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Convergence Analysis of a Chebyshev Wavelet and Its Applications to Differential Equations through Operational Matrices of Integration

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ABSTRACT: In this paper, convergence analysis of the Chebyshev wavelet of first kind is thoroughly carried out. Operational matrices for integration and product operations of the first kind Chebyshev wavelets are constructed and these matrices are utilized to obtain solutions to the differential equations. A theorem related to the proposed operational matrix method is establised. Solutions of the differential equations considered in this paper, resemble with their exact solutions. The characteristics of first kind Chebyshev wavelets are utilized to transform differential equations to the systems of algebraic equations, which are solved very efficiently using appropriate methods.

Keywords: Chebyshev wavelets, convergence analysis, operation matrix of integration (OMI), product operation matrix (POM), Lane–Emden type differential equation, third order singular differential equation.

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1. Introduction

Wavelets are compact or short-lived waves. Rather than continuing to oscillate indefinitely, they return to zero. Wavelets allow for the precise representation of various functions. They are considered as basis functions $\varphi_{i,j}(t)$ in continuous time. A distinct characteristic of the wavelet basis is that each function $\varphi_{i,j}(t)$ is derived from $\varphi(t)$, which is known as mother wavelet. Typically, a collection of linearly independent (L.I.) functions is generated through the translation and dilation of the mother wavelet.

The Chebyshev wavelets of the first kind form an orthogonal and complete basis in $L^2[-1,1]$, and therefore any square–integrable function can be approximated by a finite number of wavelet coefficients. Owing to the smoothness and polynomial structure inherited from the Chebyshev polynomials of the

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first kind, the corresponding wavelet expansions exhibit faster convergence when the target function is sufficiently smooth. This wavelet provides a highly accurate and convergent framework for approximating smooth as well as moderately regular functions, making it suitable for operational matrix–based numerical solutions of differential equations.

Applications of the Chebyshev wavelets of first kind span a wide range of technological fields [27], especially in signal analysis [24], time-frequency analysis, and efficient algorithms for straight forward implementation [1]. Linear differential equations arise when we model a real world phenomenon. These models appear widely in science, engineering, economics, medicine and many other fields, for examples, motion with resistence, electrical systems, fluid mechanics, quantum mechanics etc.

In numerous instances, obtaining analytical solutions for linear initial value problems is not feasible. For these situations, we employ numerical method like operational matrix of integrations (OMI). The operational matrix of integration (OMI) is a highly effective numerical method for solving linear differential equations. This technique is based on converting differential equations into integral equations using operational matrix of integration by eliminating the integral operator in order to reduce the problem into a system of algebraic equations, which is further solved by suitable methods.

Employing OMI of Haar wavelets to address differential equations, a limitation arises due to a jump discontinuity at $x = 1/2$. Therefore, exploring new approaches to solve and examine differential equations has become a fascinating topic within the realm of wavelets.

Since Chebyshev wavelets are very useful wavelet methods, therefore, in this study, we develop the operational matrix of integration (OMI) and the product operation matrix (POM) for first kind Chebyshev wavelets and use this technique to solve some of the most important linear differential equations. The suggested technique for solving these differential equations utilizes a limited number of bases and takes advantage of the orthogonality of Chebyshev wavelets to transform the linear differential equations into a straightforward system of algebraic equations. One can find more details on orthogonal functions and polynomials in [2,4,6,14,25].

Further, in case of linear differential equations, we consider first order linear and Lane Emden differential equations, third order linear and singular differential equation which are solved using operational matrix of integration for $k = 3$ and $M = 4$ and product operational matrix. linear differential equation has applications in the field of motion with resistance; charging of capacitors in the field of electrical systems, while Lane-Emden differential equation has applications in astrophysics and theoretical physics. It is primarily used to describe the structure of self-gravitating, spherically symmetric polytropic gas spheres, such as stars and gaseous planets. and third order singular differential equation has applications in models physical systems with time-dependent or spatially varying parameters, such as unsteady fluid flow or mechanical oscillations. Its analysis provides insights into variable-coefficient dynamical systems frequently encountered in engineering and applied sciences.

It can be noted that in recent past, particular emphesis has been placed on the use of Legendre wavelets [6,5,10,26] and Hybrid functions [9,8,11,12]. Some most recent work in the context of present paper can also be found in [19,20,21,22,23,16,17,3,6,26].

It can be noted that in recent past, particular emphesis has been placed on the use of Legendre wavelets [6,5,10,26] and Hybrid functions [9,8,11,12]. Some most recent work in the context of present paper can also be found in [19,20,21,22,23,16,17,18,1,6,26].

The organization of this paper is outlined as follows: section 2 contains key definitions pertinent to the current study. In section 3, The convergence of the proposed method is thoroughly analyzed. In section 4, we present a format operational matrices related to integration and the product operation matrix (POM) for Chebyshev wavelets (CW) of the first kind. Moreover, We establish a theorem related to the proposed operational matrix method. In section 5, we find solutions of linear differential equation and Lane Emden differential equations using proposed method and compare these solutions with their exact solutions. In section 6, conclusion is given.

2. Preliminaries

2.1. Wavelets and Chebyshev wavelets (CW)

Wavelets consist of a set of functions obtained from the dilation and translation of a particular function referred to as the mother wavelet. By continuously varying the parameters for dilation and translation, represented by a and b, we generate a continuous family of wavelets.

$$\varphi_{a,b}(t) = \frac{1}{\sqrt{|a|}}\varphi\left(\frac{t-b}{a}\right), \qquad a, b \in \mathbb{R},$$

where $\varphi_{a,b}(t)$ forms a wavelet basis for $L^2(\mathbb{R})$ provided $a \neq 0$ [13]. We consider the parameters a and b as discrete values $a = a_0^{-n},\ b = mb_0a_0^{-n}, a_0 > 1, b_0 > 0$, where n and m are positive integers. Specifically, when $a_0 = 2$ and $b_0 = 1$, the set $\{\varphi_{n,m}(t)\}$ constitutes an orthonormal basis.

The Chebyshev wavelets denoted as $\varphi_{n,m}(t) = \varphi(k, n, m, t)$ rely on four parameters. Here, n takes the values $1, 2, 3, ..., 2^{k-1}$, where k belongs to positive integer, t is time and $m = 0, 1, ..., M-1$, which is the degree of Chebyshev polynomials (C.P).
Now, we define $\varphi_{n,m}(t)$ as

$$\varphi_{n,m}(t) = \begin{cases} 2^{\frac{k}{2}}\tilde{T_m}(2^kt - 2n + 1), & \frac{n-1}{2^{k-1}} \leq t < \frac{n}{2^{k-1}}; \\ 0, & \text{otherwise}, \end{cases} \tag{2.1}$$

where m and n are as defined above and $\tilde{T_m}$ is given as follows:

$$\tilde{T_m} = \begin{cases} \frac{1}{\sqrt{\pi}}, & m = 0; \\ \sqrt{\frac{2}{\pi}}T_m(t), & m > 0, \end{cases}$$

The Chebyshev polynomials of first kind with degree m are represented by polynomials $T_m(t)$. They follow this recursive connection and show orthogonality with respect to weight function $w(t) = \frac{1}{\sqrt{1-t^2}}$ over the interval [0,1], and they follow this recursive relationship:

$$T_m(t) = 2tT_m(t) - T_{m-1}(t); \qquad m = 1, 2, ...$$

We note that

$$T_0(t) = 1, T_1(t) = t, T_2 = 2t^2 - 1, T_3(t) = 4t^3 - 3t, ...$$

It is noted that when working with Chebyshev wavelets, the weight function $w(t)$ must undergo dilation and translation, given by $w_n(t) = w(2^kt - 2n + 1)$, in order to obtain orthogonal wavelets.

