or tRs , then $s \in Z$.
- iii. Z is strongly open.

Proof:

1. The proof of Part (1) is similar to the proof of Lemma 3.5 and Theorem 3.6 of [6].
2. (i) \Rightarrow (ii). Suppose Z is a non-empty strongly closed subset and for each $s \in A$ there exists $t \in Z$ such that sRt or tRs . Let R' be the final structure on A/Z induced by the epi map $Q : U(A, R) = A \rightarrow A/Z = (A \setminus Z) \cup \{*\}$, identifying Z with a point $*$. If sRt , then $Q(s)RQ(t) = *$ and since $(A/Z, R')$ is T_1 at $*$, then $Q(s) = *$, i.e., $s \in Z$.
If tRs , then $* = Q(t)RQ(s)$ and since $(A/Z, R')$ is T_1 at $*$, $Q(s) = *$, i.e., $s \in Z$.

(ii) \Rightarrow (iii). Suppose (ii) holds. Let R'' be the final structure on A/Z^C induced by the epi map $q : U(A, R) = A \rightarrow A/Z^C = (A \setminus Z^C) \cup \{w\}$, identifying Z^C with a point w and for each $a \in A/Z^C$, $aR''w$ or $*R''w$. There exist $s \in A$ and $t \in Z^C$ such that sRt or tRs with $q(s) = a$ and $w = q(t)$. By assumption, $s \in Z^C$ and thus, $a = q(s) = w$, i.e., $(A/Z^C, R'')$ is T_1 at w . Hence, Z^C is strongly closed and Z is strongly open.

(iii) \Rightarrow (i). Suppose Z is strongly open and Let R' be the final structure on A/Z induced by the epi map $Q : U(A, R) = A \rightarrow A/Z = (A \setminus Z) \cup \{*\}$, identifying Z with a point $*$ and for each $a \in A/Z$, $aR'*$ or $*R'a$. If $aR'*$, then there exist $s \in A$ and $t \in Z$ such that sRt with $Q(s) = a$ and $* = Q(t)$. Let R'' be the final structure on A/Z^C induced by the epi map $q : U(A, R) = A \rightarrow A/Z^C = (A \setminus Z^C) \cup \{w\}$, identifying Z^C with a point w . sRt implies $q(s)R''q(t) = t$. If $s \in Z^C$, then $w = q(s)R''t$ and thus, $w = t$, a contradiction since Z is strongly open, i.e., $(A/Z^C, R'')$ T_1 at w . Hence, $s \in Z$ and $* = Q(s) = a$.
Similarly, if $*R'a$, then we get $a = *$ and by (1), $(A/Z, R')$ is T_1 at $*$. Hence, by Definition 3.1, Z is strongly closed.

□

Let (A, R) be a reflexive space and $M \subset X$. The up-closure of M is given by $\uparrow_X(M) = \{x \in A : \exists a \in M, aRx\}$ and the down-closure of M given by $\downarrow_X(M) = \{x \in A : \exists a \in M, xRa\}$ [10]. Note that (A, R) be a reflexive space and $M \subset A$. M is strongly closed if and only if M is both upper closed ($\uparrow M = M$) and lower closed ($\downarrow M = M$)(Theorem 8.1.1, [10]).

Examples :

1. Let $A = \{a, b\}$, $R_1 = \{(a, a), (b, b)\}$, $R_2 = A \times A$, $R_3 = \{(a, a), (b, b), (a, b)\}$ and $R_4 = \{(a, a), (b, b), (b, a)\}$ be relations on A .
 - i. By Theorem 3.1, all subsets of (A, R_1) are strongly closed (open).
 - ii. the only strongly closed (open) subset of (A, R_i) , $i = 2, 3, 4$ are \emptyset and A . By Theorem 3.1, $\{a\}$ is not strongly closed since $b \in A$, aR_ib and $b \notin \{a\}$. Similarly, $\{b\}$ is not strongly closed (open).
2. By Theorem 3.1, \emptyset and A are both strongly open and strongly closed subset of a reflexive space (A, R) .
3. Every topological space is reflexive space. Indeed, let (A, τ) be a topological space and $a, b \in A$. aRb if and only if every open set containing a contains b , i.e., $a \in \overline{\{b\}} = \downarrow b$. R is called specialization

relation. If R is transitive, then the reflexive space (A, R) is a topological space. Note that $\mathcal{B} = \{\uparrow a : a \in A\}$ is a basis for A and $\tau_{\mathcal{B}}$ is a topology on A induced from \mathcal{B} [10,14].

