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Stability of the Sine Addition-Subtraction Law

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ABSTRACT: In this paper, we investigate the stability of the functional equation

$$f(xy) = f(x)g(y) + \beta\, g(x)f(y) + \gamma\, f(x)f(y), \qquad x, y \in S,$$

where S is a semigroup, $f, g : S \to \mathbb{C}$ are two unknown functions, $\beta \in \mathbb{C} \setminus \{0\}$ and $\gamma \in \mathbb{C}$ are fixed constants. We extend our analysis to the functional equation

$$f(x\sigma(y)) = f(x)g(y) + \beta\, g(x)f(y) + \gamma\, f(x)f(y), \qquad x, y \in S,$$

where $\sigma : S \to S$ is an involutive automorphism.

Keywords: Stability, sine functional equation, sine addition-subtraction law, multiplicative function.

Contents

1. Introduction

The study of the stability of functional equations originates from a problem posed by Ulam in 1940 concerning the stability of group homomorphisms [23]. Thereafter, Hyers [15] provided a first partial affirmative answer for Banach spaces. This result was extended by Aoki [4] for additive mappings and by Rassias [18] for linear applications by allowing an unbounded Cauchy difference. In [20] Székelyhidi proved the stability of the sine functional equation

$$f(xy) = f(x)g(y) + g(x)f(y) \tag{1.1}$$

and cosine

$$f(xy) = f(x)f(y) + g(x)g(y) \tag{1.2}$$

on amenable groups

Chung, choi and kim [12] have established the stability of the functional equation

$$f(x\sigma(y)) = f(x)g(y) - g(x)f(y) \tag{1.3}$$

on 2-divisible group. Chang and chung [10] prouved Hyres-Ulam stability of the functional equations (1.1) and (1.2) in a generalized functions spaces. Chang et al. [9] proved the stability of functional equation

$$f(x\sigma(y)) = f(x)f(y) - g(x)g(y) \tag{1.4}$$

on abelien group

The theory of stability for functional equations has evolved into a rich and active field of research. A comprehensive account of its progress and main developments can be found in [1,2,3,5,6,7,8,11,13,14,16,17,21].

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The study of the stability of trigonometric functional equations with more than three unknowns or more than two terms is not well suited to the calculations used for trigonometric functional equations of the types above.

As a contribution to a more general form of trigonometric functional equations, we deel in this paper with the stability of the functional equation

$$f(xy) = f(x)g(y) + \beta g(x)f(y) + \gamma f(x)f(y), \qquad x, y \in S \tag{1.5}$$

where S is a semi group, $f, g : S \to \mathbb{C}$ are unknown functions, $\beta \in \mathbb{C} \setminus \{0\}$ and $\gamma \in \mathbb{C}$ are fixed constants. As an application we get the stability of the functional equation

$$f(x\sigma(y)) = f(x)g(y) + \beta g(x)f(y) + \gamma f(x)f(y), \qquad x, y \in S \tag{1.6}$$

where $\sigma : S \to S$ is an involutive automorphism.
Notice that Stetkær [19] solved (1.6) for $\sigma : S \to S$ a surjective homomophism and $f, g : S \to \mathbb{F}$ where $\mathbb{F}$ a field of characteristic different from 2, $\beta \in \mathbb{F}^*$ and $\gamma \in \mathbb{F}$.

2. Notations and definitions

Throughtout this paper S denotes a semigroup (a set with an associative composition), $\sigma : S \to S$ an involutive automorphism, that is, $\sigma(\sigma(x)) = x$ for all $x \in S$ and $\mathcal{T}$ a linear space of complex-valued functions on S.
We denote by $\mathcal{B}(S)$ the linear space of all bounded complex-valued functions on S.

Definition 2.1 *Let $f : S \to \mathbb{C}$ a function. We define the even part and the odd part of f with respect to σ by*

$$f_e(x) = \frac{f(x) + f \circ \sigma(x)}{2} \quad \text{and} \quad f_o(x) = \frac{f(x) - f \circ \sigma(x)}{2}, \qquad x \in S.$$

Hence, $f = f_e + f_o$.

Definition 2.2 *Let $m : S \to \mathbb{C}$ be a function.*
We say that m is multiplicative function if $m(xy) = m(x)m(y)$ for all $x, y \in S$.