2.2. Approximation of a function using CW

A function $f \in L^2_{\tilde{w}}[0, 1)$, where $\tilde{w}(t) = w(2t-1)$, can be expended as the following wavelet series

$$f(t) = \sum_{n=1}^{\infty}\sum_{m=0}^{\infty} c_{nm}\varphi_{nm}(t), \tag{2.2}$$

where $c_{nm} = \langle f(t), \varphi_{nm}\rangle$. $\langle , \rangle$ denotes the inner product with respect the weight function $w(t)$. After truncating (2.2), we have

$$\begin{aligned} f(t) &\simeq \sum_{n=1}^{2^{k-1}}\sum_{m=0}^{M-1} c_{n,m}\varphi_{n,m}(t) \\ &= C^T\varphi(t), \end{aligned} \tag{2.3}$$

where

$$C = [c_{1,0},\ c_{1,1}\ c_{1,2}, ..., c_{1,M},\ c_{2,0}, ..., c_{2,M},\ ...c_{2^{k-1},0}, ..., c_{2^{k-1},M-1}]^T \tag{2.4}$$

and

$$\varphi(t) = [\varphi_{1,0}, \varphi_{1,1}..., \varphi_{1,M}, \varphi_{2,0}, ...\varphi_{2,M}, ..., \varphi_{2,0}, ..., \varphi_{2^{k-1},M-1}]^T. \tag{2.5}$$

In (2.4) and (2.5), the order of matrices is $2^{k-1}M \times 1$ and T stands for transposition.

3. Convergence Analysis

In this section, we established the following convergence theorems for the Chebyshev wavelet of first kind:

3.1. Theorem 3.1

Theorem 3.1 *Assume that $g(t) \in L^2_w[0,1]$ possesses a bounded first derivative, that is,*

$$|g'(t)| \leq L \qquad \text{for all } t \in [0,1].$$

Suppose further that g admits the first–kind Chebyshev wavelet expansion

$$g(t) = \sum_{n=1}^{\infty} \sum_{m=0}^{\infty} c_{nm}\, \varphi_{nm}(t).$$

Then each wavelet coefficient c_{nm} obeys the estimate

$$|c_{nm}| \ \leq \ \frac{L\sqrt{2\pi}}{(n+1)^{3/2}m}. \tag{3.1}$$

Consequently, the resulting first–kind Chebyshev wavelet series converges uniformly to $g(t)$.

Proof: From the definition of the coefficient c_{nm},

$$\begin{aligned} c_{nm} &= \int_0^1 g(t)\varphi_{nm}(t)w_n(t)\,dt \\ &= 2^{\frac{k}{2}}\sqrt{\frac{2}{\pi}}\int_{\frac{n-1}{2^{k-1}}}^{\frac{n}{2^{k-1}}} g(t)T_m(2^k t - 2n + 1)w_n(t)\,dt, \end{aligned}$$

For $m > 0$. Employing the substitution $2^k t - 2n + 1 = \cos\theta$, we get

$$c_{nm} = \frac{\sqrt{2}}{2^{\frac{k}{2}}\sqrt{\pi}}\int_0^{\pi} g\left(\frac{\cos\theta + 2n - 1}{2^k}\right)\cos m\theta d\theta.$$

using integration by part we obtain

$$c_{nm} = \frac{L\sqrt{2}}{2^{3k/2}m\sqrt{\pi}}\int_0^{\pi} \sin m\theta \sin\theta \ d\theta$$

so we have

$$|c_{nm}| \leq \frac{L\sqrt{2}}{2^{3k/2}m\sqrt{\pi}}\int_0^{\pi} d\theta \leq \frac{L\sqrt{2\pi}}{2^{3k/2}m}$$

Since $n \leq 2^k - 1$, we obtain

$$\begin{aligned} |c_{nm}| &\leq \frac{L\sqrt{2\pi}}{2^{3k/2}m} \\ &\leq \frac{L\sqrt{2\pi}}{(n+1)^{3/2}m} \end{aligned}$$

□

3.2. Theorem 3.2

Theorem 3.2 *Let $g \in L^2_w[0,1]$ be continuous, where the associated weight is*

$$w(t) = \frac{1}{\sqrt{1-t^2}}.$$

Assume further that the Jth derivative of g is uniformly bounded on $[0,1]$, i.e.,

$$\sup_{t\in[0,1]} |g^{(J)}(t)| < \infty.$$

Under these conditions, the approximation of g by the Chebyshev wavelet partial sum of order $(2^{k-1}, J)$,

$$S_{2^{k-2},J}(t) = \sum_{n=1}^{2^{k-2}} \sum_{m=0}^{J-1} c_{n,m}\, \varphi_{n,m}(t),$$

yields the following error representation:

$$E_{2^k-1,J}(g) = \|g - S_{2^{k-2},J}\|_2 = \left\| g - \sum_{n=1}^{2^{k-2}} \sum_{m=0}^{J-1} c_{n,m}\, \varphi_{n,m}(t) \right\|_2 .$$

Moreover, the approximation error satisfies the asymptotic behavior

$$E_{2^k-1,J}(g) = O\left(\frac{1}{J!\, 2^{\,J(k+1)}} \right),$$

which establishes the convergence rate of the corresponding Chebyshev wavelet expansion in $L^2[0,1]$.

Proof: Since a function g is J times differentiable, by Taylor's expansion we have

$$g(b+h) = g_{J-1} = g(b) + \frac{h}{1!} g'(b) + \cdots + \frac{h^{J-1}}{(J-1)!} g^{(J-1)}(b) + \frac{h^J}{J!} g^{(J)}(b+\zeta h),$$

where $0 < \zeta < 1$, and

$$g_J = g(b) + \frac{h}{1!} g'(b) + \cdots + \frac{h^{J-1}}{(J-1)!} g^{(J-1)}(b).$$

Now we write

$$g_{J+1} - g_J = \frac{h^J}{J!} g^{(J)}(b+\zeta h), \qquad 0 < \zeta < 1. \tag{3.2}$$

Using (3.2) and dividing the interval $[0,1]$ into subintervals $\left[\frac{l-1}{2^k}, \frac{l}{2^k}\right]$, we get

$$\begin{aligned}
\|g - S_{2^k-2,J-1}\|_2^2 &= \int_0^1 \left| g(t) - \sum_{l=1}^{2^k-2} \sum_{m=0}^{J-1} c_{lm}\varphi_{lm}(t) \right|^2 dt \\
&= \sum_{l=0}^{2^k-1} \int_{\frac{l-1}{2^k}}^{\frac{l}{2^k}} \left| g(t) - \sum_{l=1}^{2^k-2} \sum_{m=0}^{J-1} c_{lm}\varphi_{lm}(t) \right|^2 dt \\
&\le \sum_{l=1}^{2^k} \int_{\frac{l-1}{2^k}}^{\frac{l}{2^k}} \left(\frac{1}{J!} \left(\frac{1}{2^{k-1}} \right)^J \sup_{x\in[0,1]} |g^{(J)}(t)| \right)^2 dt \\
&= \int_0^1 \left(\frac{1}{J!} \right)^2 \left(\frac{1}{2^{J(k-1)}} \right)^2 \sup_{x\in[0,1]} |g^{(J)}(t)|^2\, dt.
\end{aligned}$$

