Bol. Soc. Paran. Mat.
©SPM – E-ISSN-2175-1188
SPM: www.spm.uem.br/bspm

(3s.) **v. 2026 (44) 5** : 1–12.
ISSN-0037-8712
doi:10.5269/bspm.81072

Dirichlet Boundary Value Problems for Cauchy-Riemann and Polyanalytic Equations in a Half-Ring Domain

Bahriye Karaca

ABSTRACT: In this work, we study the Dirichlet boundary value problem for several classes of complex partial differential equations in a half-ring domain. We begin with the homogeneous Cauchy-Riemann equation and derive an explicit integral representation that satisfies the given boundary conditions. The approach is then extended to the inhomogeneous Cauchy-Riemann equation, where the presence of a nonhomogeneous term necessitates modifications in the solution technique. Finally, we investigate the Dirichlet problem for a polyanalytic equation of nth order. By employing advanced tools from complex analysis, we construct solution formulas that account for the higher-order structure of the equation and ensure the boundary conditions are met.

Keywords: Dirichlet problem, half-ring domain, Cauchy-Riemann equation.

Contents

1. Introduction

Boundary value problems for complex partial differential equations have garnered considerable attention due to their wide range of applications in mathematical physics and engineering. Among the most studied equations in this field are the Cauchy-Riemann and Bitsadze equations, which serve as fundamental models for analytic and polyanalytic function theory [1,4,7,19]. In particular, iterated and higher-order boundary value problems for inhomogeneous equations have been extensively explored in various geometries, such as the upper half plane, quarter plane, ring domains, and triangular regions [1,5,6,10,11,17,18,20].

Recent studies have extended classical approaches by employing integral representations, Green's functions, and complex iteration techniques to address both linear and nonlinear systems [2,3,8,14,15]. These methods have been successfully applied to Schwarz, Dirichlet, Neumann, and Robin boundary conditions, revealing rich structural properties of bi- and tri-analytic functions and their generalizations [9,12,13,16,19].

In this article, we focus on the upper half-ring domain in the complex plane, defined as

$$R^+ := \{z \in \mathbb{C} : r < |z| < 1,\ \mathrm{Im}(z) > 0\},$$

where $r \in (0,1)$. This domain, denoted R^+, consists of two circular arcs of radii r and 1 in the upper half-plane, as well as the corresponding intervals on the real axis connecting these arcs. The boundary of this domain, ∂R^+, is composed of these arcs and the real axis, forming a structure that is critical for solving boundary value problems within this region.

To facilitate our analysis of complex functions defined in R^+, we utilize the complex differential operators that describe analytic and generalized analytic functions. These operators are given by:

$$\frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right), \quad \frac{\partial}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right),$$

2020 *Mathematics Subject Classification:* 35C15, 35F15.
Submitted December 29, 2025. Published April 09, 2026

where $z = x + iy$ is a complex variable and $\bar{z} = x - iy$ is its conjugate. These operators play a pivotal role in the formulation of boundary value problems involving complex functions.

Let ω be a complex-valued function defined in R^+, which can be expressed as $\omega = u + iv$, where $u(x, y)$ and $v(x, y)$ are real-valued functions. The Cauchy-Riemann equations governing the analyticity of ω take the form:

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x},$$

and they compactly reduce to the condition $\omega_{\bar{z}} = 0$, ensuring that ω is holomorphic in R^+. This condition serves as the foundation for solving boundary value problems for analytic functions.

An important tool for solving inhomogeneous Cauchy-Riemann equations is the Pompeiu integral operator, defined as:

$$T[f](z) = -\frac{1}{\pi} \iint_{R^+} \frac{f(\varsigma)}{\varsigma - z} \, d\xi d\eta, \quad \varsigma = \xi + i\eta.$$

For $f \in L_p(R^+; \mathbb{C})$ with $p > 2$, this operator T is weakly differentiable in R^+ and satisfies the equation:

$$\frac{\partial}{\partial \bar{z}} T[f](z) = f(z).$$

Thus, $T[f](z)$ acts as a right inverse for the operator $\frac{\partial}{\partial \bar{z}}$ in the weak sense, providing a valuable tool for solving boundary value problems in complex analysis. This operator allows for the construction of solutions to inhomogeneous equations that arise in boundary value problems, including those defined on complex domains such as R^+.

The Cauchy-Pompeiu representation formula is fundamental for solving complex boundary value problems. It provides integral representations for functions that are continuously differentiable within a domain, allowing the solution of such problems to be expressed in terms of boundary conditions. The classical Cauchy integral formula is a special case of the Cauchy-Pompeiu representation when the function is holomorphic.

Theorem 1.1 *Let $D \subset \mathbb{C}$ be a regular domain with a piecewise smooth boundary, and let $\omega \in C^1(D; \mathbb{C}) \cap C(\overline{D}; \mathbb{C})$. Then, the function ω admits the following integral representations, known as the Cauchy-Pompeiu formulas:*

$$\omega(z) = \frac{1}{2\pi i} \int_{\partial D} \frac{\omega(\varsigma)}{\varsigma - z} \, d\varsigma - \frac{1}{\pi} \iint_{D} \frac{\omega_{\bar{\varsigma}}(\varsigma)}{\varsigma - z} \, d\xi d\eta, \tag{1.1}$$

or equivalently,

$$\omega(z) = -\frac{1}{2\pi i} \int_{\partial D} \frac{\omega(\varsigma)}{\overline{\varsigma - z}} \, d\bar{\varsigma} - \frac{1}{\pi} \iint_{D} \frac{\omega_{\bar{\varsigma}}(\varsigma)}{\overline{\varsigma - z}} \, d\xi d\eta, \tag{1.2}$$

where $\varsigma = \xi + i\eta$, and $z \in D$.

These identities express the value of a continuously differentiable function ω inside the domain D in terms of its boundary values and its antiholomorphic derivative.

In the case where ω is analytic in D, meaning that $\omega_{\bar{\varsigma}} = 0$, the first formula reduces to the classical Cauchy integral formula. This highlights the versatility of the Cauchy-Pompeiu representation, which can handle both holomorphic and generalized analytic functions within a given domain. In particular, the formula enables the solution of boundary value problems in a wide range of complex domains, including those encountered in this article.

2. A Cauchy-Pompeiu Type Representation for Functions in a Half Ring

To adapt the Cauchy-Pompeiu formula for the domain $R^+ := \{z \in \mathbb{C} : r < |z| < 1, \ \text{Im}(z) > 0\}$. The standard Cauchy kernel is first replaced by a suitable kernel specific to this domain, as discussed in [21]. The resulting integral representation is then enhanced by adding expressions evaluated at the reflected points $\frac{1}{\bar{z}}$, $\bar{z}$, and $\frac{1}{z}$. These four expressions are appropriately combined to derive the final formula.

