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Infinite Families of Sextic Number Fields with all Possible Indices

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ABSTRACT: For each rational prime $p \in \{2,3,5\}$, we construct infinite families of sextic number fields K such that the p-adic valuation of the index $i(K)$ satisfies $\nu_p(i(K)) = \nu_p$, for every possible positive integer ν_p. We illustrate our results by some computational examples.

Keywords: Theorem of Dedekind, Theorem of Ore, prime ideal factorization, Newton polygon, index of a number field, monogenic.

Contents

1. Introduction

Let $K = \mathbb{Q}(\alpha)$ be a number field of degree n, where α is a primitive integer of K, and let $\mathbb{Z}_K$ denotes the ring of integers of K. The index of α, denoted by $(\mathbb{Z}_K : \mathbb{Z}[\alpha])$, is the index of the Abelian group $\mathbb{Z}[\alpha]$ in $\mathbb{Z}_K$. A well-known formula linking this index with the discriminants is given by:

$$\Delta(\alpha) = (\mathbb{Z}_K : \mathbb{Z}[\alpha])^2 \cdot d_K, \tag{1.1}$$

where d_K is the absolute discriminant of K and $\Delta(\alpha)$ is the discriminant of the minimal polynomial of α over $\mathbb{Q}$. The index of K, denoted by $i(K)$, is defined as the greatest common divisor of the indices of all primitive integers of K. That is, $i(K) = \gcd\{(\mathbb{Z}_K : \mathbb{Z}[\theta]) \mid K = \mathbb{Q}(\theta) \text{ and } \theta \in \mathbb{Z}_K\}$. It is well known that if K is monogenic, then its index is trivial; $i(K) = 1$. Therefore, a number field with non-trivial index is not monogenic. Dedekind was the first to discover a number field with non-trivial index ([3]). In 1930, for every number field K of degree $n \leq 7$ and every rational prime p, Engstrom established a connection between the prime ideal factorization of $p\mathbb{Z}_K$ and $\nu_p(i(K))$. This motivated a very important question, stated as problem 22 in Narkiewicz's book ([19]), which asks for an explicit formula of the highest power $\nu_p(i(K))$ for a given rational prime p dividing $i(K)$. In [24], Śliwa extended Engstrom's results to number fields up to degree 12, under the condition that p is unramified in K. These results were further generalized by Nart ([20]), who developed a p-adic characterization of the index of a number field. In [18], Nakahara studied the indices of non-cyclic but abelian biquadratic number fields. In [8], Funakura showed that $i(K) = 1$ or 2 for every pure quartic number field K. In [10], Gaál et al. characterized the field indices of biquadratic number fields. In [25], Spearman and Williams characterized the indices of cyclic quartic number fields. In [23], Pethö and Pohst studied the index divisors of multiquadratic number fields. Recently, many authors are interested in the characterization of the prime power decomposition of the indices of number fields, especially those defined by trinomials and quadrinomials of fixed degrees (see [2,?,4,5,6,8,10,15,16,17,18,23,25]). In all the former papers, for a given number field K, the authors try to calculate the index $i(K)$. In contrast, the present paper introduces a new approach. Namely, for each rational prime $p \in \{2,3,5\}$ and every possible natural integer ν_p, we construct infinite families of sextic number fields, with p-indices ν_p, where the p-index of a number field K is defined as the p-valuation of its index. Namely, $\nu_p = \nu_p(i(K))$. According to Engstrom's results ([?]), the index of any sextic number

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field K is of the form $i(K) = 2^{\nu_2} \cdot 3^{\nu_3} \cdot 5^{\nu_5}$, where $\nu_2 \in \{0,1,2,3,4,8\}$, $\nu_3 \in \{0,1,2,3\}$ and $\nu_5 \in \{0,1\}$. These results exhibit infinite families of sextic number fields for each of the possible non-trivial index values.

2. Main Results

Given that the index of sextic number field is of the form $i = 2^{\nu_2} \cdot 3^{\nu_3} \cdot 5^{\nu_5}$, where $\nu_2 \in \{0,1,2,3,4,8\}$, $\nu_3 \in \{0,1,2,3\}$ and $\nu_5 \in \{0,1\}$, in the remainder, for any rational prime $p \in \{2,3,5\}$, we provide infinite families of sextic number fields with p-indices ν_p, where $\nu_2 \in \{1,2,3,4,8\}$, $\nu_3 \in \{1,2,3\}$ and $\nu_5 = 1$.

In the reminder, $K = \mathbb{Q}(\alpha)$ is a sextic number field generated by a root θ of a monic irreducible polynomial, $F(x) = x^6 + a_5x^5 + a_4x^4 + a_3x^3 + a_2x^2 + a_1x + a_0 \in \mathbb{Z}[x]$.
For $p = 2$ and $\nu_2 = 1$, Theorem 2.1 provides sufficient conditions on $F(x)$, which guarantee that each sextic number field of these infinite families has 2-index $\nu_2 = 1$.

Theorem 2.1 *Suppose that for every $i = 0, \ldots, 5$, $\nu_2(a_i) \geq 1$. Then each of the following conditions guarantees that $\nu_2(i(K)) = 1$.*

1. $a_3 \equiv 2 \pmod 4$, $a_2 \equiv 0 \pmod 4$, $\nu_2(a_0) < 2\nu_2(a_1) - \nu_2(a_2)$, $\nu_2(a_0) > 3\nu_2(a_2) - 2$ *and* $\nu_2(a_0) \not\equiv \nu_2(a_2) \pmod 2$.
2. $a_4 \equiv 2 \pmod 4$, $a_3 \equiv 0 \pmod 4$, $a_2 \equiv 4 \pmod 8$, $a_1 \equiv 0 \pmod 8$ *and* $\nu_2(a_0) = 2\nu_2(a_1) - 2$.

Example 2.1 Let K be a sextic number field defined by the monic irreducible polynomial $F(x) = x^6 + 12x^4 + 14x^3 + 12x^2 + 48x + 32$. Since $a_3 \equiv 2 \pmod 4$, $a_2 \equiv 4 \pmod 8$, $a_1 \equiv 16 \pmod{32}$ and $a_0 \equiv 32 \pmod{64}$, by Theorem 2.1 (1), we conclude that $\nu_2(i(K)) = 1$.

Recall that, for every rational integer $z \in \mathbb{Z}$, the $(x - z)$-Taylor expansion of every polynomial $F(x)$ of degree 6 is given by the following:

$$F(x) = \sum_{k=0}^{6} \frac{F^{(k)}(z)}{k!} (x - z)^k.$$

In the remainder, we shall denote $A_k(z) = \dfrac{F^{(k)}(z)}{k!}$.
The following theorem provides infinite family of sextic number fields with 2-indices $\nu_2 = 2$.

Theorem 2.2 *Suppose that $\nu_2(a_4) = 0$, $\nu_2(a_i) \geq 1$ for every $i \neq 4$ and $\nu_2(A_0(1)) = 2\nu_2(A_1(1))$. Then each of the following conditions guarantees that $\nu_2(i(K)) = 2$.*

1. $\nu_2(a_0) > \frac{4}{3}\nu_2(a_1)$, $\nu_2(a_1) > \frac{3}{2}\nu_2(a_2)$, $\nu_2(a_1) > 3\nu_2(a_3)$ *and* $\nu_2(a_1) \equiv 0 \pmod 3$.
2. *If* $a_3 \equiv 2 \pmod 4$, $a_2 \equiv 4 \pmod 8$, $a_1 \equiv 0 \pmod{16}$ *and* $\nu_2(a_0) = 2\nu_2(a_1) - 2$.

