



Solvability Analysis of $(k - \ell)$ -Hilfer Fractional Differential Equations through Generalized Weak Wardowski Contractions

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ABSTRACT: In this work, we introduce the concept of generalized weak Wardowski contractions and establish the existence and uniqueness of fixed points for such mappings. Furthermore, we apply weak Wardowski contraction to investigate the existence of solutions for a novel $(k - \ell)$ -Hilfer fractional differential equation of order $2 < \alpha \leq 3$ subject to specific boundary conditions. Finally, an example is provided to illustrate the applicability and effectiveness of the obtained theoretical results.

Key Words: Generalized weak Wardowski contractions, Hilfer fractional derivative, $(k - \ell)$ -Hilfer fractional differential equation, fractional inclusions.

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1. Abbreviations

FD: Fractional derivative
 R-L.f.i.: Riemann–Liouville fractional integral
 FDE: fractional differential equation
 R-L.f.d.: Riemann–Liouville fractional derivative
 K.f.d.: Katugampola fractional derivative
 H.f.d.: Hadamard fractional derivative
 Hilfer f.d.: Hilfer fractional derivative
 R.H.S.: right hand side
 F.P.: fixed point
 M.S.: metric space
 C.M.S.: complete metric space
 $\mathcal{I} = [a, b]$
 $\mathbb{R} = (-\infty, \infty)$

2. Introduction

Fractional calculus (FC) extends classical calculus by focusing on integration and differentiation of non-integer (fractional) orders. The idea of fractional operators emerged almost concurrently with the development of classical calculus. One of the earliest references to the notion of a semi-derivative can be traced back to a correspondence in 1695 between G. W. Leibniz and the Marquis de l’Hospital, wherein

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they discussed its interpretation. Over time, this intriguing subject attracted the attention of numerous renowned mathematicians, including Euler, Liouville, Laplace, Riemann, Grünwald, Letnikov, and others.

Since the 19th century, the rapid development of FC theory has significantly influenced various applied fields in mathematics, including fractional integral equations, fractional differential equations (FDE), etc. Today, FC is utilized across a broad spectrum of disciplines. It can be confidently stated that nearly every branch of modern engineering and related scientific research has, in some capacity, benefited from FC. Notable examples include applications in optics, chemistry and physics, robotics, control theory, electrical and mechanical engineering, and bioengineering. In fact, fractional-order systems often provide more accurate representations of real-world phenomena compared to their integer-order counterparts. For more comprehensive studies on this subject, the reader is referred to [2], [7], [9], [13], [24], [30], and [31]. Among the numerous fractional operators, the Riemann–Liouville and Caputo integro-differential operators have been the most widely utilized in the literature.

The concept of the " ℓ -Caputo fractional derivative" (ℓ -C.f.d.) represents a f.d. defined with respect to another strictly increasing, differentiable function. This innovative approach was first introduced in [4] and subsequently applied in studies such as [8] and [15]. Numerous researchers have since employed this operator across diverse areas (see, for instance, [1], [4], [5], [16], [29], and [32]). Notably, the ℓ -C.f.d. generalizes several classical fractional operators, including those proposed by Caputo, Hadamard, and Erdlyi–Kober. In parallel, the Hilfer f.d., introduced in [14], further expanded the field.

The $(k - \ell)$ -Hilfer operator, presented in [18], offers a broad generalization of many well-known f.d.s. More recently, Tariboon et al. [33] investigated a multi-point boundary value problem (b.v.p.) of $(k - \ell)$ -Hilfer fractional differential type, utilizing Banach's contraction principle and Nadler's contraction to establish key results. Distinctly, they studied the following multi-point b.v.p. of fractional differential inclusion of order $1 < \zeta < 2$:

$$\begin{cases} {}^k\mathfrak{D}_{a^+}^{\zeta, \zeta'; \ell} \theta(w) \in \kappa(w, \theta(w)), w \in \mathcal{I} \\ \theta(a) = 0, \theta(b) = \sum_{i=1}^m \lambda_i \theta(\varepsilon_i), \end{cases} \quad (2.1)$$

where ${}^k\mathfrak{D}_{a^+}^{\zeta, \zeta'; \ell}$ is the $(k - \ell)$ -H-f.d. operator introduced in [18], $\kappa : \mathcal{I} \times (-\infty, \infty) \rightarrow \mathcal{P}((-\infty, \infty))$ is a multi-valued continuous function, $1 < \zeta < 2$, the increasing function $\ell \in C^1(\mathcal{I})$ is such that $\ell'(t) \neq 0$, $t \in \mathcal{I}$, $a < \varepsilon_i < b$, $i = 1, 2, 3, \dots, m$ and $\lambda_i \in (-\infty, \infty)$.

This paper pursues two main objectives: First, we introduce the concept of generalized weak Wardowski contractions and establish the existence of F.P.s for such mappings. Second, we apply this novel contraction to show that the following $(k - \ell)$ -H-f.d. of order $2 < \zeta \leq 3$, subject to the specified boundary value conditions, admits a solution:

$$\begin{cases} {}^{k,H}\mathfrak{D}_{a^+}^{\zeta, \zeta'; \ell} \theta(w) = \kappa(w, \theta(w)); w \in \mathcal{I}, \\ \theta(a) = 0, \\ \theta(b) = d_0, [(\frac{k}{\ell'(w)} \frac{d}{dw}) \theta(w)]_{w=b} = d_1, \end{cases} \quad (2.2)$$

$\zeta' \in [0, 1]$, $k > 0$, $2 < \frac{\zeta}{k} \leq 3$, $\theta_k = \zeta + \zeta'(nk - \zeta)$ when the R.H.S. function $\kappa : \mathcal{I} \times (-\infty, \infty) \rightarrow (-\infty, \infty)$ is a continuous function and the Banach's contraction can not apply necessarily for it, while it leads to generalized weak Wardowski contraction. We present a clear numerical example to better understand the calculations of the relationships between the parameters.

3. Preliminaries and auxiliary notions

The fundamental definitions related to FDEs are summarized below (see [26] and [17] for detailed discussions).

Consider a continuous function $h : \mathcal{I} \rightarrow (-\infty, \infty)$. The R-L.f.i. of order $\zeta \geq 0$ is defined as:

$$\mathfrak{J}_{a^+}^{\zeta} h(w) = \frac{1}{\Gamma(\zeta)} \int_a^w (w - \beta)^{\zeta-1} h(\beta) d\beta; \zeta > 0, \quad (3.1)$$

and $\mathfrak{J}_{a^+}^0 h(w) = h(w)$ where $\zeta = 0$ for agreement.

The Caputo fractional derivative (C.f.d.) of order ζ is formally defined as:

$${}^C\mathfrak{D}_{a^+}^\zeta \hbar(w) = \mathfrak{J}_{a^+}^{n-\zeta} \hbar^{(n)}(w) \quad (n-1 < \zeta \leq n, n = \lceil \zeta \rceil), \quad (3.2)$$

where $\lceil \cdot \rceil : (-\infty, \infty) \rightarrow \mathbb{Z}$ stands for the ceiling function.

The Riemann–Liouville fractional derivative (R–L.f.d.) of order ζ is formally defined as:

$${}^{RL}\mathfrak{D}_{a^+}^\zeta \hbar(w) = \left(\frac{d}{dw}\right)^n \mathfrak{J}_{a^+}^{n-\zeta} \hbar(w) \quad (n-1 < \zeta \leq n, n = \lceil \zeta \rceil). \quad (3.3)$$

Definition 3.1 Suppose that ℓ is an increasing function satisfying $\ell'(s) > 0$ for every $s \in \mathcal{I}$. Then, the ℓ -R-L.f.i. of order ζ for an integrable function $\hbar : \mathcal{I} \rightarrow (-\infty, \infty)$ with respect to ℓ is given by:

$$\mathfrak{J}_{a^+}^{\zeta;\ell} \hbar(w) = \begin{cases} \frac{1}{\Gamma(\zeta)} \int_{a^+}^w \ell'(\beta) (\ell(w) - \ell(\beta))^{\zeta-1} \hbar(\beta) d\beta, & \zeta > 0 \\ \hbar(w), & \zeta = 0, \end{cases} \quad (3.4)$$

provided that the R.H.S. of the given equality takes finite values.