Theorem 2.1 *[21] Any $\omega \in C^1(R^+;\mathbb{C}) \cap C(\overline{R^+};\mathbb{C})$ can be represented as*

$$\omega(z) = \frac{1}{2\pi i} \int_{\partial R^+} \omega(\varsigma) \left(\frac{\varsigma}{\varsigma - z} + \sum_{n=1}^{\infty} r^{2n} \left[\frac{\varsigma}{r^{2n}\varsigma - z} + \frac{z}{\varsigma - r^{2n} z} \right] \right) \frac{d\varsigma}{\varsigma}$$

$$-\frac{1}{\pi} \iint_{R^+} w_{\bar\varsigma}(\varsigma) \left(\frac{1}{\varsigma - z} + \sum_{n=1}^{\infty} r^{2n} \left[\frac{1}{r^{2n}\varsigma - z} + \frac{z}{\varsigma(\varsigma - r^{2n} z)} \right] \right) d\xi d\eta \tag{2.1}$$

and

$$\omega(z) = \frac{1}{2\pi i} \int_{\substack{|\varsigma|=1,\\ 0<\operatorname{Im}\varsigma}} Re\omega(\varsigma) \left(\frac{\varsigma + z}{\varsigma - z} - \frac{\bar\varsigma + z}{\bar\varsigma - z} + 2\sum_{n=1}^{\infty} r^{2n} \left[\frac{\varsigma}{r^{2n}\varsigma - z} - \frac{z}{r^{2n}z - \varsigma} - \frac{\bar\varsigma}{r^{2n}\bar\varsigma - z} + \frac{z}{r^{2n}z - \bar\varsigma} \right] \right) \frac{d\varsigma}{\varsigma}$$

$$-\frac{1}{2\pi i} \int_{\substack{|\varsigma|=r,\\ 0<\operatorname{Im}\varsigma}} Re\omega(\varsigma) \left(\frac{\varsigma + z}{\varsigma - z} - \frac{\bar\varsigma + z}{\bar\varsigma - z} + 2\sum_{n=1}^{\infty} r^{2n} \left[\frac{\varsigma}{r^{2n}\varsigma - z} - \frac{z}{r^{2n}z - \varsigma} - \frac{\bar\varsigma}{r^{2n}\bar\varsigma - z} + \frac{z}{r^{2n}z - \bar\varsigma} \right] \right) \frac{d\varsigma}{\varsigma}$$

$$+\frac{1}{\pi i} \int_{[-1,r]\cup[r,1]} Re\omega(t) \left(\frac{1}{t - z} - \frac{z}{1 - zt} + \sum_{n=1}^{\infty} r^{2n} \left[\frac{1}{r^{2n}t - z} - \frac{z}{t(r^{2n}z - t)} - \frac{1}{t(r^{2n} - zt)} + \frac{z}{r^{2n}zt - 1} \right] \right) dt$$

$$+\frac{1}{\pi} \int_{\substack{|\varsigma|=1,\\ 0<\operatorname{Im}\varsigma}} \operatorname{Im}\omega(\varsigma) \frac{d\varsigma}{\varsigma}$$

$$-\frac{1}{\pi} \iint_{R^+} \left(w_{\bar\varsigma} \left[\frac{1}{\varsigma - z} - \frac{z}{1 - z\varsigma} + \sum_{n=1}^{\infty} r^{2n} \left(\frac{1}{r^{2n}\varsigma - z} + \frac{z}{\varsigma(\varsigma - r^{2n}z)} - \frac{1}{\varsigma(z\varsigma - r^{2n})} + \frac{z}{r^{2n}z\varsigma - 1} \right) \right] \right.$$
$$\left. -\overline{w_{\bar\varsigma}} \left[\frac{1}{\bar\varsigma - z} - \frac{z}{1 - z\bar\varsigma} + \sum_{n=1}^{\infty} r^{2n} \left(\frac{1}{r^{2n}\bar\varsigma - z} + \frac{z}{\bar\varsigma(\bar\varsigma - r^{2n}z)} - \frac{1}{\bar\varsigma(r^{2n} - z\bar\varsigma)} + \frac{z}{r^{2n}z\bar\varsigma - 1} \right) \right] \right) d\xi d\eta. \tag{2.2}$$

The second representation theorem is obtained from the first by adding specific kernel terms; these terms, when integrated over the domain, vanish due to the Cauchy-Pompeiu formula. Thus, the second formula yields an equivalent representation, which will be employed in solving the Dirichlet problem.

Theorem 2.2 *Any $\omega \in C^1(R^+;\mathbb{C}) \cap C(\overline{R^+};\mathbb{C})$ can be represented as*

$$\omega(z) = \frac{1}{2\pi i} \int_{\partial R^+} \omega(\varsigma) \left(\frac{\varsigma}{\varsigma - z} + \frac{z\varsigma}{z\varsigma - 1} + \sum_{n=1}^{\infty} r^{2n} \left[\frac{\varsigma}{r^{2n}\varsigma - z} + \frac{z}{\varsigma - r^{2n}z} - \frac{1}{r^{2n} - z\varsigma} + \frac{z\varsigma}{r^{2n}z\varsigma - 1} \right] \right) \frac{d\varsigma}{\varsigma}$$

$$-\frac{1}{\pi} \iint_{R^+} w_{\bar\varsigma}(\varsigma) \left(\frac{1}{\varsigma - z} + \frac{z}{z\varsigma - 1} + \sum_{n=1}^{\infty} r^{2n} \left[\frac{1}{r^{2n}\varsigma - z} + \frac{z}{\varsigma(\varsigma - r^{2n}z)} - \frac{1}{\varsigma(r^{2n} - z\varsigma)} + \frac{z}{r^{2n}z\varsigma - 1} \right] \right) d\xi d\eta \tag{2.3}$$

Proof: The integral representation is derived from the previous theorem:

$$\omega(z) = \frac{1}{2\pi i} \int_{\partial R^+} \omega(\varsigma) \left(\frac{\varsigma}{\varsigma - z} + \sum_{n=1}^{\infty} r^{2n} \left[\frac{\varsigma}{r^{2n}\varsigma - z} + \frac{z}{\varsigma - r^{2n}z} \right] \right) \frac{d\varsigma}{\varsigma}$$

$$-\frac{1}{\pi}\iint_{R^+} w_{\bar{\varsigma}}(\varsigma)\left(\frac{1}{\varsigma - z}+\sum_{n=1}^{\infty} r^{2n}\left[\frac{1}{r^{2n}\varsigma - z}+\frac{z}{\varsigma(\varsigma - r^{2n}z)}\right]\right)d\xi d\eta \tag{2.4}$$

Applying both parts of the Cauchy-Pompeiu formula for $z \in R^+$, we obtain:

$$0=\frac{1}{2\pi i}\int_{\partial R^+}\omega(\varsigma)\frac{z}{z\varsigma - 1}d\varsigma - \frac{1}{\pi}\iint_{R^+} w_{\bar{\varsigma}}(\varsigma)\frac{z}{z\varsigma - 1}d\xi d\eta \tag{2.5}$$

and

$$0=\frac{1}{2\pi i}\int_{\partial R^+}\omega(\varsigma)\sum_{n=1}^{\infty} r^{2n}\left[\frac{z}{r^{2n}z\varsigma - 1}+\frac{1}{\varsigma(z\varsigma - r^{2n})}\right]d\varsigma$$

$$-\frac{1}{\pi}\iint_{R^+} w_{\bar{\varsigma}}(\varsigma)\sum_{n=1}^{\infty} r^{2n}\left[\frac{z}{r^{2n}z\varsigma - 1}+\frac{1}{\varsigma(z\varsigma - r^{2n})}\right]d\xi d\eta \tag{2.6}$$