Example 2.2 Let K be a sextic number field defined by the monic irreducible polynomial $F(x) = x^6 + 51x^4 + 48x^3 + 96x^2 + 24x + 384$. Since $a_0 \equiv 0 \pmod{128}$, $a_1 \equiv 8 \pmod{16}$, $a_2 \equiv a_3 \equiv 0 \pmod{16}$, a_4 is odd, $A_0(1) = 604 \equiv 4 \pmod 8$ and $A_1(1) = 570 \equiv 2 \pmod 4$, then by Theorem 2.2 (1), we obtain $\nu_2(i(K)) = 2$.

In the next theorem, we provide infinite family of sextic number fields with 2-indices $\nu_2 = 3$.

Theorem 2.3 *Suppose that $\nu_2(a_4) = 0$ and $\nu_2(a_i) \geq 1$ for every $i \neq 4$. Then the following conditions guarantee that $\nu_2(i(K)) = 3$.*

1. $\nu_2(A_0(1)) < 2\nu_2(A_1(1))$ *and* $\nu_2(A_0(1))$ *is odd.*

2. $\nu_2(a_2) < 2\nu_2(a_3)$, $\nu_2(a_2)$ *is odd,* $\nu_2(a_1) > \nu_2(a_2) + \nu_2(a_3)$, $2\nu_2(a_2) < \nu_2(a_0) < 2\nu_2(a_1) - \nu_2(a_2)$ *and* $\nu_2(a_0)$ *is even.*

Example 2.3 Let K be a sextic number field defined by the monic irreducible polynomial $F(x) = x^6 + 3x^4 + 10x^2 + 24x + 16$. Since $a_0 \equiv 16 \pmod{32}$, $a_1 \equiv 8 \pmod{16}$, $a_2 \equiv 2 \pmod 4$, a_4 is odd. On the other hand, $A_0(1) = 54 \equiv 2 \pmod 4$. Therefore, by Theorem 2.3, we obtain $\nu_2(i(K)) = 3$.

Theorems 2.4 and 2.5 provide sufficient conditions on $F(x)$, which guarantee that each sextic number field of these infinite families has 2-index $\nu_2 = 4$.

Theorem 2.4 *Suppose that* 2 *does not divide* a_4 *and* $\nu_2(a_i) \geq 1$ *for every* $i \neq 4$. *Then the following conditions guarantee that* $\nu_2(i(K)) = 4$.

1. $\nu_2(A_0(1)) < 2\nu_2(A_1(1))$ *and* $\nu_2(A_0(1))$ *is odd.*

2. $\nu_2(a_{i-1}) > 2\nu_2(a_i) - \nu(a_{i+1})$ *for every* $i = 1, 2, 3$.

Theorem 2.5 *Suppose that* 2 *does not divide* a_4 *and* $\nu_2(a_i) \geq 1$ *for every* $i \neq 4$. *Then the following conditions guarantee that* $\nu_2(i(K)) = 4$.

1. $\nu_2(A_0(1)) > 2\nu_2(A_1(1))$.

2. $\nu_2(a_2) < 2\nu_2(a_3)$, $\nu_2(a_2)$ *is odd,* $\nu_2(a_1) > \nu_2(a_2) + \nu_2(a_3)$ *and* $\nu_2(a_0) > 2\nu_2(a_1)$.

Example 2.4 Let K be a sextic number field defined by the monic irreducible polynomial $F(x) = x^6 + 12x^5 + 3x^4 + 6x^3 + 2x^2 + 8x + 128$. Since $a_0 \equiv 128 \pmod{256}$, $a_1 \equiv 8 \pmod{16}$, $a_2 \equiv 2 \pmod 4$ a_4 is odd, $A_0(1) = 160 \equiv 32 \pmod{64}$ and $A_1(1) = 108 \equiv 4 \pmod 8$, by theorem 2.5, we conclude that $\nu_2(i(K)) = 4$.

The following theorem provides infinite families of sextic number fields with 2-indices $\nu_2 = 8$.

Theorem 2.6 *Each one of the following conditions guarantee that* $\nu_2(i(K)) = 8$.

1. $\nu_2(a_2) = 0$, $\nu_2(a_i) \geq 1$ *for every* $i \neq 2$, $\nu_2(a_0) > 2\nu_2(a_1)$ *and* $\nu_2(A_{i-1}) > 2\nu_2(A_i) - \nu_2(A_{i+1})$ *for every* $i = 1, 2, 3$.

2. $\nu_2(a_i) \geq 1$ *for* $i = 0, 1, 2$, $\nu_2(a_i) = 0$ *for* $i = 3, 4, 5$, $\nu_2(a_{i-1}) > 2\nu_2(a_i) - \nu_2(a_{i+1})$ *and* $\nu_2(A_{i-1}) > 2\nu_2(A_i) - \nu_2(A_{i+1})$ *for every* $i = 1, 2$.

3. $\nu_2(a_4) = 0$, $\nu_2(a_i) \geq 1$ *for every* $i \neq 4$, $\nu_2(A_0(1)) \geq 2\nu_2(A_1(1))$ *and* $\nu_2(a_{i-1}) > 2\nu_2(a_i) - \nu_2(a_{i+1})$ *for every* $i = 1, 2, 3$.

Example 2.5 Let K be a sextic number field defined by the monic irreducible polynomial $F(x) = x^6 + x^5 + x^4 + 7x^3 + 14x^2 + 40x + 64$. Since a_3, a_4, a_5 are odd, $a_0 \equiv 64 \pmod{128}$, $a_1 \equiv 8 \pmod{16}$, $a_2 \equiv 2 \pmod 4$, $A_0(1) = 128$, $A_1(1) = 104 \equiv 8 \pmod{16}$ and $A_2(1) = 66 \equiv 2 \pmod 4$, then by Theorem 2.6 (2), we conclude that $\nu_2(i(K)) = 8$.

For $p = 3$ and $\nu_3 = 1$, Theorems 2.7 and 2.8 provide sufficient conditions on $F(x)$, which guarantee that each sextic number field of these infinite families has 3-index $\nu_3 = 1$.

Theorem 2.7 *Suppose that* $a_4 \equiv -1 \pmod 3$ *and* $\nu_3(a_i) \geq 1$ *for every* $i \neq 4$. *Then the following conditions guarantee that* $\nu_3(i(K)) = 1$.

1. $\nu_3(a_1) \le \nu_3(a_i)$ *for every* $i = 2, 3$.
2. $\nu_3(a_1) \not\equiv 0 \pmod 3$ *and* $\nu_3(a_0) > \dfrac{3}{2}\nu_3(a_1)$.

Example 2.6 Let K be a sextic number field defined by the monic irreducible polynomial $F(x) = x^6 + 12x^5 - x^4 + 9x^3 + 27x^2 + 24x + 18$. Since $a_4 \equiv -1 \pmod 3$, $\nu_3(a_0) = 2$, $\nu_3(a_1) = 1$. $\nu_3(a_2) = 3$ and $\nu(a_3) = 2$, by Theorem 2.7, we conclude that $\nu_3(i(K)) = 1$.

In the remainder of this section, for every rational prime $p \in \{3, 5\}$ and every, we shall denote $(a_i)_p = \dfrac{a_i}{p^{\nu_p(a_i)}}$ for every rational integer $a_i \in \mathbb{Z}$.

Theorem 2.8 *Suppose that* $a_3 \equiv -1 \pmod 3$ *and* $a_i \equiv 0 \pmod 3$ *for every* $i \ne 3$. *Then each one of the following conditions guarantees that* $\nu_3(i(K)) = 1$.