Definition 2.3 *(see [20]).*
We say that the functions $f, g : S \longrightarrow \mathbb{C}$ are linearly independent modulo $\mathcal{T}$ if $\lambda f + \mu g \in \mathcal{T}$ implies that $\lambda = \mu = 0$ for any $\lambda, \mu \in \mathbb{C}$.

Definition 2.4 *(see [20]).*
We say that the linear space $\mathcal{T}$ is
- left invariant if $f \in \mathcal{T}$ implies that the function $x \mapsto f(yx)$, $x \in S$ belongs to $\mathcal{T}$ for any $y \in S$.
- right invariant if $f \in \mathcal{T}$ implies that the function $x \mapsto f(xy)$, $x \in S$ belongs to $\mathcal{T}$ for any $y \in S$.
- tow sided invariant if $\mathcal{T}$ left and right invariant.

The set of all bounded complex-valued functions on S is an example of two sided invariant linear space of complex-valued functions on S.

3. Preliminaries

In this section, we give some useful results to prove our main results. In Lemma 3.1 the functional equation (1.5) holds in the case where $\mathcal{T}$ is two sided invariant, and f and g are linearly independent modulo $\mathcal{T}$.

Lemma 3.1 *Let $f, g : S \longrightarrow \mathbb{C}$ a solutions of (1.5) such that f and g are linearly independent modulo $\mathcal{T}$. Assume that $\mathcal{T}$ is two sided invariant. If the function*

$$x \longrightarrow f(xy) - f(x)g(y) - \beta g(x)f(y) - \gamma f(x)f(y)$$

belongs to $\mathcal{T}$ for all y in S, then

$$f(xy) = f(x)g(y) + \beta g(x)f(y) - \gamma f(x)f(y), \qquad \text{for all } x, y \in S$$

Proof: Define

$$\phi(x,y) := f(xy) - f(x)g(y) - \beta g(x)f(y) - \gamma f(x)f(y), \text{ for all } x,y \in S \tag{3.1}$$

Then

$$f(xy) = f(x)g(y) + \beta g(x)f(y) + \gamma f(x)f(y) + \phi(x,y). \tag{3.2}$$

Since f and g are linearly independent modulo $\mathcal{T}$, there exists y_0 in S such that $f(y_0) \neq 0$. Therefore $f(xy_0) = f(x)g(y_0) + \beta g(x)f(y_0) + \gamma f(x)f(y_0) + \phi(x,y_0)$

So, $g(x) = \frac{1}{\beta f(y_0)}f(xy_0) - \frac{g(y_0)}{\beta f(y_0)}f(x) - \frac{\gamma}{\beta}f(x) - \frac{1}{\beta f(y_0)}\phi(x,y_0)$

Let $\alpha_0 := \frac{1}{f(y_0)}$ and $\alpha_1 := \frac{g(y_0)}{f(y_0)}$

Then

$$g(x) = \frac{1}{\beta}[\alpha_0 f(xy_0) - (\alpha_1+\gamma)f(x) - \alpha_0\phi(x,y_0)] \tag{3.3}$$

Let $x,y,z \in S$ be arbitrary. We compute $f(xyz)$ using the associativity of the composition in S, and applying (3.2) and (3.3). We obtain:

$$\begin{aligned}
f((xy)z) =& f((xy)z) = f(xy)g(z) + \beta g(xy)f(z) + \gamma f(xy)f(z) + \phi(xy,z)\\
=& [g(z)+\gamma f(z)][f(x)g(y) + \beta g(x)f(y) + \gamma f(x)f(y) + \phi(x,y)]\\
&+ [\alpha_0 f(xyy_0) - (\alpha_1+\gamma)f(xy) - \alpha_0\phi(xy,y_0)]f(z) + \phi(xy,z)\\
=& [f(x)g(y) + \beta g(x)f(y) + \gamma f(x)f(y) + \phi(x,y)]g(z)\\
&+ \gamma[f(x)g(y) + \beta g(x)f(y) + \gamma f(x)f(y) + \phi(x,y)]f(z)\\
&+ \alpha_0[f(x)g(yy_0) + \beta g(x)f(yy_0) + \gamma f(x)f(yy_0) + \phi(x,yy_0)]f(z)\\
&- (\alpha_1+\gamma)[f(x)g(y) + \beta g(x)f(y) + \gamma f(x)f(y) + \phi(x,y)]f(z)\\
&- \alpha_0\phi(xy,y_0)f(z) + \phi(xy,z)\\
=& f(x)[g(y)g(z) + \gamma f(y)g(z) + \gamma g(y)f(z) + \gamma^2 f(y)f(z) + \alpha_0 g(yy_0)f(z)\\
&+ \alpha_0\gamma f(yy_0)f(z) - (\alpha_1+\gamma)g(y)f(z) - (\alpha_1\gamma+\gamma^2)f(y)f(z)]\\
&+ g(x)[\alpha_0\beta f(yy_0)f(z) + \beta f(y)g(z) + \gamma\beta f(y)f(z) - (\alpha_1\beta+\gamma\beta)f(y)f(z)]\\
&+ f(z)[\gamma\phi(x,y) + \alpha_0\phi(x,yy_0) - (\alpha_1+\gamma)\phi(x,y) - \alpha_0\phi(xy,y_0)]\\
&+ g(z)\phi(x,y) + \phi(xy,z)\\
=& f(x)[g(y)g(z) + \gamma f(y)g(z) + \alpha_0 g(yy_0)f(z) + \alpha_0\gamma f(yy_0)f(z)\\
&- \alpha_1 g(y)f(z) - \alpha_1\gamma f(y)f(z)] + g(x)[\alpha_0\beta f(yy_0)f(z) + \beta f(y)g(z)\\
&- \alpha_1\beta f(y)f(z)] + f(z)[\alpha_0\phi(x,yy_0) - \alpha_1\phi(x,y) - \alpha_0\phi(xy,y_0)]\\
&+ g(z)\phi(x,y) + \phi(xy,z)\\
=& f(x)[g(y)g(z) + \alpha_0 g(yy_0)f(z) - \alpha_1 g(y)f(z)\\
&+ \gamma(f(y)g(z) + \alpha_0 f(yy_0)f(z) - \alpha_1 f(y)f(z))]\\
&+ \beta g(x)[\alpha_0 f(yy_0)f(z) + f(y)g(z) - \alpha_1 f(y)f(z)]\\
&+ f(z)[\alpha_0\phi(x,yy_0) - \alpha_1\phi(x,y) - \alpha_0\phi(xy,y_0)]\\
&+ g(z)\phi(x,y) + \phi(xy,z)
\end{aligned}$$

On the other hand, by associativity:

$$f((xy)z) = f(x(yz)) = f(x)g(yz) + \beta g(x)f(yz) + \gamma f(x)f(yz) + \phi(x,yz). \tag{3.4}$$

By using the fact that f and g are linearly independent modulo $\mathcal{T}$ and that $\mathcal{T}$ is right invariant, we derive that

$$f(yz) = f(y)g(z) + \alpha_0 f(yy_0)f(z) - \alpha_1 f(y)f(z)$$

and

$$g(yz) = g(y)g(z) + \alpha_0 g(yy_0)f(z) - \alpha_1 g(y)f(z)$$

for all $y, z \in S$. Then

$$\phi(xy, z) - \phi(x, yz) = \big[\alpha_1\phi(x, y) + \alpha_0\phi(xy, y_0) - \alpha_0\phi(x, yy_0)\big] f(z) + \phi(x, y)g(z) \tag{3.5}$$

for all $x, y, z \in S$.

The function $z \mapsto \phi(xy, z) - \phi(x, yz)$ belongs to $\mathcal{T}$ for any $x, y \in S$. Hence the left hand side of (3.5) belongs to $\mathcal{T}$ as a function of z for any $x, y \in S$.