It should be emphasized that the ℓ -R-L.f.i. defined in (3.4) coincides with the standard R-L.f.i. given in (3.1) in the special case where $\ell(\beta) = \beta$.

Definition 3.2 ([15]) Let $n = \lceil \zeta \rceil$. For a real mapping $\hbar \in C(\mathcal{I}, (-\infty, \infty))$, the ℓ -R-L.f.d. of order ζ is formulated as

$${}^{RL}\mathfrak{D}_{a^+}^{\zeta;\ell} \hbar(w) = \left(\frac{1}{\ell'(w)} \frac{d}{dw}\right)^n \mathfrak{J}_{a^+}^{n-\zeta;\ell} \hbar(w), \quad (3.5)$$

provided that the R.H.S. of the above expression is finite.

Similarly, it is clear that the ℓ -R-L.f.d. in (3.5) reduces to the standard R-L.f.d. in (3.3) when $\ell(\beta) = \beta$.

Motivated by these operators, Almeida introduced a generalized ℓ -version of the Caputo fractional derivative (C.f.d.), given in the following form:

Definition 3.3 ([4]) Let $n = \lceil \zeta \rceil$ and $\ell \in AC^n(\mathcal{I}, (-\infty, \infty))$ be an increasing map with $\ell'(\beta) > 0$ for any $\beta \in \mathcal{I}$. The ℓ -C.f.d. of order ζ of \hbar with respect to ℓ is

$${}^C\mathfrak{D}_{a^+}^{\zeta;\ell} \hbar(w) = \mathfrak{J}_{a^+}^{n-\zeta;\ell} \left(\frac{1}{\ell'(w)} \frac{d}{dw}\right)^n \hbar(w), \quad (3.6)$$

provided that the R.H.S. of the aforementioned equality is finite.

It is important to observe that the ℓ -C.f.d. of order ζ in (3.6) coincides with the standard C.f.d. of order ζ in (3.2) when $\ell(\beta) = \beta$.

In the subsequent discussion, we present several useful specifications of the ℓ -Caputo and ℓ -Riemann–Liouville integro-differential operators. We define $AC(\mathcal{I}, (-\infty, \infty))$ as the set of absolutely continuous functions mapping \mathcal{I} to $(-\infty, \infty)$.

Define $AC_\ell^n(\mathcal{I}, (-\infty, \infty))$ by

$$AC_\ell^n(\mathcal{I}, (-\infty, \infty)) = \left\{ w : \mathcal{I} \rightarrow (-\infty, \infty) \mid \delta_\ell^{n-1} w \in AC(\mathcal{I}, (-\infty, \infty)), \delta_\ell = \frac{1}{\ell'(y)} \frac{d}{dy} \right\}.$$

Lemma 3.1 ([15]) Let $n = \lceil \zeta \rceil + 1$. For a real mapping $\hbar \in AC^n(\mathcal{I}, (-\infty, \infty))$,

$$\mathfrak{J}_{a^+}^{\zeta;\ell C} \mathfrak{D}_{a^+}^{\zeta;\ell} \hbar(t) = \hbar(t) - \sum_{j=0}^{n-1} \frac{(\delta_\ell^j \hbar)(a)}{j!} (\ell(t) - \ell(a))^j, \quad (3.7)$$

where $\delta_\ell^j = \underbrace{\delta_\ell \delta_\ell \cdots \delta_\ell}_{j \text{ times}}$.

Lemma 3.2 ([3]) Let $n = [\zeta] + 1$, $\zeta, \zeta' > 0$. For a real mapping $h \in C(\mathcal{I}, (-\infty, \infty))$ we have:

- (i) $\mathfrak{J}_{a^+}^{\zeta; \ell} \mathfrak{J}_{a^+}^{\zeta'; \ell} h(t) = \mathfrak{J}_{a^+}^{\zeta + \zeta'; \ell} h(t)$,
- (ii) ${}^C \mathfrak{D}_{a^+}^{\zeta; \ell} \mathfrak{J}_{a^+}^{\zeta; \ell} h(t) = h(t)$,
- (iii) ${}^C \mathfrak{D}_{a^+}^{\zeta; \ell} (\ell(t) - \ell(a))^{\zeta' - 1} = \frac{\Gamma(\zeta')}{\Gamma(\zeta' - \zeta)} (\ell(t) - \ell(a))^{\zeta' - \zeta - 1}$,
- (iv) $\mathfrak{J}_{a^+}^{\zeta; \ell} (\ell(t) - \ell(a))^{\zeta' - 1} = \frac{\Gamma(\zeta')}{\Gamma(\zeta' + \zeta)} (\ell(t) - \ell(a))^{\zeta' + \zeta - 1}$,
- (v) ${}^C \mathfrak{D}_{a^+}^{\zeta; \ell} (\ell(t) - \ell(a))^k = 0$, $k = 0, 1, 2, \dots, n - 1$.

Mubeen and Habibullah [22] broadened the scope of the Riemann-Liouville fractional integral (R-L.f.i.) operator, defining the k-R-L.f.i. operator of order ζ as

$${}^k \mathfrak{J}_{a^+}^{\zeta} h(w) = \frac{1}{k \Gamma_k(\zeta)} \int_a^w (w - \beta)^{\frac{\zeta}{k} - 1} h(\beta) d\beta; \quad \zeta > 0, \quad (3.8)$$

and ${}^k \mathfrak{J}_{a^+}^0 h(w) = h(w)$ when $\zeta = 0$ for agreement, where $h \in L^1(\mathcal{I}, (-\infty, \infty))$ and Γ_k is the k-Gamma function which is defined in [10] by

$$\Gamma_k(\mu) = \int_0^\infty t^{\mu-1} e^{-\frac{t^k}{k}} dt$$

where $\mu \in \mathbb{C}$ such that $\Re(\mu) > 0$ and $0 < k \in (-\infty, \infty)$.

It is widely acknowledged that the following equalities hold:

- (i) $\Gamma_k(\mu) = k^{\frac{\mu}{k} - 1} \Gamma\left(\frac{\mu}{k}\right)$,
- (ii) $\Gamma_k(\mu + k) = \mu \Gamma_k(\mu)$,
- (iii) $\lim_{k \rightarrow 1} \Gamma_k(\mu) = \Gamma(\mu)$.

Dorrego in [11] introduced the k-R-L.f.d. of $h \in C(\mathcal{I}, (-\infty, \infty))$ of order $n - 1 < \zeta \leq n$ as

$${}^{k, RL} \mathfrak{D}_{a^+}^{\zeta} h(w) = \left(k \frac{d}{dw}\right)^n \mathfrak{J}_{a^+}^{nk - \zeta} h(w), \quad n = \lceil \frac{\zeta}{k} \rceil. \quad (3.9)$$

Kucche and Mali in [18] introduced the k-C.f.d. as

$${}^{k, C} \mathfrak{D}_{a^+}^{\zeta} h(w) = {}^k \mathfrak{J}_{a^+}^{nk - \zeta} \left(k \frac{d}{dw}\right)^n h(w), \quad n = \lceil \frac{\zeta}{k} \rceil. \quad (3.10)$$

Sousa and Oliveira in [32] defined the ℓ -H.f.d. of the function $h \in C(\mathcal{I}, (-\infty, \infty))$ of order $n - 1 < \bar{\zeta} \leq n$ and type $\zeta' \in [0, 1]$ for $\ell \in C^n(\mathcal{I}, (-\infty, \infty))$, $\ell'(w) \neq 0$, $w \in \mathcal{I}$ as

$${}^H \mathfrak{D}_{a^+}^{\bar{\zeta}, \zeta'; \ell} h(w) = \mathfrak{J}_{a^+}^{\zeta'(n - \bar{\zeta}); \ell} \left(\frac{1}{\ell'(w)} \frac{d}{dw}\right)^n \mathfrak{J}_{a^+}^{(1 - \zeta')(n - \bar{\zeta}); \ell} h(w), \quad n = \lceil \frac{\bar{\zeta}}{k} \rceil. \quad (3.11)$$