The contributions from (7) and (8), when combined with the initial integral representation, complete the expression of $\omega(z)$ as stated. Therefore, we arrive at the following expression:

$$\omega(z)=\frac{1}{2\pi i}\int_{\partial R^+}\omega(\varsigma)\left(\frac{\varsigma}{\varsigma - z}+\frac{z\varsigma}{z\varsigma - 1}+\sum_{n=1}^{\infty} r^{2n}\left[\frac{\varsigma}{r^{2n}\varsigma - z}+\frac{z}{\varsigma - r^{2n}z}-\frac{1}{r^{2n}-z\varsigma}+\frac{z\varsigma}{r^{2n}z\varsigma - 1}\right]\right)\frac{d\varsigma}{\varsigma}$$

$$-\frac{1}{\pi}\iint_{R^+} w_{\bar{\varsigma}}(\varsigma)\left(\frac{1}{\varsigma - z}+\frac{z}{z\varsigma - 1}+\sum_{n=1}^{\infty} r^{2n}\left[\frac{1}{r^{2n}\varsigma - z}+\frac{z}{\varsigma(\varsigma - r^{2n}z)}-\frac{1}{\varsigma(r^{2n}-z\varsigma)}+\frac{z}{r^{2n}z\varsigma - 1}\right]\right)d\xi d\eta \tag{2.7}$$

□

3. Dirichlet Problem for the Cauchy-Riemann Equation in the Half Ring

In this section, we undertake a detailed analysis of the Dirichlet boundary value problem for the Cauchy-Riemann equations, considering both the homogeneous and inhomogeneous cases. Our primary goal is to construct explicit solutions in a half-ring domain by employing complex analysis techniques. We begin by focusing on the homogeneous Cauchy-Riemann equation, for which we derive an integral representation that satisfies the prescribed boundary conditions. The methods developed in this part will also serve as a foundation for addressing the inhomogeneous case in the subsequent discussion. Finally, we consider the Dirichlet problem for the polyanalytic equation, where the presence of higher-order derivatives introduces additional challenges. In this setting, we extend the developed techniques and present an appropriate representation formula that meets the boundary requirements.

Theorem 3.1 *The Dirichlet problem for the homogeneous Cauchy-Riemann equation*

$$\omega_{\bar{z}}=0 \text{ in } R^+,\ w=\gamma \text{ on } \partial R^+,\ \gamma\in C(\partial R^+;\mathbb{C}),\ \gamma(-1)=\gamma(1)=\gamma(-r)=\gamma(r)=0 \tag{3.1}$$

is solvable if and only if

$$\frac{1}{2\pi i}\int_{\partial R^+}\gamma(\varsigma)\left[\frac{-1}{\varsigma - \bar{z}}+\frac{\bar{z}}{1-\bar{z}\varsigma}\right]d\varsigma$$

$$+\frac{1}{2\pi i}\int_{\partial R^+}\gamma(\varsigma)\sum_{n=1}^{\infty} r^{2n}\left[\frac{-1}{r^{2n}\varsigma - \bar{z}}+\frac{\bar{z}}{\varsigma(r^{2n}\bar{z}-\varsigma)}+\frac{1}{\varsigma(r^{2n}-\bar{z}\varsigma)}+\frac{-\bar{z}}{r^{2n}\bar{z}\varsigma - 1}\right]d\varsigma = 0 \tag{3.2}$$

and the unique solution can be presented as

$$\omega(z)=\frac{1}{2\pi i}\int\limits_{\partial R^+}\gamma(\varsigma)\left[\frac{1}{\varsigma-z}+\frac{z}{\varsigma z-1}\right]d\varsigma$$

$$+\frac{1}{2\pi i}\int\limits_{\partial R^+}\gamma(\varsigma)\sum_{n=1}^{\infty}r^{2n}\left[\frac{1}{r^{2n}\varsigma-z}-\frac{z}{\varsigma(r^{2n}z-\varsigma)}-\frac{1}{\varsigma(r^{2n}-z\varsigma)}+\frac{z}{r^{2n}z\varsigma-1}\right]d\varsigma. \tag{3.3}$$

Remark In this theorem, the boundary data γ is required to satisfy the following condition at the corner points $1,-1,r,-r$:

$$\gamma(-1)=\gamma(1)=\gamma(-r)=\gamma(r)=0. \tag{3.4}$$

This condition excludes certain important cases-particularly when the boundary data is a nonzero constant function, i.e. $\gamma(\varsigma)=c,\ c\neq 0,\ c\in\mathbb{R}$. Nevertheless, we conjecture that the theorem may still hold even when condition (3.4) is not satisfied.

Proof: Let ω defined by (3.3) be a solution to the Dirichlet problem. Then the following condition holds:

$$\omega(z)=\gamma(\varsigma),\ \varsigma\in\partial R^+ \tag{3.5}$$

Adding (3.2) to (3.3) leads to

$$\omega(z)=\frac{1}{2\pi i}\int\limits_{\partial R^+}\gamma(\varsigma)\left[\frac{1}{\varsigma-z}+\frac{z}{\varsigma z-1}+\frac{-1}{\varsigma-\bar{z}}+\frac{\bar{z}}{1-\bar{z}\varsigma}\right]d\varsigma$$

$$+\frac{1}{2\pi i}\int\limits_{\partial R^+}\gamma(\varsigma)\sum_{n=1}^{\infty}r^{2n}\Bigg[\frac{1}{r^{2n}\varsigma-z}-\frac{z}{\varsigma(r^{2n}z-\varsigma)}-\frac{1}{\varsigma(r^{2n}-z\varsigma)}+\frac{z}{r^{2n}z\varsigma-1}$$
$$-\frac{1}{r^{2n}\varsigma-\bar{z}}+\frac{\bar{z}}{\varsigma(r^{2n}\bar{z}-\varsigma)}+\frac{1}{\varsigma(r^{2n}-\bar{z}\varsigma)}-\frac{\bar{z}}{r^{2n}\bar{z}\varsigma-1}\Bigg]d\varsigma.$$

The initial representation of the solution $\omega(z)$ involves Cauchy-type integrals over the boundary ∂R^+. In the second formulation, the boundary integrals are explicitly expressed over semicircular arcs in the upper half-plane, namely $|\varsigma|=1$ and $|\varsigma|=r$, with $\operatorname{Im}\varsigma>0$, using parametrizations adapted to the geometry of R^+. The kernel functions are then reorganized using identities such as

$$\frac{z}{\varsigma z-1}=\frac{1}{z-\bar{\varsigma}},\quad \frac{\bar{z}}{1-\bar{z}\varsigma}=\frac{1}{\bar{\varsigma}-\bar{z}},$$

to express the integral in terms of singularities reflected across the real axis. This symmetrization facilitates the application of complex analytic techniques and highlights the analytic structure of the solution in R^+.