Now, using that f and g are linearly independent modulo $\mathcal{T}$ and the fact that $\mathcal{T}$ is tow sided invariant, we deduce from (3.5) that $\phi = 0$. Hence

$$f(xy) = f(x)g(y) + \beta g(x)f(y) + \gamma f(x)f(y), \qquad \text{for all } x, y \in S.$$

Completing the proof of Lemma 3.1 □

4. Main results

Theorem 4.1 *Let $f, g : S \longrightarrow \mathbb{C}$ be functions and let $\beta \in \mathbb{C} \setminus \{0\}$ and $\gamma \in \mathbb{C}$ be a constants such that the function*

$$(x, y) \mapsto f(xy) - f(x)g(y) - \beta g(x)f(y) - \gamma f(x)f(y)$$

is bounded. Then the pair $\{f, g\}$ falls into one of the following families where $\lambda \in \mathbb{C} \setminus \{0\}$ is a constant, $m : S \to \mathbb{C}$ is a multiplicative function.

1. *f and g are bounded functions.*
2. *$g + \gamma f$ multiplicative and $g \in \mathcal{B}(S)$.*
3. *$f = \lambda g - \lambda m$ and $g \notin \mathcal{B}(S)$, where $m \in \mathcal{B}(S)$ and $1 + \beta + \gamma\lambda = 0$,*
4. *$1 + \beta + \gamma\lambda \neq 0$,*
$$f = \frac{\lambda}{1 + \beta + \gamma\lambda} m + \frac{1}{1 + \beta + \gamma\lambda} b$$
and
$$g = \frac{1}{1 + \beta + \gamma\lambda} m - \frac{\beta + \gamma\lambda}{(1 + \beta)\lambda + \gamma\lambda^2} b,$$
5. *$f(xy) = f(x)g(y) + \beta g(x)f(y) + \gamma f(x)f(y)$ for all $x, y \in S$.*

Conversely, if one of the assertions $1 - 5$ is satisfied, then the function

$$(x, y) \mapsto f(xy) - f(x)g(y) - \beta g(x)f(y) - \gamma f(x)f(y)$$

is bounded.

Proof: We check by elementary computations, that if one of the assertions $1 - 5$ in Theorem 4.1 is satisfied then the function

$$(x, y) \mapsto f(xy) - f(x)f(y) - g(x)g(y) - \alpha f(x)g(y)$$

is bounded.
Conversely, let $f, g : S \to \mathbb{C}$ such that the function

$$(x, y) \mapsto f(xy) - f(x)f(y) - g(x)g(y) - \alpha f(x)g(y)$$

is bounded.
Define

$$\varphi(x, y) := f(xy) - f(x)g(y) - \beta g(x)f(y) - \gamma f(x)f(y), \text{ for all } x, y \in S \tag{4.1}$$

Case 1: If f and g are linearly independent modulo $\mathcal{B}(S)$ then, by applying Lemma 3.1 for $\mathcal{T} = \mathcal{B}(S)$, we get part 5 of Theorem 4.1.
If $g \in \mathcal{B}(S)$ then the function $x \mapsto f(xy) - f(x)(g(y) + \gamma f(y))$ belongs to $\mathcal{B}(S)$ for any $y \in S$. So, according to [22, Theorem], we deduce that $f \in \mathcal{B}(S)$ or $g + \gamma f$ is multiplicative. The results occur in parts 1 and 2 of Theorem 4.1.
Case 2: we assume that f and g are linearly dependent modulo $\mathcal{B}(S)$ and that $g \notin \mathcal{B}(S)$.
There exist $\lambda \in \mathbb{C} \setminus \{0\}$ and $b \in \mathcal{B}(S)$ such that $f = \lambda g + b$.

Now, substituting f by $\lambda g + b$ in (4.1) we obtain

$$\begin{aligned}\varphi(x,y) =& f(xy) - f(x)g(y) - \beta g(x)f(y) - \gamma f(x)f(y)\\ =&\lambda g(xy) + b(xy) - \lambda g(x)g(y) - b(x)g(y) - \lambda\beta g(x)g(y)\\ &- \beta g(x)b(y) - \gamma\lambda^2 g(x)g(y) - \gamma\lambda g(x)b(y) - \gamma\lambda b(x)g(y) - \gamma b(x)b(y)\\ =&\lambda g(xy) + b(xy) - (\lambda + \lambda\beta + \gamma\lambda^2)g(x)g(y) - \beta g(x)b(y)\\ &- \gamma\lambda g(x)b(y) - b(x)g(y) - \gamma\lambda b(x)g(y) - \gamma b(x)b(y)\\ =&\lambda g(xy) - ((\beta+1)\lambda + \gamma\lambda^2)g(x)g(y) - (\beta + \gamma\lambda)g(x)b(y)\\ &- (1+\gamma\lambda)b(x)g(y) + b(xy) - \gamma b(x)b(y)\end{aligned}$$

