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Families of Sextic Number Fields with Prescribed Indices

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ABSTRACT: For each prime number $p \in \{2, 3, 5\}$, we establish sufficient conditions for a sextic number field to have a prescribed p-adic valuation of its index, and we illustrate our results with explicit computational examples.

Keywords: Dedekind's Theorem, Ore's Theorem, Prime ideals factorization, Newton polygon, Index of a number field, Monogenicity.

Contents

1. Introduction

Let $K = \mathbb{Q}(\alpha)$ be a number field of degree n, where α is a primitive integer of K, and let $\mathbb{Z}_K$ denote the ring of integers of K. The index of α, denoted by $(\mathbb{Z}_K : \mathbb{Z}[\alpha])$, is the index of the abelian group $\mathbb{Z}[\alpha]$ in $\mathbb{Z}_K$. A well-known formula linking this index with the discriminants is given by:

$$\Delta(\alpha) = (\mathbb{Z}_K : \mathbb{Z}[\alpha])^2 \cdot d_K, \tag{1.1}$$

where d_K is the absolute discriminant of K and $\Delta(\alpha)$ is the discriminant of the minimal polynomial of α over $\mathbb{Q}$. The index of K, denoted by $i(K)$, is defined as the greatest common divisor of the indices of all primitive integers of K. That is, $i(K) = \gcd\{(\mathbb{Z}_K : \mathbb{Z}[\theta]) \mid K = \mathbb{Q}(\theta) \text{ and } \theta \in \mathbb{Z}_K\}$. It is well known that if K is monogenic, then its index is trivial; $i(K) = 1$. Therefore, a number field with non-trivial index is not monogenic. Dedekind was the first to discover a number field with non-trivial index ([3]). In 1930, for every number field K of degree $n \le 7$ and every rational prime p, Engstrom established a connection between the prime ideals factorization of $p\mathbb{Z}_K$ and $\nu_p(i(K))$. This motivated a very important question, stated as problem 22 in Narkiewicz's book ([17]), which asks for an explicit formula of the highest power $\nu_p(i(K))$ for a given rational prime p dividing $i(K)$. In [21], Śliwa extended Engstrom's results to number fields up to degree 12, under the condition that p is unramified in K. These results were further generalized by Nart ([18]), who developed a p-adic characterization of the index of a number field. In [16], Nakahara studied the indices of non-cyclic but abelian biquadratic number fields. In [9], Funakura showed that $i(K) = 1$ or 2 for every pure quartic number field K. In [10], Gaál et al. characterized the field indices of biquadratic number fields. In [22], Spearman and Williams characterized the indices of cyclic quartic number fields. In [20], Pethö and Pohst studied the index divisors of multiquadratic number fields. Recently, many authors are interested in the characterization of the prime power decomposition of the indices of number fields, especially those defined by trinomials and quadrinomials of fixed degrees (see [2,4,5,6,9,13,14,15,16,20,22]). In all the former papers, for a given number field K, the authors try to calculate the index $i(K)$. In contrast, the present paper introduces a new approach. Namely, for each rational prime $p \in \{2, 3, 5\}$ and every possible natural integer ν_p, we give sufficient conditions for a sextic number field to have p-indices ν_p, where the p-index of a number field K is defined as the p-valuation of its index. Namely, $\nu_p = \nu_p(i(K))$. According to Engstrom's results in [8, Table, p. 234; Degree of field:

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6], the index of any sextic number field K is of the form $i(K) = 2^{\nu_2} \cdot 3^{\nu_3} \cdot 5^{\nu_5}$, where $\nu_2 \in \{0, 1, 2, 3, 4, 8\}$, $\nu_3 \in \{0, 1, 2, 3\}$ and $\nu_5 \in \{0, 1\}$. These results can be used to exhibit families of sextic number fields for each of the possible non-trivial index values.

2. Main Results

Given that the index of a sextic number field K is of the form

$$i(K) = 2^{\nu_2} \cdot 3^{\nu_3} \cdot 5^{\nu_5},$$

where $\nu_2 \in \{0, 1, 2, 3, 4, 8\}$, $\nu_3 \in \{0, 1, 2, 3\}$, and $\nu_5 \in \{0, 1\}$, in this section we provide, for each rational prime $p \in \{2, 3, 5\}$, sufficient conditions for a sextic number field to have p-adic index ν_p, where $\nu_2 \in \{1, 2, 3, 4, 8\}$, $\nu_3 \in \{1, 2, 3\}$, and $\nu_5 = 1$.

Throughout this section, let $K = \mathbb{Q}(\alpha)$ be a sextic number field, where α is a root of the monic irreducible polynomial

$$F(x) = x^6 + a_5x^5 + a_4x^4 + a_3x^3 + a_2x^2 + a_1x + a_0 \in \mathbb{Z}[x].$$

Theorem 2.1 *Suppose that for every $i = 0, \dots, 5$, $\nu_2(a_i) \geq 1$. Then each one of the following conditions guarantees that $\nu_2(i(K)) = 1$.*

1. $a_3 \equiv 2 \pmod 4$, $a_2 \equiv 0 \pmod 4$, $\nu_2(a_0) < 2\nu_2(a_1) - \nu_2(a_2)$, $\nu_2(a_0) > 3\nu_2(a_2) - 2$ *and* $\nu_2(a_0) \not\equiv \nu_2(a_2) \pmod 2$.
2. $a_4 \equiv 2 \pmod 4$, $a_3 \equiv 0 \pmod 4$, $a_2 \equiv 4 \pmod 8$, $a_1 \equiv 0 \pmod 8$ *and* $\nu_2(a_0) = 2\nu_2(a_1) - 2$.

Example 2.1 Let K be a sextic number field defined by the monic irreducible polynomial $F(x) = x^6 + 12x^4 + 14x^3 + 12x^2 + 48x + 32$. Since $a_3 \equiv 2 \pmod 4$, $a_2 \equiv 4 \pmod 8$, $a_1 \equiv 16 \pmod{32}$ and $a_0 \equiv 32 \pmod{64}$, by Theorem 2.1 (1), we conclude that $\nu_2(i(K)) = 1$.

Recall that, for every rational integer $z \in \mathbb{Z}$, the $(x - z)$-Taylor expansion of every polynomial $F(x)$ of degree 6 is given by the following:

$$F(x) = \sum_{k=0}^{6} \frac{F^{(k)}(z)}{k!} (x - z)^k.$$

In the sequel, set $A_k(z) := \frac{F^{(k)}(z)}{k!}$.

Theorem 2.2 *Suppose that $\nu_2(a_4) = 0$, $\nu_2(a_i) \geq 1$ for every $i \neq 4$ and $\nu_2(A_0(1)) = 2\nu_2(A_1(1))$. Then each one of the following conditions guarantees that $\nu_2(i(K)) = 2$.*

1. $\nu_2(a_0) > \frac{4}{3}\nu_2(a_1)$, $\nu_2(a_1) > \frac{3}{2}\nu_2(a_2)$, $\nu_2(a_1) > 3\nu_2(a_3)$ *and* $\nu_2(a_1) \equiv 0 \pmod 3$.
2. *If* $a_3 \equiv 2 \pmod 4$, $a_2 \equiv 4 \pmod 8$, $a_1 \equiv 0 \pmod{16}$ *and* $\nu_2(a_0) = 2\nu_2(a_1) - 2$.