$$\omega(z)=\frac{1}{2\pi i}\int\limits_{\substack{|\varsigma|=1,\\ 0<\operatorname{Im}\varsigma}}\gamma(\varsigma)\left[\frac{\varsigma}{\varsigma-z}+\frac{z\varsigma}{\varsigma z-1}+\frac{-\varsigma}{\varsigma-\bar{z}}+\frac{\bar{z}\varsigma}{1-\bar{z}\varsigma}\right]\frac{d\varsigma}{\varsigma}$$

$$+\frac{1}{2\pi i}\int\limits_{\substack{|\varsigma|=1,\\ 0<\operatorname{Im}\varsigma}}\gamma(\varsigma)\sum_{n=1}^{\infty}r^{2n}\Bigg[\frac{\varsigma}{r^{2n}\varsigma-z}-\frac{z}{r^{2n}z-\varsigma}-\frac{1}{r^{2n}-z\varsigma}+\frac{z\varsigma}{r^{2n}z\varsigma-1}$$
$$-\frac{\varsigma}{r^{2n}\varsigma-\bar{z}}+\frac{\bar{z}}{r^{2n}\bar{z}-\varsigma}+\frac{1}{r^{2n}-\bar{z}\varsigma}-\frac{\bar{z}\varsigma}{r^{2n}\bar{z}\varsigma-1}\Bigg]\frac{d\varsigma}{\varsigma}$$

$$-\frac{1}{2\pi i}\int\limits_{\substack{|\varsigma|=r,\\0<\operatorname{Im}\varsigma}}\gamma(\varsigma)\left[\frac{\varsigma}{\varsigma-z}+\frac{z\varsigma}{\varsigma z-1}+\frac{-\varsigma}{\varsigma-\bar z}+\frac{\bar z\varsigma}{1-\bar z\varsigma}\right]\frac{d\varsigma}{\varsigma}$$

$$-\frac{1}{2\pi i}\int\limits_{\substack{|\varsigma|=r,\\0<\operatorname{Im}\varsigma}}\gamma(\varsigma)\sum_{n=1}^{\infty}r^{2n}\left[\frac{\varsigma}{r^{2n}\varsigma-z}-\frac{z}{r^{2n}z-\varsigma}-\frac{1}{r^{2n}-z\varsigma}+\frac{z\varsigma}{r^{2n}z\varsigma-1}\right.$$
$$\left.-\frac{\varsigma}{r^{2n}\varsigma-\bar z}+\frac{\bar z}{r^{2n}\bar z-\varsigma}+\frac{1}{r^{2n}-\bar z\varsigma}-\frac{\bar z\varsigma}{r^{2n}\bar z\varsigma-1}\right]\frac{d\varsigma}{\varsigma}$$

$$+\frac{1}{2\pi i}\int\limits_{\partial R^+\cap\mathbb{R}}\gamma(t)\left[\frac{1}{t-z}+\frac{z}{tz-1}+\frac{-1}{t-\bar z}+\frac{\bar z}{1-\bar z t}\right]dt$$

$$+\frac{1}{2\pi i}\int\limits_{\partial R^+\cap\mathbb{R}}\gamma(t)\sum_{n=1}^{\infty}r^{2n}\left[\frac{1}{r^{2n}t-z}-\frac{z}{t(r^{2n}z-t)}-\frac{1}{t(r^{2n}-zt)}+\frac{z}{r^{2n}zt-1}\right.$$
$$\left.-\frac{1}{r^{2n}t-\bar z}+\frac{\bar z}{t(r^{2n}\bar z-t)}+\frac{1}{t(r^{2n}-\bar z t)}-\frac{\bar z}{r^{2n}\bar z t-1}\right]dt$$

$$=\frac{1}{2\pi i}\int\limits_{\substack{|\varsigma|=1,\\0<\operatorname{Im}\varsigma}}\gamma(\varsigma)\left[\frac{\varsigma}{\varsigma-z}+\frac{z}{z-\bar\varsigma}-\frac{\varsigma}{\varsigma-\bar z}+\frac{\bar z}{\bar\varsigma-\bar z}\right]\frac{d\varsigma}{\varsigma}$$

$$+\frac{1}{2\pi i}\int\limits_{\substack{|\varsigma|=1,\\0<\operatorname{Im}\varsigma}}\gamma(\varsigma)\sum_{n=1}^{\infty}r^{2n}\left[\frac{\varsigma}{r^{2n}\varsigma-z}-\frac{z}{r^{2n}z-\varsigma}-\frac{1}{r^{2n}-z\varsigma}+\frac{z\varsigma}{r^{2n}z\varsigma-1}\right.$$
$$\left.-\frac{\varsigma}{r^{2n}\varsigma-\bar z}+\frac{\bar z}{r^{2n}\bar z-\varsigma}+\frac{1}{r^{2n}-\bar z\varsigma}-\frac{\bar z\varsigma}{r^{2n}\bar z\varsigma-1}\right]\frac{d\varsigma}{\varsigma}$$

$$-\frac{1}{2\pi i}\int\limits_{\substack{|\varsigma|=r,\\0<\operatorname{Im}\varsigma}}\gamma(\varsigma)\left[\frac{\varsigma}{\varsigma-z}+\frac{z}{z-\bar\varsigma}+\frac{-\varsigma}{\varsigma-\bar z}+\frac{\bar z}{\bar\varsigma-\bar z}\right]\frac{d\varsigma}{\varsigma}$$

$$-\frac{1}{2\pi i}\int\limits_{\substack{|\varsigma|=r,\\0<\operatorname{Im}\varsigma}}\gamma(\varsigma)\sum_{n=1}^{\infty}r^{2n}\left[\frac{\varsigma}{r^{2n}\varsigma-z}-\frac{z}{r^{2n}z-\varsigma}-\frac{1}{r^{2n}-z\varsigma}+\frac{z\varsigma}{r^{2n}z\varsigma-1}\right.$$
$$\left.-\frac{\varsigma}{r^{2n}\varsigma-\bar z}+\frac{\bar z}{r^{2n}\bar z-\varsigma}+\frac{1}{r^{2n}-\bar z\varsigma}-\frac{\bar z\varsigma}{r^{2n}\bar z\varsigma-1}\right]\frac{d\varsigma}{\varsigma}$$

$$+\frac{z-\bar z}{2\pi i}\int\limits_{\partial R^+\cap\mathbb{R}}\gamma(t)\left[\frac{1}{|t-z|^2}-\frac{1}{|1-tz|^2}\right]dt$$

$$+\frac{z-\bar z}{2\pi i}\int\limits_{\partial R^+\cap\mathbb{R}}\gamma(t)\sum_{n=1}^{\infty}r^{2n}\left[\frac{1}{|r^{2n}t-z|^2}+\frac{1}{|r^{2n}z-t|^2}-\frac{1}{|r^{2n}-zt|^2}-\frac{1}{|r^{2n}zt-1|^2}\right]dt. \tag{3.6}$$

$$= \frac{1}{2\pi i}\int\limits_{\substack{|\varsigma|=1,\\ 0<\mathrm{Im}\,\varsigma}} \gamma(\varsigma)\left[\frac{\varsigma}{\varsigma-z}+\frac{\bar\varsigma}{\bar\varsigma-\bar z}-\frac{\bar\varsigma}{\bar\varsigma-z}-\frac{\varsigma}{\varsigma-\bar z}\right]\frac{d\varsigma}{\varsigma}$$