A reflexive space may not be a topological space. Take $A = \{a, b, c\}$ and $R = \{(a, a), (b, b), (c, c), (a, c), (c, b)\}$. If (A, R) were a topological space then the topology would be $\tau_R = \{\emptyset, A, \{b\}, \{b, c\}, \{a, c\}\}$. But the specialization order $R_{\tau_R} = \{(a, a), (b, b), (c, c), (a, c)\}$ which is not equal to R , a contradiction.

Theorem 3.2 *Let (B, R) be a reflexive space, $\mathbf{M}_{\mathbf{B}} = \{\mathcal{K} : M \in \mathcal{K} \text{ if and only if } M \text{ is a strongly open subset of } B\}$, and $\mathbf{Eq}(\mathbf{B})$ be the set of all equivalence relations on B . Then there is a bijection from $\mathbf{M}_{\mathbf{B}}$ onto $\mathbf{Eq}(\mathbf{B})$.*

Proof: Let (B, R) be a reflexive space and $a_R = \{b \in B : aRb\}$ be the equivalence class of $a \in B$. Define functions $f : \mathbf{Eq}(\mathbf{B}) \rightarrow \mathbf{M}_{\mathbf{B}}$ by $f(R) = B/R = \{a_R : a \in B\}$. We show that $B/R \in \mathbf{M}_{\mathbf{B}}$. Suppose for each $s \in B$ there exist $t \in a_R$ for some $a \in B$ such that sRt or tRs . Since tRa , sRt , and R is the equivalence relation on B , we have sRa , i.e., $s \in a_R$. By Theorem 3.1, a_R is a strongly open and thus, $B/R \in \mathbf{M}_{\mathbf{B}}$.

Let $\mathcal{K} \in \mathbf{M}_{\mathbf{B}}$ and define the function $f : \mathbf{M}_{\mathbf{B}} \rightarrow \mathbf{Eq}(\mathbf{B})$ by $g(\mathcal{K}) = R_{\mathcal{K}}$ where $aR_{\mathcal{K}}b$ for all $a, b \in B$ if and only if $\{a, b\} \subset M$ for some $M \in \mathcal{K}$. Note that $R_{\mathcal{K}}$ is reflexive and symmetric. Suppose $aR_{\mathcal{K}}b$ and $bR_{\mathcal{K}}c$ for $a, b, c \in B$. $aR_{\mathcal{K}}b$ implies $\{a, b\} \subset M$ for some $M \in \mathcal{K}$. Since M is strongly open, $bR_{\mathcal{K}}c$, and $b \in M$, by Theorem 3.1, $c \in M$. Hence, $\{a, c\} \subset M$, i.e., $aR_{\mathcal{K}}c$, i.e., $R_{\mathcal{K}}$ is transitive, i.e., $R_{\mathcal{K}} \in \mathbf{Eq}(\mathbf{B})$. Observe that a_R is a strongly open and $M = a_R$.

Finally, we need to verify that $g \circ f = id_{\mathbf{Eq}(\mathbf{B})}$ and $f \circ g = id_{\mathbf{M}_{\mathbf{B}}}$. Let $R \in \mathbf{Eq}(\mathbf{B})$ and $(g \circ f)(R) = g(B/R) = S_{B/R}$. Then for each $a, b \in B$, $aS_{B/R}b$ if and only if $\{a, b\} \subset c_R$ for some $c_R \in B/R$ if and only if cRa , cRb , and so, aRb since R is the equivalence relation on B which shows that $g \circ f = id_{\mathbf{Eq}(\mathbf{B})}$.