Example 2.2 Let K be a sextic number field defined by the monic irreducible polynomial $F(x) = x^6 + 51x^4 + 48x^3 + 96x^2 + 24x + 384$. Since $a_0 \equiv 0 \pmod{128}$, $a_1 \equiv 8 \pmod{16}$, $a_2 \equiv a_3 \equiv 0 \pmod{16}$, a_4 is odd, $A_0(1) = 604 \equiv 4 \pmod 8$ and $A_1(1) = 570 \equiv 2 \pmod 4$, it follows from Theorem 2.2(1) that $\nu_2(i(K)) = 2$.

Theorem 2.3 *Suppose that $\nu_2(a_4) = 0$ and $\nu_2(a_i) \geq 1$ for every $i \neq 4$. If all of the following conditions are satisfied, then $\nu_2(i(K)) = 3$.*

1. $\nu_2(A_0(1)) < 2\nu_2(A_1(1))$ *and* $\nu_2(A_0(1))$ *is odd.*

2. $\nu_2(a_2) < 2\nu_2(a_3)$, $\nu_2(a_2)$ *is odd,* $\nu_2(a_1) > \nu_2(a_2)+\nu_2(a_3)$, $2\nu_2(a_2) < \nu_2(a_0) < 2\nu_2(a_1)-\nu_2(a_2)$ *and* $\nu_2(a_0)$ *is even.*

Example 2.3 Let K be a sextic number field defined by the monic irreducible polynomial $F(x) = x^6+3x^4+10x^2+24x+16$. Since $a_0 \equiv 16 \pmod{32}$, $a_1 \equiv 8 \pmod{16}$, $a_2 \equiv 2 \pmod 4$, a_4 is odd, and $A_0(1) = 54 \equiv 2 \pmod 4$, using Theorem 2.3, we obtain $\nu_2(i(K)) = 3$.

Theorem 2.4 *Suppose that* 2 *does not divide* a_4 *and* $\nu_2(a_i) \ge 1$ *for every* $i \ne 4$. *If all of the following conditions are satisfied, then* $\nu_2(i(K)) = 4$.

1. $\nu_2(A_0(1)) < 2\nu_2(A_1(1))$ *and* $\nu_2(A_0(1))$ *is odd.*

2. $\nu_2(a_{i-1}) > 2\nu_2(a_i) - \nu_2(a_{i+1})$ *for every* $i = 1,2,3$.

Theorem 2.5 *Suppose that* 2 *does not divide* a_4 *and* $\nu_2(a_i) \ge 1$ *for every* $i \ne 4$. *If all of the following conditions are satisfied, then* $\nu_2(i(K)) = 4$.

1. $\nu_2(A_0(1)) > 2\nu_2(A_1(1))$.

2. $\nu_2(a_2) < 2\nu_2(a_3)$, $\nu_2(a_2)$ *is odd,* $\nu_2(a_1) > \nu_2(a_2)+\nu_2(a_3)$ *and* $\nu_2(a_0) > 2\nu_2(a_1)$.

Example 2.4 Let K be a sextic number field defined by the monic irreducible polynomial $F(x) = x^6+12x^5+3x^4+6x^3+2x^2+8x+128$. Since $a_0 \equiv 128 \pmod{256}$, $a_1 \equiv 8 \pmod{16}$, $a_2 \equiv 2 \pmod 4$ a_4 is odd, $A_0(1) = 160 \equiv 32 \pmod{64}$ and $A_1(1) = 108 \equiv 4 \pmod 8$, by Theorem 2.5, we conclude that $\nu_2(i(K)) = 4$.

Theorem 2.6 *Each one of the following conditions guarantees that* $\nu_2(i(K)) = 8$.

1. $\nu_2(a_2) = 0$, $\nu_2(a_i) \ge 1$ *for every* $i \ne 2$, $\nu_2(a_0) > 2\nu_2(a_1)$ *and* $\nu_2(A_{i-1}) > 2\nu_2(A_i) - \nu_2(A_{i+1})$ *for every* $i = 1,2,3$.

2. $\nu_2(a_i) \ge 1$ *for* $i = 0,1,2$, $\nu_2(a_i) = 0$ *for* $i = 3,4,5$, $\nu_2(a_{i-1}) > 2\nu_2(a_i) - \nu_2(a_{i+1})$ *and* $\nu_2(A_{i-1}) > 2\nu_2(A_i) - \nu_2(A_{i+1})$ *for every* $i = 1,2$.

3. $\nu_2(a_4) = 0$, $\nu_2(a_i) \ge 1$ *for every* $i \ne 4$, $\nu_2(A_0(1)) \ge 2\nu_2(A_1(1))$ *and* $\nu_2(a_{i-1}) > 2\nu_2(a_i) - \nu_2(a_{i+1})$ *for every* $i = 1,2,3$.

Example 2.5 Let K be a sextic number field defined by the monic irreducible polynomial $F(x) = x^6+x^5+x^4+7x^3+14x^2+40x+64$. Since a_3, a_4, a_5 are odd, $a_0 \equiv 64 \pmod{128}$, $a_1 \equiv 8 \pmod{16}$, $a_2 \equiv 2 \pmod 4$, $A_0(1) = 128$, $A_1(1) = 104 \equiv 8 \pmod{16}$ and $A_2(1) = 66 \equiv 2 \pmod 4$, from Theorem 2.6 (2) we deduce that $\nu_2(i(K)) = 8$.

Theorem 2.7 *Suppose that* $a_4 \equiv -1 \pmod 3$ *and* $\nu_3(a_i) \ge 1$ *for every* $i \ne 4$. *If all of the following conditions are satisfied, then* $\nu_3(i(K)) = 1$.

1. $\nu_3(a_1) \le \nu_3(a_i)$ *for every* $i = 2,3$.

2. $\nu_3(a_1) \not\equiv 0 \pmod 3$ *and* $\nu_3(a_0) > \dfrac{3}{2}\nu_3(a_1)$.

Example 2.6 Let K be a sextic number field defined by the monic irreducible polynomial $F(x) = x^6 + 12x^5 + 2x^4 + 18x^3 + 54x^2 + 24x + 18$. Since $a_4 \equiv -1 \pmod{3}$, $\nu_3(a_0) = 2$, $\nu_3(a_1) = 1$, $\nu_3(a_2) = 3$ and $\nu_3(a_3) = 2$, by Theorem 2.7, we conclude that $\nu_3(i(K)) = 1$.

In the sequel, for each $p \in \{3, 5\}$, we define $(a_i)_p := \frac{a_i}{p^{\nu_p(a_i)}}$ for all $a_i \in \mathbb{Z}$.