So that

$$\begin{aligned}\varphi(x,y) =& \lambda\big[g(xy) - ((\beta + 1 + \gamma\lambda)g(y) + \frac{\beta+\gamma\lambda}{\lambda}b(y))g(x)\big]\\ &- (1+\gamma\lambda)b(x)g(y) + b(xy) - \gamma b(x)b(y)\end{aligned} \tag{4.2}$$

we deduce from (4.2) that the function

$$x \mapsto g(xy) - ((\beta+1) + \gamma\lambda)g(y) + \frac{\beta+\gamma\lambda}{\lambda}b(y))g(x)$$

belongs to $\mathcal{B}(S)$ for any $y \in S$. So, by using [22, Theorem] and taking into account that $g \notin \mathcal{B}(S)$, we derive that

$$(\beta + 1 + \gamma\lambda)g + \frac{\beta+\gamma\lambda}{\lambda}b = m, \tag{4.3}$$

where $m : S \to \mathbb{C}$ is a multiplicative function. We split the discussion into the following subcases:

Subcase 2.1: $\beta + 1 + \gamma\lambda = 0$
Then (4.3) becomes:

$$\frac{\beta+\gamma\lambda}{\lambda}b = m$$

Note that $\beta + 1 + \gamma\lambda = 0$ implies $\beta + \gamma\lambda = -1$, so $b = -\lambda m$ and $f = \lambda g - \lambda m$, $m \in \mathcal{B}(S)$. The result occurs in part 3 of Theorem 4.1.

Subcase 2.2: $\beta + \gamma\lambda + 1 \neq 0$, we obtain

$$f = \frac{\lambda}{1+\beta+\gamma\lambda}m + \frac{1}{1+\beta+\gamma\lambda}b$$

and

$$g = \frac{1}{1+\beta+\gamma\lambda}m - \frac{\beta+\gamma\lambda}{(1+\beta)\lambda+\gamma\lambda^2}b.$$

The result occurs in part 4 of Theorem 4.1. This completes the proof of Theorem 4.1.

□

Corollary 4.1 *Let S be a monoid with identity element e, $f, g : S \longrightarrow \mathbb{C}$ two functions. Assume that $\mathcal{T}$ is a linear space of $\mathbb{C}$ valued functions right invariant on S. We suppose that f and g are linearly independent modulo $\mathcal{T}$ and $f(e) \neq 0$. If the function*

$$x \longrightarrow f(x\sigma(y)) - f(x)g(y) - \beta g(x)f(y) - \gamma f(x)f(y)$$

belongs to $\mathcal{T}$ for all y in S, then $fo\sigma = f$ and $go\sigma = g$

Proof: Define

$$\psi(x,y) := f(x\sigma(y)) - f(x)g(y) - \beta g(x)f(y) - \gamma f(x)f(y), \text{ for all } x,y \in S \tag{4.4}$$

Then

$$f(x\sigma(y)) = f(x)g(y) + \beta g(x)f(y) + \gamma f(x)f(y) + \psi(x,y) \tag{4.5}$$

Since f and g are linearly independent modulo $\mathcal{T}$, there exists y_0 in S such as $f(y_0) \neq 0$. Therefore $f(x\sigma(y_0)) = f(x)g(y_0) + \beta g(x)f(y_0) + \gamma f(x)f(y_0) + \psi(x,y_0)$

So, $g(x) = \dfrac{1}{\beta f(y_0)} f(x\sigma(y_0)) - \dfrac{g(y_0)}{\beta f(y_0)} f(x) - \dfrac{\gamma}{\beta} f(x) - \dfrac{1}{\beta f(y_0)}\psi(x,y_0)$

Let $\alpha_0 := \dfrac{1}{f(y_0)}$ and $\alpha_1 := \dfrac{g(y_0)}{f(y_0)}$

Then

$$g(x) = \frac{\alpha_0}{\beta} f(x\sigma(y_0)) - \frac{\alpha_1+\gamma}{\beta} f(x) - \frac{\alpha_0}{\beta}\psi(x,y_0) \tag{4.6}$$