Let $\mathcal{K} \in \mathbf{M}_{\mathbf{B}}$ and $(f \circ g)(\mathcal{K}) = f(R_{\mathcal{K}}) = B/R_{\mathcal{K}}$. Then $a_{R_{\mathcal{K}}} \in B/R_{\mathcal{K}}$ if and only if $bR_{\mathcal{K}}c$ for all $b, c \in B$ with $\{b, c\} \subset a_{R_{\mathcal{K}}}$ for some $a \in B$ if and only if $a_{R_{\mathcal{K}}} \in \mathcal{K}$ ($a_{R_{\mathcal{K}}}$ is strongly open). Hence, $B/R_{\mathcal{K}} = \mathcal{K}$ and $f \circ g = id_{\mathbf{M}_{\mathbf{B}}}$. \square

4. ST_D Reflexive Spaces

In this section, we characterize each of T_D , T_1 and T_2 reflexive spaces and compare them.

Recall that a topological space (B, τ) is called T_D if there exists an open set U containing x with $U \setminus \{x\}$ is open for all $x \in B$ [3]. A topological space is $preT_2$ (pre-Hausdorff), for any two distinct points, if there is a neighborhood of one missing the other, then the two points have disjoint neighborhoods [4]. A point (x, y) in $B^2 \vee_{\Delta} B^2$ is denoted by $(x, y)_1$ (resp. $(x, y)_2$) if (x, y) is in the first (resp. second) component of $B^2 \vee_{\Delta} B^2$. Note that $S(x, y)_1 = (x, y, y)$, $S(x, y)_2 = (x, x, y) = A(x, y)_2$, $A(x, y)_1 = (x, y, x)$, and $\nabla(x, y)_i = (x, y)$ for $i = 1, 2$ [4].

By Theorem 6.2.6 of [10], (B, τ) is $preT_2$ if and only if the initial topology of the skewed axis map $S : B^2 \vee_{\Delta} B^2 \rightarrow (B^3, \tau_*)$ and initial topology of the principle axis map $A : B^2 \vee_{\Delta} B^2 \rightarrow (B^3, \tau_*)$ coincide.

By Theorem 6.2.6 of [10], (B, τ) is T_0 if and only if initial topology of the principle axis map $A : B^2 \vee_{\Delta} B^2 \rightarrow (B^3, \tau_*)$ and the fold map $\nabla : B^2 \vee_{\Delta} B^2 \rightarrow (B^2, P(B^2))$ is discrete.

Definition 4.1 1. Let $U : \mathcal{E} \rightarrow \mathbf{Set}$ be a topological functor and $A \in Ob(\mathcal{E})$. A is called ST_D if there exists a strongly open set U containing x such that $U \setminus \{x\}$ is strongly open for all $x \in A$.

2. A is called ST_1 if $\{x\}$ is strongly closed for all $x \in A$.

3. A is called ST_2 if the diagonal Δ is strongly closed.

If a topological space is T_1 then, by Remark 3.1, the notion of ST_D and ST_1 coincide with usual T_D and T_1 , respectively. Moreover, a topological space is ST_2 if and only if it is T_2 (Hausdorff) [4].

Theorem 4.1 *For a reflexive space (A, R) , the followings are equivalent:*

- i. (A, R) is ST_1 .
- ii. (A, R) is ST_D .
- iii. $R = \Delta$.
- iv. (A, R) is ST_2 .

Proof: (i) \Rightarrow (ii). Suppose there exists a strongly open set U containing x . Since (A, R) is ST_1 , $\{x\}$ is strongly closed for all $x \in A$. Then $U \setminus \{x\}$ is strongly open for all $x \in A$. Hence (A, R) is ST_D .

(ii) \Rightarrow (iii). Let $x, y \in A$ and xRy with $x \neq y$. Since $y \in A \setminus \{x\}$, xRy and $A \setminus \{x\}$ is strongly open. By Theorem 3.1, $x \in A \setminus \{x\}$, a contradiction. Hence, $x = y$ and $R = \Delta$.

(iii) \Rightarrow (iv). Suppose $R = \Delta$, then by Theorem 3.1, all subsets of A are strongly closed. Let $(x, y)R^2(t, t)$ or $(t, t)R^2(x, y)$ for all $x, y, t \in A$. If $(x, y)R^2(t, t)$ then, by Proposition 3.1, xRt and yRt . Since $R = \Delta$ we have $x = y = t$, i.e., $(x, y) \in \Delta$. Hence by Theorem 3.1, $\Delta \subset A^2$ is strongly closed and (A, R) is ST_2 .