1. $\nu_3(a_1) < 2\nu_3(a_2)$, $\nu_3(a_0) > 2\nu_3(a_1)$, $\nu_3(a_1) \not\equiv 0 \pmod 2$, $\nu_3(A_1(1)) < 2\nu_3(A_2(1))$, $\nu_3(A_1(1)) \not\equiv 0 \pmod 2$ *and* $\nu_3(A_0(1)) > 2\nu_3(A_1(1))$.
2. $\nu_3(a_1) > 2\nu_3(a_2)$, $\nu_3(a_0) > 2\nu_3(a_1) - \nu_3(a_2)$, $\nu_3(A_0(1)) < \dfrac{3}{2}\nu_3(A_1(1))$, $\nu_3(A_0(1)) < 3\nu_3(A_2(1))$ *and* $\nu_3(A_0(1)) \not\equiv 0 \pmod 3$.
3. $\nu_3(a_1) < 2\nu_3(a_2)$, $\nu_3(a_1)$ *is even,* $(a_1)_3 \equiv -1 \pmod 3$, $\nu_3(a_0) > \dfrac{3}{2}\nu_3(a_1)$, $\nu_3(A_1(1)) > 2\nu_3(A_2(1))$ *and* $\nu_3(A_0(1)) > 2\nu_3(A_1(1)) - \nu_3(A_2(1))$.

Example 2.7 Let K be a sextic number field defined by the monic irreducible polynomial $F(x) = x^6 + 12x^4 + 8x^3 + 3x^2 + 3x + 27$. Since $\nu_3(a_0) = 3$, $\nu_3(a_1) = 1$, $\nu_3(a_2) = 1$, $a_3 \equiv -1 \pmod 3$, $A_0(1) = 54 \equiv 54 \pmod{81}$ and $A_1(1) = 87 \equiv -3 \pmod 9$, then by Theorem 2.8 (1), we conclude $\nu_3(i(K)) = 1$.

The following theorem provides infinite families of sextic number fields with 3-indices $\nu_3 = 2$.

Theorem 2.9 *Suppose that* $a_4 \equiv -1 \pmod 3$ *and* $\nu_3(a_i) \ge 1$ *for every* $i \ne 4$. *Then the following conditions guarantee that* $\nu_3(i(K)) = 2$.

1. $\nu_3(a_2) < 2\nu_3(a_3)$ *and* $\nu_3(a_2)$ *is odd.*
2. $\nu_3(a_1) > 2\nu_3(a_2)$ *and* $\nu_3(a_0) > 2\nu_3(a_1)$.

Example 2.8 Let K be a sextic number field defined by the monic irreducible polynomial $F(x) = x^6 + 2x^4 + 6x^3 + 12x^2 + 54x + 2187$. Since $\nu_3(a_0) = 7$, $\nu_3(a_1) = 3$, $\nu_3(a_2) = 1$ and $\nu_3(a_3) = 1$. By theorem 2.9, we get $\nu_3(i(K)) = 2$.

In the next theorem, we provide infinite families of sextic number fields with 3-indices $\nu_3 = 3$.

Theorem 2.10 *Suppose that* $a_3 \equiv -1 \pmod 3$ *and* $a_i \equiv 0 \pmod 3$ *for every* $i \ne 3$. *Then each one of the following conditions guarantees that* $\nu_3(i(K)) = 3$.

1. $\nu_3(a_{i-1}) > 2\nu_3(a_i) - \nu_3(a_{i+1})$ *and* $\nu_3(A_{i-1}(1)) > 2\nu_3(A_i(1)) - \nu_3(A_{i+1}(1))$ *for every* $i = 1, 2$.
2. $\nu_3(a_1) < 2\nu_3(a_2)$, $\nu_3(a_1)$ *is even,* $(a_1)_3 \equiv 1 \pmod 3$, $\nu_3(a_0) > 2\nu_3(a_1)$, $\nu_3(A_1(1)) < 2\nu_3(A_2(1))$, $\nu_3(A_1(1))$ *is even,* $(A_1(1))_3 \equiv 1 \pmod 3$ *and* $\nu_3(A_0(1)) > 2\nu_3(A_1(1))$.

Example 2.9 Let K be a sextic number field defined by the monic irreducible polynomial $F(x) = x^6 + 50x^3 + 3x^2 + 675x + 729$. Since $\nu_3(a_0) = 6$, $\nu_3(a_1) = 3$, $\nu_3(a_2) = 1$, $\nu_3(A_0(1)) = 6$, $\nu_3(A_1(1)) = 3$ and $\nu_3(A_2(1)) = 1$, then by theorem 2.5 (1), we conclude $\nu_3(i(K)) = 3$.

For $p = 5$ and $\nu_5 = 1$, Theorem 2.11 provides sufficient conditions on $F(x)$, which guarantee that each sextic number field of these infinite families has 5-index $\nu_5 = 1$.

Theorem 2.11 *Suppose that $a_2 \equiv -1 \pmod 5$ and $a_i \equiv 0 \pmod 5$ for every $i \neq 2$. Then each one of the following conditions guarantees that $\nu_5(i(K)) = 1$.*

1. $\nu_5(a_0) > 2\nu_5(a_1)$.
2. $\nu_5(a_0) = 2\nu_5(a_1)$ *and* $((a_0)_5, (a_1)_5) \in \{(2, 1), (2, 4)(3, 2), (3, 3)\} \pmod 5$.
3. $\nu_5(a_0) < 2\nu_5(a_1)$, $\nu_5(a_0)$ *even,* $\nu_5(a_0) < \nu_5(a_1)$ *and* $(a_0)_5 \equiv \pm 1 \pmod 5$.

Example 2.10 Let K be a sextic number field defined by the monic irreducible polynomial $F(x) = x^6 + 10x^4 - x^2 + 500x + 25$. Since $\nu_5(a_0) = 2$, $\nu_5(a_1) = 3$ and $(a_0)_5 \equiv 1 \pmod 5$, by Theorem 2.11 (3), we conclude $\nu_5(i(K)) = 1$.

The following example provides a sextic number field with index $i(K) = 240$.

Example 2.11 Let K be a sextic number field defined by the monic irreducible polynomial $F(x) = x^6 + 2x^5 + 5x^4 + 6x^3 + 24x^2 + 1920x + 4608000$.

1. For $p = 2$, since a_4 is odd, $a_0 \equiv 4096 \pmod{8192}$, $a_1 \equiv 128 \pmod{256}$, $a_2 \equiv 8 \pmod{16}$, $A_0(1) = 4609958 \equiv 2 \pmod 4$ and $A_1(1) = 2022 \equiv 2 \pmod 4$. Then by theorem 2.4, we get $\nu_2(i(K)) = 4$.
2. For $p = 3$, $a_4 \equiv -1 \pmod 3$, $\nu_3(a_0) = 2$, $\nu_3(a_1) = 1$. $\nu_3(a_2) = 3$ and $\nu(a_3) = 2$. By Theorem 2.7, we get $\nu_3(i(K)) = 1$.
3. Finally, for $p = 5$, we have $\nu_5(a_0) = 3$ and $\nu_5(a_1) = 1$. Then by theorem 2.11 (1), we get $\nu_5(i(K)) = 1$.

We conclude that $i(K) = 240$.

3. Preliminaries

Our proofs are based on Newton polygon techniques applied on prime ideal factorization, which is rather technical but very efficient to apply. We have introduced the corresponding concepts in several former papers. Here we only give the theorem of index of Ore which plays a key role for proving our main results. For more details, we refer to [7] and [12].