In [19], the $(k - \ell)$ -R-L.f.i. of order $\bar{\zeta} \geq 0$ of a function $h \in L^1(\mathcal{I}, (-\infty, \infty))$, $k > 0$ is defined as

$${}^k \mathfrak{J}_{a^+}^{\bar{\zeta}; \ell} h(w) = \begin{cases} \frac{1}{k \Gamma_k(\bar{\zeta})} \int_a^w \ell'(\beta) (\ell(w) - \ell(\beta))^{\frac{\bar{\zeta}}{k} - 1} h(\beta) d\beta, & \bar{\zeta} > 0, \\ h(w), & \bar{\zeta} = 0. \end{cases} \quad (3.12)$$

Kucche and Mali in [18] defined the $(k - \ell)$ -H.f.d. ($k > 0$) of the function $h \in C(\mathcal{I}, (-\infty, \infty))$ of order $\bar{\zeta} \geq 0$ and type $\zeta' \in [0, 1]$ for $\ell \in C^n(\mathcal{I}, (-\infty, \infty))$ where $\ell'(w) \neq 0$ for all $w \in \mathcal{I}$ as

$${}^{k, H} \mathfrak{D}_{a^+}^{\bar{\zeta}, \zeta'; \ell} h(w) = {}^k \mathfrak{J}_{a^+}^{\zeta'(nk - \bar{\zeta}); \ell} \left(\frac{k}{\ell'(w)} \frac{d}{dw}\right)^n \mathfrak{J}_{a^+}^{(1 - \zeta')(nk - \bar{\zeta}); \ell} h(w), \quad n = \lceil \frac{\bar{\zeta}}{k} \rceil. \quad (3.13)$$

Remark 3.1

(i) Taking $\zeta' = 0$, (3.13) reduces to the $(k - \ell)$ -R-L.f.d.

$${}^{k,RL}\mathfrak{D}_{a^+}^{\bar{\zeta};\ell}\bar{h}(w) = \left(\frac{k}{\ell'(w)}\frac{d}{dw}\right)^n {}^k\mathfrak{J}_{a^+}^{(nk-\bar{\zeta});\ell}\bar{h}(w). \quad (3.14)$$

Also, taking $\ell(w) = w$ in (5.8), it reduces to the k -R-L.f.d. (3.9).

(ii) Taking $\zeta' = 1$, (3.13) reduces to the $(k - \ell)$ -C.f.d.

$${}^{k,C}\mathfrak{D}_{a^+}^{\bar{\zeta};\ell}\bar{h}(w) = {}^k\mathfrak{J}_{a^+}^{(nk-\bar{\zeta});\ell}\left(\frac{k}{\ell'(w)}\frac{d}{dw}\right)^n\bar{h}(w) \quad (3.15)$$

and taking $\ell(w) = w$ in (3.15), it reduces to the k -C.f.d. (3.10).

(iii) Taking $\ell(w) = w^p$, then, (3.13) coincides with the k -Hilfer-K.f.d. as defined in [23]:

(a) If $\ell(w) = w^p$ and $\zeta' = 0$, then, equation (3.13) transforms into the k -K.f.d. outlined in [23],

(b) If $\ell(w) = w^p$ and $\zeta' = 1$, then, (3.13) takes the form of the k -Caputo-K.f.d. defined in [23].

(iv) When $\ell(w) = \log w$, equation (3.13) simplifies to the k -Hilfer-H.f.d., as defined in [18]:

(a) If $\ell(w) = \log w$ and $\zeta' = 0$, thus, (3.13) can be represented by the k -H.f.d. as defined in [18],

(b) If $\ell(w) = \log w$ and $\zeta' = 1$, thus, equation (3.13) transforms into the k -Caputo-H.f.d. presented in [18].

Remark 3.2 If $\theta_k = \bar{\zeta} + \zeta'(nk - \bar{\zeta})$, then $\zeta'(nk - \bar{\zeta}) = \theta_k - \bar{\zeta}$ and $(1 - \zeta')(nk - \bar{\zeta}) = nk - \theta_k$. Therefore the $(k - \ell)$ -H.f.d. will get the $(k - \ell)$ -R-L.f.d. form as

$$\begin{aligned} {}^{k,H}\mathfrak{D}_{a^+}^{\bar{\zeta},\zeta';\ell}\bar{h}(w) &= {}^k\mathfrak{J}_{a^+}^{\theta_k-\bar{\zeta};\ell}\left(\frac{k}{\ell'(w)}\frac{d}{dw}\right)^n {}^k\mathfrak{J}_{a^+}^{nk-\theta_k;\ell}\bar{h}(w) \\ &= {}^k\mathfrak{J}_{a^+}^{\theta_k-\bar{\zeta};\ell}\left({}^{k,RL}\mathfrak{D}_{a^+}^{\theta_k;\ell}\bar{h}\right)(w). \end{aligned} \quad (3.16)$$

Note that for $n - 1 < \frac{\bar{\zeta}}{k} \leq n$, we have $n - 1 < \frac{\theta_k}{k} \leq n$.

Lemma 3.3 ([18]) Let $\mu, k \in (0, \infty)$ and $n = \lceil \frac{\mu}{k} \rceil$. Assume that $\bar{h} \in C^n(\mathcal{I}, (-\infty, \infty))$ and ${}^k\mathfrak{J}_{a^+}^{nk-\mu;\ell}\bar{h} \in C^n(\mathcal{I}, (-\infty, \infty))$. Then

$${}^k\mathfrak{J}_{a^+}^{\mu;\ell}\left({}^{k,RL}\mathfrak{D}_{a^+}^{\mu;\ell}\bar{h}\right)(w) = \bar{h}(w) - \sum_{j=1}^{n-1} \frac{(\ell(w)-\ell(a))^{\frac{\mu}{k}-j}}{\Gamma_k(\mu-jk+k)} \left[\left(\frac{k}{\ell'(w)}\frac{d}{dw}\right)^{n-j} {}^k\mathfrak{J}_{a^+}^{nk-\mu;\ell}\bar{h}(w) \right]_{w=a}. \quad (3.17)$$

Lemma 3.4 ([18]) Let $\zeta, k \in (0, \infty)$ with $\zeta < k$ and $\zeta' \in [0, 1]$. Assume that $\theta_k = \zeta + \zeta'(k - \zeta)$ and $\bar{h} \in C^n(\mathcal{I}, (-\infty, \infty))$. Then

$${}^k\mathfrak{J}_{a^+}^{\zeta;\ell}({}^{k,H}\mathfrak{D}_{a^+}^{\zeta,\zeta';\ell}\bar{h})(w) = {}^k\mathfrak{J}_{a^+}^{\theta_k;\ell}\left({}^{k,RL}\mathfrak{D}_{a^+}^{\theta_k;\ell}\bar{h}\right)(w). \quad (3.18)$$

For a comprehensive discussion on relations involving the k -R-L.f.d., we refer the reader to [2, 6, 12, 20, 21, 27] and references cited therein. Furthermore, the work in [18] investigates key properties of the $(k - \ell)$ -Hilfer f.d..