Theorem 2.8 *Suppose that $a_3 \equiv -1 \pmod{3}$ and $a_i \equiv 0 \pmod{3}$ for every $i \neq 3$. Then each one of the following conditions guarantees that $\nu_3(i(K)) = 1$.*

1. $\nu_3(a_1) < 2\nu_3(a_2)$, $\nu_3(a_0) > 2\nu_3(a_1)$, $\nu_3(a_1) \not\equiv 0 \pmod{2}$, $\nu_3(A_1(1)) < 2\nu_3(A_2(1))$, $\nu_3(A_1(1)) \not\equiv 0 \pmod{2}$ *and* $\nu_3(A_0(1)) > 2\nu_3(A_1(1))$.
2. $\nu_3(a_1) > 2\nu_3(a_2)$, $\nu_3(a_0) > 2\nu_3(a_1) - \nu_3(a_2)$, $\nu_3(A_0(1)) < \frac{3}{2}\nu_3(A_1(1))$, $\nu_3(A_0(1)) < 3\nu_3(A_2(1))$ *and* $\nu_3(A_0(1)) \not\equiv 0 \pmod{3}$.
3. $\nu_3(a_1) < 2\nu_3(a_2)$, $\nu_3(a_1)$ *is even,* $(a_1)_3 \equiv -1 \pmod{3}$, $\nu_3(a_0) > \frac{3}{2}\nu_3(a_1)$, $\nu_3(A_1(1)) > 2\nu_3(A_2(1))$ *and* $\nu_3(A_0(1)) > 2\nu_3(A_1(1)) - \nu_3(A_2(1))$.

Example 2.7 Let K be a sextic number field defined by the monic irreducible polynomial $F(x) = x^6 + 12x^4 + 8x^3 + 3x^2 + 3x + 27$. Since $\nu_3(a_0) = 3$, $\nu_3(a_1) = 1$, $\nu_3(a_2) = 1$, $a_3 \equiv -1 \pmod{3}$, $A_0(1) = 54 \equiv 54 \pmod{81}$ and $A_1(1) = 87 \equiv -3 \pmod{9}$, Theorem 2.8 (1) implies $\nu_3(i(K)) = 1$.

Theorem 2.9 *Suppose that $a_4 \equiv -1 \pmod{3}$ and $\nu_3(a_i) \geq 1$ for every $i \neq 4$. If all the following conditions are satisfied, then $\nu_3(i(K)) = 2$.*

1. $\nu_3(a_2) < 2\nu_3(a_3)$ *and* $\nu_3(a_2)$ *is odd.*
2. $\nu_3(a_1) > 2\nu_3(a_2)$ *and* $\nu_3(a_0) > 2\nu_3(a_1)$.

Example 2.8 Let K be a sextic number field defined by the monic irreducible polynomial $F(x) = x^6 + 2x^4 + 6x^3 + 12x^2 + 54x + 2187$. Since $\nu_3(a_0) = 7$, $\nu_3(a_1) = 3$, $\nu_3(a_2) = 1$ and $\nu_3(a_3) = 1$, by Theorem 2.9 we get $\nu_3(i(K)) = 2$.

Theorem 2.10 *Suppose that $a_3 \equiv -1 \pmod{3}$ and $a_i \equiv 0 \pmod{3}$ for every $i \neq 3$. Then each one of the following conditions guarantees that $\nu_3(i(K)) = 3$.*

1. $\nu_3(a_{i-1}) > 2\nu_3(a_i) - \nu_3(a_{i+1})$ *and* $\nu_3(A_{i-1}(1)) > 2\nu_3(A_i(1)) - \nu_3(A_{i+1}(1))$ *for every* $i = 1, 2$.
2. $\nu_3(a_1) < 2\nu_3(a_2)$, $\nu_3(a_1)$ *is even,* $(a_1)_3 \equiv 1 \pmod{3}$, $\nu_3(a_0) > 2\nu_3(a_1)$, $\nu_3(A_1(1)) < 2\nu_3(A_2(1))$, $\nu_3(A_1(1))$ *is even,* $(A_1(1))_3 \equiv 1 \pmod{3}$ *and* $\nu_3(A_0(1)) > 2\nu_3(A_1(1))$.

Example 2.9 Let K be a sextic number field defined by the monic irreducible polynomial $F(x) = x^6 + 50x^3 + 3x^2 + 675x + 729$. We have $\nu_3(a_0) = 6$, $\nu_3(a_1) = 3$, $\nu_3(a_2) = 1$, $\nu_3(A_0(1)) = 6$, $\nu_3(A_1(1)) = 3$ and $\nu_3(A_2(1)) = 1$. Hence, $\nu_3(i(K)) = 3$ by Theorem 2.10(1).

Theorem 2.11 *Suppose that $a_2 \equiv -1 \pmod{5}$ and $a_i \equiv 0 \pmod{5}$ for every $i \neq 2$. Then each one of the following conditions guarantees that $\nu_5(i(K)) = 1$.*

1. $\nu_5(a_0) > 2\nu_5(a_1)$.

2. $\nu_5(a_0) = 2\nu_5(a_1)$ *and* $((a_0)_5, (a_1)_5) \in \{(2,1),(2,4),(3,2),(3,3)\} \pmod 5$.

3. $\nu_5(a_0) < 2\nu_5(a_1)$, $\nu_5(a_0)$ *is even and* $(a_0)_5 \equiv \pm 1 \pmod 5$.

Example 2.10 Let K be a sextic number field defined by the monic irreducible polynomial $F(x) = x^6 + 10x^4 - x^2 + 500x + 25$. It follows from $\nu_5(a_0) = 2$, $\nu_5(a_1) = 3$ and $(a_0)_5 \equiv 1 \pmod 5$, and Theorem 2.11 (3) that $\nu_5(i(K)) = 1$.

The following example provides a sextic number field with index $i(K) = 240$.

Example 2.11 Let K be a sextic number field defined by the monic irreducible polynomial $F(x) = x^6 + 2x^5 + 5x^4 + 6x^3 + 24x^2 + 1920x + 4608000$.

1. For $p = 2$, as a_4 is odd, $a_0 \equiv 4096 \pmod{8192}$, $a_1 \equiv 128 \pmod{256}$, $a_2 \equiv 8 \pmod{16}$, $A_0(1) = 4609958 \equiv 2 \pmod 4$ and $A_1(1) = 2022 \equiv 2 \pmod 4$, we have $\nu_2(i(K)) = 4$ by Theorem 2.4.

2. For $p = 3$, given that $a_4 \equiv -1 \pmod 3$, $\nu_3(a_0) = 2$, $\nu_3(a_1) = 1$, $\nu_3(a_2) = 3$ and $\nu_3(a_3) = 2$. Using Theorem 2.7, we obtain $\nu_3(i(K)) = 1$.

3. Finally, for $p = 5$, we have $\nu_5(a_0) = 3$ and $\nu_5(a_1) = 1$. Consequently $\nu_5(i(K)) = 1$ thanks to Theorem 2.11 (1).

We then have $i(K) = 2^4 \cdot 3^1 \cdot 5^1 = 240$.

3. Preliminaries

Our proofs are based on Newton polygon techniques applied to prime ideals factorization, which is rather technical but very efficient to apply. We have introduced the corresponding concepts in several former papers. In this section, we state Ore's index theorem, which plays a key role in the proof of our main results. For further details, see [7,11].

We start by introducing some preliminaries required for the statement of Ore's index theorem.