Let $x,y,z \in S$ be arbitrary. We compute $f(x\sigma(y)\sigma(z))$ using the associativity of the composition in S, and applying (4.5) and (4.6) we obtain:

$$\begin{aligned}
f(x\sigma(y)\sigma(z)) =& f(x\sigma(y))g(z) + \beta g(x\sigma(y))f(z) + \gamma f(x\sigma(y))f(z) + \psi(x\sigma(y),z) \\
=& [f(x)g(y) + \beta g(x)f(y) + \gamma f(x)f(y) + \psi(x,y)]g(z) \\
&+ \beta[\frac{\alpha_0}{\beta} f(x\sigma(y)\sigma(y_0)) - \frac{\alpha_1+\gamma}{\beta} f(x\sigma(y)) - \frac{\alpha_0}{\beta}\psi(x\sigma(y),y_0)]f(z) \\
&+ \gamma f(x\sigma(y)f(z) + \psi(x\sigma(y),z) \\
=& [f(x)g(y) + \beta g(x)f(y) + \gamma f(x)f(y) + \psi(x,y)]g(z) \\
&+ [\alpha_0 f(x\sigma(y)\sigma(y_0)) - (\alpha_1+\gamma) f(x\sigma(y)) - \alpha_0\psi(x\sigma(y),y_0)]f(z) \\
&+ \gamma f(x\sigma(y)f(z) + \psi(x\sigma(y),z) \\
=& [f(x)g(y) + \beta g(x)f(y) + \gamma f(x)f(y) + \psi(x,y)]g(z) \\
&+ [\alpha_0 f(x\sigma(y)\sigma(y_0)) - \alpha_1 f(x\sigma(y)) - \alpha_0\psi(x\sigma(y),y_0)]f(z) \\
&+ \psi(x\sigma(y),z) \\
=& [f(x)g(y) + \beta g(x)f(y) + \gamma f(x)f(y) + \psi(x,y)]g(z) \\
&+ \alpha_0[f(x)g(yy_0) + \beta g(x)f(yy_0) + \gamma f(x)f(yy_0) + \psi(x,yy_0)]f(z) \\
&- \alpha_1[f(x)g(y) + \beta g(x)f(y) + \gamma f(x)f(y) + \psi(x,y)]f(z) \\
&- \alpha_0\phi(x\sigma(y),y_0)f(z) + \psi(x\sigma(y),z)
\end{aligned}$$

$$\begin{aligned}
f(x\sigma(y)\sigma(z)) =& f(x)[g(y)g(z) + \gamma f(y)g(z) + \alpha_0 g(yy_0)f(z) + \alpha_0\gamma f(yy_0)f(z) - \alpha_1 g(y)f(z) - \alpha_1\gamma f(y)f(z)] \\
&+ g(x)[\beta f(y)g(z) + \alpha_0\beta f(yy_0)f(z) - \alpha_1\beta f(y)f(z)] \\
&+ f(z)[\alpha_0\psi(x,yy_0) - \alpha_1\psi(x,y) - \alpha_0\psi(x\sigma(y),y_0)] \\
&+ g(z)\psi(x,y) + \psi(x\sigma(y),z)
\end{aligned} \tag{4.7}$$

On the other hand:

$$f(x\sigma(y)\sigma(z)) = f(x\sigma(yz)) = f(x)g(yz) + \beta g(x)f(yz) + \gamma f(x)f(yz) + \psi(x,yz) \tag{4.8}$$

From (4.7) and (4.8), and using the linear independence of f and g modulo $\mathcal{T}$ and that $\beta \neq 0$ we get:

$$f(yz) = f(y)g(z) + \alpha_0 f(yy_0)f(z) - \alpha_1 f(y)f(z)$$

Then we write

$$f(xy) = f(x)g(y) + \alpha_0 f(xy_0)f(y) - \alpha_1 f(x)f(y) \tag{4.9}$$

Now, applying (4.5) for $(x, \sigma(y))$ we obtain:

$$f(xy) = f(x)g(\sigma(y)) + \beta g(x)f(\sigma(y)) + \gamma f(x)f(\sigma(y)) + \psi(x, \sigma(y)) \tag{4.10}$$