In addition, by employing the well-known Banach contraction principle, they investigated the solvability of the following initial value problem (i.v.p.) involving the $(k - \ell)$ -Hilfer f.d.:

$${}^{k,H}\mathfrak{D}_{a^+}^{\bar{\zeta},\zeta';\ell}\vartheta(t) = \kappa(t, \vartheta(t)), t \in \mathcal{I}, \quad {}^k\mathfrak{J}_{a^+}^{k-\theta_k;\ell}\vartheta(a) = J_a \in (-\infty, \infty), \quad \text{with } \theta_k = \bar{\zeta} + \zeta'(k - \bar{\zeta}), \quad (3.19)$$

where, ${}^k\mathfrak{D}_{a^+}^{\bar{\zeta},\zeta';\ell}$ denotes the $(k - \ell)$ -H-f.d. operator of order $0 < \bar{\zeta} \leq 1$ and type $\zeta' \in [0, 1]$. The function $\kappa : \mathcal{I} \times (-\infty, \infty) \rightarrow (-\infty, \infty)$ is assumed to be continuous, and $\ell \in C^1(\mathcal{I})$ is a strictly increasing function with $\ell'(t) \neq 0$ for all $t \in \mathcal{I}$.

Recently, using the Nadler's contraction and Banach contraction principles, Tariboon et al. [33] studied multi-point b.v.p. of $(k - \ell)$ -Hilfer fractional differential type at form (2.1).

The following sections make up this paper:

In Section 4, we introduce the concept of generalized weak Wardowski contractions and establish the uniqueness of F.P.s for these mappings. In Section 5, we apply this contraction framework to prove the existence of solutions for the $(k - \ell)$ -Hilfer FDEs of order $2 < \zeta \leq 3$ under the boundary conditions given in (2.2). In this setting, the R.H.S. function $\kappa : \mathcal{I} \times (-\infty, \infty) \rightarrow \mathcal{P}((-\infty, \infty))$ does not necessarily satisfy the classical Banach contraction property but instead aligns with the structure of generalized weak Wardowski contractions. Furthermore, an example is provided to illustrate the practical applicability of our theoretical results.

The weak Wardowski contractions were recently presented by Paunović et al. [25], who also established several F.P. theorems for these contractions.

Following the approach in [34], let Ω denote the class of all functions $\mathcal{U} : (0, \infty) \rightarrow (-\infty, \infty)$ that satisfy the following properties:

(U1) \mathcal{U} is strictly increasing,

(U2) For any sequence λ_n in $(0, +\infty)$, $\lim_{n \rightarrow \infty} \lambda_n = 0$ if and only if $\lim_{n \rightarrow \infty} \mathcal{U}(\lambda_n) = -\infty$,

(U3) $\lim_{\lambda \rightarrow 0^+} \lambda^k \mathcal{U}(\lambda) = 0$, where $k \in (0, 1)$.

Additionally, let Ω' represent the collection of all functions $\mathcal{U} : (0, \infty) \rightarrow (-\infty, \infty)$ that satisfy the following condition:

(U₁) \mathcal{U} is continuous and strictly increasing.

Examples of functions belonging to the set Ω' include:

(i) $\mathcal{U}_1(t) = \ln(t)$,

(ii) $\mathcal{U}_2(t) = \ln(t) + t$,

(iii) $\mathcal{U}_3(t) = -\frac{1}{\sqrt{t}} + 1$

(iv) $\mathcal{U}_4(t) = -\frac{1}{t} + 1$

Let Θ denote the set of all functions $\vartheta : (-\infty, \infty) \rightarrow (0, \infty)$ satisfying the following conditions:

(ϑ_1) ϑ is continuous,

(ϑ_2) For every sequence $s_n \subseteq (-\infty, \infty)$, the series $\sum_{i=1}^{\infty} \vartheta(s_n)$ diverges to infinity.

Examples of functions belonging to the class Θ include:

(i) $\vartheta_1(s) = \tau$, $\tau > 0$,

(ii) $\vartheta_2(s) = \tau + s^2$, $\tau > 0$,

(iii) $\vartheta_3(s) = \tau + e^s$, $\tau > 0$,

(iv) $\vartheta_4(s) = \tau e^{s^2}$, $\tau > 0$.

Definition 3.4 ([25]) *If $\mathfrak{L} : \nabla \rightarrow \nabla$ be a self-mapping, it is called a weak Wardowski contraction whenever there exist $\vartheta \in \Theta$ and $\mathfrak{U} \in \Omega'$ such that*

$$\mathfrak{U}(\sigma(\mathfrak{L}j, \mathfrak{L}j')) \leq \mathfrak{U}(\sigma(j, j')) - \vartheta(\mathfrak{U}(\sigma(j, j'))) \quad (3.20)$$

for all $j, j' \in \nabla$ with $\mathfrak{L}j \neq \mathfrak{L}j'$ where (∇, σ) is a M.S..

Theorem 3.1 ([25]) *There exists a unique F.P. for any weak Wardowski contraction $\mathfrak{L} : \nabla \rightarrow \nabla$ where (∇, σ) is a C.M.S..*

4. Main results

We impose a weaker condition on the function ϑ . Specifically, let Θ' denote the set of all continuous functions $\vartheta : (-\infty, \infty) \rightarrow (0, \infty)$. Note that $\Theta \subset \Theta'$.

A few representative elements of Θ' are as follows:

(i) $\vartheta_1(t) = \tau, \tau > 0,$

(ii) $\vartheta_2(t) = e^{-t},$

(iii) $\vartheta_3(t) = \tau e^{-t}, \tau > 0,$

(iv) $\vartheta_4(t) = \sin \frac{(|t|+1)\pi}{(t^2+1)^2}.$

Definition 4.1 [28] *Let ∇ be a non-empty set, $\mathfrak{h} : \nabla \rightarrow \nabla$ and $\zeta : \nabla \times \nabla \rightarrow [0, \infty)$. \mathfrak{h} is said to be ζ -admissible if, for all $j, j' \in \nabla$, $\zeta(j, j') \geq 1$ implies that $\zeta(\mathfrak{h}j, \mathfrak{h}j') \geq 1$.*

The function $\zeta : \nabla \times \nabla \rightarrow [0, \infty)$ will be assumed to be triangular for the rest of this section, that is, $\zeta(j, j') \geq 1$ and $\zeta(j', z) \geq 1$ imply that $\zeta(j, z) \geq 1$.

Definition 4.2 *Let (∇, σ) be a M.S. and $\mathfrak{L} : \nabla \rightarrow \nabla$ be a self-mapping. We call \mathfrak{L} is a ζ -admissible generalized weak Wardowski contraction whenever there exist $\zeta : \nabla \times \nabla \rightarrow [0, \infty)$, $\vartheta \in \Theta'$ and $\mathfrak{U} \in \Omega'$ such that*

$$\mathfrak{U}(\sigma(\mathfrak{L}j, \mathfrak{L}j')) \leq \mathfrak{U}(\Upsilon_\sigma(j, j')) - \vartheta(\mathfrak{U}(\Upsilon_\sigma(j, j'))) \quad (4.1)$$

for all $j, j' \in \nabla$ with $\zeta(j, j') \geq 1$ and $\mathfrak{L}j \neq \mathfrak{L}j'$, where

$$\Upsilon_\sigma(j, j') = \max\{\sigma(j, j'), \sigma(j, \mathfrak{L}j), \sigma(j', \mathfrak{L}j'), \frac{\sigma(j, \mathfrak{L}j') + \sigma(j', \mathfrak{L}j)}{2}\}.$$

Theorem 4.1 *Let (∇, σ) be a C.M.S. and $\mathfrak{L} : \nabla \rightarrow \nabla$ be a ζ -admissible generalized weak Wardowski contraction self-mapping. Moreover, let*

(i) *there exists $j_0 \in \nabla$ such that $\zeta(j_0, \mathfrak{L}j_0) \geq 1,$*

(ii) *for each sequence $\{j_n\}$ in ∇ with $j_n \rightarrow j$ and $\zeta(j_n, j_{n+1}) \geq 1$ for all n , one has $\zeta(j_n, j) \geq 1$ for all $n,$*

Then, \mathfrak{L} has a F.P.. Moreover, if $\zeta(j, j') \geq 1$, for all F.P.s j, j' , then the F.P. is unique.