Now

$$\|f - S_{2^k-2,J-1}\|_2^2 \leq \left(\frac{1}{J!}\right)^2 \left(\frac{1}{2^{J(k-1)}}\right)^2 \sup_{x\in[0,1]} |g^{(J)}(t)|^2.$$

Hence

$$\|f - S_{2^k-2,J-1}\|_2 \leq \left(\frac{1}{J!}\right) \left(\frac{1}{2^{J(k-1)}}\right) \sup_{x\in[0,1]} |g^{(J)}(t)|.$$

Thus

$$E_{2^k-1}(f) = \|f - S_{2^k-2,J-1}\|_2 \leq \left(\frac{1}{J!}\right) \left(\frac{1}{2^{J(k-1)}}\right) \sup_{x\in[0,1]} |g^{(J)}(t)|$$

$$= O\left(\frac{1}{J!2^{J(k-1)}}\right).$$

□

4. Operational matrices of integration (OMI)

In this section, we present the construction of the operational matrix of integration as well as the matrix associated with the product operation for first–kind Chebyshev wavelets.

4.1. Operational matrix of integration of first-order Chebyshev wavelets for $k = 3$ and $M = 4$:

Here, we introduce the structure of OMI for first-kind Chebyshev wavelets, in particular for $k = 3$ and $M = 4$. The following is an analysis of sixteen basis functions defined on the interval [0,1):

$$\left.\begin{aligned} \varphi_{1,0}(t) &= 2\sqrt{\tfrac{2}{\pi}}, \\ \varphi_{1,1}(t) &= \tfrac{4}{\sqrt{\pi}}(8t-1), \\ \varphi_{1,2}(t) &= \tfrac{4}{\sqrt{\pi}}(128t^2-32t+1), \\ \varphi_{1,3}(t) &= \tfrac{4}{\sqrt{\pi}}(2048t^3-768t^2+72t-1), \end{aligned}\right\} 0 \leq t < \frac{1}{4}; \tag{4.1a}$$

$$\left.\begin{aligned} \varphi_{2,0}(t) &\;\; 2\sqrt{\tfrac{2}{\pi}}, \\ \varphi_{2,1}(t) &= \tfrac{4}{\sqrt{\pi}}(8t-3), \\ \varphi_{2,2}(t) &= \tfrac{4}{\sqrt{\pi}}(128t^2-96t+17), \\ \varphi_{2,3}(t) &= 2\sqrt{\tfrac{2}{\pi}}(2048t^3-2304t^2+840t-99), \end{aligned}\right\} \frac{1}{4} \leq t < \frac{1}{2}; \tag{4.1b}$$

$$\left.\begin{aligned} \varphi_{3,0}(t) &= 2\sqrt{\tfrac{2}{\pi}}, \\ \varphi_{3,1}(t) &= \tfrac{4}{\sqrt{\pi}}(8t-5), \\ \varphi_{3,2}(t) &= \tfrac{4}{\sqrt{\pi}}(128t^2-160t+49), \\ \varphi_{3,3}(t) &= 2\sqrt{\tfrac{2}{\pi}}(2048t^3-3840t^2+2376t-485), \end{aligned}\right\} \frac{1}{2} \leq t < \frac{3}{4}; \tag{4.1c}$$

$$\left.\begin{aligned} \varphi_{4,0}(t) &= 2\sqrt{\tfrac{2}{\pi}}, \\ \varphi_{4,1}(t) &= \tfrac{4}{\sqrt{\pi}}(8t-7), \\ \varphi_{4,2}(t) &= \tfrac{4}{\sqrt{\pi}}(128t^2-224t+97), \\ \varphi_{4,3}(t) &= \tfrac{4}{\sqrt{\pi}}(2048t^3-5376t^2+4680t-1351), \end{aligned}\right\} \frac{3}{4} \leq t < 1; \tag{4.1d}$$

Let

$$\varphi_{16}(t) = [\varphi_{1,0}(t)\ \varphi_{1,1}(t)\ \varphi_{1,2}(t)\ \varphi_{1,3}(t)\ ...\varphi_{4,0}(t)\ \varphi_{4,1}(t)\ \varphi_{4,2}(t)\ \varphi_{4,3}]^T. \tag{4.2}$$

By integrating the first basis function between 0 and t, we get

$$\int_0^t \varphi_{1,0}(t')dt' = \begin{cases} 2\sqrt{\frac{2}{\pi}}t & ,0 \le t < \frac{1}{4} \\ \frac{1}{\sqrt{2\pi}} & ,\frac{1}{4} \le t < \frac{1}{2} \\ \frac{1}{\sqrt{2\pi}} & ,\frac{1}{2} \le t < \frac{3}{4} \\ \frac{1}{\sqrt{2\pi}} & ,\frac{3}{4} \le t < 1 \end{cases} \tag{4.3}$$

Expanding L.H.S of (4.3) in form of the basis function, we have

$$\begin{aligned} \int_0^t \varphi_{1,0}(t')\,dt' = a_{1,0}\varphi_{1,0} + a_{1,1}\varphi_{1,1} + a_{1,2}\varphi_{1,2} + a_{1,3}\varphi_{1,3} + a_{2,0}\varphi_{2,0} + a_{2,1}\varphi_{2,1} + a_{2,2}\varphi_{2,2} \\ + a_{2,3}\varphi_{2,3} + a_{3,0}\varphi_{3,0} + a_{3,1}\varphi_{3,1} + a_{3,2}\varphi_{3,2} + a_{3,3}\varphi_{3,3} \\ + a_{4,0}\varphi_{4,0} + a_{4,1}\varphi_{4,1} + a_{4,2}\varphi_{4,2} + a_{4,3}\varphi_{4,3}, \end{aligned} \tag{4.4}$$

where the first coefficient is

$$\begin{aligned} a_{1,0} &= \left\langle \int_0^t \varphi_{1,0}(t')dt', \varphi_{1,0}(t) \right\rangle_{w_n} \\ &= \frac{1}{8} \end{aligned}$$

Other coefficients of (4.4) can be calculated in the same manner, which are as follows:

$a_{1,1} = \frac{1}{8\sqrt{2}}, a_{1,2} = a_{1,3} = 0, a_{2,0} = \frac{1}{4}, a_{2,1} = a_{2,2} = a_{2,3} = 0, a_{3,0} = \frac{1}{4}, a_{3,1} = a_{3,2} = a_{3,3} = 0,$
$a_{4,0} = \frac{1}{4}, a_{4,1} = a_{4,2} = a_{4,3} = 0$

Thus, (4.4) is defined as

$$\begin{aligned} \int_0^t \varphi_{1,0}(t')dt' &= \frac{1}{8}\varphi_{1,0}(t) + \frac{1}{8\sqrt{2}}\varphi_{1,1}(t) + \frac{1}{4}\varphi_{2,0}(t) + \frac{1}{4}\varphi_{3,0}(t) + \frac{1}{4}\varphi_{2,0}(t) \\ &= \left[\frac{1}{8}\ \frac{1}{8\sqrt{2}}\ 0\ 0\ \frac{1}{4}\ 0\ 0\ 0\ \frac{1}{4}\ 0\ 0\ 0\ \frac{1}{4}\ 0\ 0\ 0\right]\varphi_{16}(t) \end{aligned} \tag{4.5}$$