(iv) \Rightarrow (i). Let $x \in A$ and xRy . By Proposition 3.1, $(x, y)R^2(y, y)$. Since $\Delta \subset A^2$ is strongly closed, by Theorem 3.1, $(x, y) \in \Delta$, i.e., $x = y$ and $\{x\}$ is strongly closed. Hence (A, R) is ST_1 . □

Examples : Let $A = \{a, b\}$, $R_1 = \{(a, a), (b, b)\}$, $R_2 = A \times A$, $R_3 = \{(a, a), (b, b), (a, b)\}$ and $R_4 = \{(a, a), (b, b), (b, a)\}$ be relations on A . By Theorem 3.1,

- iii. (A, R_1) is ST_1 , ST_2 and ST_D .
- iv. For $i = 2, 3, 4$, (A, R_i) is not ST_D , because $a \in U = A$ strongly open but by Theorem 3.1, $U \setminus \{a\} = \{b\}$ is not strongly open.

5. Comments

In **Top**,

1. If (A, τ) finite topological space then by Proposition 1.5.4 in [16] and by Theorem 8.2.2 in [10] the partitions of A are in one-to-one correspondence with $preT_2$ topologies on A .
2. If (A, τ) Alexandroff space, i.e., the intersection of any family of open sets is open, the partitions of A are in one-to-one correspondence with $preT_2$ topologies on A [10].
3. By Theorem 4.3 in [8] and Remark 8.7.1.6 in [11], in the realm of $preT_2$ objects each of T_0 , T_1 , T_2 and T_D are equivalent.
4. There is no relations between $preT_2$ and each of T_0 , T_1 spaces.
5. By Proposition 8.7.1.4 of [11] and Theorem 6.2.4 of [10], $T_2 \Rightarrow T_1 \Rightarrow T_D \Rightarrow T_0$.
6. If the topological space T_1 then, $T_D = ST_D$.
7. By (5) and (6), $T_2 = ST_2 \Rightarrow ST_1 = T_1 \Rightarrow ST_D \Rightarrow T_D$.
8. By (2), the full subcategory **AlexTop** comprising all Alexandroff topological spaces and the full subcategory **preT₂Top** comprising all $preT_2$ topological spaces are isomorphic.

9. By (7), the full subcategory $T_2\mathbf{Top}$ comprising all Hausdorff topological spaces and the full subcategory $ST_2\mathbf{Top}$ comprising all ST_2 topological spaces are isomorphic. Furthermore, $T_2\mathbf{Top} \subset ST_1\mathbf{Top} \subset ST_D\mathbf{Top}$.

In \mathbf{RRel} ,

1. By Theorem 3.2, a reflexive space (A, R) has a partition consisting of strongly open subsets of A .
2. By Theorem 4.1, $ST_2 = ST_1 = ST_D$.
3. By Theorem 3.2 of [7], (A, R) is $preT_2$ if and only if R is an equivalence relation.
4. Let \mathbf{FM}_A be the set of all strongly closed subsets of a reflexive space (A, R) , $\mathbf{Eq}(A)$ be the set of all equivalence relations on A and $preT_2$ be the set of all $preT_2$ reflexive relations on A .
By (1) and (3), $\mathbf{Eq}(A) \simeq preT_2(A) \simeq \mathbf{M}_B \simeq \mathbf{FM}_A$.
5. Let \mathbf{TRRel} be the full subcategory of \mathbf{RRel} comprising all T reflexive spaces, where $T = ST_D, ST_1$ or ST_2 . By (2), the categories $ST_D\mathbf{RRel}$, $ST_1\mathbf{RRel}$ and $ST_2\mathbf{RRel}$ are pairwise isomorphic.

Acknowledgments

The author, Rudrapati Bhuvanewara Prasad, would like to express sincere gratitude to his research guide Prof. A. Mallikarjuna Reddy (Rtd.) for his constant support, valuable guidance, and encouragement throughout the course of this research. The author also extends heartfelt thanks to Dr. G. Shobhalatha, co-supervisor, for her insightful suggestions and continued support. This work was carried out at the Department of Mathematics, Sri Krishnadevaraya University, Ananthapuramu, Andhra Pradesh.

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