Proof: Construct the sequence $\{j_n\}$ by $j_n = \mathfrak{L}j_{n-1}$ for all $n \in \mathbb{N}$. If $j_n = j_{n-1}$ for some $n \in \mathbb{N}$, then j_{n-1} is a F.P. of \mathfrak{L} . So, we may assume $j_n \neq j_{n-1}$ for all $n \in \mathbb{N}$. Then, from (4.1), for any $n \in \mathbb{N}$, since $\zeta(j_n, j_{n+1}) \geq 1$ we have

$$\mathfrak{U}(\sigma(j_n, j_{n+1})) = \mathfrak{U}(\sigma(\mathfrak{L}j_{n-1}, \mathfrak{L}j_n)) \leq \mathfrak{U}(\Upsilon_\sigma(j_{n-1}, j_n)) - \vartheta(\mathfrak{U}(\Upsilon_\sigma(j_{n-1}, j_n))) \quad (4.2)$$

where

$$\begin{aligned}
& \max \left\{ \sigma(j_{n-1}, j_n), \sigma(j_n, j_{n+1}) \right\} \leq \Upsilon(j_{n-1}, j_n) \\
& = \max \left\{ \sigma(j_{n-1}, j_n), \sigma(j_{n-1}, \mathfrak{L}j_{n-1}), \sigma(j_n, \mathfrak{L}j_n), \frac{\sigma(j_n, \mathfrak{L}j_{n-1}) + \sigma(j_{n-1}, \mathfrak{L}j_n)}{2} \right\} \\
& \leq \max \left\{ \sigma(j_{n-1}, j_n), \sigma(j_n, j_{n+1}) \right\}.
\end{aligned} \tag{4.3}$$

Thus

$$\Upsilon(j_{n-1}, j_n) = \max \left\{ \sigma(j_{n-1}, j_n), \sigma(j_n, j_{n+1}) \right\}.$$

If

$$\max \left\{ \sigma(j_{n-1}, j_n), \sigma(j_n, j_{n+1}) \right\} = \sigma(j_n, j_{n+1}),$$

then, by (4.2), we have

$$\mathfrak{U}(\sigma(j_n, j_{n+1})) \leq \mathfrak{U}(\sigma(j_n, j_{n+1})) - \vartheta(\mathfrak{U}(\sigma(j_n, j_{n+1}))) < \mathfrak{U}(\sigma(j_n, j_{n+1}))$$

which gives a contradiction. Thus,

$$\max \left\{ \sigma(j_{n-1}, j_n), \sigma(j_n, j_{n+1}) \right\} = \sigma(j_{n-1}, j_n).$$

Therefore, from (4.2),

$$\mathfrak{U}(\sigma(j_n, j_{n+1})) \leq \mathfrak{U}(\sigma(j_{n-1}, j_n)) - \vartheta(\mathfrak{U}(\sigma(j_{n-1}, j_n))) < \mathfrak{U}(\sigma(j_{n-1}, j_n)). \tag{4.4}$$

Since, \mathfrak{U} is increasing, thus we have

$$\sigma(j_n, j_{n+1}) < \sigma(j_{n-1}, j_n), \quad \text{for each } n \geq 0. \tag{4.5}$$

So, $\{\sigma(j_{n-1}, j_n)\}$ is a decreasing sequence in $[0, \infty)$ and so there is $r \geq 0$ so that $\sigma(j_{n-1}, j_n) \rightarrow r^+$. We now demonstrate that $r = 0$. Suppose, however, that $r > 0$. Taking the limit in (4.4),

$$\mathfrak{U}(r) \leq \mathfrak{U}(r) - \vartheta(\mathfrak{U}(r)) < \mathfrak{U}(r),$$

which is a contradiction. So, $\lim_{n \rightarrow \infty} \sigma(j_{n-1}, j_n) = r = 0$. We claim that $\{j_n\}$ is Cauchy. If $\{j_n\}$ is not Cauchy, then there are $\varepsilon > 0$ and subsequences $\{j_{m_i}\}$ and $\{j_{n_i}\}$ of $\{j_n\}$ so that

$$n_i > m_i > i, \sigma(j_{m_i}, j_{n_i}) \geq \varepsilon \tag{4.6}$$

and

$$\sigma(j_{m_i}, j_{n_i-1}) < \varepsilon.$$

Using (4.6), we get

$$\varepsilon \leq \sigma(j_{m_i}, j_{n_i}) \leq \sigma(j_{m_i}, j_{n_i-1}) + \sigma(j_{n_i-1}, j_{n_i}) < \varepsilon + \sigma(j_{n_i-1}, j_{n_i}).$$

As $i \rightarrow \infty$, we find

$$\lim_{i \rightarrow \infty} \sigma(j_{m_i}, j_{n_i}) = \varepsilon.$$

Also, we have

$$\begin{aligned}
& \sigma(j_{m_i}, j_{n_i}) - \sigma(j_{m_i}, j_{m_i+1}) - \sigma(j_{n_i}, j_{n_i+1}) \\
& \leq \sigma(j_{m_i+1}, j_{n_i+1}) \\
& \leq \sigma(j_{m_i}, j_{m_i+1}) + \sigma(j_{m_i}, j_{n_i}) + \sigma(j_{n_i}, j_{n_i+1}).
\end{aligned}$$

As $i \rightarrow \infty$, we find

$$\lim_{i \rightarrow \infty} \sigma(J_{m_i+1}, J_{n_i+1}) = \varepsilon.$$

Since ζ is triangular, we find $\zeta(J_{m_i}, J_{n_i}) \geq 1$, for all $i \in \mathbb{N}$. From (4.1), we get

$$\mathcal{U}(\sigma(J_{m_i+1}, J_{n_i+1})) \leq \mathcal{U}(\Upsilon_\sigma(J_{m_i}, J_{n_i})) - \vartheta(\mathcal{U}(\Upsilon_\sigma(J_{m_i}, J_{n_i}))). \quad (4.7)$$

On the other hand

$$\begin{aligned} & \sigma(J_{m_i}, J_{n_i}) \leq \Upsilon_\sigma(J_{m_i}, J_{n_i}) \\ &= \max\{\sigma(J_{m_i}, J_{n_i}), \sigma(J_{m_i}, \mathfrak{L}J_{m_i}), \sigma(J_{n_i}, \mathfrak{L}J_{n_i}), \frac{\sigma(J_{m_i}, \mathfrak{L}J_{n_i}) + \sigma(J_{n_i}, \mathfrak{L}J_{m_i})}{2}\} \\ &\leq \max\{\sigma(J_{m_i}, J_{n_i}), \sigma(J_{m_i}, J_{m_i+1}), \sigma(J_{n_i}, J_{n_i+1}), \\ &\quad \frac{2\sigma(J_{m_i+1}, J_{n_i+1}) + \sigma(J_{n_i}, J_{n_i+1}) + \sigma(J_{m_i}, J_{m_i+1})}{2}\}. \end{aligned}$$

Considering the limit $i \rightarrow \infty$ in the aforementioned inequality, we have $\lim_{i \rightarrow \infty} \Upsilon_\sigma(J_{m_i}, J_{n_i}) = \varepsilon$. Taking limit in both sides of (4.7),

$$\mathcal{U}(\varepsilon) \leq \mathcal{U}(\varepsilon) - \vartheta(\mathcal{U}(\varepsilon)) < \mathcal{U}(\varepsilon),$$

a contradiction.

Thus, $\{J_n\}$ is a Cauchy sequence in the C.M.S. (∇, σ) , which yields that there is $j \in \nabla$ so that

$$\lim_{n \rightarrow \infty} J_n = j. \quad (4.8)$$

In conclusion, we assert that $\mathfrak{L}j = j$. To show this, we have two cases:

- (1) $\mathfrak{L}J_n \neq \mathfrak{L}j$ for each $n \geq N$ where $N \in \mathbb{N}$,
- (2) $\mathfrak{L}J_{n_k} = \mathfrak{L}j$ for each $k \geq 0$ where $\{J_{n_k}\} \subseteq \{J_n\}$.