Let $K = \mathbb{Q}(\alpha)$ be a number field generated by a complex root α of a monic irreducible polynomial $F(x) \in \mathbb{Z}[x]$. We shall use Dedekind's theorem [19, Chapter I, Proposition 8.3] and Dedekind's criterion [1, Theorem 6.1.4]. Let $\phi \in \mathbb{Z}_p[x]$ be a monic lift to an irreducible factor of $F(x)$ modulo p, $F(x) = a_0(x) + a_1(x)\phi(x) + \cdots + a_k(x)\phi(x)^k$ the ϕ-expansion of $F(x)$ and $N_\phi^+(F)$ the principal ϕ-Newton polygon of $F(x)$, which can be obtained only by considering the principal ϕ-expansion of $F(x)$. As defined in [7, Def. 1.3], the ϕ-index of $F(x)$, denoted by $\mathrm{ind}_\phi(F)$, is $\deg(\phi)$ multiplied by the number of points with natural integer coordinates that lie below or on the polygon $N_\phi^+(F)$, strictly above the horizontal axis and strictly beyond the vertical axis. Let $\mathbb{F}_\phi$ be the field $\mathbb{F}_p[x]/(\overline{\phi})$ and $u_i = \nu_p(a_i(x))$, then to every side S of $N_\phi^+(F)$ with initial point (s, u_s), length $l = l(S)$ and every $i = 0, \ldots, l$, let the residue coefficient $c_i \in \mathbb{F}_\phi$ be defined as follows:

$$c_i = \begin{cases} 0, & \text{if } (s+i, u_{s+i}) \text{ lies strictly above } S, \\ \left(\dfrac{a_{s+i}(x)}{p^{u_{s+i}}}\right) \mod (p, \phi(x)), & \text{if } (s+i, u_{s+i}) \text{ lies on } S. \end{cases}$$

Let $-\lambda = -h/e$ be the slope of S, where h and e are two positive coprime integers and $l = l(S)$ its length. Then $d = l/e$ is the degree of S. Hence, if i is not a multiple of e, then $(s+i, u_{s+i})$ does not lie on S and $c_i = 0$. Let $R_\lambda(F)(y) = t_d y^d + t_{d-1}y^{d-1} + \cdots + t_1 y + t_0 \in \mathbb{F}_\phi[y]$ be the residual polynomial of $F(x)$ associated to the side S, where for every $i = 0, \ldots, d$, $t_i = c_{s+ie}$. If $R_\lambda(F)(y)$ is square-free for each side of the polygon $N_\phi^+(F)$, then we say that $F(x)$ is ϕ-regular.

Let $\overline{F(x)} = \prod_{i=1}^r \overline{\phi_i}^{k_i}$ be the factorization of $F(x)$ into powers of monic irreducible coprime polynomials over $\mathbb{F}_p$, we say that the polynomial $F(x)$ is p-regular if $F(x)$ is a ϕ_i-regular polynomial with respect to p for every $i = 1, \ldots, r$. Let $N_{\phi_i}^+(F) = S_{i1} + \cdots + S_{ir_i}$ be the ϕ_i-principal Newton polygon of $F(x)$ with respect to p. For every $j = 1, \ldots, r_i$, let $R_{\lambda_{ij}}(F)(y) = \prod_{s=1}^{s_{ij}} \psi_{ijs}^{a_{ijs}}(y)$ be the factorization of $R_{\lambda_{ij}}(F)(y)$ in $\mathbb{F}_{\phi_i}[y]$. Then we have the following Ore's index theorem:

Theorem 3.1 *Ore's index theorem ([7, Theorem 1.7 and Theorem 1.9])*
Under the above hypotheses, we have the following:

1.
$$\nu_p((\mathbb{Z}_K : \mathbb{Z}[\alpha])) \geq \sum_{i=1}^{r} ind_{\phi_i}(F).$$

The equality holds if $F(x)$ is p-regular.

2. *If $F(x)$ is p-regular, then*
$$p\mathbb{Z}_K = \prod_{i=1}^{r}\prod_{j=1}^{r_i}\prod_{s=1}^{s_{ij}} \mathfrak{p}_{ijs}^{e_{ij}}$$
is the factorization of $p\mathbb{Z}_K$ into powers of prime ideals of $\mathbb{Z}_K$, where e_{ij} is the smallest positive integer satisfying $e_{ij}\lambda_{ij} \in \mathbb{Z}$ and the residue degree of $\mathfrak{p}_{ijs}$ over p is given by $f_{ijs} = \deg(\phi_i)\deg(\psi_{ijs})$ for every (i,j,s).

For the proof of our results, we need the following lemma, which characterizes the prime divisors of $i(K)$.

Lemma 3.1 *([8]) Let p be a rational prime and K a number field. For every positive integer f, let $\mathcal{P}_f$ be the number of distinct prime ideals of $\mathbb{Z}_K$ lying above p, with residue degree f, and $\mathcal{N}_f$ the number of monic irreducible polynomials of $\mathbb{F}_p[x]$ of degree f. Then p divides the index $i(K)$ if and only if $\mathcal{P}_f > \mathcal{N}_f$ for some positive integer f.*

For every number field of degree $n \leq 7$ and every rational prime p, Engstrom established a connection between $\nu_p = \nu_p(i(K))$ and the prime ideals factorization of $p\mathbb{Z}_K$. That is, from the factorization of $p\mathbb{Z}_K$, one can determine explicitly ν_p (for more details, see [8, Table, p. 234]).

4. Proofs of Main Results

In the proofs of the main results, we first determine the decomposition of $p\mathbb{Z}_K$; we then repeatedly use Lemma 3.1 to establish that $p \mid i(K)$, and Engstrom's result [8, Table, p. 234; Degree of field: 6] to compute $\nu_p(i(K))$. The latter will be stated once and not repeated. For simplicity of presentation, we provide only the decomposition of $p\mathbb{Z}_K$ in the proofs.

Recall that, according to the factorization given in Theorem 3.1, we use the triple indices in the factorization of $p\mathbb{Z}_K$. Namely $p\mathbb{Z}_K = \prod_{i=1}^{r}\prod_{j=1}^{r_i}\prod_{s=1}^{s_{ij}} \mathfrak{p}_{ijs}^{e_{ij}}$. Here e_{ij} is the ramification index of $\mathfrak{p}_{ijs}$ and $f_{ijs} = \deg(\phi_i)\deg(\psi_{ijs})$ is its residue degree for every (i,j,s).

Proof of Theorem 2.1.
As for every $i = 1, \ldots, 6$, $\nu_2(a_i) \geq 1$, we have $F(x) \equiv x^6$. Let $\phi_1 = x$. Then
$$F(x) = \phi_1^6 + a_5\phi_1^5 + \cdots + a_1\phi_1 + a_0.$$

1. If $a_3 \equiv 2 \pmod 4$, $a_2 \equiv 0 \pmod 4$, $\nu_2(a_1) > \nu_2(a_2) + 1$ and $\nu_2(a_0) > \nu_2(a_2) + 2$, then $N_{\phi_1}^+(F) = S_{11} + S_{12} + S_{13}$ has three sides joining $(0, \nu_2(a_0))$, $(2, \nu_2(a_2))$, $(3, 1)$ and $(6, 0)$ with $d(S_2) = d(S_3) = 1$. We have $d(S_1) = 1$ because $\nu_2(a_0) \not\equiv \nu_2(a_2) \pmod 2$. By Theorem 3.1, $2\mathbb{Z}_K = \mathfrak{p}_{111}^2\mathfrak{p}_{121}\mathfrak{p}_{131}^3$, with residue degree 1 for each prime ideal factor. It follows from Lemma 3.1 that 2 divides $i(K)$. Applying Engstrom's results, we obtain $\nu_2(i(K)) = 1$.