Subtracting (4.9) from (4.10) we obtain

$$\psi(x, \sigma(y)) = f(x)(g(y) - g(\sigma(y)) + \alpha_0 f(xy_0)f(y) - \beta g(x)f(\sigma(y)) + f(x)(\alpha_1 f(y) - \gamma f(\sigma(y)))$$

So,

$$\psi(x, \sigma(y)) = 2f(x)g_o(y) + \alpha_0 f(xy_0)f(y) - \beta g(x)f(\sigma(y)) + f(x)(\alpha_1 f(y) - \gamma f(\sigma(y))) \tag{4.11}$$

Replacing y by $\sigma(y)$ in (4.11), we obtain

$$\psi(x, y) = -2f(x)g_o(y) + \alpha_0 f(xy_0)f(\sigma(y)) - \beta g(x)f(y) + f(x)(\alpha_1 f(\sigma(y)) - \gamma f(y)) \tag{4.12}$$

Adding (4.11) and (4.12) we have

$$\begin{aligned}\psi(x, \sigma(y)) + \psi(x, y) =& \alpha_0 f(xy_0)(f(y) + fo\sigma(y)) - \beta g(x)(f(y) + fo\sigma(y)) \\ &+ \alpha_1 f(x)(f(y) + fo\sigma(y)) - \gamma f(x)(f(y) + fo\sigma(y)) \\ =& 2\alpha_0 f(xy_0)f_e(y) - 2\beta g(x)f_e(y) + 2\alpha_1 f(x)f_e(y) \\ &- 2\gamma f(x)f_e(y) \\ =& 2f_e(y)[\alpha_0 f(xy_0) - \beta g(x) + (\alpha_1 - \gamma)f(x)]\end{aligned}$$

Thus,

$$\psi(x, \sigma(y)) + \psi(x, y) = f_e(y)h(x) \tag{4.13}$$

where $h(x) := \alpha_0 f(xy_0) - \beta g(x) + (\alpha_1 - \gamma)f(x)$, $x \in S$

Subtracting (4.12) from (4.11) we have

$$\begin{aligned}\psi(x, \sigma(y)) - \psi(x, y) =& 4f(x)g_o(y) + \alpha_0 f(xy_0)(f(y) - f(\sigma(y)) \\ &+ \beta g(x)(f(y) - f(\sigma(y)) + \alpha_1 f(x)(f(y) - f(\sigma(y)) \\ &+ \gamma f(x)(f(y) - f(\sigma(y)) \\ =& 4f(x)g_o(y) + 2\alpha_0 f(xy_0)f_o(y) + 2\beta g(x)f_o(y) \\ &+ 2\alpha_1 f(x)f_o(y) + 2\gamma f(x)f_o(y) \\ =& 4f(x)g_o(y) + 2f_o(y)[\alpha_0 f(xy_0) - \beta g(x) + (\alpha_1 - \gamma)f(x)] \\ &+ 4\beta g(x)f_o(y) + 4\gamma f(x)f_o(y) \\ =& 2f_o(y)h(x) + 4f(x)g_o(y) + 4\gamma f(x)f_o(y) + 4\beta g(x)f_o(y)\end{aligned}$$

So,

$$\psi(x, \sigma(y)) - \psi(x, y) = 2f_o(y)h(x) + 4f(x)[g_o(y) + \gamma f_o(y)] + 4\beta g(x)f_o(y) \tag{4.14}$$

We discus tow cases $f_e = 0$ and $f_e \neq 0$.
Case 1: $f_e \neq 0$ (i.e., $fo\sigma \neq -f$). Then there exists $y_1 \in S$ such that $f_e(y_1) \neq 0$
Putting $y = y_1$ in (4.13) we get: $\psi(x, \sigma(y_1)) + \psi(x, y_1) = f_e(y_1)h(x)$, then $h \in \mathcal{T}$ because $x \mapsto \psi(x, \sigma(y))$ and $x \mapsto \psi(x; y)$ belong to $\mathcal{T}$, for any $y \in S$.