In the case (1), if $\sigma(j, \mathfrak{L}j) \neq 0$, according to assumption (ii), we have

$$\mathcal{U}(\sigma(J_{n+1}, \mathfrak{L}j)) = \mathcal{U}(\sigma(\mathfrak{L}J_n, \mathfrak{L}j)) \leq \mathcal{U}(\Upsilon_\sigma(J_n, j)) - \vartheta(\mathcal{U}(\Upsilon_\sigma(J_n, j))). \quad (4.9)$$

On the other hand

$$\begin{aligned} & \sigma(j, \mathfrak{L}j) \leq \Upsilon_\sigma(J_n, j) \\ &= \max\{\sigma(J_n, j), \sigma(J_n, \mathfrak{L}J_n), \sigma(j, \mathfrak{L}j), \frac{\sigma(J_n, \mathfrak{L}j) + \sigma(j, \mathfrak{L}J_n)}{2}\} \\ &\leq \max\{\sigma(J_n, j), \sigma(J_n, J_{n+1}), \sigma(j, \mathfrak{L}j), \\ &\quad \frac{\sigma(J_n, j) + \sigma(j, \mathfrak{L}j) + \sigma(j, J_{n+1})}{2}\}. \end{aligned}$$

Considering the limit $i \rightarrow \infty$ in the aforementioned inequality, we have $\lim_{i \rightarrow \infty} \Upsilon_\sigma(J_n, j) = \sigma(j, \mathfrak{L}j)$.

Passing to the limit through (4.9), we obtain

$$\mathcal{U}(\sigma(j, \mathfrak{L}j)) \leq \mathcal{U}(\sigma(j, \mathfrak{L}j)) - \vartheta(\mathcal{U}(\sigma(j, \mathfrak{L}j))) < \mathcal{U}(\sigma(j, \mathfrak{L}j)),$$

a contradiction. Thus $\sigma(j, \mathfrak{L}j) = 0$ and so $j = \mathfrak{L}j$. In the case 2,

$$\sigma(j, \mathfrak{L}j) = \lim_{n \rightarrow \infty} \sigma(J_{n_k+1}, \mathfrak{L}j) = \lim_{n \rightarrow \infty} \sigma(\mathfrak{L}J_{n_k}, \mathfrak{L}j) = 0.$$

From this, we conclude that $\mathfrak{L}j = j$. Assume that ω, ω' are two different F.P.s of \mathfrak{L} to illustrate the F.P.'s uniqueness. Using (4.1), we have

$$\mathcal{U}(\sigma(\omega, \omega')) = \mathcal{U}(\sigma(\mathfrak{L}\omega, \mathfrak{L}\omega')) \leq \mathcal{U}(\sigma(\omega, \omega')) - \vartheta(\mathcal{U}(\sigma(\omega, \omega'))) < \mathcal{U}(\sigma(\omega, \omega')),$$

a contradiction. Thus $\omega = \omega'$. □

Similarly, one can demonstrate the outcome that follows:

Theorem 4.2 Let (∇, σ) be a C.M.S. and $\mathfrak{L} : \nabla \rightarrow \nabla$ be a ζ -admissible generalized weak Wardowski contraction self-mapping. Moreover, let

- (i) $\zeta(j_0, \mathfrak{L}j_0) \geq 1$ for some $j_0 \in \nabla$,
- (ii) \mathfrak{L} is continuous.

Consequently, there exists a F.P. for \mathfrak{L} . Moreover, if $\zeta(j, j') \geq 1$, for all F.P.s j, j' , then the F.P. is unique.

Taking $\vartheta(t) = \tau(\tau > 0)$, we arrive to the following conclusion, which broadens Wardowski's conclusion [34].

Corollary 4.1 Let $\mathfrak{L} : \nabla \rightarrow \nabla$ be a ζ -admissible mapping where (∇, σ) is a C.M.S. and there exist $\tau > 0$ and $\mathfrak{U} \in \Omega'$ such that

$$\mathfrak{U}(\sigma(\mathfrak{L}j, \mathfrak{L}j')) \leq \mathfrak{U}(\Upsilon_\sigma(j, j')) - \tau \quad (4.10)$$

for all $j, j' \in \nabla$ with $\zeta(j, j') \geq 1$ and $\mathfrak{L}j \neq \mathfrak{L}j'$, where

$$\Upsilon_\sigma(j, j') = \max\{\sigma(j, j'), \sigma(j, \mathfrak{L}j), \sigma(j', \mathfrak{L}j'), \frac{\sigma(j, \mathfrak{L}j') + \sigma(j', \mathfrak{L}j)}{2}\}.$$

Moreover, let

- (i) $\zeta(j_0, \mathfrak{L}j_0) \geq 1$ for some $j_0 \in \nabla$,
- (ii) \mathfrak{L} is continuous or for each sequence $\{j_n\}$ in ∇ with $\zeta(j_n, j_{n+1}) \geq 1$ for all n and $j_n \rightarrow j$, then, $\zeta(j_n, j) \geq 1$ for all n .

Consequently, there exists a F.P. for \mathfrak{L} . Moreover, if $\zeta(j, j') \geq 1$, for all F.P.s j, j' , then the F.P. is unique.

Taking $\vartheta(t) = -\ln k$ ($0 < k < 1$) and $\mathfrak{U}(t) = \ln t$, we get the following outcome which is an extension of Banach's finding.

Corollary 4.2 Let $\mathfrak{L} : \nabla \rightarrow \nabla$ be a ζ -admissible mapping where (∇, σ) is a C.M.S. and there exist $0 < k < 1$ and $\mathfrak{U} \in \Omega'$ such that

$$\sigma(\mathfrak{L}j, \mathfrak{L}j') \leq k\Upsilon_\sigma(j, j') \quad (4.11)$$

for all $j, j' \in \nabla$ with $\zeta(j, j') \geq 1$ and $\mathfrak{L}j \neq \mathfrak{L}j'$, where

$$\Upsilon_\sigma(j, j') = \max\{\sigma(j, j'), \sigma(j, \mathfrak{L}j), \sigma(j', \mathfrak{L}j'), \frac{\sigma(j, \mathfrak{L}j') + \sigma(j', \mathfrak{L}j)}{2}\}.$$

Moreover, let

- (i) $\zeta(j_0, \mathfrak{L}j_0) \geq 1$ for some $j_0 \in \nabla$,
- (ii) \mathfrak{L} is continuous or for each sequence $\{j_n\}$ in ∇ with $\zeta(j_n, j_{n+1}) \geq 1$ for all n and $j_n \rightarrow j$, then, $\zeta(j_n, j) \geq 1$ for all n .

Consequently, there exists a F.P. for \mathfrak{L} . Moreover, the F.P. is unique if $\zeta(j, j') \geq 1$, for all F.P.s j, j' .

5. Application to f.d.e.s

Assume moving forward that the Banach space of continuous nonnegative real valued functions equipped with the norm

$$\|z\| = \sup_{t \in \mathcal{I}} |z(t)|.$$

is $\nabla = C(\mathcal{I}, (-\infty, \infty))$.

Define $\sigma(z_1, z_2) = \|z_1 - z_2\|$ for all $z_1, z_2 \in \nabla$. Then (∇, σ) is a C.M.S..