2. If $a_4 \equiv 2 \pmod 4$, $a_3 \equiv 0 \pmod 4$, $a_2 \equiv 4 \pmod 8$, $a_1 \equiv 0 \pmod 8$ and $\nu_2(a_0) = 2\nu_2(a_1) - 2$, then $N_{\phi_1}^+(F) = S_{11} + S_{12}$ has two sides joining $(0, \nu_2(a_0))$, $(2, 2)$ and $(6, 0)$. Thus the degree of each side of $N_{\phi_1}^+(F)$ is 2 (see Figure 1). Therefore $R_{\lambda_{11}}(F)(y) = R_{\lambda_{21}}(F)(y) = y^2 + y + 1$, which are irreducible over $\mathbb{F}_{\phi_1}$. By Theorem 3.1, $2\mathbb{Z}_K = \mathfrak{p}_{111}\mathfrak{p}_{211}^2$ with residue degree 2 for each prime ideal factor.

□

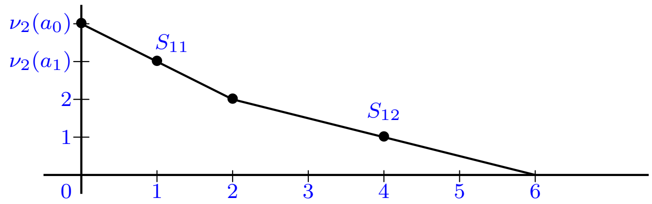

Figure 1: $N^+_{\phi_1}(F)$

Proof of Theorem 2.2.
Given that $\nu_2(a_4) = 0$ and $\nu_2(a_i) \geq 1$ for every $i \neq 4$, we have $F(x) \equiv x^4(x-1)^2 \pmod 2$. Let $\phi_1 = x$ and $\phi_2 = x - 1$. Then

$$\begin{aligned} F(x) &= \cdots + a_4\phi_1^4 + a_3\phi_1^3 + a_2\phi_1^2 + a_1\phi_1 + a_0, \\ &= \cdots + A_1(1)\phi_2 + A_0(1). \end{aligned}$$

As $\nu_2(A_0(1)) = 2\nu_2(A_1(1))$, $N^+_{\phi_2}(F) = S_{21}$ has a single side of degree 2 with $R_{\lambda_{21}}(F)(y) = y^2 + y + 1$, which is irreducible over $\mathbb{F}_{\phi_2}$. Hence ϕ_2 provides a unique prime ideal of $\mathbb{Z}_K$ lying above 2 with residue degree 2. For ϕ_1, we have the following:

1. If $\nu_2(a_0) > \dfrac{4}{3}\nu_2(a_1)$, $\nu_2(a_1) > \dfrac{3}{2}\nu_2(a_2)$ and $\nu_2(a_1) > 3\nu_2(a_3)$, then $N^+_{\phi_1}(F) = S_{11} + S_{12}$ has two sides joining $(0, \nu_2(a_0))$, $(1, \nu_2(a_1))$ and $(4, 0)$ with $d(S_{11}) = 1$ (see Figure 2). Since $\nu_2(a_1) \equiv 0 \pmod 3$, we have $d(S_{12}) = 3$ with $R_{\lambda_{12}}(F)(y) = y^3 + 1 = (y+1)(y^2+y+1) \in \mathbb{F}_{\phi_1}[y]$. Thus $2\mathbb{Z}_K = \mathfrak{p}_{111}\mathfrak{p}_{121}\mathfrak{p}_{122}\mathfrak{p}_{211}$ with $f_{111} = f_{122} = 1$ and $f_{121} = f_{211} = 2$.

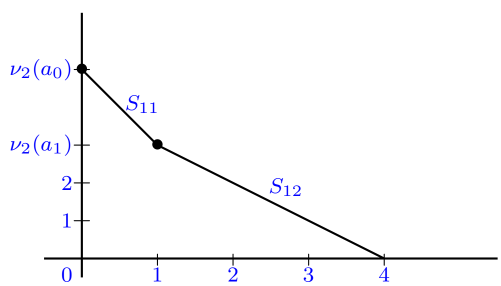

Figure 2: $N^+_{\phi_1}(F)$

2. If $a_3 \equiv 2 \pmod 4$, $a_2 \equiv 4 \pmod 8$, $a_1 \equiv 0 \pmod{16}$ and $\nu_2(a_0) = 2\nu_2(a_1) - 2$, then $N^+_{\phi_1}(F) = S_{11} + S_{12}$ has two sides joining $(0, \nu_2(a_0))$, $(2, 2)$ and $(4, 0)$ with $d(S_{11}) = d(S_{12}) = 2$ and $R_{\lambda_{11}}(F)(y) = R_{\lambda_{12}}(F)(y) = y^2 + y + 1$ which are irreducible over $\mathbb{F}_{\phi_1}$ (see Figure 3). Thus $2\mathbb{Z}_K = \mathfrak{p}_{111}\mathfrak{p}_{121}\mathfrak{p}_{211}$, with residue degree 2 for each prime ideal factor.

□

Proof of Theorem 2.3.
Since $\nu_2(a_4) = 0$ and $\nu_2(a_i) \geq 1$ for every $i \neq 4$, one deduces that $F(x) \equiv x^4(x-1)^2 \pmod 2$. Let $\phi_1 = x$ and $\phi_2 = x - 1$. Then

$$\begin{aligned} F(x) &= \cdots + a_4\phi_1^4 + a_3\phi_1^3 + a_2\phi_1^2 + a_1\phi_1 + a_0 \\ &= \cdots + A_1(1)\phi_2 + A_0(1). \end{aligned}$$

Thanks to $\nu_2(A_0(1)) < 2\nu_2(A_1(1))$ and $\nu_2(A_0(1))$ is odd, $N^+_{\phi_2}(F) = S_{21}$ has a single side joining $(0, \nu_2(A_0(1))$ and $(2, 0)$ with $d(S_{21}) = 1$. For ϕ_1, as $\nu_2(a_2) < 2\nu_2(a_3)$, $\nu_2(a_2)$ is odd, $\nu_2(a_1) > \nu_2(a_2) +$

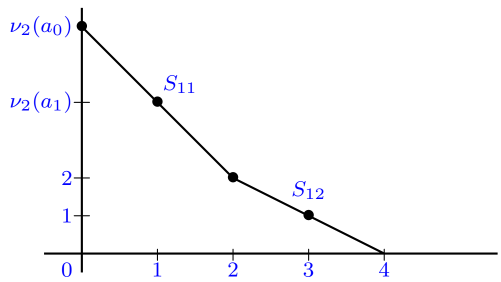

Figure 3: $N^+_{\phi_1}(F)$

$\nu_2(a_3)$, $2\nu_2(a_2) < \nu_2(a_0) < 2\nu_2(a_1) - \nu_2(a_2)$ and $\nu_2(a_0)$ is even, we have $N^+_{\phi_1}(F) = S_{11} + S_{12}$ has two sides joining $(0, \nu_2(a_0))$, $(2, \nu_2(a_2))$ and $(4, 0)$ and the degree of each side of $N^+_{\phi_1}(F)$ is 1 (see Figure 4). Thus $2\mathbb{Z}_K = \mathfrak{p}^2_{111}\mathfrak{p}^2_{121}\mathfrak{p}^2_{211}$, with residue degree 1 for each prime ideal factor.