$$+\frac{1}{2\pi i}\int\limits_{\substack{|\varsigma|=1,\\ 0<\mathrm{Im}\,\varsigma}} \gamma(\varsigma)\sum_{n=1}^{\infty} r^{2n}\left[\frac{1}{r^{2n}-z\bar\varsigma}-\frac{|z|^2}{r^{2n}|z|^2-\bar z\varsigma}-\frac{1}{r^{2n}-z\varsigma}+\frac{|z|^2}{r^{2n}|z|^2-\bar z\bar\varsigma}\right.$$

$$\left.-\frac{1}{r^{2n}-\bar z\bar\varsigma}+\frac{|z|^2}{r^{2n}|z|^2-z\varsigma}+\frac{1}{r^{2n}-\bar z\varsigma}-\frac{|z|^2}{r^{2n}|z|^2-z\bar\varsigma}\right]\frac{d\varsigma}{\varsigma}$$

$$-\frac{1}{2\pi i}\int\limits_{\substack{|\varsigma|=r,\\ 0<\mathrm{Im}\,\varsigma}} \gamma(\varsigma)\left[\frac{\varsigma}{\varsigma-z}+\frac{\bar\varsigma}{\bar\varsigma-\bar z}-\frac{\bar\varsigma}{\bar\varsigma-z}-\frac{\varsigma}{\varsigma-\bar z}\right]\frac{d\varsigma}{\varsigma}$$

$$-\frac{1}{2\pi i}\int\limits_{\substack{|\varsigma|=r,\\ 0<\mathrm{Im}\,\varsigma}} \gamma(\varsigma)\sum_{n=1}^{\infty} r^{2n}\left[\frac{r^2}{r^{2n+2}-z\bar\varsigma}-\frac{|z|^2}{r^{2n}|z|^2-\bar z\varsigma}-\frac{r^2}{r^{2n+2}-z\varsigma}+\frac{|z|^2}{r^{2n}|z|^2-\bar z\bar\varsigma}\right.$$

$$\left.-\frac{r^2}{r^{2n+2}-\bar z\bar\varsigma}+\frac{|z|^2}{r^{2n}|z|^2-z\varsigma}+\frac{r^2}{r^{2n+2}-\bar z\varsigma}-\frac{|z|^2}{r^{2n}|z|^2-z\bar\varsigma}\right]\frac{d\varsigma}{\varsigma}$$

$$+\frac{z-\bar z}{2\pi i}\int\limits_{\partial R^+\cap\mathbb{R}} \gamma(t)\left[\frac{1}{|t-z|^2}-\frac{1}{|1-tz|^2}\right]dt$$

$$+\frac{z-\bar z}{2\pi i}\int\limits_{\partial R^+\cap\mathbb{R}} \gamma(t)\sum_{n=1}^{\infty} r^{2n}\left[\frac{1}{|r^{2n}t-z|^2}+\frac{1}{|r^{2n}z-t|^2}-\frac{1}{|r^{2n}-zt|^2}-\frac{1}{|r^{2n}zt-1|^2}\right]dt. \tag{3.7}$$

Thus for $|\varsigma_0|=1,\ Im\varsigma_0>0,\ z\in R^+$,

$$\lim_{z\to\varsigma_0}\frac{1}{2\pi i}\int\limits_{\substack{|\varsigma|=1,\\ 0<\mathrm{Im}\,\varsigma}} \gamma(\varsigma)\left[\frac{\varsigma}{\varsigma-z}+\frac{\bar\varsigma}{\bar\varsigma-\bar z}-\frac{\bar\varsigma}{\bar\varsigma-z}-\frac{\varsigma}{\varsigma-\bar z}\right]\frac{d\varsigma}{\varsigma}$$

$$-\lim_{z\to\varsigma_0}\frac{1}{2\pi i}\int\limits_{\substack{|\varsigma|=r,\\ 0<\mathrm{Im}\,\varsigma}} \gamma(\varsigma)\left(\sum_{n=1}^{\infty} r^{2n}\left[\frac{\varsigma}{r^{2n}\varsigma-z}+\frac{\bar\varsigma}{r^{2n}\bar\varsigma-\bar z}-\frac{\bar\varsigma}{r^{2n}\bar\varsigma-z}-\frac{\varsigma}{r^{2n}\varsigma-\bar z}\right]\right.$$

$$\left.-\sum_{n=1}^{\infty} r^{2n}\left[\frac{|z|^2\varsigma}{r^{2n}|z|^2\varsigma-r^2z}+\frac{|z|^2\bar\varsigma}{r^{2n}|z|^2\bar\varsigma-r^2\bar z}-\frac{|z|^2\bar\varsigma}{r^{2n}|z|^2\bar\varsigma-r^2z}-\frac{|z|^2\varsigma}{r^{2n}|z|^2\varsigma-r^2\bar z}\right]\right)\frac{d\varsigma}{\varsigma}$$

$$=\lim_{z\to\varsigma_0}\frac{1}{2\pi i}\int\limits_{|\varsigma|=1}\Gamma(\varsigma)\left[\frac{\bar\varsigma}{\overline{\overline{\varsigma-z}}}+\frac{\varsigma}{\varsigma-z}-1\right]\frac{d\varsigma}{\varsigma}=\gamma(\varsigma_0) \tag{3.8}$$

with

$$\Gamma(\varsigma)=\begin{cases}\gamma(\varsigma), & Im\varsigma\geqslant 0\\ -\gamma(\bar\varsigma), & Im\varsigma<0\end{cases} \tag{3.9}$$

For $|\varsigma_0|=r,\ Im\varsigma_0>0,\ z\in R^+$,

$$\lim_{z\to\varsigma_0}\frac{1}{2\pi i}\int\limits_{\substack{|\varsigma|=1,\\ 0<\operatorname{Im}\varsigma}}\gamma(\varsigma)\left(\sum_{n=1}^{\infty}r^{2n}\left[\frac{z}{r^{2n}z-|z|^2\varsigma}+\frac{\bar z}{r^{2n}\bar z-|z|^2\bar\varsigma}-\frac{\bar z}{r^{2n}\bar z-|z|^2\varsigma}-\frac{z}{r^{2n}z-|z|^2\bar\varsigma}\right]\right.$$

$$\left.-\sum_{n=1}^{\infty}r^{2n}\left[\frac{z}{r^{2n}z-\varsigma}+\frac{\bar z}{r^{2n}\bar z-\bar\varsigma}-\frac{\bar z}{r^{2n}\bar z-\varsigma}-\frac{z}{r^{2n}z-\bar\varsigma}\right]\right)\frac{d\varsigma}{\varsigma}$$

$$-\lim_{z\to\varsigma_0}\frac{1}{2\pi i}\int\limits_{\substack{|\varsigma|=r,\\ 0<\operatorname{Im}\varsigma}}\gamma(\varsigma)\left[\frac{\varsigma}{\varsigma-z}+\frac{\bar\varsigma}{\bar\varsigma-\bar z}-\frac{\bar\varsigma}{\bar\varsigma-z}-\frac{\varsigma}{\varsigma-\bar z}\right]\frac{d\varsigma}{\varsigma}$$