Let $K = \mathbb{Q}(\alpha)$ be a number field generated by a complex root α of a monic irreducible polynomial $F(x) \in \mathbb{Z}[x]$. We shall use Dedekind's theorem [21, Chapter I, Proposition 8.3] and Dedekind's criterion [1, Theorem 6.1.4]. Let $\phi \in \mathbb{Z}_p[x]$ be a monic lift to an irreducible factor of $F(x)$ modulo p, $F(x) = a_0(x) + a_1(x)\phi(x) + \cdots + a_k(x)\phi(x)^k$ the ϕ-expansion of $F(x)$ and $N_\phi^+(F)$ the principal ϕ-Newton polygon of $F(x)$, which can be obtained only by considering the principal ϕ-expansion of $F(x)$. As defined in [7, Def. 1.3], the ϕ-index of $F(x)$, denoted $\mathrm{ind}_\phi(F)$, is $\deg(\phi)$ multiplied by the number of points with natural integer coordinates that lie below or on the polygon $N_\phi^+(F)$, strictly above the horizontal axis and strictly beyond the vertical axis. Let $\mathbb{F}_\phi$ be the field $\mathbb{F}_p[x]/(\overline{\phi})$ and $u_i = \nu_p(a_i(x))$, then to every side S of $N_\phi^+(F)$ with initial point (i, u_i), length $l = l(S)$ and every $i = 0, \ldots, l$, let the residue coefficient $c_i \in \mathbb{F}_\phi$ defined as follows:

$$c_i = \begin{cases} 0, & \text{if } (s+i, u_{s+i}) \text{ lies strictly above } S, \\ \left(\dfrac{a_{s+i}(x)}{p^{u_{s+i}}}\right) \mod (p, \phi(x)), & \text{if } (s+i, u_{s+i}) \text{ lies on } S. \end{cases}$$

Let $-\lambda = -h/e$ be the slope of S, where h and e are two positive coprime integers and $l = l(S)$ its length. Then $d = l/e$ is the degree of S. Hence, if i is not a multiple of e, then $(s+i, u_{s+i})$ does not lie on S and $c_i = 0$. Let $R_\lambda(F)(y) = t_d y^d + t_{d-1}y^{d-1} + \cdots + t_1 y + t_0 \in \mathbb{F}_\phi[y]$ be the residual polynomial of $F(x)$ associated to the side S, where for every $i = 0, \ldots, d$, $t_i = c_{s+ie}$. If $R_\lambda(F)(y)$ is square-free for each side of the polygon $N_\phi^+(F)$, then we say that $F(x)$ is ϕ-regular.

Let $\overline{F(x)} = \prod_{i=1}^r \overline{\phi_i}^{k_i}$ be the factorization of $F(x)$ into powers of monic irreducible coprime polynomials over $\mathbb{F}_p$, we say that the polynomial $F(x)$ is p-regular if $F(x)$ is a ϕ_i-regular polynomial with respect to p for every $i = 1, \ldots, r$. Let $N_{\phi_i}^+(F) = S_{i1} + \cdots + S_{ir_i}$ be the ϕ_i-principal Newton polygon of $F(x)$ with respect to p. For every $j = 1, \ldots, r_i$, let $R_{\lambda_{ij}}(F)(y) = \prod_{s=1}^{s_{ij}} \psi_{ijs}^{a_{ijs}}(y)$ be the factorization of $R_{\lambda_{ij}}(F)(y)$ in $\mathbb{F}_{\phi_i}[y]$. Then we have the following theorem of index of Ore:

Theorem 3.1 *([7, Theorem 1.7 and Theorem 1.9])*
Under the above hypothesis, we have the following:

1.
$$\nu_p((\mathbb{Z}_K : \mathbb{Z}[\alpha])) \geq \sum_{i=1}^r ind_{\phi_i}(F).$$
The equality holds if $F(x)$ is p-regular.

2. *If $F(x)$ is p-regular, then*
$$p\mathbb{Z}_K = \prod_{i=1}^r \prod_{j=1}^{r_i} \prod_{s=1}^{s_{ij}} \mathfrak{p}_{ijs}^{e_{ij}}$$
is the factorization of $p\mathbb{Z}_K$ into powers of prime ideals of $\mathbb{Z}_K$, where e_{ij} is the smallest positive integer satisfying $e_{ij}\lambda_{ij} \in \mathbb{Z}$ and the residue degree of $\mathfrak{p}_{ijs}$ over p is given by $f_{ijs} = \deg(\phi_i)\cdot\deg(\psi_{ijs})$ for every (i, j, s).

For the proof of our results, we need the following lemma, which characterizes the prime divisors of $i(K)$.

Lemma 3.1 *([?])*
Let p be a rational prime and K a number field. For every positive integer f, let $\mathcal{P}_f$ be the number of distinct prime ideals of $\mathbb{Z}_K$ lying above p with residue degree f and $\mathcal{N}_f$ the number of monic irreducible polynomials of $\mathbb{F}_p[x]$ of degree f. Then p divides the index $i(K)$ if and only if $\mathcal{P}_f > \mathcal{N}_f$ for some positive integer f.

For every number field of degree $n \leq 7$ and every rational prime p, Engstrom established a connection between $\nu_p = \nu_p(i(K))$ and the prime ideal factorization of $p\mathbb{Z}_K$. That is, from the factorization of $p\mathbb{Z}_K$, one can determine explicitly ν_p (for more details, see [?]).

4. Proofs of Main Results

Recall that, according to the factorization given in Theorem 3.1, we use the triple indices in the factorization of $p\mathbb{Z}_K$. Namely $p\mathbb{Z}_K = \prod_{i=1}^r \prod_{j=1}^{r_i} \prod_{s=1}^{s_{ij}} \mathfrak{p}_{ijs}^{e_{ij}}$. Here e_{ij} is the ramification index of $\mathfrak{p}_{ijs}$ and $f_{ijs} = \deg(\phi_i)\cdot\deg(\psi_{ijs})$ is its residue degree for every (i, j, s).

Proof of Theorem 2.1.
Since for every $i = 1, \ldots, 6$, $\nu_2(a_i) \geq 1$, then $F(x) \equiv x^6$. Let $\phi_1 = x$. Then
$$F(x) = \phi_1^6 + a_5\phi_1^5 + \cdots + a_1\phi_1 + a_0.$$

1. If $a_3 \equiv 2 \pmod 4$, $a_2 \equiv 0 \pmod 4$, $\nu_2(a_1) > \nu_2(a_2) + 1$ and $\nu_2(a_0) > \nu_2(a_2) + 2$, then $N_{\phi_1}^+(F) = S_{11} + S_{12} + S_{13}$ has three sides joining $(0, \nu_2(a_0))$, $(2, \nu_2(a_2))$, $(3, 1)$ and $(6, 0)$ with $d(S_2) = d(S_3) = 1$. Since $\nu_2(a_0) \not\equiv \nu_2(a_2) \pmod 2$, then $d(S_1) = 1$ also. By Theorem 3.1, $2\mathbb{Z}_K = \mathfrak{p}_{111}^2\mathfrak{p}_{121}\mathfrak{p}_{131}^3$ with residue degree 1 each ideal. By Lemma 3.1, 2 divides $i(K)$. Applying Engstrom's results [?], we obtain $\nu_2(i(K)) = 1$.

2. If $a_4 \equiv 2 \pmod 4$, $a_3 \equiv 0 \pmod 4$, $a_2 \equiv 4 \pmod 8$, $a_1 \equiv 0 \pmod 8$ and $\nu_2(a_0) = 2\nu_2(a_1) - 2$, then $N^+_{\phi_1}(F) = S_{11} + S_{12}$ has two sides joining $(0, \nu_2(a_0))$, $(2, 2)$ and $(6, 0)$. Thus the degree of each side of $N^+_{\phi_1}(F)$ is 2 (see Figure 1). Therefore $R_{\lambda_{11}}(F)(y) = R_{\lambda_{21}}(F)(y) = y^2 + y + 1$, which are irreducible over $\mathbb{F}_{\phi_1}$. By Theorem 3.1, $2\mathbb{Z}_K = \mathfrak{p}_{111}\mathfrak{p}_{211}^2$ with residue degree 2 each ideal. By Lemma 3.1, 2 divides $i(K)$. Applying Engstrom's results [?], we obtain $\nu_2(i(K)) = 1$.