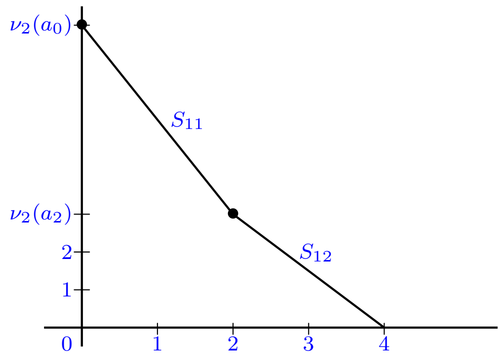

Figure 4: $N^+_{\phi_1}(F)$

□

Proof of Theorem 2.4.
Given that $\nu_2(a_4) = 0$ and $\nu_2(a_i) \geq 1$ for every $i \neq 4$, we conclude that $F(x) \equiv x^4(x-1)^2 \pmod 2$. Let $\phi_1 = x$ and $\phi_2 = x - 1$. Then

$$\begin{aligned} F(x) &= \cdots + a_4\phi_1^4 + a_3\phi_1^3 + a_2\phi_1^2 + a_1\phi_1 + a_0 \\ &= \cdots + A_1(1)\phi_2 + A_0(1). \end{aligned}$$

Since $\nu_2(A_0(1)) < 2\nu_2(A_1(1))$ and $\nu_2(A_0(1))$ is odd, $N^+_{\phi_2}(F) = S_{21}$ has a single side of degree 1. Thus ϕ_2 provides a unique prime ideal of $\mathbb{Z}_K$ lying above 2 with residue degree 1. Given that $\nu_2(a_{i-1}) > 2\nu_2(a_i) - \nu_2(a_{i+1})$ for every $i = 1, 2, 3$, $N^+_{\phi_1}(F) = S_{11} + S_{12} + S_{13} + S_{14}$ has four sides joining $(0, \nu_2(a_0))$, $(1, \nu_2(a_1))$, $(2, \nu_2(a_2))$, $(3, \nu_2(a_3))$ and $(4, 0)$. Thus the degree of each side of $N^+_{\phi_1}(F)$ is 1. Hence $2\mathbb{Z}_K = \mathfrak{p}_{111}\mathfrak{p}_{121}\mathfrak{p}_{131}\mathfrak{p}_{141}\mathfrak{p}^2_{211}$, with residue degree 1 for each prime ideal factor.

□

Proof of Theorem 2.5.
We have $F(x) \equiv x^4(x-1)^2 \pmod 2$ because $\nu_2(a_4) = 0$ and $\nu_2(a_i) \geq 1$ for every $i \neq 4$. Let $\phi_1 = x$ and $\phi_2 = x - 1$. Then

$$\begin{aligned} F(x) &= \cdots + A_1(1)\phi_2 + A_0(1), \\ &= \cdots + a_4x^4 + a_3x^3 + a_2x^2 + a_1x + a_0. \end{aligned}$$

Given that $\nu_2(A_0(1)) > 2\nu_2(A_1(1))$, $N^+_{\phi_2}(F) = S_{21}+S_{22}$ has two sides, each of degree 1. Thus ϕ_2 provides two prime ideals of $\mathbb{Z}_K$ lying above 2 with residue degree 1 for each prime ideal factor. As $\nu_2(a_2) < 2\nu_2(a_3)$, $\nu_2(a_2)$ is odd, $\nu_2(a_1) > \nu_2(a_2) + \nu_2(a_3)$ and $\nu_2(a_0) > 2\nu_2(a_1)$, we have $N^+_{\phi_1}(F) = S_{11} + S_{12} + S_{13}$ has three sides joining $(0, \nu_2(a_0))$, $(2, \nu_2(a_2))$, $(3, \nu_2(a_3))$ and $(4, 0)$. Thus the degree of each side of $N^+_{\phi_1}(F)$ is 1. So, $2\mathbb{Z}_K = \mathfrak{p}_{111}\mathfrak{p}_{121}\mathfrak{p}^2_{131}\mathfrak{p}_{211}\mathfrak{p}_{221}$, with residue degree 1 for each prime ideal factor.

□

Proof of Theorem 2.6.

1. Since 2 does not divide a_2 and $\nu_2(a_i) \geq 1$ for every $i \neq 2$, $F(x) \equiv x^2(x-1)^4 \pmod 2$. Let $\phi_1 = x$ and $\phi_2 = x - 1$. Then

$$\begin{aligned} F(x) &= \cdots + a_2\phi_1^2 + a_1\phi_1 + a_0, \\ &= \cdots + A_4(1)\phi_2^4 + A_3(1)\phi_2^3 + A_2(1)\phi_2^2 + A_1(1)\phi_2 + A_0(1). \end{aligned}$$

   If $\nu_2(a_0) > 2\nu_2(a_1)$, then $N^+_{\phi_1}(F) = S_{11} + S_{12}$ has two sides,each of degree 1. Hence ϕ_1 provides two prime ideals of $\mathbb{Z}_K$ lying above 2 with residue degree 1 for each prime ideal factor. Given that $\nu_2(A_{i-1}) > 2\nu_2(A_i) - \nu_2(A_{i+1})$ for every $i = 1, 2, 3$, $N^+_{\phi_2}(F) = S_{21} + S_{22} + S_{23} + S_{24}$ has four sides joining $(0, \nu_2(A_0(1)))$, $(1, \nu_2(A_1(1))$, $(2, \nu_2(A_2(1))$, $(3, \nu_2(A_3(1))$ and $(4, 0)$. Thus the degree of each side of $N^+_{\phi_2}(F)$ is 1. By Theorem 3.1, we obtain $2\mathbb{Z}_K = \mathfrak{p}_{111}\mathfrak{p}_{121}\mathfrak{p}_{211}\mathfrak{p}_{221}\mathfrak{p}_{231}\mathfrak{p}_{241}$, with residue degree 1 for each prime ideal factor.

2. From $\nu_2(a_3) = \nu_2(a_4) = \nu_2(a_5) = 0$ and $\nu_2(a_i) \geq 1$ for every $i \in \{0, 1, 2\}$, we obtain $F(x) \equiv x^3(x-1)^3 \pmod 2$. Let $\phi_1 = x$ and $\phi_2 = x - 1$. Then

$$\begin{aligned} F(x) &= \cdots + a_3\phi_1^3 + a_2\phi_1^2 + a_1\phi_1 + a_0, \\ &= \cdots + A_3(1)\phi_2^3 + A_2(1)\phi_2^2 + A_1(1)\phi_2 + A_0(1). \end{aligned}$$

   If $\nu_2(a_{i-1}) > 2\nu_2(a_i) - \nu_2(a_{i+1})$ for every $i = 1, 2$, then $N^+_{\phi_1}(F) = S_{11} + S_{12} + S_{13}$ has three sides joining $(0, \nu_2(a_0))$, $(1, \nu_2(a_1))$, $(2, \nu_2(a_2))$ and $(3, 0)$. Thus the degree of each side of $N^+_{\phi_1}(F)$ is 1. The assertion $\nu_2(A_{i-1}) > 2\nu_2(A_i) - \nu_2(A_{i+1})$ for every $i = 1$ or 2, implies that $N^+_{\phi_2}(F) = S_{21} + S_{22} + S_{23}$ has three sides joining $(0, \nu_2(A_0(1)))$, $(1, \nu_2(A_1(1))$, $(2, \nu_2(A_2(1))$ and $(3, 0)$. Thus the degree of each side of $N^+_{\phi_2}(F)$ is 1. By Theorem 3.1, the rational prime 2 splits completely in K.