$$-\lim_{z\to\varsigma_0}\frac{1}{2\pi i}\int\limits_{\substack{|\varsigma|=r,\\ 0<\operatorname{Im}\varsigma}}\gamma(\varsigma)\sum_{n=1}^{\infty}r^{2n}\left[\frac{|z|^2\varsigma}{r^{2n}|z|^2\varsigma-r^2z}+\frac{|z|^2\bar\varsigma}{r^{2n}|z|^2\bar\varsigma-r^2\bar z}-\frac{|z|^2\bar\varsigma}{r^{2n}|z|^2\bar\varsigma-r^2z}-\frac{|z|^2\varsigma}{r^{2n}|z|^2\varsigma-r^2\bar z}\right]\frac{d\varsigma}{\varsigma}$$

$$=-\lim_{z\to\varsigma_0}\frac{1}{2\pi i}\int\limits_{|\varsigma|=1}\Gamma(\varsigma)\left[\frac{\bar\varsigma}{\overline{\overline{\varsigma-z}}}+\frac{\varsigma}{\varsigma-z}-1\right]\frac{d\varsigma}{\varsigma}=\gamma(\varsigma_0) \tag{3.10}$$

with

$$\Gamma(\varsigma)=\begin{cases}\gamma(\varsigma), & Im\varsigma\geqslant 0\\ -\gamma(\bar\varsigma), & Im\varsigma<0\end{cases} \tag{3.11}$$

For $t_0\in(-1,-r)\bigcup(r,1),\ z\in R^+$

$$\lim_{z\to t_0}\frac{1}{2\pi i}\int\limits_{\partial R^+\cap\mathbb{R}}\gamma(t)\left(\frac{z-\bar z}{|t-z|^2}-\frac{z-\bar z}{|1-tz|^2}+\sum_{n=1}^{\infty}r^{2n}\left[\frac{z-\bar z}{|r^{2n}t-z|^2}+\frac{z-\bar z}{|r^{2n}z-t|^2}-\frac{z-\bar z}{|r^{2n}-zt|^2}-\frac{z-\bar z}{|1-r^{2n}zt|^2}\right]\right)dt$$

$$=\gamma(t_0). \tag{3.12}$$

□

Theorem 3.2 *Consider the Dirichlet problem for the inhomogeneous Cauchy-Riemann equation*

$$\omega_{\bar z}=g\quad in\quad R^+,\ \omega=\gamma\quad on\ \partial R^+, \tag{3.13}$$

where $\gamma\in C(\partial R^+;\mathbb{C})$, $g\in L_p(R^+;\mathbb{C})$ is solvable if and only if

$$\frac{1}{2\pi i}\int\limits_{\partial R^+}\gamma(\varsigma)\left[\frac{-1}{\varsigma-\bar z}+\frac{\bar z}{1-\bar z\varsigma}\right]d\varsigma$$

$$+\frac{1}{2\pi i}\int\limits_{\partial R^+}\gamma(\varsigma)\sum_{n=1}^{\infty}r^{2n}\left[\frac{-1}{r^{2n}\varsigma-\bar z}+\frac{\bar z}{\varsigma(r^{2n}\bar z-\varsigma)}+\frac{1}{\varsigma(r^{2n}-\bar z\varsigma)}+\frac{-\bar z}{r^{2n}\bar z\varsigma-1}\right]d\varsigma=0$$

$$-\frac{1}{\pi}\iint\limits_{R^+}g(\varsigma)\left[\frac{-1}{\varsigma-\bar z}+\frac{\bar z}{1-\bar z\varsigma}\right]d\varsigma$$

$$-\frac{1}{\pi}\iint\limits_{R^+}g(\varsigma)\sum_{n=1}^{\infty}r^{2n}\left[\frac{-1}{r^{2n}\varsigma-\bar z}+\frac{\bar z}{\varsigma(r^{2n}\bar z-\varsigma)}+\frac{1}{\varsigma(r^{2n}-\bar z\varsigma)}+\frac{-\bar z}{r^{2n}\bar z\varsigma-1}\right]d\xi d\eta=0 \tag{3.14}$$

the unique solution is represented by the expression

$$\omega(z)=\frac{1}{2\pi i}\int\limits_{\partial R^+}\gamma(\varsigma)\left[\frac{1}{\varsigma-z}+\frac{z}{\varsigma z-1}\right]d\varsigma$$

$$+\frac{1}{2\pi i}\int\limits_{\partial R^+}\gamma(\varsigma)\sum_{n=1}^{\infty}r^{2n}\left[\frac{1}{r^{2n}\varsigma-z}-\frac{z}{\varsigma(r^{2n}z-\varsigma)}-\frac{1}{\varsigma(r^{2n}-z\varsigma)}+\frac{z}{r^{2n}z\varsigma-1}\right]d\varsigma$$

$$-\frac{1}{\pi}\iint\limits_{R^+}g(\varsigma)\left[\frac{1}{\varsigma-z}+\frac{z}{\varsigma z-1}\right]d\varsigma$$

$$-\frac{1}{\pi}\iint\limits_{R^+}g(\varsigma)\sum_{n=1}^{\infty}r^{2n}\left[\frac{1}{r^{2n}\varsigma-z}-\frac{z}{\varsigma(r^{2n}z-\varsigma)}-\frac{1}{\varsigma(r^{2n}-z\varsigma)}+\frac{z}{r^{2n}z\varsigma-1}\right]d\xi d\eta. \tag{3.15}$$

Proof: The area integral in (3.15) defines an analytic function in R^+ up to the term $Tg(z)$, which is introduced earlier and captures the non-analytic component associated with the inhomogeneous term g. Moreover,the boundary integral terms in (3.15) were verified that the boundary values of the expression in (3.3) coincide with the prescribed function γ on $\partial R^{+.}$ Therefore, the expression in (3.15) satisfies the boundary conditions and provides a weak solution to the inhomogeneous Cauchy-Riemann equation.
Define

$$\theta(z)=\omega(z)-Tg(z).$$

Then θ satisfies

$$\theta_{\bar z}=\omega_{\bar z}-(Tg)_{\bar z}=g-g=0 \quad \text{in } R^+,$$

and

$$\theta=\omega-Tg \quad \text{on } \partial R^+.$$

Thus, θ solves the homogeneous Cauchy-Riemann equation with nonhomogeneous Dirichlet boundary condition:

$$\theta_{\bar z}=0 \quad \text{in } R^+, \quad \theta=\gamma-Tg \quad \text{on } \partial R^+.$$