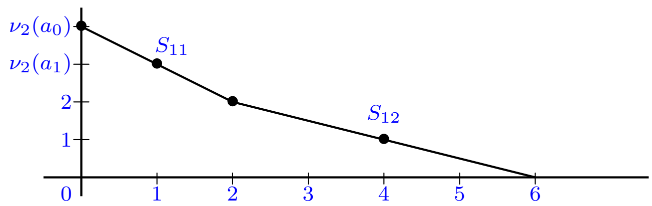

Figure 1: $N^+_{\phi_1}(F)$

□

Proof of Theorem 2.2.
Since $\nu_2(a_4) = 0$ and $\nu_2(a_i) \geq 1$ for every $i \neq 4$. Then $F(x) \equiv x^4(x-1)^2 \pmod 2$. Let $\phi_1 = x$ and $\phi_2 = x - 1$. Then

$$\begin{aligned} F(x) &= \cdots + a_4\phi_1^4 + a_3\phi_1^3 + a_2\phi_1^2 + a_1\phi_1 + a_0, \\ &= \cdots + A_1(1)\phi_2 + A_0(1). \end{aligned}$$

Since $\nu_2(A_0(1)) = 2\nu_2(A_1(1))$, then $N^+_{\phi_2}(F) = S_{21}$ has a single side of degree 2 with $R_{\lambda_{21}}(F)(y) = y^2 + y + 1$, which is irreducible over $\mathbb{F}_{\phi_2}$. Hence ϕ_2 provides a unique prime ideal of $\mathbb{Z}_K$ lying above 2 with residue degree 2. For ϕ_1, we have the following:

1. If $\nu_2(a_0) > \frac{4}{3}\nu_2(a_1)$, $\nu_2(a_1) > \frac{3}{2}\nu_2(a_2)$ and $\nu_2(a_1) > 3\nu_2(a_3)$, then $N^+_{\phi_1}(F) = S_{11} + S_{12}$ has two sides joining $(0, \nu_2(a_0))$, $(1, \nu_2(a_1))$ and $(4, 0)$ with $d(S_{11}) = 1$ (see Figure 2). Since $\nu_2(a_1) \equiv 0 \pmod 3$, then $d(S_{12}) = 3$ with $R_{\lambda_{12}}(F)(y) = y^3 + 1 = (y+1)(y^2+y+1) \in \mathbb{F}_2[y]$. Thus $2\mathbb{Z}_K = \mathfrak{p}_{111}\mathfrak{p}_{121}\mathfrak{p}_{122}\mathfrak{p}_{211}$ with $f_{111} = f_{122} = 1$ and $f_{121} = f_{211} = 2$. Hence 2 divides $i(K)$. Applying Engstrom's results [?], we obtain $\nu_2(i(K)) = 2$.

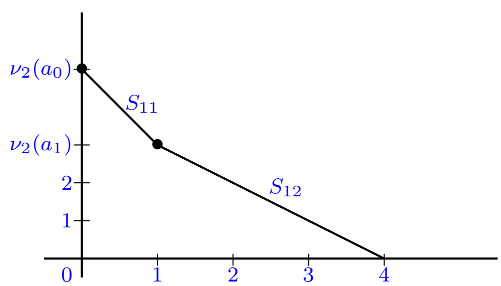

Figure 2: $N^+_{\phi_1}(F)$

2. If $a_3 \equiv 2 \pmod 4$, $a_2 \equiv 4 \pmod 8$, $a_1 \equiv 0 \pmod{16}$ and $\nu_2(a_0) = 2\nu_2(a_1) - 2$, then $N^+_{\phi_1}(F) = S_{11} + S_{12}$ has two sides joining $(0, \nu_2(a_0))$, $(2, 2)$ and $(4, 0)$ with $d(S_{11}) = d(S_{12}) = 2$ and $R_{\lambda_{11}}(F)(y) = R_{\lambda_{12}}(F)(y) = y^2 + y + 1$ which are irreducible over $\mathbb{F}_{\phi_1}$ (see Figure 3). Thus $2\mathbb{Z}_K = \mathfrak{p}_{111}\mathfrak{p}_{121}\mathfrak{p}_{211}$, with residue degree 2 each prime ideal. Hence 2 divides $i(K)$. Applying Engstrom's results [?], we obain $\nu_2(i(K)) = 2$.

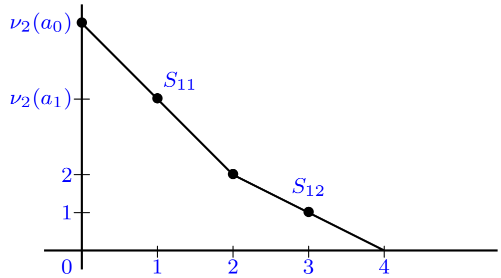

Figure 3: $N_{\phi_1}^+(F)$

□

Proof of Theorem 2.3.
Since $\nu_2(a_4) = 0$ and $\nu_2(a_i) \geq 1$ for every $i \neq 4$, then $F(x) \equiv x^4(x-1)^2 \pmod 2$. Let $\phi_1 = x$ and $\phi_2 = x - 1$. Then

$$\begin{array}{rcl} F(x) & = & \cdots + a_4\phi_1^4 + a_3\phi_1^3 + a_2\phi_1^2 + a_1\phi_1 + a_0 \\ & = & \cdots + A_1(1)\phi_2 + A_0(1). \end{array}$$

Since $\nu_2(A_0(1)) < 2\nu_2(A_1(1))$ and $\nu_2(A_0(1))$ is odd, then $N_{\phi_2}^+(F) = S_{21}$ has a single side joining $(0, \nu_2(A_0(1))$ and $(2, \nu_2(A_1(1))$ with $d(S_{21}) = 1$. For ϕ_1, since $\nu_2(a_2) < 2\nu_2(a_3)$, $\nu_2(a_2)$ is odd, $\nu_2(a_1) > \nu_2(a_2) + \nu_2(a_3)$, $2\nu_2(a_2) < \nu_2(a_0) < 2\nu_2(a_1) - \nu_2(a_2)$ and $\nu_2(a_0)$ is even, then $N_{\phi_1}^+(F) = S_{11} + S_{12}$ has two sides joining $(0, \nu_2(a_0))$, $(2, \nu_2(a_2))$ and $(4, 0)$ and the degree of each side of $N_{\phi_1}^+(F)$ is 1 (see Figure 4). Thus $2\mathbb{Z}_K = \mathfrak{p}_{111}^2\mathfrak{p}_{121}^2\mathfrak{p}_{211}^2$, with residue degree 1 each ideal. By Lemma 3.1, 2 divides $i(K)$. Applying Engstrom's results [?], we obtain $\nu_2(i(K)) = 3$.