By integrating the second basis function from 0 to t, we get

$$\int_0^t \varphi_{1,1}(t')dt' = \begin{cases} \frac{4}{\sqrt{\pi}}(4t^2 - t), & 0 \le t < \frac{1}{4} \\ 0, & \frac{1}{4} \le t < \frac{1}{2} \\ 0, & \frac{1}{2} \le t < \frac{3}{4} \\ 0, & \frac{3}{4} \le t < 1 \end{cases} \tag{4.6}$$

Expanding L.H.S of (4.6) in form of the basis function, we have

$$\begin{aligned} \int_0^t \varphi_{1,1}(t)dt' &= \frac{-1}{16\sqrt{2}}\varphi_{1,0} + \frac{1}{32}\varphi_{1,2} \\ &= \left[\frac{-1}{16\sqrt{2}}\ 0\ \frac{1}{32}\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\right]\varphi_{16}(t) \end{aligned} \tag{4.7}$$

Adopting the similar procedure, we obtain

$$\int_0^t \varphi_{1,2}(t')dt' = \left[\frac{-1}{12\sqrt{2}}\ \frac{-1}{16}\ 0\ \frac{1}{48}\ \frac{-1}{6\sqrt{2}}\ 0\ 0\ 0\ \frac{-1}{6\sqrt{2}}\ 0\ 0\ 0\ \frac{-1}{6\sqrt{2}}\ 0\ 0\ 0\right]\varphi_{16}(t)$$
$$\int_0^t \varphi_{1,3}(t')dt' = \left[\frac{1}{32\sqrt{2}}\ 0\ \frac{-1}{32}\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\right]\varphi_{16}(t)$$
$$\int_0^t \varphi_{2,0}(t')dt' = \left[0\ 0\ 0\ 0\ \frac{1}{8}\ \frac{1}{8\sqrt{2}}\ 0\ 0\ \frac{1}{4}\ 0\ 0\ 0\ \frac{1}{4}\ 0\ 0\ 0\right]\varphi_{16}(t)$$
$$\int_0^t \varphi_{2,1}(t')dt = \left[0\ 0\ 0\ 0\ \frac{-1}{16\sqrt{2}}\ 0\ \frac{1}{32}\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\right]\varphi_{16}(t)$$
$$\int_0^t \varphi_{2,2}(t')dt' = \left[0\ 0\ 0\ 0\ \frac{-1}{12\sqrt{2}}\ \frac{-1}{16}\ 0\ \frac{1}{48}\ \frac{-1}{6\sqrt{2}}\ 0\ 0\ 0\ \frac{-1}{\sqrt{2}}\ 0\ 0\ 0\right]\varphi_{16}(t)$$
$$\int_0^t \varphi_{2,3}(t')dt' = \left[0\ 0\ 0\ 0\ \frac{1}{32\sqrt{2}}\ 0\ \frac{-1}{32}\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\right]\varphi_{16}(t)$$
$$\int_0^t \varphi_{3,0}(t')dt' = \left[0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ \frac{1}{8}\ \frac{1}{8\sqrt{2}}\ 0\ 0\ \frac{1}{4}\ 0\ 0\ 0\right]\varphi_{16}(t)$$
$$\int_0^t \varphi_{3,1}(t')dt' = \left[0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ \frac{-1}{16\sqrt{2}}\ 0\ \frac{1}{32}\ 0\ 0\ 0\ 0\ 0\right]\varphi_{16}(t)$$
$$\int_0^t \varphi_{3,2}(t')dt' = \left[0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ \frac{1}{12\sqrt{2}}\ \frac{-1}{16}\ 0\ \frac{1}{48}\ \frac{-1}{6\sqrt{2}}\ 0\ 0\ 0\right]\varphi_{16}(t)$$
$$\int_0^t \varphi_{3,3}(t')dt' = \left[0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ \frac{1}{32\sqrt{2}}\ 0\ \frac{-1}{32}\ 0\ 0\ 0\ 0\ 0\right]\varphi_{16}(t)$$
$$\int_0^t \varphi_{4,0}(t')dt' = \left[0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ \frac{1}{8}\ \frac{1}{8\sqrt{2}}\ 0\ 0\right]\varphi_{16}(t)$$
$$\int_0^t \varphi_{4,1}(t')dt' = \left[0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ \frac{-1}{16\sqrt{2}}\ 0\ \frac{1}{32}\ 0\ 0\right]\varphi_{16}(t)$$
$$\int_0^t \varphi_{4,2}(t')dt = \left[0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ \frac{-1}{12\sqrt{2}}\ \frac{-1}{16}\ 0\ \frac{1}{48}\right]\varphi_{16}(t)$$
$$\int_0^t \varphi_{4,3}(t')dt = \left[0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ \frac{1}{32\sqrt{2}}\ 0\ \frac{-1}{32}\ 0\right]\varphi_{16}(t)$$

Now, we write

$$\int_0^t \varphi_{16}(t')dt' = P_{16\times16}\varphi_{16}(t), \tag{4.8}$$

where $P_{16\times 16}$ is an operational matrix of integration (OMI), given by

$$P_{16\times 16}=\begin{bmatrix}
\frac{1}{8} & \frac{1}{8\sqrt{2}} & 0 & 0 & \frac{1}{4} & 0 & 0 & 0 & \frac{1}{4} & 0 & 0 & 0 & \frac{1}{4} & 0 & 0 & 0\\
\frac{-1}{16\sqrt{2}} & 0 & \frac{1}{32} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
\frac{-1}{12\sqrt{2}} & \frac{-1}{16} & 0 & \frac{1}{48} & \frac{-1}{6\sqrt{2}} & 0 & 0 & 0 & \frac{-1}{6\sqrt{2}} & 0 & 0 & 0 & \frac{-1}{6\sqrt{2}} & 0 & 0 & 0\\
\frac{1}{32\sqrt{2}} & 0 & \frac{-1}{32} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & \frac{1}{8} & \frac{1}{8\sqrt{2}} & 0 & 0 & \frac{1}{4} & 0 & 0 & 0 & \frac{1}{4} & 0 & 0 & 0\\
0 & 0 & 0 & 0 & \frac{-1}{16\sqrt{2}} & 0 & \frac{1}{32} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & \frac{-1}{12\sqrt{2}} & \frac{-1}{16} & 0 & \frac{1}{48} & \frac{-1}{6\sqrt{2}} & 0 & 0 & 0 & \frac{-1}{6\sqrt{2}} & 0 & 0 & 0\\
0 & 0 & 0 & 0 & \frac{1}{32\sqrt{2}} & 0 & \frac{-1}{32} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{8} & \frac{1}{8\sqrt{2}} & 0 & 0 & \frac{1}{4} & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{-1}{16\sqrt{2}} & 0 & \frac{1}{32} & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{-1}{12\sqrt{2}} & \frac{-1}{16} & 0 & \frac{1}{48} & \frac{-1}{6\sqrt{2}} & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{32\sqrt{2}} & 0 & \frac{-1}{32} & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{8} & \frac{1}{8\sqrt{2}} & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{-1}{16\sqrt{2}} & 0 & \frac{1}{32} & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{-1}{12\sqrt{2}} & \frac{-1}{16} & 0 & \frac{1}{48}\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{32\sqrt{2}} & 0 & \frac{-1}{32} & 0
\end{bmatrix}_{16\times 16} \tag{4.9}$$