Lemma 5.1 For a function $g \in L^1(\mathcal{I}, (-\infty, \infty))$, a function $\theta \in C(\mathcal{I}, (-\infty, \infty))$ is a solution of the equation

$$\begin{cases} {}^{k,H}\mathfrak{D}_{a^+}^{\zeta, \zeta'; \ell} \theta(w) = g(w); w \in \mathcal{I}, 2 < \zeta, \frac{\zeta}{k} \leq 3, \\ \theta(a) = 0, \\ \theta(b) = b_0, [(\frac{k}{\ell'(w)} \frac{d}{dw}) \theta(w)]_{w=b} = b_1, \theta_k = \zeta + \zeta'(nk - \zeta) \end{cases} \quad (5.1)$$

if and only if

$$\begin{aligned} \theta(w) &= {}^k\mathfrak{J}_{a^+}^{\zeta; \ell} g(w) + \frac{1}{k} ({}^k\mathfrak{J}_{a^+}^{\zeta-k; \ell} g(b) - b_1) \left(\frac{(\ell(w) - \ell(a))^{\frac{\theta_k}{k} - 2} (\ell(b) - \ell(w))}{(\ell(b) - \ell(a))^{\frac{\theta_k}{k} - 2}} \right) \\ &+ \frac{1}{k} (b_0 - {}^k\mathfrak{J}_{a^+}^{\zeta; \ell} g(b)) \left(\frac{(\ell(w) - \ell(a))^{\frac{\theta_k}{k} - 2}}{(\ell(b) - \ell(a))^{\frac{\theta_k}{k} - 2}} \right) \left((\theta_k - k) \frac{\ell(b) - \ell(w)}{\ell(b) - \ell(a)} + k \frac{\ell(w) - \ell(a)}{\ell(b) - \ell(a)} \right) \end{aligned} \quad (5.2)$$

for all $w \in \mathcal{I}$.

Proof: Assume that $\theta \in \nabla$ is a solution of the equation (5.1). Then, from Remark 3.2,

$${}^{k,H}\mathfrak{D}_{a^+}^{\zeta, \zeta'; \ell} \theta(w) = {}^k\mathfrak{J}_{a^+}^{\theta_k - \zeta; \ell} \left({}^{k,RL}\mathfrak{D}_{a^+}^{\theta_k; \ell} \theta \right) (w) = g(w). \quad (5.3)$$

Applying ${}^k\mathfrak{J}_{a^+}^{\zeta; \ell}$ on both sides of the above equality, we get

$${}^k\mathfrak{J}_{a^+}^{\theta_k; \ell} \left({}^{k,RL}\mathfrak{D}_{a^+}^{\theta_k; \ell} \theta \right) (w) = {}^k\mathfrak{J}_{a^+}^{\zeta; \ell} g(w). \quad (5.4)$$

From Lemma 3.3, we get

$$\theta(w) - \sum_{j=1}^3 \frac{(\ell(w) - \ell(a))^{\frac{\theta_k}{k} - j}}{\Gamma_k(\theta_k - jk + k)} \left[\left(\frac{k}{\ell'(w)} \frac{d}{dw} \right)^{n-j} {}^k\mathfrak{J}_{a^+}^{nk - \theta_k; \ell} \theta(w) \right]_{w=a} = {}^k\mathfrak{J}_{a^+}^{\zeta; \ell} g(w).$$

Thus

$$\theta(w) = {}^k\mathfrak{J}_{a^+}^{\zeta; \ell} g(w) + \sum_{j=1}^3 d_j \frac{(\ell(w) - \ell(a))^{\frac{\theta_k}{k} - j}}{\Gamma_k(\theta_k - jk + k)} \quad (5.5)$$

where

$$d_j = \left[\left(\frac{k}{\ell'(w)} \frac{d}{dw} \right)^{n-j} {}^k\mathfrak{J}_{a^+}^{nk - \theta_k; \ell} \theta(w) \right]_{w=a}, j = 1, 2, 3.$$

Applying boundary value $\theta(a) = 0$, we get $d_3 = 0$. Taking $\theta(b) = b_0$ in (5.5), we obtain

$$b_0 = {}^k\mathfrak{J}_{a^+}^{\zeta; \ell} g(b) + d_1 \frac{(\ell(b) - \ell(a))^{\frac{\theta_k}{k} - 1}}{\Gamma_k(\theta_k)} + d_2 \frac{(\ell(b) - \ell(a))^{\frac{\theta_k}{k} - 2}}{\Gamma_k(\theta_k - k)}. \quad (5.6)$$

Applying $[(\frac{k}{\ell'(w)} \frac{d}{dw}) \theta(w)]_{w=b} = b_1$ in (5.5), we obtain

$$b_1 = {}^k\mathfrak{J}_{a^+}^{\zeta-k; \ell} g(b) + d_1 \frac{(\ell(b) - \ell(a))^{\frac{\theta_k}{k} - 2}}{\Gamma_k(\theta_k - k)} + d_2 \frac{(\ell(b) - \ell(a))^{\frac{\theta_k}{k} - 3}}{\Gamma_k(\theta_k - 2k)}. \quad (5.7)$$

From (5.6) and (5.7), we get the following system:

$$\begin{cases} d_1 \frac{\ell(b)-\ell(a)}{\theta_k-2k} + d_2 = \frac{\Gamma_k(\theta_k-2k)}{\frac{\theta_k}{k}-3} (b_1 - {}^k\mathfrak{J}_{a^+}^{\zeta-k;\ell} g(b)), \\ d_1 \frac{\ell(b)-\ell(a)}{\theta_k-k} + d_2 = \frac{\Gamma_k(\theta_k-k)}{(\ell(b)-\ell(a))\frac{\theta_k}{k}-2} (b_0 - {}^k\mathfrak{J}_{a^+}^{\zeta;\ell} g(b)). \end{cases} \quad (5.8)$$

Solving (5.8), we obtain

$$\begin{cases} d_1 = \frac{\Gamma_k(\theta_k)}{k(\ell(b)-\ell(a))\frac{\theta_k}{k}-2} [(b_1 - {}^k\mathfrak{J}_{a^+}^{\zeta-k;\ell} g(b)) - \frac{\theta_k-2k}{\ell(b)-\ell(a)} (b_0 - {}^k\mathfrak{J}_{a^+}^{\zeta;\ell} g(b))], \\ d_2 = \frac{\Gamma_k(\theta_k-k)}{k(\ell(b)-\ell(a))\frac{\theta_k}{k}-3} [\frac{\theta_k-k}{\ell(b)-\ell(a)} (b_0 - {}^k\mathfrak{J}_{a^+}^{\zeta;\ell} g(b)) - (b_1 - {}^k\mathfrak{J}_{a^+}^{\zeta-k;\ell} g(b))]. \end{cases} \quad (5.9)$$

Substituting $d_3 = 0$ and (5.9) in (5.5), we get (5.2). The converse of the proof can be obtained easily from an inverse direct computation. \square

In what follows we always assume that the mapping $\mathfrak{U}^{-1}[\mathfrak{U}(\cdot) - \vartheta(\mathfrak{U}(\cdot))]$ is nondecreasing.

Theorem 5.1 *Let $\kappa : \mathcal{I} \times (-\infty, \infty) \rightarrow (-\infty, \infty)$ be a continuous mapping. Suppose that*

$$|\kappa(w, u) - \kappa(w, v)| \leq \frac{k\Gamma_k(\zeta + k)}{(\theta_k + \zeta)(\ell(b) - \ell(a))\frac{\zeta}{k}} \mathfrak{U}^{-1}[\mathfrak{U}(|u - v|) - \vartheta(\mathfrak{U}(|u - v|))], \quad (5.10)$$

for all $w \in \mathcal{I}$ and $u, v \in (-\infty, \infty)$ where $\vartheta \in \Theta'$ and $\mathfrak{U} \in \Omega'$. Then, there is a unique solution for the problem (2.2).