3. As $\nu_2(a_4) = 0$ and $\nu_2(a_i) \geq 1$ for every $i \neq 4$, we have $F(x) \equiv x^4(x-1)^2 \pmod 2$. Let $\phi_1 = x$ and $\phi_2 = x - 1$. Then

$$\begin{aligned} F(x) &= \cdots + a_4x^4 + a_3x^3 + a_2x^2 + a_1x + a_0, \\ &= \cdots + A_1(1)\phi_2 + A_0(1). \end{aligned}$$

   Given that $\nu_2(A_0(1) > 2\nu_2(A_1(1))$, $N^+_{\phi_2}(F) = S_{21} + S_{22}$ has two sides, each of degree 1. Thus ϕ_2 provides two prime ideals of $\mathbb{Z}_K$ lying above 2 with residue degree 1 for each prime ideal factor. Since $\nu_2(a_{i-1}) > 2\nu_2(a_i) - \nu_2(a_{i+1})$ for every $i = 1, 2, 3$, $N^+_{\phi_1}(F) = S_{11} + S_{12} + S_{13} + S_{14}$ has four sides joining $(0, \nu_2(a_0))$, $(1, \nu_2(a_1))$, $(2, \nu_2(a_2))$, $(3, \nu_2(a_3))$ and $(4, 0)$. Thus the degree of each side of $N^+_{\phi_1}(F)$ is 1. By Theorem 3.1, the rational prime 2 splits completely in K.

□

Proof of Theorem 2.7.

Thanks to $a_4 \equiv -1 \pmod 3$ and $\nu_3(a_i) \geq 1$ for every $i \neq 4$, we obtain $F(x) \equiv x^4(x-1)(x-2) \pmod 3$. For every $k = 0, 1, 2$, let $\phi_k = x - k$. Then, for every $k = 1, 2$, ϕ_k provides a unique prime ideal of $\mathbb{Z}_K$ lying above 3 with residue degree 1. For $\phi_0 = x$, as $\nu_3(a_1) \leq \nu_3(a_i)$ for every $i = 2, 3$, $\nu_3(a_1) \not\equiv 0 \pmod 3$ and $\nu_3(a_0) > \dfrac{3}{2}\nu_3(a_1)$, $N^+_{\phi_0}(F) = S_{01} + S_{02}$ has two sides joining $(0, \nu_3(a_0))$, $(1, \nu_3(a_1))$, and $(4, 0)$. Thus the degree of each side of $N^+_{\phi_0}(F)$ is 1. Therefore, $3\mathbb{Z}_K = \mathfrak{p}_{011}\mathfrak{p}^3_{021}\mathfrak{p}_{111}\mathfrak{p}_{211}$, with residue degree 1 for each prime ideal factor.

□

Proof of Theorem 2.8.
Given that $a_3 \equiv -1 \pmod 3$ and $a_i \equiv 0 \pmod 3$ for every $i \neq 3$, we have $F(x) \equiv x^3(x-1)^3 \pmod 3$. Let $\phi_1 = x$ and $\phi_2 = x-1$. Then

$$\begin{aligned} F(x) &= \cdots + a_3\phi_1^3 + a_2\phi_1^2 + a_1\phi_1 + a_0, \\ &= \cdots + A_2(1)\phi_2^2 + A_1(1)\phi_2 + A_0(1). \end{aligned}$$

1. If $\nu_3(a_1) < 2\nu_3(a_2)$, $\nu_3(a_0) > 2\nu_3(a_1)$ and $\nu_3(a_1) \not\equiv 0 \pmod 2$, then $N_{\phi_1}^{+} = S_{11} + S_{12}$ has two sides joining $(0, \nu_3(a_0))$, $(1, \nu_3(a_1))$ and $(3, 0)$. Thus the degree of each side of $N_{\phi_1}^{+}(F)$ is 1. Since $\nu_3(A_1(1)) < 2\nu_3(A_2(1))$, $\nu_3(A_1(1)) \not\equiv 0 \pmod 2$ and $\nu_3(A_0(1)) > 2\nu_3(A_1(1))$, $N_{\phi_2}^{+}(F) = S_{21} + S_{22}$ has two sides joining $(0, \nu_3(A_0(1)))$, $(1, \nu_3(A_1(1)))$ and $(3, 0)$. Thus the degree of each side of $N_{\phi_2}^{+}(F)$ is 1. Hence $3\mathbb{Z}_K = \mathfrak{p}_{111}\mathfrak{p}_{121}^2\mathfrak{p}_{211}\mathfrak{p}_{221}^2$, with residue degree 1 for each prime ideal factor.
2. If $\nu_3(a_1) > 2\nu_3(a_2)$ and $\nu_3(a_0) > 2\nu_3(a_1) - \nu_3(a_2)$, then $N_{\phi_1}^{+}(F) = S_{11} + S_{12} + S_{13}$ has three sides joining $(0, \nu_3(a_3))$, $(1, \nu_3(a_1))$, $(2, \nu_3(a_2))$ and $(3, 0)$. Thus the degree of each side of $N_{\phi_1}^{+}(F)$ is 1. Because $\nu_3(A_0(1)) < \dfrac{3}{2}\nu_3(A_1(1))$, $\nu_3(A_0(1)) < 3\nu_3(A_2(1))$ and $\nu_3(A_0(1)) \not\equiv 0 \pmod 3$, we have that $N_{\phi_2}^{+}(F) = S_{21}$ has a single side joining $(0, \nu_3(A_0(1)))$ and $(3, 0)$, with $d(S_{21}) = 1$. Thus $3\mathbb{Z}_K = \mathfrak{p}_{111}\mathfrak{p}_{121}\mathfrak{p}_{131}\mathfrak{p}_{211}^3$, with residue degree 1 for each prime ideal factor.
3. If $\nu_3(a_1) < 2\nu_3(a_2)$, $\nu_3(a_1)$ is even, $(a_1)_3 \equiv -1 \pmod 3$ and $\nu_3(a_0) > \dfrac{3}{2}\nu_3(a_1)$, then $N_{\phi_1}^{+}(F) = S_{11} + S_{12}$ has two sides joining $(0, \nu_3(a_0))$, $(1, \nu_3(a_1))$ and $(3, 0)$, with $d(S_{11}) = 1$ and $R_{\lambda_{12}}(F)(y) = -y^2 - 1$ which is irreducible over $\mathbb{F}_{\phi_1}$. Given that $\nu_3(A_1(1)) > 2\nu_3(A_2(1))$ and $\nu_3(A_0(1)) > 2\nu_3(A_1(1)) - \nu_3(A_2(1))$, $N_{\phi_2}^{+}(F) = S_{21} + S_{22} + S_{23}$ has three sides joining $(0, \nu_3(A_0(1)))$, $(1, \nu_3(A_1(1)))$, $(2, \nu_3(A_2(1)))$ and $(0, 3)$. Thus the degree of each side of $N_{\phi_2}^{+}(F)$ is 1. Hence $3\mathbb{Z}_K = \mathfrak{p}_{111}\mathfrak{p}_{121}\mathfrak{p}_{211}\mathfrak{p}_{221}\mathfrak{p}_{231}$, where $f_{121} = 2$ and $f_{111} = f_{211} = f_{221} = f_{231} = 1$.