The desired result follows from the previous theorem through the following computations:

$$\frac{1}{2\pi i}\int\limits_{\partial R^+}(\gamma(\varsigma)-Tg(\varsigma))\left[\frac{-1}{\varsigma-\bar z}+\frac{\bar z}{1-\bar z\varsigma}\right]d\varsigma$$

$$+\frac{1}{2\pi i}\int\limits_{\partial R^+}(\gamma(\varsigma)-Tg(\varsigma))\sum_{n=1}^{\infty}r^{2n}\left[\frac{-1}{r^{2n}\varsigma-\bar z}+\frac{\bar z}{\varsigma(r^{2n}\bar z-\varsigma)}+\frac{1}{\varsigma(r^{2n}-\bar z\varsigma)}+\frac{-\bar z}{r^{2n}\bar z\varsigma-1}\right]d\varsigma=0 \tag{3.16}$$

and

$$\theta(z)=\frac{1}{2\pi i}\int\limits_{\partial R^+}(\gamma(\varsigma)-Tg(\varsigma))\left[\frac{1}{\varsigma-z}+\frac{z}{\varsigma z-1}\right]d\varsigma$$

$$+\frac{1}{2\pi i}\int\limits_{\partial R^+}(\gamma(\varsigma)-Tg(\varsigma))\sum_{n=1}^{\infty}r^{2n}\left[\frac{1}{r^{2n}\varsigma-z}-\frac{z}{\varsigma(r^{2n}z-\varsigma)}-\frac{1}{\varsigma(r^{2n}-z\varsigma)}+\frac{z}{r^{2n}z\varsigma-1}\right]d\varsigma. \tag{3.17}$$

We now investigate the uniqueness of the solution. Suppose that ω and ϖ are two solutions of the problem. Then, they satisfy the following:

$$\omega_{\bar z}=g \ \textit{in } R^+, \ w=\gamma \ \textit{on } \partial R^+$$

$$\varpi_{\bar z}=g \ \textit{in } R^+, \ \varpi=\gamma \ \textit{on } \partial R^+$$

Subtracting these two sets of equations yields:

$$(\omega-\varpi)_{\bar z}=0 \ in\ R^+,\ \omega-\varpi=\gamma-\gamma=0\ on\ \partial R^+.$$

Hence, $\omega-\varpi$ satisfies a homogeneous problem with zero boundary conditions, which implies that $\omega=\varpi$ in R^+. This completes the proof. □

Following our analysis of the Dirichlet problem for both the homogeneous and inhomogeneous Cauchy-Riemann equations, we now extend our attention to a more general setting involving polyanalytic functions. These functions satisfy higher-order inhomogeneous Cauchy-Riemann equations. In the following theorem, we study the Dirichlet problem for the inhomogeneous polyanalytic equation in the domain R^+, and we provide conditions for solvability along with an explicit representation of the solution.

Theorem 3.3 *The Dirichlet problem for the inhomogeneous polyanalytic equation in R^+*

$$\partial_{\bar z}^{n}\omega=g\ in\ R^+,\ \partial_{\bar z}^{\iota}\omega=\gamma_\iota\ on\ \partial R^+,\ 0\leqslant\iota\leqslant n-1 \tag{3.18}$$

is uniquely solvable for $g\in L_1(R^+;\mathbb{C}),\ \gamma_\iota\in C(\partial R^+;\mathbb{C}),\ 0\leqslant\iota\leqslant n-1$ if and only if

$$\begin{aligned}
&\sum_{\kappa=\iota}^{n-1}\frac{(-1)^{\kappa-\iota}}{(\kappa-\iota)!}\cdot\frac{1}{2\pi i}\int_{\partial R^+}\gamma_\kappa(\varsigma)\Big\{\Big[\frac{-1}{\varsigma-\bar z}+\frac{\bar z}{1-\bar z\varsigma}\Big]\\
&+\sum_{n=1}^{\infty}r^{2n}\Big[\frac{-1}{r^{2n}\varsigma-\bar z}+\frac{\bar z}{\varsigma(r^{2n}\bar z-\varsigma)}+\frac{1}{\varsigma(r^{2n}-\bar z\varsigma)}-\frac{\bar z}{r^{2n}\bar z\varsigma-1}\Big]\Big\}\overline{(\varsigma-z)}^{\kappa-\iota}d\varsigma\\
&+\frac{(-1)^{n-\iota}}{\pi}\iint_{R^+}\frac{g(\varsigma)}{(n-1-\iota)!}\Big\{\Big[\frac{-1}{\varsigma-\bar z}+\frac{\bar z}{1-\bar z\varsigma}\Big]\\
&+\sum_{n=1}^{\infty}r^{2n}\Big[\frac{-1}{r^{2n}\varsigma-\bar z}+\frac{\bar z}{\varsigma(r^{2n}\bar z-\varsigma)}+\frac{1}{\varsigma(r^{2n}-\bar z\varsigma)}-\frac{\bar z}{r^{2n}\bar z\varsigma-1}\Big]\Big\}\overline{(\varsigma-z)}^{n-1-\iota}\,d\xi d\eta=0
\end{aligned}\tag{3.19}$$

The solution is

$$\begin{aligned}
\omega(z)=&\sum_{\iota=0}^{n-1}\frac{(-1)^{\iota}}{2\pi i}\int_{\partial R^+}\frac{\gamma(\varsigma)}{\iota!}\Big\{\Big[\frac{1}{\varsigma-z}+\frac{z}{\varsigma z-1}\Big]\\
&+\sum_{n=1}^{\infty}r^{2n}\Big[\frac{1}{r^{2n}\varsigma-z}-\frac{z}{\varsigma(r^{2n}z-\varsigma)}-\frac{1}{\varsigma(r^{2n}-z\varsigma)}+\frac{z}{r^{2n}z\varsigma-1}\Big]\Big\}\overline{(\varsigma-z)}^{\iota}\,d\varsigma\\
&+\frac{(-1)^{n}}{\pi(n-1)!}\iint_{R^+}g(\varsigma)\Big\{\Big[\frac{1}{\varsigma-z}+\frac{z}{\varsigma z-1}\Big]\\
&+\sum_{n=1}^{\infty}r^{2n}\Big[\frac{1}{r^{2n}\varsigma-z}-\frac{z}{\varsigma(r^{2n}z-\varsigma)}-\frac{1}{\varsigma(r^{2n}-z\varsigma)}+\frac{z}{r^{2n}z\varsigma-1}\Big]\Big\}\overline{(\varsigma-z)}^{n-1}\,d\xi d\eta.
\end{aligned}\tag{3.20}$$

Proof: For $n=1$ condition (3.19) coincides with (3.14) and equation (3.20) becomes (3.15). Now, assume that the theorem holds for $n-1$; we will prove it for n by induction. We decompose the problem into the following system:

$$\partial_{\bar z}^{n-1}\omega=h\ in\ R^+,\partial_{\bar z}^{\iota}\omega=\gamma_\iota\ on\ \partial R^+,\ 0\leqslant\iota\leqslant n-2$$

$$\partial_{\bar z}h=g\ in\ R^+,\ h=\gamma_{n-1}\ on\ \partial R^+,$$

The solvability conditions include condition (3.19) for $0 \leqslant \iota \leqslant n-2$ now applied to h instead of g, and the additional condition:

$$\frac{1}{2\pi i}\int\limits_{\partial R^+}\gamma_{n-1}(\varsigma)\left[\frac{-1}{\varsigma-\bar z}+\frac{\bar z}{1-\bar z\varsigma}\right]d\varsigma$$