Proof: Define a mapping $A : C(\mathcal{I}, (-\infty, \infty)) \rightarrow C(\mathcal{I}, (-\infty, \infty))$ with

$$\begin{aligned} A(\theta)(w) &= {}^k\mathfrak{J}_{a^+}^{\zeta;\ell} \kappa(w, \theta(w)) + \frac{1}{k} \left({}^k\mathfrak{J}_{a^+}^{\zeta-k;\ell} \kappa(b, \theta(b)) - b_1 \right) \left(\frac{(\ell(w)-\ell(a))\frac{\theta_k}{k}-2(\ell(b)-\ell(w))}{(\ell(b)-\ell(a))\frac{\theta_k}{k}-2} \right) \\ &+ \frac{1}{k} \left(b_0 - {}^k\mathfrak{J}_{a^+}^{\zeta;\ell} \kappa(b, \theta(b)) \right) \left(\frac{(\ell(w)-\ell(a))\frac{\theta_k}{k}-2}{(\ell(b)-\ell(a))\frac{\theta_k}{k}-2} \right) \left((\theta_k - k) \frac{\ell(b)-\ell(w)}{\ell(b)-\ell(a)} + k \frac{\ell(w)-\ell(a)}{\ell(b)-\ell(a)} \right). \end{aligned} \quad (5.11)$$

We'll demonstrate that there is a unique F.P. for the mapping A . Let $\theta, \bar{\theta} \in C(\mathcal{I}, (-\infty, \infty))$. For all $w \in \mathcal{I}$ from (5.10), we have

$$\begin{aligned} &|A(\theta)(w) - A(\bar{\theta})(w)| \\ &\leq \frac{1}{k\Gamma_k(\zeta)} \int_a^w \ell'(\beta) (\ell(w) - \ell(\beta))^{\frac{\zeta}{k}-1} |\kappa(\beta, \theta(\beta)) - \kappa(\beta, \bar{\theta}(\beta))| d\beta \\ &+ \frac{1}{k^2\Gamma_k(\zeta - k)} (\ell(b) - \ell(a)) \int_a^b \ell'(\beta) (\ell(w) - \ell(\beta))^{\frac{\zeta}{k}-2} |\kappa(\beta, \theta(\beta)) - \kappa(\beta, \bar{\theta}(\beta))| d\beta \\ &+ \frac{1}{k^2\Gamma_k(\zeta)} \int_a^b \ell'(\beta) (\ell(w) - \ell(\beta))^{\frac{\zeta}{k}-1} |\kappa(\beta, \theta(\beta)) - \kappa(\beta, \bar{\theta}(\beta))| d\beta \\ &\leq \frac{k\Gamma_k(\zeta + k)}{(\theta_k + \zeta)(\ell(b) - \ell(a))\frac{\zeta}{k}} \mathfrak{U}^{-1}[\mathfrak{U}(\|\theta - \bar{\theta}\|) - \vartheta(\mathfrak{U}(\|\theta - \bar{\theta}\|))] \\ &(\ell(b) - \ell(a))\frac{\zeta}{k} \left[\frac{1}{\Gamma_k(\zeta + k)} + \frac{1}{k\Gamma_k(\zeta)} + \frac{\theta_k - k}{k\Gamma_k(\zeta + k)} \right] \\ &= \mathfrak{U}^{-1}[\mathfrak{U}(\|\theta - \bar{\theta}\|) - \vartheta(\mathfrak{U}(\|\theta - \bar{\theta}\|))] \end{aligned}$$

Taking the sup on $w \in \mathcal{I}$, we get

$$\|A(\theta) - A(\bar{\theta})\| \leq \mathfrak{U}^{-1}[\mathfrak{U}(\|\theta - \bar{\theta}\|) - \vartheta(\mathfrak{U}(\|\theta - \bar{\theta}\|))]$$

and so

$$\mathfrak{U}(\|A(\theta) - A(\bar{\theta})\|) \leq \mathfrak{U}(\|\theta - \bar{\theta}\|) - \vartheta(\mathfrak{U}(\|\theta - \bar{\theta}\|)).$$

Taking $\zeta(\theta, \bar{\theta}) = 1$ for all $\theta, \bar{\theta} \in C(\mathcal{I}, (-\infty, \infty))$, we can observe that Theorem (4.1)'s requirements are all met. As a result, the issue (2.2) has a unique solution as the mapping A has a unique F.P. \square

Example 5.1 Consider the f.d.e.:

$$\begin{cases} {}_{\frac{36}{35}}^H D_{2^+}^{\frac{12}{5}, \frac{1}{5}; w^2} \theta(w) = e^w + \frac{486}{35(479)} \sqrt[3]{\frac{36}{175}} \sqrt{\pi} \frac{3|\theta(w)|}{3|\theta(w)|+1}, & w \in [2, 3], \\ \theta(2) = 0, \\ \theta(3) = 4, \\ \theta'(3) = \frac{35}{6}. \end{cases} \quad (5.12)$$

Note that

$$\kappa(w, u) = e^w + \frac{486}{35(479)} \sqrt[3]{\frac{36}{175}} \sqrt{\pi} \frac{3|u|}{3|u|+1}.$$

Obviously, κ is continuous. Here,

$$k = \frac{36}{35}, \zeta = \frac{12}{5}, \zeta' = \frac{1}{5}, a = 2, b = 3, \ell(w) = w^2,$$

$$b_0 = 4, b_1 = [(\frac{k}{\ell'(w)} \frac{d}{dw})\theta(w)]_{w=3} = \frac{36}{6} \theta'(3) = 1$$

and

$$\theta_k = \zeta + \zeta'(nk - \zeta) = \frac{444}{175}, n = \lceil \frac{\theta_k}{k} \rceil = \lceil \frac{37}{15} \rceil = 3.$$

Thus

$$\frac{k\Gamma_k(\zeta + k)}{(\theta_k + \zeta)(\ell(b) - \ell(a))^{\frac{\zeta}{k}}} = \frac{k^{\frac{\zeta}{k}} \Gamma(\frac{\zeta+k}{k})}{(\theta_k + \zeta)5^{\frac{\zeta}{k}}} = \frac{(\frac{36}{35})^{\frac{7}{3}} \Gamma(\frac{7}{2})}{(\frac{479}{175})5^{\frac{7}{3}}} = \frac{486}{35(479)} \sqrt[3]{\frac{36}{175}} \sqrt{\pi},$$

and

$$\begin{aligned} |\kappa(w, u) - \kappa(w, v)| &= \frac{486}{35(479)} \sqrt[3]{\frac{36}{175}} \sqrt{\pi} \left| \frac{3|u|}{3|u|+1} - \frac{4\sqrt{\pi}}{15\sqrt{15}} \frac{3|v|}{3|v|+1} \right| \\ &= \frac{486}{35(479)} \sqrt[3]{\frac{36}{175}} \sqrt{\pi} \left| \frac{3(|u| - |v|)}{(3|u|+1)(3|v|+1)} \right| \\ &\leq \frac{486}{35(479)} \sqrt[3]{\frac{36}{175}} \sqrt{\pi} \frac{3|u-v|}{3|u-v|+1} \\ &= \frac{k\Gamma_k(\zeta + k)}{(\theta_k + \zeta)(\ell(b) - \ell(a))^{\frac{\zeta}{k}}} \mathcal{U}^{-1}[\mathcal{U}(|u-v|) - \vartheta(\mathcal{U}(|u-v|))], \end{aligned}$$

where $\mathcal{U}(t) = \frac{-1}{t} + 1$ and $\vartheta(t) = \frac{1}{3}$.

As a result, Theorem 5.1's requirements are all met. Hence, by virtue of this theorem, problem (5.12) admits a unique solution.

6. Conclusion

In this study, we developed generalized weak Wardowski contractions and demonstrate that such mappings have F.P.s. Then, using new boundary value conditions, we leverage our new contraction to show that some $(k - \ell)$ -Hilfer f.d.e.s of order $2 < \zeta \leq 3$ are solvable. In order to arrive at our conclusions, we draw on some prior research on $(k - \ell)$ -Hilfer f.d.e.s. To illustrate the applicability of our main findings, a concrete example is presented. Moreover, as a potential direction for future research, we propose the development of coupled F.P. theorems for two-variable mappings that satisfy a generalized weak Wardowski contraction condition.

Then, when the right side acts as a generalized weak Wardowski contraction, we suggest looking into the solvability of the $(k - \ell)$ -Hilfer fractional differential systems of equations.

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