□

Proof of Theorem 2.9.
As $a_4 \equiv -1 \pmod 3$ and $\nu_3(a_i) \geq 1$ for every $i \neq 4$, we have $F(x) \equiv x^4(x-1)(x-2) \pmod 3$. For every $k = 0, 1, 2$, let $\phi_k = x - k$. Then, for every $k = 1, 2$, ϕ_k provides a unique prime ideal of $\mathbb{Z}_K$ lying above 3 with residue degree 1. For $\phi_0 = x$, since $\nu_3(a_2) < 2\nu_3(a_3)$, $\nu_3(a_1) > 2\nu_3(a_2)$ and $\nu_3(a_0) > 2\nu_3(a_1)$, $N_{\phi_0}^{+}(F) = S_{01} + S_{02} + S_{03}$ has three sides joining $(0, \nu_3(a_0))$, $(1, \nu_3(a_1))$, $(2, \nu_3(a_2))$ and $(4, 0)$. Thus the degree of each side of $N_{\phi_0}^{+}(F)$ is 1. Therefore, $3\mathbb{Z}_K = \mathfrak{p}_{011}\mathfrak{p}_{021}\mathfrak{p}_{031}^2\mathfrak{p}_{111}\mathfrak{p}_{211}$, with residue degree 1 for each prime ideal factor.

□

Proof of Theorem 2.10.
From $a_3 \equiv -1 \pmod 3$ and $a_i \equiv 0 \pmod 3$ for every $i \neq 3$, it follows that $F(x) \equiv x^3(x-1)^3 \pmod 3$. Let $\phi_1 = x$ and $\phi_2 = x - 1$. Then

$$\begin{aligned} F(x) &= \cdots + a_3\phi_1^3 + a_2\phi_1^2 + a_1\phi_1 + a_0, \\ &= \cdots + A_2(1)\phi_2^2 + A_1(1)\phi_2 + A_0(1). \end{aligned}$$

1. If $\nu_3(a_{i-1}) > 2\nu_3(a_i) - \nu_3(a_{i+1})$ for every $i = 1, 2$, then $N_{\phi_1}^{+}(F) = S_{11} + S_{12} + S_{13}$ has three sides joining $(0, \nu_3(a_0))$, $(1, \nu_3(a_1))$,$(2, \nu_3(a_2))$ and $(3, 0)$. Thus $d(S_{11}) = d(S_{12}) = d(S_{13}) = 1$. As $\nu_3(A_{i-1}(1)) > 2\nu_3(A_i(1)) - \nu_3(A_{i+1}(1))$ for every $i = 1, 2$, $N_{\phi_2}^{+}(F) = S_{21} + S_{22} + S_{23}$ has three sides joining $(0, \nu_3(A_0(1)))$, $(1, \nu_3(A_1(1)))$, $(2, \nu_3(A_2(1)))$ and $(3, 0)$. Thus $d(S_{21}) = d(S_{22}) = d(S_{23}) = 1$. Hence $3\mathbb{Z}_K = \mathfrak{p}_{111}\mathfrak{p}_{121}\mathfrak{p}_{131}\mathfrak{p}_{211}\mathfrak{p}_{221}\mathfrak{p}_{231}$, with residue degree 1 for each prime ideal factor.

2. If $\nu_3(a_1) < 2\nu_3(a_2)$, $\nu_3(a_1)$ is even, $(a_1)_3 \equiv 1 \pmod 3$ and $\nu_3(a_0) > 2\nu_3(a_1)$, then $N^+_{\phi_1}(F) = S_{11}+S_{12}$ has two sides joining $(0, \nu_3(a_0))$, $(1, \nu_3(a_1))$ and $(3,0)$, with $d(S_{11}) = 1$ and $R_{\lambda_{12}}(F)(y) = -y^2+1 = -(y+1)(y-1) \in \mathbb{F}_{\phi_1}[y]$. Since $\nu_3(A_1(1)) < 2\nu_3(A_2(1))$, $\nu_3(A_1(1))$ is even, $(A_1(1))_3 \equiv 1 \pmod 3$ and $\nu_3(A_0(1)) > 2\nu_3(A_1(1))$, we have that $N^+_{\phi_2}(F) = S_{21} + S_{22}$ has two sides joining $(0, \nu_3(A_0(1)))$, $(2, \nu_3(A_1(1)))$ and $(3,0)$, with $d(S_{21}) = 1$ and $R_{\lambda_{22}}(F)(y) = -y^2+1 = -(y+1)(y-1) \in \mathbb{F}_{\phi_2}[y]$. Hence the rational prime 3 splits completely in K.

$\square$

Proof of Theorem 2.11.
As $a_2 \equiv -1 \pmod 5$ and $a_i \equiv 0 \pmod 5$ for every $i \neq 2$, it can be seen that $F(x) \equiv x^2(x-1)(x-2)(x-3)(x-4) \pmod 5$. For every $k = 0, 1, \ldots, 4$, let $\phi_k = x - k$. Then, for every $k = 1, \ldots, 4$, ϕ_k provides a unique prime ideal of $\mathbb{Z}_K$ lying above 5 with residue degree 1. For $\phi_0 = x$, we have the following:

1. If $\nu_5(a_0) > 2\nu_5(a_1)$, then $N^+_{\phi_0}(F) = S_{01} + S_{02}$ has two sides joining $(0, \nu_5(a_0))$, $(1, \nu_5(a_1))$ and $(2,0)$. Thus the degree of each side of $N^+_{\phi_0}(F)$ is 1. Therefore, $5\mathbb{Z}_K = \mathfrak{p}_{011}\mathfrak{p}_{021}\mathfrak{p}_{111}\mathfrak{p}_{211}\mathfrak{p}_{311}\mathfrak{p}_{411}$, with residue degree 1 for each prime ideal factor.

2. If $\nu_5(a_0) = 2\nu_5(a_1)$, then $N^+_{\phi_0}(F) = S_{01}$, where $d(S_{01}) = 2$. Since $a_2 \equiv -1 \pmod 5$ and $((a_0)_5, (a_1)_5) \in \{(2,1), (2,4)(3,2), (3,3)\} \pmod 5$, then $R_{\lambda_{01}}(F)(y)$ can be factorized in $\mathbb{F}_{\phi_0}[y]$ by two distinct linear polynomials (see Table 1). In all these cases, the rational prime 5 splits completely in K.

$((a_0)_5, (a_1)_5) \pmod 5$	$R_{\lambda_{01}}(F)(y) \bmod (5, \phi_0)$
$(2,1)$	$-(y+1)(y+3)$
$(2,4)$	$-(y-1)(y+2)$
$(3,2)$	$-(y+1)(y+2)$
$(3,3)$	$-(y-1)(y+3)$

Table 1: $R_{\lambda_{01}}(F)(y)$ in $\mathbb{F}_{\phi_0}[y]$

3. If $\nu_5(a_0)$ even and $\nu_5(a_0) < 2\nu_5(a_1)$, then $N^+_{\phi_0}(F) = S_{01}$ where $d(S_{01}) = 2$. If $(a_0)_5 \equiv 1 \pmod 5$, then $R_{\lambda_{01}}(F)(y) = -(y-1)(y+1)$. If $(a_0)_5 \equiv -1 \pmod 5 \in \mathbb{F}_{\phi_0}[y]$, then $R_{\lambda_{01}}(F)(y) = -(y+2)(y+3) \in \mathbb{F}_{\phi_0}[y]$. In both cases, the rational prime 5 splits completely in K.

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