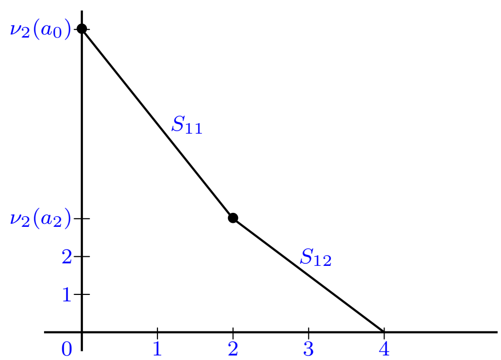

Figure 4: $N_{\phi_1}^+(F)$

□

Proof of Theorem 2.4.
Since $\nu_2(a_4) = 0$ and $\nu_2(a_i) \geq 1$ for every $i \neq 4$, then $F(x) \equiv x^4(x-1)^2 \pmod 2$. Let $\phi_1 = x$ and $\phi_2 = x - 1$. Then

$$\begin{array}{rcl} F(x) & = & \cdots + a_4\phi_1^4 + a_3\phi_1^3 + a_2\phi_1^2 + a_1\phi_1 + a_0 \\ & = & \cdots + A_1(1)\phi_2 + A_0(1). \end{array}$$

Since $\nu_2(A_0(1)) < 2\nu_2(A_1(1))$, then $N_{\phi_2}^+(F) = S_{21}$ has single side of degree 1. Thus ϕ_2 provides a unique prime ideal of $\mathbb{Z}_K$ lying above 2 with residue degree 1. On the other hand, since $\nu_2(a_{i-1}) >$

$2\nu_2(a_i) - \nu(a_{i+1})$ for every $i = 1, 2, 3$, then $N_{\phi_1}^+(F) = S_{11} + S_{12} + S_{13} + S_{14}$ has four sides joining $(0, \nu_2(a_0))$, $(1, \nu_2(a_1))$, $(2, \nu_2(a_2))$, $(3, \nu_2(a_3))$ and $(4, 0)$. Thus the degree of each side of $N_{\phi_1}^+(F)$ is 1. Hence $2\mathbb{Z}_K = \mathfrak{p}_{111}\mathfrak{p}_{121}\mathfrak{p}_{131}\mathfrak{p}_{141}\mathfrak{p}_{211}^2$, with residue degree 1 each ideal. Applying Engstrom's results [?], we obtain $\nu_2(i(K)) = 4$.

□

Proof of Theorem 2.5.
Since $\nu_2(a_4) = 0$ and $\nu_2(a_i) \geq 1$ for every $i \neq 4$, $F(x) \equiv x^4(x-1)^2 \pmod 2$. Let $\phi_1 = x$ and $\phi_2 = x - 1$. Then

$$\begin{aligned} F(x) &= \cdots + A_1(1)\phi_2 + A_0(1), \\ &= \cdots + a_4x^4 + a_3x^3 + a_2x^2 + a_1x + a_0. \end{aligned}$$

Since $\nu_2(A_0(1)) > 2\nu_2(A_1(1))$, then $N_{\phi_2}^+(F) = S_{21} + S_{22}$ has two sides of degree 1 each side. Thus ϕ_2 provides two prime ideals of $\mathbb{Z}_K$ lying above 2 with residue degree 1 each prime ideal factor. On the other hand, since $\nu_2(a_2) < 2\nu_2(a_3)$, $\nu_2(a_2)$ is odd, $\nu_2(a_1) > \nu_2(a_2) + \nu_2(a_3)$ and $\nu_2(a_0) \geq 2\nu_2(a_1)$, then $N_{\phi_1}^+(F) = S_{11} + S_{12} + S_{13}$ has three sides joining $(0, \nu_2(a_0))$, $(2, \nu_2(a_2))$, $(3, \nu_2(a_3))$ and $(4, 0)$. Thus the degree of each side of $N_{\phi_1}^+(F)$ is 1. So, $2\mathbb{Z}_K = \mathfrak{p}_{111}\mathfrak{p}_{121}\mathfrak{p}_{131}^2\mathfrak{p}_{211}\mathfrak{p}_{221}$, with residue degree 1 each ideal. Applying Engstrom's results [?], we obtain $\nu_2(i(K)) = 4$.

□

Proof of Theorem 2.6.

1. Since 2 does not divide a_2 and $\nu_2(a_i) \geq 1$ for every $i \neq 2$? then $F(x) \equiv x^2(x-1)^4 \pmod 2$. Let $\phi_1 = x$ and $\phi_2 = x - 1$. Then

$$\begin{aligned} F(x) &= \cdots + a_2\phi_1^2 + a_1\phi_1 + a_0, \\ &= \cdots + A_4(1)\phi_2^4 + A_3(1)\phi_2^3 + A_2(1)\phi_2^2 + A_1(1)\phi_2 + A_0(1). \end{aligned}$$

   If $\nu_2(a_0) > 2\nu_2(a_1)$, then $N_{\phi_1}^+(F) = S_{11} + S_{12}$ has two sides of degree 1 each. Hence ϕ_1 provides two prime ideals of $\mathbb{Z}_K$ lying above 2 with residue degree 1 each prime ideal factor. On the other hand, since $\nu_2(A_{i-1}) > 2\nu_2(A_i) - \nu_2(A_{i+1})$ for every $i = 1, 2, 3$, then $N_{\phi_2}^+(F) = S_{21} + S_{22} + S_{23} + S_{24}$ has four sides joining $(0, \nu_2(A_0(1))$, $(1, \nu_2(A_1(1))$, $(2, \nu_2(A_2(1))$, $(3, \nu_2(A_3(1))$ and $(4, 0)$. Thus the degree of each side of $N_{\phi_2}^+(F)$ is 1. By Theorem 3.1, we obtain $2\mathbb{Z}_K = \mathfrak{p}_{111}\mathfrak{p}_{121}\mathfrak{p}_{211}\mathfrak{p}_{221}\mathfrak{p}_{231}\mathfrak{p}_{241}$, with residue degree 1 each ideal factor. By Lemma 3.1, 2 divide $i(K)$. Using Engstrom's results [?], we obtain $\nu_2(i(K)) = 8$.

2. Since $\nu_2(a_3) = \nu_2(a_4) = \nu_2(a_5) = 0$ and $\nu_2(a_i) \geq 1$ for every $i \in \{0, 1, 2\}$. Then $F(x) \equiv x^3(x-1)^3 \pmod 2$. Let $\phi_1 = x$ and $\phi_2 = x - 1$. Then

$$\begin{aligned} F(x) &= \cdots + a_3\phi_1^3 + a_2\phi_1^2 + a_1\phi_1 + a_0, \\ &= \cdots + A_3(1)\phi_2^3 + A_2(1)\phi_2^2 + A_1(1)\phi_2 + A_0(1). \end{aligned}$$

   If $\nu_2(a_{i-1}) > 2\nu_2(a_i) - \nu_2(a_{i+1})$ for every $i = 1, 2$, then $N_{\phi_1}^+(F) = S_{11} + S_{12} + S_{13}$ has three sides joining $(0, \nu_2(a_0))$, $(1, \nu_2(a_1))$, $(2, \nu_2(a_2))$ and $(3, 0)$. Thus the degree of each side of $N_{\phi_1}^+(F)$ is 1. On the other hand, since $\nu_2(A_{i-1}) > 2\nu_2(A_i) - \nu_2(A_{i+1})$ for every $i = 1, 2$. Then $N_{\phi_2}^+(F) = S_{21} + S_{22} + S_{23}$ has three sides joining $(0, \nu_2(A_0(1))$, $(1, \nu_2(A_1(1))$, $(2, \nu_2(A_2(1))$ and $(3, 0)$. Thus the degree of each side of $N_{\phi_2}^+(F)$ is 1. By Theorem 3.1, the rational prime 2 splits completely in K. Applying Engstrom's [?] results, we obtain $\nu_2(i(K)) = 8$.