This matrix can be written in the block form

$$P_{16\times 16}=\frac{1}{2^k}\begin{bmatrix} M_{4\times 4} & N_{4\times 4} & N_{4\times 4} & N_{4\times 4}\\ O_{4\times 4} & M_{4\times 4} & N_{4\times 4} & N_{4\times 4}\\ O_{4\times 4} & O_{4\times 4} & M_{4\times 4} & N_{4\times 4}\\ O_{4\times 4} & O_{4\times 4} & O_{4\times 4} & M_{4\times 4}\end{bmatrix},$$

where

$$M=\begin{bmatrix} 1 & \frac{1}{\sqrt{2}} & 0 & 0\\ -\frac{\sqrt{2}}{4} & 0 & \frac{1}{4} & 0\\ -\frac{\sqrt{2}}{3} & -\frac{1}{2} & 0 & \frac{1}{6}\\ \frac{1}{8\sqrt{2}} & 0 & -\frac{1}{4} & 0\end{bmatrix}_{4\times 4}, \qquad N=\begin{bmatrix} 2 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ -\frac{2\sqrt{2}}{3} & 0 & 0 & 0\\ 0 & 0 & 0 & 0\end{bmatrix}_{4\times 4}.$$

Lemma 4.1

$$A_1=\int_0^t \varphi_1(t)\,dt' = M\varphi_1(t)+N\varphi_2(t)+N\varphi_3(t)+\cdots+N\,\varphi_{2^k-1}(t), \tag{4.10}$$

where

$$M=P_{11}, \qquad N=P_{1j}, \quad j=2,3,\ldots,2^{k-1}.$$

$$M=\begin{bmatrix} 1 & \frac{1}{\sqrt{2}} & 0 & 0 & \cdots & 0\\ -\frac{\sqrt{2}}{4} & 0 & \frac{1}{4} & 0 & \cdots & 0\\ -\frac{\sqrt{2}}{3} & -\frac{1}{2} & 0 & \frac{1}{6} & \cdots & 0\\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots\\ \frac{\sqrt{2}}{2}(-1)^i\left(\frac{1}{i-2}-\frac{1}{i}\right) & \cdots & -\frac{1}{2(i-2)} & 0 & \frac{1}{2i} & 0\\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots\\ \frac{\sqrt{2}}{2}(-1)^M\left(\frac{1}{M-2}-\frac{1}{M}\right) & 0 & 0 & 0 & -\frac{1}{2(M-2)} & 0\end{bmatrix} \tag{4.11}$$

$$N = \begin{bmatrix} 2 & 0 & 0 & 0 & \cdots & 0 \\ 0 & -\frac{2\sqrt{2}}{3} & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ \frac{\sqrt{2}}{2}\left(\frac{1-(-1)^i}{i} - \frac{1-(-1)^{i-2}}{i-2}\right) & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{\sqrt{2}}{2}\left(\frac{1-(-1)^M}{M} - \frac{1-(-1)^{M-2}}{M-2}\right) & 0 & 0 & 0 & \cdots & 0 \end{bmatrix} \tag{4.12}$$

Similar to Eqs. (4.1a)–(4.1d) we can obtain

$$\int_0^t \varphi_{nm}(t')\,dt', \qquad m = 0, 1, \ldots, M-1.$$

The vector $\varphi_n(t)$, for $n = 2, 3, \ldots, 2^{k-1}$, in Eq. (4.2) is obtained by translating the vector $\varphi_1(t)$ to the interval $\left[\frac{n-1}{2^{k-1}}, \frac{n}{2^{k-1}}\right)$.

Theorem 4.1

$$A_1 = \int_0^t \varphi_1(t')\,dt' = M\varphi_1(t) + N\varphi_2(t) + N\varphi_3(t) + \cdots + N\varphi_{2^k-1}(t), \tag{4.13}$$

$$A_2 = \int_0^t \varphi_2(t')\,dt' = O\varphi_1(t) + M\varphi_2(t) + N\varphi_3(t) + \cdots + N\varphi_{2^k-1}(t), \tag{4.14}$$

$$\vdots$$

$$A_{2^k-1} = \int_0^t \varphi_{2^k-1}(t')\,dt' = O\varphi_1(t) + O\varphi_2(t) + O\varphi_3(t) + \cdots + M\varphi_{2^k-1}(t). \tag{4.15}$$

where O represents the zero matrix, and the matrices M and N are defined in Eqs. (4.11) and (4.12).

Proof: From equations. (4.1*a*) − (4.1*d*), we observe that the term A_2, has the same form as A_1 except that the Chebyshev wavelet functions are shifted to the next subinterval. Because of this shift, the corresponding coefficient blocks P_{2j} in the first position must be zero matrices. For the remaining blocks $j = 2, 3, \ldots, 2^{k-1}$ each P_{2j} appears one position below its location in the previous row; specifically, $P_{22} = M,\ P_{2j} = N,\ j = 1, \ldots, 2^{k-1}$, Repeating this shifting pattern for subsequent rows eventually produces the final row, which reveals the complete structural pattern of the matrix (OMI)

$$P = \begin{bmatrix} M & N & N & \ldots & \ldots & N \\ O & M & N & \ldots & \ldots & N \\ O & O & M & \ldots & \ldots & N \\ \vdots & & & \ldots & \ddots & N \\ O & O & O & \ldots & \ldots & M \end{bmatrix}.$$

□

4.2. Product Operation Matrix (POM)

The following relation is used to represent the product of two vectors generated by the first-kind Chebyshev wavelet functions.

$$F^T\varphi(t)\varphi^T(t) = \varphi^T(t)\tilde{F}, \tag{4.16}$$

where $\varphi(t)\varphi^T(t)$, a product operation matrix of order 16×16, is given by

$$\varphi\varphi^T(t) = \begin{bmatrix} \varphi_{10}\varphi_{10} & \varphi_{10}\varphi_{11} & \cdots & \cdots & \cdots & \varphi_{10}\varphi_{42} & \varphi_{10}\varphi_{43} \\ \varphi_{11}\varphi_{10} & \varphi_{11}\varphi_{11} & \cdots & \cdots & \cdots & \varphi_{11}\varphi_{12} & \varphi_{11}\varphi_{43} \\ \varphi_{12}\varphi_{10} & \varphi_{12}\varphi_{11} & \cdots & \cdots & \cdots & \varphi_{12}\varphi_{12} & \varphi_{12}\varphi_{43} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \varphi_{42}\varphi_{10} & \varphi_{42}\varphi_{11} & \cdots & \cdots & \cdots & \varphi_{42}\varphi_{12} & \varphi_{42}\varphi_{43} \\ \varphi_{43}\varphi_{10} & \varphi_{43}\varphi_{11} & \cdots & \cdots & \cdots & \varphi_{43}\varphi_{12} & \varphi_{43}\varphi_{43} \end{bmatrix}_{16\times 16} \tag{4.17}$$