Let $y \in S$ be arbitrary, we have $x \mapsto \psi(x, \sigma(y))$ and $x \mapsto \psi(x, y)$, $x \mapsto h(x)$ belong to $\mathcal{T}$ then from (4.14) we get that,

$$x \mapsto 4f(x)(g_o(y) + \gamma f_o(y)) + 4\beta g(x)f_o(y) \in \mathcal{T}, \text{ for any } y \in S$$

Since f and g are linearly independent modulo $\mathcal{T}$ we have: $g_o(y)+\gamma f_o(y)=0$ and $f_o(y)=0$ for all $y\in S$. Thus, $f\circ\sigma=f$ and $g\circ\sigma=g$.
Case 2: $f_e=0$ (i.e., $fo\sigma=-f$)

From (4.13) we have:

$$\psi(x,\sigma(y))+\psi(x,y)=0 \tag{4.15}$$

Applying (4.5) and (4.10) we get,

$$\begin{aligned} f(xy)+f(x\sigma(y)) =& f(x)(g(y)+go\sigma(y))+\beta g(x)(f(y)+fo\sigma(y))+\gamma f(x)(f(y)+fo\sigma(y)) \\ =& 2f(x)g_e(y)+2\beta g(x)f_e(y)+2\gamma f(x)f_e(y) \\ =& 2f(x)g_e(y). \end{aligned} \tag{4.16}$$

Put $x=e$ in (4.16), we have $2f_e(y)=2f(e)g_e(y)$ for all y in S. So, $f(e)g_e(y)=0$ for all y in S

Since $f(e)\neq 0$ we deduce that $g_e=0$.

Now set $y=e$ in (4.16) we get $f(xe)+f(x\sigma(e))=2f(x)g_e(e)=0$. But, $f(xe)+f(x\sigma(e))=f(x)+f(x)=2f(x)=0$, so $f(x)=0$ for all $x\in S$ contradictiong f and g are linearly independant modulo $\mathcal{T}$. Therefore, case 2 leads to a contradiction, and only case 1 is possible, completing the proof. □

In the next theorem, we generalize the main result of Theorem 4.1, extending the stability established for functional equation (1.5) to the functional equation (1.6) to a monoid S.

Theorem 4.2 *Let $f,g:S\longrightarrow\mathbb{C}$ be functions and let $\beta\in\mathbb{C}\setminus\{0\}$ and $\gamma\in\mathbb{C}$ be a constants such that the function*

$$(x,y)\mapsto f(x\sigma(y))-f(x)g(y)-\beta g(x)f(y)-\gamma f(x)f(y)$$

is bounded. Then the pair $\{f,g\}$ falls into one of the following families where $\lambda\in\mathbb{C}\setminus\{0\}$ is a constant, $m:S\to\mathbb{C}$ is a multiplicative function, $b\in\mathcal{B}(S)$.

1. *f and g are bounded functions and σ invariant, where $f(e)\neq 0$.*
2. *$g+\gamma f$ multiplicative and $g\in\mathcal{B}(S)$, where $(g+\gamma f)o\sigma=g+\gamma f$ and $go\sigma=g$.*
3. *$f=\lambda g-\lambda m$ and $g\notin\mathcal{B}(S)$, where $mo\sigma=m$ bounded and $fo\sigma=f$, $go\sigma=g$, $1+\beta+\gamma\lambda=0$.*
4. *$1+\beta+\gamma\lambda\neq 0$,*
$$f=\frac{\lambda}{1+\beta+\gamma\lambda}m+\frac{1}{1+\beta+\gamma\lambda}b$$
and
$$g=\frac{1}{1+\beta+\gamma\lambda}m-\frac{\beta+\gamma\lambda}{(1+\beta)\lambda+\gamma\lambda^2}b.$$
where, $m\circ\sigma=m,\quad b\circ\sigma=b$
5. *$f(x\sigma(y))=f(x)g(y)+\beta g(x)f(y)+\gamma f(x)f(y)$ for all $x,y\in S$.*

Conversely, if one of the assertions $1-5$ is satisfied, then the function

$$(x,y)\mapsto f(x\sigma(y))-f(x)g(y)-\beta g(x)f(y)-\gamma f(x)f(y)$$

is bounded.

Proof: By applying Theorem 4.1 to the pair $(x,\sigma(y))$ and corollary 4.1 the proof of Theorem 4.2 follows directly. □

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