3. Since $\nu_2(a_4) = 0$ and $\nu_2(a_i) \geq 1$ for every $i \neq 4$. Then $F(x) \equiv x^4(x-1)^2 \pmod 2$. Let $\phi_1 = x$ and $\phi_2 = x - 1$. Then

$$\begin{aligned} F(x) &= \cdots + a_4x^4 + a_3x^3 + a_2x^2 + a_1x + a_0, \\ &= \cdots + A_1(1)\phi_2 + A_0(1). \end{aligned}$$

Since $\nu_2(A_0(1) > 2\nu_2(A_1(1))$, then $N_{\phi_2}^+(F) = S_{21} + S_{22}$ has two sides of degree 1 each side. Thus ϕ_2 provides two prime ideals of $\mathbb{Z}_K$ lying above 2 with residue degree 1 each ideal factor. On the other hand, since $\nu_2(a_{i-1}) > 2\nu_2(a_i) - \nu_2(a_{i+1})$ for every $i = 1, 2, 3$, then $N_{\phi_1}^+(F) = S_{11} + S_{12} + S_{13} + S_{14}$ has four sides joining $(0, \nu_2(a_0))$, $(1, \nu_2(a_1))$, $(2, \nu_2(a_2))$, $(3, \nu_2(a_3))$ and $(4, 0)$. Thus the degree of each side of $N_{\phi_1}^+(F)$ is 1. By Theorem 3.1, the rational prime 2 splits completely in K. By Lemma 3.1, 2 divide $i(K)$. Applying Engstrom's results [?], we conclude that $\nu_2(i(K)) = 8$.

□

Proof of Theorem 2.7.
Since $a_4 \equiv -1 \pmod 3$ and $\nu_3(a_i) \geq 1$ for every $i \neq 4$. Then $F(x) \equiv x^4(x-1)(x-2) \pmod 3$. For every $k = 0, 1, 2$, let $\phi_k = x - k$. Then, for every $k = 1, 2$, ϕ_k provides a unique prime ideal of $\mathbb{Z}_K$ lying above 3 with residue degree 1. For $\phi_0 = x$, since $\nu_3(a_1) \leq \nu_3(a_i)$ for every $i = 2, 3$, $\nu_3(a_1) \not\equiv 0 \pmod 3$ and $\nu_3(a_0) > \dfrac{3}{2}\nu_3(a_1)$, then $N_{\phi_0}^+(F) = S_{01} + S_{02}$ has two sides joining $(0, \nu_3(a_0))$, $(1, \nu_3(a_1))$, and $(4, 0)$. Thus the degree of each side of $N_{\phi_0}^+(F)$ is 1. Therefore, $3\mathbb{Z}_K = \mathfrak{p}_{011}\mathfrak{p}_{021}^3\mathfrak{p}_{111}\mathfrak{p}_{211}$, with residue degree 1 each ideal factor. Hence 3 divides $i(K)$. Applying Engstrom's results [?], we obtain $\nu_3(i(K)) = 1$.

□

Proof of Theorem 2.8.
Since $a_3 \equiv -1 \pmod 3$ and $a_i \equiv 0 \pmod 3$ for every $i \neq 3$, then $F(x) \equiv x^3(x-1)^3 \pmod 3$. Let $\phi_1 = x$ and $\phi_2 = x - 1$. Then

$$\begin{aligned} F(x) &= \cdots + a_3\phi_1^3 + a_2\phi_1^2 + a_1\phi_1 + a_0, \\ &= \cdots + A_2(1)\phi_2^2 + A_1(1)\phi_2 + A_0(1). \end{aligned}$$

1. If $\nu_3(a_1) < 2\nu_3(a_2)$, $\nu_3(a_0) \geq 2\nu_3(a_1)$ and $\nu_3(a_1) \not\equiv 0 \pmod 2$, then $N_{\phi_1}^+ = S_{11} + S_{12}$ has two sides joining $(0, \nu_3(a_0))$, $(1, \nu_3(a_1))$ and $(3, 0)$. Thus the degree of each side of $N_{\phi_1}^+(F)$ is 1. On the other hand, since $\nu_3(A_1(1)) < 2\nu_3(A_2(1))$, $\nu_3(A_1(1)) \not\equiv 0 \pmod 2$ and $\nu_3(A_0(1)) > 2\nu_3(A_1(1))$, then $N_{\phi_2}^+(F) = S_{21} + S_{22}$ has two sides joining $(0, \nu_3(A_0(1)))$, $(1, \nu_3(A_1(1)))$ and $(3, 0)$. Thus the degree of each side of $N_{\phi_2}^+(F)$ is 1. Hence $3\mathbb{Z}_K = \mathfrak{p}_{111}\mathfrak{p}_{121}^2\mathfrak{p}_{211}\mathfrak{p}_{221}^2$, with residue degree 1 each ideal. By Lemma 3.1, 3 divides $i(K)$. Applying Engstrom's results [?], we obtain $\nu_3(i(K)) = 1$.

2. If $\nu_3(a_1) > 2\nu_3(a_2)$ and $\nu_3(a_0) > 2\nu_3(a_1) - \nu_3(a_2)$, then $N_{\phi_1}^+(F) = S_{11} + S_{12} + S_{13}$ has three sides joining $(0, \nu_3(a_3))$, $(1, \nu_3(a_1))$, $(2, \nu_3(a_2))$ and $(3, 0)$. Thus the degree of each side of $N_{\phi_1}^+(F)$ is 1. On the other hand, since $\nu_3(A_0(1)) < \dfrac{3}{2}\nu_3(A_1(1))$, $\nu_3(A_0(1)) < 3\nu_3(A_2(1))$ and $\nu_3(A_0(1)) \not\equiv 0 \pmod 3$, then $N_{\phi_2}^+(F) = S_{21}$ has a single side joining $(0, \nu_3(A_0(1)))$ and $(3, 0)$, with $d(S_{21}) = 1$. Thus $3\mathbb{Z}_K = \mathfrak{p}_{111}\mathfrak{p}_{121}\mathfrak{p}_{131}\mathfrak{p}_{211}^3$, with residue degree 1 each ideal factor. Hence 3 divides $i(K)$. Applying Engstrom's results [?], we obtain $\nu_3(i(K)) = 1$.

3. If $\nu_3(a_1) < 2\nu_3(a_2)$, $\nu_3(a_1)$ is even, $(a_1)_3 \equiv -1 \pmod 3$, $\nu_3(a_0) > \dfrac{3}{2}\nu_3(a_1)$, then $N_{\phi_1}^+(F) = S_{11} + S_{12}$ has two sides joining $(0, \nu_3(a_0))$, $(1, \nu_3(a_1))$ and $(3, 0)$, with $d(S_{11}) = 1$ and $R_{\lambda_{12}}(F)(y) = -y^2 - 1$ which is irreducible over $\mathbb{F}_{\phi_1}$. On the other hand, since $\nu_3(A_1(1)) > 2\nu_3(A_2(1))$ and $\nu_3(A_0(1)) > 2\nu_3(A_1(1)) - \nu_3(A_2(1))$. Thus $N_{\phi_2}^+(F) = S_{21} + S_{22} + S_{23}$ has three sides joining $(0, \nu_3(A_0(1)))$, $(1, \nu_3(A_1(1)))$, $(2, \nu_3(A_2(1)))$ and $(0, 3)$. Thus the degree of each side of $N_{\phi_2}^+(F)$ is 1. Hence $3\mathbb{Z}_K = \mathfrak{p}_{111}\mathfrak{p}_{121}\mathfrak{p}_{211}\mathfrak{p}_{221}\mathfrak{p}_{231}$, where $f_{121} = 2$ and $f_{111} = f_{211} = f_{221} = f_{231} = 1$. By Lemma 3.1, 3 divides $i(K)$. Using Engstrom's results [?], we obtain $\nu_3(i(K)) = 1$.