and

$$\tilde{F} = \begin{bmatrix} H_1 & 0 & 0 & 0 \\ 0 & H_2 & 0 & 0 \\ 0 & 0 & H_3 & 0 \\ 0 & 0 & 0 & H_4 \end{bmatrix}_{16\times 16,} \tag{4.18}$$

where $H_i, i = 1, 2, 3, 4$ are 4×4 matrices
Here, By computation, we get

$$\tilde{F} = \begin{bmatrix}
\frac{1}{16} & \frac{1}{8\sqrt{2}} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\frac{1}{8\sqrt{2}} & \frac{1}{8} & \frac{1}{16} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \frac{1}{16} & \frac{1}{8} & \frac{1}{16} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \frac{1}{16} & \frac{1}{8} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \frac{3}{8} & \frac{1}{8\sqrt{2}} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \frac{1}{8\sqrt{2}} & \frac{3}{8} & \frac{1}{16} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \frac{1}{16} & \frac{3}{8} & \frac{1}{16} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{16} & \frac{3}{8} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{5}{8} & \frac{1}{8\sqrt{2}} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{8\sqrt{2}} & \frac{5}{8} & \frac{1}{16} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{16} & \frac{5}{8} & \frac{1}{16} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{16} & \frac{5}{8} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{7}{8} & \frac{1}{8\sqrt{2}} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{8\sqrt{2}} & \frac{7}{8} & \frac{1}{16} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{16} & \frac{7}{8} & \frac{1}{16} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{16} & \frac{7}{8}
\end{bmatrix}_{16\times 16} \tag{4.19}$$

5. Solution of linear differential equations

In this section, we obtain numerical solutions of four very famous linear differential equations and compare their results with the exact solutions.

5.1. First-order linear differential equation.

Example 5.1 *First-order linear differential equation.*

$$y'(t) + y(t) = t^2; \tag{5.1}$$
$$y(0) = 0. \tag{5.2}$$

Exact solution of (5.1) with (5.2) is $y(t) = t^2 - 2(t - 1 + e^{-t})$.
Here, we solve above initial value problem using Chebyshev wavelets of first kind for $k = 3$ and $M = 4$.
Consider

$$y(t) = C^T\varphi(t) \tag{5.3}$$

In equation(5.3)

$$C = [c_{1,0}\ c_{1,1}\ c_{1,2}\ c_{1,3}\ c_{2,0}\ c_{2,1}\ c_{2,2}\ c_{2,3}\ c_{3,0}\ c_{3,1}\ c_{3,2}\ c_{3,3}\ c_{4,0}\ c_{4,1}\ c_{4,2}\ c_{4,3}] \tag{5.4}$$

and

$$\varphi(t) = [\varphi_{1,0}\ \varphi_{1,1}\ \varphi_{1,2}\ \varphi_{1,3}\ \varphi_{2,0}\ \varphi_{2,1}\ \varphi_{2,2}\ \varphi_{2,3}\ \varphi_{3,0}\ \varphi_{3,1}\ \varphi_{3,2}\ \varphi_{3,3}\ \varphi_{4,0}\ \varphi_{4,1}\ \varphi_{4,2}\ \varphi_{4,3}]. \tag{5.5}$$

The components of $\varphi(t)$ are provided in (4.1a) to (4.1d). Now, we also express

$$t^2 = \left[\frac{3\sqrt{\pi}}{256\sqrt{2}}\ \frac{\sqrt{\pi}}{128}\ \frac{\sqrt{\pi}}{512}\ 0\ \frac{19\sqrt{\pi}}{256\sqrt{2}}\ \frac{3\sqrt{\pi}}{128}\ \frac{\sqrt{\pi}}{512}\ 0\ \frac{51\sqrt{\pi}}{256\sqrt{2}}\ \frac{5\sqrt{\pi}}{128}\ \frac{\sqrt{\pi}}{512}\ 0\ \frac{99\sqrt{\pi}}{256\sqrt{2}}\ \frac{7\sqrt{\pi}}{128}\ \frac{\sqrt{\pi}}{512}\ 0\right] = E^T\varphi(t). \tag{5.6}$$

By integrating equation (5.1) from 0 to t and applying equations (5.2) and (5.6), we derive

$$C^T\varphi(t) + C^T\int_0^t \varphi(t')dt' = \int_0^t E^T\varphi(t)dt \tag{5.7}$$

Using (4.8), we have

$$\begin{aligned} C^T + C^T P &= E^T P \\ (I + P^T)C &= P^T E. \end{aligned} \tag{5.8}$$

Now, (5.8) can be written as

$$DC = F. \tag{5.9}$$

where $D = (I + P^T)$ and $F = P^T E$.
Equation (5.9) is a set of 16 algebraic equations with 16 unknowns. We solve (5.9) for C and obtain the following matrix:

$$C = \begin{bmatrix} 0.000966598513767056 \\ 0.00102155867171307 \\ 0.000402795837199018 \\ 6.37297518823087 \times 10^{-05} \\ 0.0115833118110457 \\ 0.00704910369785016 \\ 0.00107945050663595 \\ 4.96327796023727 \times 10^{-05} \\ 0.0462027260785943 \\ 0.0178693827828308 \\ 0.00160642968171731 \\ 3.86540467881775 \times 10^{-05} \\ 0.116842470177258 \\ 0.0324222468255352 \\ 0.00201684146709749 \\ 3.01038012594239 \times 10^{-05} \end{bmatrix}_{16\times 1} \tag{5.10}$$

Now, substituting the calculated value of C, we can determine $y(t)$ in Example 5.1.
In Table 1, shows a comparison between the solution estimated by the proposed method and the exact solution at different points in the interval [0, 1).

Table 1: Estimated and exact solutions for $y(t)$ in Example 5.1

t	**Estimated value of $y(t)$ using CW**	**Exact value of $y(t)$**	**Absolute error**
0.1	0.00024	0.00032	0.00008
0.2	0.00267	0.00253	0.00014
0.3	0.00825	0.00836	0.00011
0.4	0.01948	0.01935	0.00013
0.5	0.03702	0.03693	0.00009
0.6	0.06232	0.06237	0.00005
0.7	0.09690	0.09682	0.00008
0.8	0.14127	0.14134	0.00007
0.9	0.19692	0.19686	0.00006

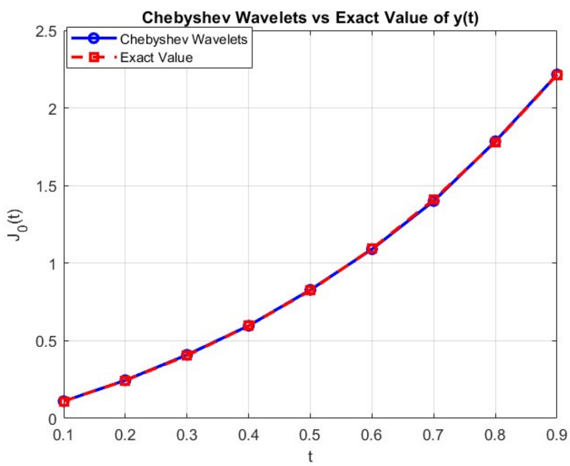

Figure 1: Graphical illustration of estimated and exact solutions for $y(t)$ in Example 5.1.