$$+\frac{1}{2\pi i}\int\limits_{\partial R^+}\gamma_{n-1}(\varsigma)\sum_{n=1}^{\infty}r^{2n}\left[\frac{-1}{r^{2n}\varsigma-\bar z}+\frac{\bar z}{\varsigma(r^{2n}\bar z-\varsigma)}+\frac{1}{\varsigma(r^{2n}-\bar z\varsigma)}+\frac{-\bar z}{r^{2n}\bar z\varsigma-1}\right]d\varsigma=0$$

$$-\frac{1}{\pi}\iint\limits_{R^+}g(\varsigma)\left[\frac{-1}{\varsigma-\bar z}+\frac{\bar z}{1-\bar z\varsigma}\right]d\varsigma$$

$$-\frac{1}{\pi}\iint\limits_{R^+}g(\varsigma)\sum_{n=1}^{\infty}r^{2n}\left[\frac{-1}{r^{2n}\varsigma-\bar z}+\frac{\bar z}{\varsigma(r^{2n}\bar z-\varsigma)}+\frac{1}{\varsigma(r^{2n}-\bar z\varsigma)}+\frac{-\bar z}{r^{2n}\bar z\varsigma-1}\right]d\xi d\eta=0 \tag{3.21}$$

Now, using the solution formula (3.20) for $n-1$ instead of n with h in place of g, we have

$$h(z)=\frac{1}{2\pi i}\int\limits_{\partial R^+}\gamma_{n-1}(\varsigma)\left[\frac{1}{\varsigma-z}+\frac{z}{\varsigma z-1}\right]d\varsigma$$

$$+\frac{1}{2\pi i}\int\limits_{\partial R^+}\gamma_{n-1}(\varsigma)\sum_{n=1}^{\infty}r^{2n}\left[\frac{1}{r^{2n}\varsigma-z}-\frac{z}{\varsigma(r^{2n}z-\varsigma)}-\frac{1}{\varsigma(r^{2n}-z\varsigma)}+\frac{z}{r^{2n}z\varsigma-1}\right]d\varsigma$$

$$-\frac{1}{\pi}\iint\limits_{R^+}g(\varsigma)\left[\frac{1}{\varsigma-z}+\frac{z}{\varsigma z-1}\right]d\varsigma$$

$$-\frac{1}{\pi}\iint\limits_{R^+}g(\varsigma)\sum_{n=1}^{\infty}r^{2n}\left[\frac{1}{r^{2n}\varsigma-z}-\frac{z}{\varsigma(r^{2n}z-\varsigma)}-\frac{1}{\varsigma(r^{2n}-z\varsigma)}+\frac{z}{r^{2n}z\varsigma-1}\right]d\xi d\eta. \tag{3.22}$$

By combining the solvability conditions and the solution formulas, we obtain the desired result for n, thus completing the induction. □

4. Conclusion

In this paper, we studied boundary value problems for various classes of complex partial differential equations in the upper half ring domain. First, we derived an explicit integral representation for functions in this domain, providing a foundation for addressing more complex equations. Using this representation, we then investigated the Dirichlet problem for both the homogeneous and nonhomogeneous Cauchy-Riemann equations. Finally, we extended our analysis to the Dirichlet problem for higher-order equations, specifically focusing on polyanalytic functions.

References

1. Abdymanapov, S. A., Begehr, H., Harutyunyan, G., Tungatarov, A. B., *Four boundary value problems for the Cauchy–Riemann equation in a quarter plane*, More Progresses in Analysis, Proc. 5th Intern. ISAAC Congress, Catania, Italy, 1137–1147, (2009).
2. Begehr, H., Hile, G. N., *A hierarchy of integral operators*, Rocky Mountain J. Math. 27, 669–706, (1997).
3. Begehr, H., *Complex Analytic Methods for Partial Differential Equations: An Introductory Text*, World Scientific, Singapore, (1994).
4. Abdymanapov, S. A., Begehr, H., Tungatarov, A. B., *Some Schwarz problems in a quarter plane*, Eurasian Math. J. 3, 22–35, (2005).
5. Kumar, A., Prakash, R., *Iterated boundary value problems for the inhomogeneous polyanalytic equation*, Complex Var. Elliptic Equ. 52, 921–932, (2007).
6. Akel, M., Begehr, H., Mohammed, A., *Integral representations in the complex plane and iterated boundary value problems*, Rocky Mountain J. Math. 52(2), 381–413, (2022).

7. Begehr, H., *Boundary value problems for the Bitsadze equation*, Memoirs Diff. Eqs. Math. Phys. 33, 5–23, (2005).

8. Begehr, H., Shupeyeva, B., *Polyanalytic boundary value problems for planar domains with harmonic Green function*, Anal. Math. Phys. 11, 1–22, (2021).

9. Karaca, B., *A note on complex combined boundary value problem for the nonhomogeneous tri-analytic equation*, Bol. Soc. Parana. Mat. 42, 1–7, (2024).

10. Chaudhary, A., Kumar, A., *Boundary value problems in upper half plane*, Complex Var. Elliptic Equ. 54(5), 441–448, (2009).

11. Begehr, H., Chaudhary, A., Kumar, A., *Bi-polyanalytic functions on upper half plane*, Complex Var. Elliptic Equ. 55(1–3), 305–316, (2010).

12. Chaudhary, A., *Neumann and mixed boundary value problems on the upper half plane*, Adv. Theory Nonlinear Anal. Appl. 6(1), 135–142, (2022).

13. Karaca, B., *Schwarz problem for model partial differential equations with one complex variable*, Sakarya Univ. J. Sci. 28(2), 410–417, (2024).

14. Xu, Y., *Generalized (λ, k) bi-analytic functions and Riemann–Hilbert problem for a class of nonlinear second order elliptic systems*, Complex Var. Theory Appl. 8(1–2), 103–121, (1987).

15. Lin, J., Xu, Y., *Riemann problem of (λ, k) bi-analytic functions*, Applicable Analysis 101(11), 3804–3815, (2021).

16. Xu, Y., *Riemann problem and inverse Riemann problem of $(\lambda, 1)$ bi-analytic functions*, Complex Var. Elliptic Equ. 52(10–11), 853–864, (2007).

17. Vaitsiakhovich, T., *Boundary value problems to first order complex partial differential equations in a ring domain*, Integral Transf. Spec. Funct. 19, 211–233, (2008).

18. Vaitsiakhovich, T., *Boundary value problems to second order complex partial differential equations in a ring domain*, Sauliai Math. Seminar 2, 117–146, (2007).

19. Linares, R. Y., Vanegas, C. J., *A Robin boundary value problem in the upper half plane for the Bitsadze equation*, J. Math. Anal. Appl. 419(1), 200–217, (2014).

20. Akel, M., Hidan, M., Abdalla, M., *Complex boundary value problems for the Cauchy–Riemann operator on a triangle*, Fractals 30(10), 1–15, (2022).

21. Begehr, H., Vaitsiakhovich, T., *Harmonic boundary value problems in half disc and half ring*, Funct. Approximatio 40(2), 251–282, (2009).

Bahriye Karaca,

Department of Fundamental Sciences,

Faculty of Engineering and Architecture, Izmir Bakircay University,

Izmir, Turkey.

E-mail address: bahriye.karaca@bakircay.edu.tr, bahriyekaraca@gmail.com