□

Proof of Theorem 2.9.
Since $a_4 \equiv -1 \pmod 3$ and $\nu_3(a_i) \geq 1$ for every $i \neq 4$, then $F(x) \equiv x^4(x-1)(x-2) \pmod 3$. For every $k = 0, 1, 2$, let $\phi_k = x - k$. Then, for every $k = 1, 2$, ϕ_k provides a unique prime ideal of $\mathbb{Z}_K$ lying above

3 with residue degree 1. For $\phi_0 = x$, since $\nu_3(a_2) < 2\nu_3(a_3)$, $\nu_3(a_1) > 2\nu_3(a_2)$ and $\nu_3(a_0) > 2\nu_3(a_1)$, then $N^+_{\phi_0}(F) = S_{01} + S_{02} + S_{03}$ has three sides joining $(0, \nu_3(a_0))$, $(1, \nu_3(a_1))$, $(2, \nu_3(a_2))$ and $(4, 0)$. Thus the degree of each side of $N^+_{\phi_0}(F)$ is 1. Therefore, $3\mathbb{Z}_K = \mathfrak{p}_{011}\mathfrak{p}_{021}\mathfrak{p}_{031}^2\mathfrak{p}_{111}\mathfrak{p}_{211}$, with residue degree 1 each prime ideal. Hence 3 divides $i(K)$. Applying Engstrom's results [?], we obtain $\nu_3(i(K)) = 2$.

□

Proof of Theorem 2.10.
Since $a_3 \equiv -1 \pmod 3$ and $a_i \equiv 0 \pmod 3$ for every $i \neq 3$, then $F(x) \equiv x^3(x-1)^3 \pmod 3$. Let $\phi_1 = x$ and $\phi_2 = x - 1$. Then

$$\begin{aligned} F(x) &= \cdots + a_3\phi_1^3 + a_2\phi_1^2 + a_1\phi_1 + a_0, \\ &= \cdots + A_2(1)\phi_2^2 + A_1(1)\phi_2 + A_0(1). \end{aligned}$$

1. If $\nu_3(a_{i-1}) > 2\nu_3(a_i) - \nu_3(a_{i+1})$ for every $i = 1, 2$, then $N^+_{\phi_1}(F) = S_{11} + S_{12} + S_{13}$ has three sides joining $(0, \nu_3(a_0))$, $(1, \nu_3(a_1))$,$(2, \nu_3(a_2))$ and $(3, 0)$. Thus $d(S_{11}) = d(S_{12}) = d(S_{13}) = 1$. On the other hand, since $\nu_3(A_{i-1}(1)) > 2\nu_3(A_i(1)) - \nu_3(A_{i+1}(1))$ for every $i = 1, 2$, then $N^+_{\phi_2}(F) = S_{21} + S_{22} + S_{23}$ has three sides joining $(0, \nu_3(A_0(1)))$, $(1, \nu_3(A_1(1)))$, $(2, \nu_3(A_2(1)))$ and $(3, 0)$. Thus $d(S_{21}) = d(S_{22}) = d(S_{23}) = 1$. Hence $3\mathbb{Z}_K = \mathfrak{p}_{111}\mathfrak{p}_{121}\mathfrak{p}_{131}\mathfrak{p}_{211}\mathfrak{p}_{221}\mathfrak{p}_{231}$, with residue degree 1 each ideal. By Lemma 3.1, 3 divides $i(K)$. Applying Engstrom's results [?], we obtain $\nu_3(i(K)) = 3$.

2. If $\nu_3(a_1) < 2\nu_3(a_2)$, $\nu_3(a_1)$ is even, $(a_1)_3 \equiv 1 \pmod 3$ and $\nu_3(a_0) > 2\nu_3(a_1)$, then $N^+_{\phi_1}(F) = S_{11}+S_{12}$ has two sides joining $(0, \nu_3(a_0))$, $(1, \nu_3(a_1))$ and $(3, 0)$, with $d(S_{11}) = 1$ and $R_{\lambda_{12}}(F)(y) = -y^2+1 = -(y+1)(y-1) \in \mathbb{F}_{\phi_1}[y]$. On the other hand, since $\nu_3(A_1(1)) < 2\nu_3(A_2(1))$, $\nu_3(A_1(1))$ is even, $(A_1(1))_3 \equiv 1 \pmod 3$ and $\nu_3(A_0(1)) > 2\nu_3(A_1(1))$. Thus $N^+_{\phi_2}(F) = S_{21} + S_{22}$ has two sides joining $(0, \nu_3(A_0(1)))$, $(2, \nu_3(A_1(1)))$ and $(3, 0)$, with $d(S_{21}) = 1$ and $R_{\lambda_{22}}(F)(y) = -y^2 + 1 = -(y+1)(y-1) \in \mathbb{F}_{\phi_2}[y]$. Hence the rational prime 3 splits completely in K. By Lemma 3.1, 3 divides $i(K)$. Using Engstrom's results [?], we obtain $\nu_3(i(K)) = 3$.

□

Proof of Theorem 2.11.
Since $a_2 \equiv -1 \pmod 5$ and $a_i \equiv 0 \pmod 5$ for every $i \neq 2$, then $F(x) \equiv x^2(x-1)(x-2)(x-3)(x-4) \pmod 5$. For every $k = 0, 1, \ldots, 4$, let $\phi_k = x - k$. Then, for every $k = 1, \ldots, 4$, ϕ_k provides a unique prime ideal of $\mathbb{Z}_K$ lying above 5 with residue degree 1. For $\phi_0 = x$, we have the following:

1. If $\nu_5(a_0) > 2\nu_5(a_1)$, then $N^+_{\phi_0}(F) = S_{01} + S_{02}$ has two sides joining $(0, \nu_5(a_0))$, $(1, \nu_5(a_1))$ and $(2, 0)$. Thus the degree of each side $N^+_{\phi_0}(F)$ is 1. Therefore, $5\mathbb{Z}_K = \mathfrak{p}_{011}\mathfrak{p}_{021}\mathfrak{p}_{111}\mathfrak{p}_{211}\mathfrak{p}_{311}\mathfrak{p}_{411}$, with residue degree 1 each ideal factor. Hence 5 divides $i(K)$. Using Engstrom's results [?], we obtain $\nu_5(i(K)) = 1$.

2. If $\nu_5(a_0) = 2\nu_5(a_1)$, then $N^+_{\phi_0}(F) = S_{01}$, where $d(S_{01}) = 2$. Since $a_2 \equiv -1 \pmod 5$ and $((a_0)_5, (a_1)_5) \in \{(2,1), (2,4)(3,2), (3,3)\} \pmod 5$, then $R_{\lambda_{01}}(F)(y)$ can be factorized in $\mathbb{F}_{\phi_0}[y]$ by two distinct linear polynomials (see Table 1). In all these cases, the rational prime 5 splits completely in K. By Lemma 3.1, 5 divides $i(K)$. Applying Engstrom's results [?], we obtain $\nu_5(i(K)) = 1$.

$((a_0)_5, (a_1)_5) \pmod 5$	$R_{\lambda_{01}}(F)(y) \bmod (5, \phi_0)$
$(2, 1)$	$-(y+1)(y+3)$
$(2, 4)$	$-(y-1)(y+2)$
$(3, 2)$	$-(y+1)(y+2)$
$(3, 3)$	$-(y-1)(y+3)$

Table 1: $R_{\lambda_{01}}(F)(y)$ in $\mathbb{F}_{\phi_0}[y]$

3. If $\nu_5(a_0)$ even and $\nu_5(a_0) < \nu_5(a_1)$, then $N^+_{\phi_0}(F) = S_{01}$ where $d(S_{01}) = 2$. If $(a_0)_5 \equiv 1 \pmod 5$, then $R_{\lambda_{01}}(F)(y) = -(y-1)(y+1)$ and if $(a_0)_5 \equiv -1 \pmod 5 \in \mathbb{F}_{\phi_0}[y]$, then $R_{\lambda_{01}}(F)(y) = -(y+2)(y+3) \in \mathbb{F}_{\phi_0}[y]$. In both cases, the rational prime 5 splits completely in K. By Lemma 3.1, 5 divides $i(K)$. Using Engstrom's results [?], we obtain $\nu_5(i(K)) = 1$.

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