5.2. Lane–Emden differential equation

Example 5.2 *Consider the Lane–Emden type differential equation*

$$y''(t) + \frac{2}{t}y'(t) + y(t) = 0; \tag{5.11}$$

$$y(0) = 1, y'(0) = 0. \tag{5.12}$$

The exact solution of (5.11) is $y(t) = \frac{sint}{t}$.
Here, we solve the problem using Chebyshev wavelets of first kind for $k = 3$ and $M = 4$. Initially, The unknown function $y(t)$ is treated as

$$y''(t) = C^T\varphi(t) \quad , \tag{5.13}$$

where C and $\varphi(t)$ are given by (5.4) and (5.5) respectively and the elements of $\varphi(t)$ are given in (4.1a) to (4.1d)
Integrating (5.13) from 0 to t and utilizing (4.8) and (5.12), we obtain

$$
\begin{aligned}
y'(t) &\simeq C^T P\varphi(t) + y'(0) \\
&= C^T P\varphi(t) + A^T\varphi(t) \\
y'(t) &= C^T P\varphi(t) \qquad (5.14)\\
y(t) &\simeq C^T P^2\varphi(t) + ty'(0) + y(0) \\
y(t) &= C^T P^2\varphi(t) + B^T\varphi(t), \qquad (5.15)
\end{aligned}
$$

where

$$B^T = \left[\frac{\sqrt{\pi}}{2\sqrt{2}}\ 0\ 0\ 0\ \frac{\sqrt{\pi}}{2\sqrt{2}}\ 0\ \ 0\ 0\ \frac{\sqrt{\pi}}{2\sqrt{2}}\ 0\ 0\ 0\ \frac{\sqrt{\pi}}{2\sqrt{2}}\ 0\ 0\ 0\right] \qquad (5.16)$$

and P is the 16×16 first kind Chebyshev wavelets OMI given by (4.9).
Also, we express t as $F^T\varphi(t)$ in the following manner

$$
\begin{aligned}
t &= \left[\frac{\sqrt{\pi}}{16\sqrt{2}}\ \frac{\sqrt{\pi}}{32}\ 0\ 0\ \frac{3\sqrt{\pi}}{16\sqrt{2}}\ \frac{\sqrt{\pi}}{32}\ 0\ 0\ \frac{5\sqrt{\pi}}{16\sqrt{2}}\ \frac{\sqrt{\pi}}{32}\ 0\ 0\ \frac{7\sqrt{\pi}}{16\sqrt{2}}\ \frac{\sqrt{\pi}}{32}\ 0\ 0\right] \\
&= F^T\varphi(t). \qquad (5.17)
\end{aligned}
$$

Substituting (5.13) to (5.17) in (5.11), we get

$$F^T\varphi(t)\varphi^T(t)C + 2\varphi^T(t)P^TC + F^T\varphi(t)\varphi^T(t)P^{2T}C + F^T\varphi(t)\varphi^T(t)B = 0. \qquad (5.18)$$

Now, using (4.16) in (5.18) we get

$$(\tilde{F} + 2P^T + \tilde{F}P^{2T})C + \tilde{F}B = 0 \qquad (5.19)$$

Equation (5.19) gives a set of 16 algebraic equations with 16 unknowns. We solve (5.19) for C and obtain the following matrix

$$\begin{bmatrix} -0.207420940900039 \\ 0.00138022839961823 \\ 0.000343932696170102 \\ -5.9931758536383 \times 10^{-07} \\ -0.199681923545239 \\ 0.00407932926203381 \\ 0.000328651371034389 \\ -1.88167498596669 \times 10^{-06} \\ -0.184559502063885 \\ 0.00659332170509406 \\ 0.000212088153286713 \\ -3.52190131002023 \times 10^{-05} \\ -0.162703174216956 \\ 0.00882698287083203 \\ 0.000255374338199167 \\ -4.00295308387157 \times 10^{-05} \end{bmatrix}_{16\times 1.} \qquad (5.20)$$

Now, substituting the calculated value of C, we can determine $y(t)$ in Example 5.2.
In Table 2, shows a comparison between the solution estimated by the proposed method and the exact solution at different points in the interval [0, 1).

Table 2: Estimated and exact solutions for $y(t)$ in Example 5.2

t	**Estimated value of $y(t)$ using CW**	**Exact value of $y(t)$**	**Absolute error**
0.1	0.998332	0.998334	0.000002
0.2	0.993348	0.993346	0.000002
0.3	0.985061	0.985067	0.000006
0.4	0.973549	0.973540	0.000009
0.5	0.958860	0.958850	0.000010
0.6	0.941065	0.941070	0.000005
0.7	0.920320	0.920310	0.000010
0.8	0.896685	0.896690	0.000005
0.9	0.870373	0.870360	0.000013

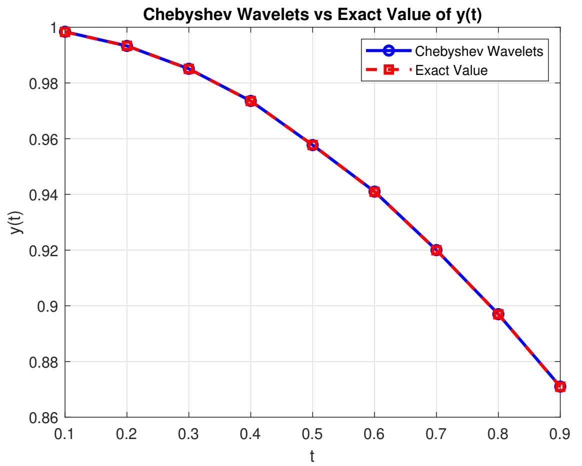

Figure 2: Graphical illustration of estimated and exact solutions for $y(t)$ in Example 5.2.

5.3. Solution of differential equation of third order

Example 5.3 *Consider the following third-order differential equation*

$$y'''(t) - y''(t) = 0; \tag{5.21}$$

$$y(0) = 0, y'(0) = 1, y''(0) = 2. \tag{5.22}$$

The exact solution of (5.21) is $y(t) = 2e^t - t - 2$.
Here, we solve the problem using Chebyshev wavelets of first kind for $k = 3$ and $M = 4$. Initially, The unknown function $y(t)$ is treated as

$$y'''(t) = C^T \varphi(t), \tag{5.23}$$

where C and $\varphi(t)$ are given by (5.4) and (5.5) respectively and the elements of $\varphi(t)$ are given in (4.1a) to (4.1d).
Integrating (5.23) over the interval 0 to t, and employing the identities given in (4.8) and (5.22), we obtain

$$\begin{aligned} y''(t) &\simeq C^T P\varphi(t) + y''(0) \\ &= C^T P\varphi(t) + 2B^T \varphi(t) \end{aligned} \tag{5.24}$$

Now, integrating (5.24) over the interval 0 to t

$$\begin{aligned} y'(t) &\simeq C^T P^2 \varphi(t) + 2B^T P \varphi(t) + y'(0) \\ y'(t) &= C^T P^2 \varphi(t) + 2B^T P \varphi(t) + B^T \varphi(t), \end{aligned} \tag{5.25}$$

Now, integrating (5.25) over the interval

$$\begin{aligned} y(t) &\simeq C^T P^3 \varphi(t) + 2B^T P^2 \varphi(t) + B^T P \varphi(t) + y(0) \\ y(t) &= C^T P^3 \varphi(t) + 2B^T P^2 \varphi(t) + B^T P \varphi(t), \end{aligned} \tag{5.26}$$

using (5.23) and (5.24) in (5.21), we get

$$\varphi^T(t)C - \varphi^T(t)P^T C - 2\varphi^T B = 0. \tag{5.27}$$

we get

$$(I - P^T)C - 2B = 0 \tag{5.28}$$

Equation (5.28) gives a set of 16 algebraic equations with 16 unknowns. We solve (5.28) for C and obtain the following matrix

$$C = \begin{bmatrix} 1.42574403457769 \\ 0.125773667516836 \\ 0.00392786990293307 \\ 8.18306229777723 \times 10^{-05} \\ 1.83069161750059 \\ 0.161496589318418 \\ 0.00504348489738716 \\ 0.000105072602028899 \\ 2.35065461759385 \\ 0.207365729857489 \\ 0.00647596293634317 \\ 0.000134915894507149 \\ 3.01830033982416 \\ 0.26626287341924 \\ 0.00831530118680775 \\ 0.000173235441391828 \end{bmatrix}_{16\times 1.} \tag{5.29}$$

Now, substituting the calculated value of C, we can determine $y(t)$ in Example 5.3.
In Table 3, shows a comparison between the solution estimated by the proposed method and the exact solution at different points in the interval [0, 1).

Table 3: Estimated and exact solutions for $y(t)$ in Example 5.3

t	**Estimated value of $y(t)$ using CW**	**Exact value of $y(t)$**	**Absolute error**
0.1	0.11023	0.11034	0.00011
0.2	0.24298	0.24280	0.00018
0.3	0.39972	0.39971	0.00001
0.4	0.58364	0.58364	0.00000
0.5	0.79743	0.79744	0.00001
0.6	1.04423	1.04423	0.00000
0.7	1.32750	1.32750	0.00000
0.8	1.65109	1.65108	0.00001
0.9	2.01920	2.01920	0.00000

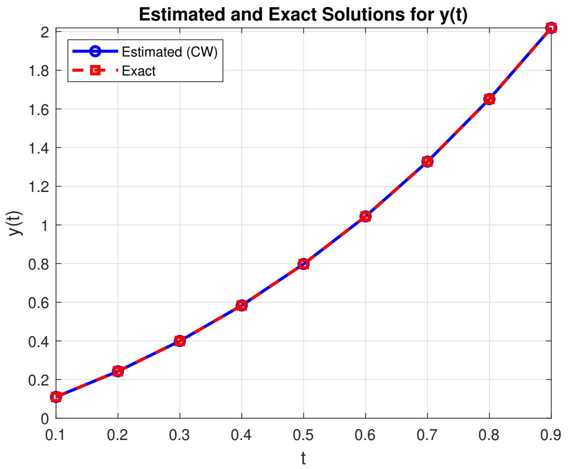

Figure 3: Graphical illustration of estimated and exact solutions for $y(t)$ in Example 5.3.

5.4. Solution of singular differential equation of third order

Example 5.4 *Consider the singular initial value problem governed by.*

$$y'''(t) - \frac{2}{t}y''(t) - y'(t) = 0; \tag{5.30}$$

$$y(0) = 0, y'(0) = 1, y''(0) = 2. \tag{5.31}$$

The exact solution of (5.30) is $y(t) = te^t$.
Here, we solve the problem using Chebyshev wavelets of first kind for $k = 3$ and $M = 4$. Substituting (5.23) to (5.26) in (5.30), we get

$$\begin{aligned} F^T\varphi(t)\varphi^T(t)C - 2\varphi^T(t)P^{3T}C - 2\varphi^T(t)P^{2T}B - \varphi^T(t)P^T B - F^T\varphi(t)\varphi^T(t)P^{2T}C \\ -2F^T\varphi(t)\varphi^T(t)P^T B - F^T\varphi(t)\varphi^T(t)B = 0. \end{aligned} \tag{5.32}$$

Now with using $\tilde{F}$ of (4.16) and (5.16), (5.17) in (5.32), we get

$$(\tilde{F} - 2P^{3T} - \tilde{F}P^{2T})C = (2P^T + 4P^{2T} + \tilde{F} + 2\tilde{F}P^T)B. \tag{5.33}$$

Equation (5.33) gives a set of 16 algebraic equations with 16 unknowns. We solve (5.33) for C and obtain the following matrix

$$C = \begin{bmatrix} 2.2332836684442 \\ 0.259654622831473 \\ 0.0100684738298114 \\ 0.000253321583249678 \\ 3.09643048123376 \\ 0.353588975582964 \\ 0.013560806182609 \\ 0.000335805691822889 \\ 4.26975881607091 \\ 0.47964873675553 \\ 0.0209315257531573 \\ 0.000634848602845755 \\ 5.85963497695811 \\ 0.649494735781432 \\ 0.0244356062158953 \\ 0.00059628252564063 \end{bmatrix}_{16\times 1}. \tag{5.34}$$

Now, substituting the calculated value of C, we can determine $y(t)$ in Example 5.4.
In Table 4, shows a comparison between the solution estimated by the proposed method and the exact solution at different points in the interval [0, 1).

Table 4: Estimated and exact solutions for $y(t)$ in Example 5.4

t	**Estimated value of** $y(t)$ **using CW**	**Exact value of** $y(t)$	**Absolute error**
0.1	0.11051	0.11052	0.00001
0.2	0.24428	0.24428	0.00000
0.3	0.40496	0.40496	0.00000
0.4	0.59672	0.59673	0.00001
0.5	0.79435	0.79436	0.00001
0.6	1.09327	1.09327	0.00000
0.7	1.40964	1.40963	0.00001
0.8	1.78044	1.78043	0.00001
0.9	2.21362	2.21364	0.00002

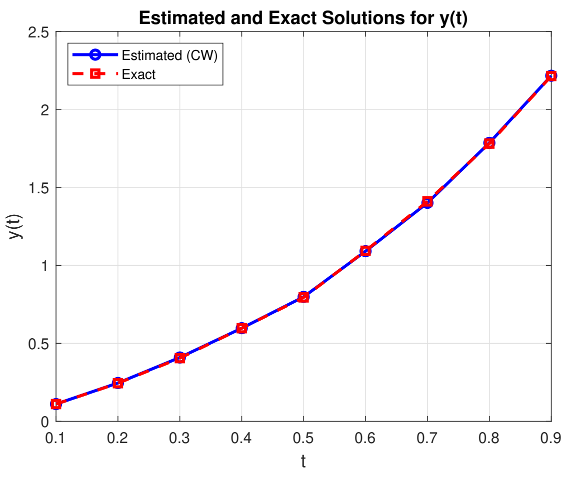

Figure 4: Graphical illustration of estimated and exact solutions for $y(t)$ in Example 5.4.

6. Conclusion

In this work, an efficient and accurate method for solving linear differential equations has been developed. first-kind Chebyshev wavelets operational matrices for integration have been constructed, along with a matrix for product operations. We have considered linear differential equations, Lane–Emden type and a third order singular differential equations . These differential equations were solved using the operational matrix of first-kind Chebyshev wavelets derived for $k = 3$ and $M = 4$. The estimated solutions obtained using the present approach have been compared with their exact solutions, and the convergence behavior of the proposed wavelets is examined.

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