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LRS Bianchi Type II Bulk Viscous Fluid String Dust Magnetized Dark Energy Cosmological Model in Rosens Bimetric Theory of Gravitation with the Term-Λ

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ABSTRACT: This work investigates the LRS (Locally Rotationally Symmetric) Bianchi Type II cosmological framework that incorporates bulk-viscous fluid, string dust, a magnetized component, and a time-dependent cosmological term (Λ) within Rosen's bimetric theory of gravitation (BTG). By solving field equations developed by Rosen's, we construct exact solutions that describe the dynamical evolution of an anisotropic universe influenced by bulk viscosity, cosmic strings, magnetic fields, and dark energy. The analysis highlights the role of Λ in driving the late-time acceleration of cosmic expansion, examines the transformation from a decelerated to an accelerated phase, and explores anisotropic dark energy dominated epochs. The results of our model provide insights into how magnetic fields, cosmic strings, and dark energy influence the universe's structure, background radiation, and evolution in alternative gravity theories. Furthermore, the geometric and physical properties of the model are examined under both power-law and exponential-law expansions.

Keywords: Bimetric theory of gravitation, dark energy, cosmology.

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1. Introduction

Recently, string cosmology has garnered significant attention. Cosmic strings are seen as topologically stable things that likely formed during a stage change in the early cosmos, they assume a vital part in the investigation of the early universe [1]. As the universe cooled after the Big Bang, a phase transition occurred when the temperature dropped below a crucial limit, leading to the emergence of these phenomena, as predicted by unified theoretical models [1,2,3,4,5]. One widely recognized theory suggests that cosmic strings play a role in creating density fluctuations, which may influence the process of galaxy formation [6]. Dark energy plays a vital role in comprehending the universe. Cosmologists theorized that around seven billion years ago, a mysterious repulsive force counteracting gravity began

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accelerating the universe's expansion. This phenomenon is referred to as dark energy (DE). Initially, the accelerated expansion of the universe was discovered through observations of high-redshift Type Ia supernovae [6,7,8,9,10,11,12] and was later validated through independent analyses of cosmic microwave background (CMB) radiation [13,14] and large-scale structure data [15,16,17,18,19].It presently appears that the cosmos is composed of 68.3% dark energy, 4.9% ordinary matter, and 26.8% dark matter [20,21,22,23].

The theory of General Relativity (GR) proposed by Einstein which is the most successful theories of gravitation and is satisfies observations as well as experimental data but it has some drawbacks that it shows singularities in big-bang theory, therefore many researchers develops theories of gravitation which act as replacement to the theory of GR. Out of all the theories, most significant is bimetric theory of gravitation (BTG) proposed by Rosen's [24,25]. This BTG satisfies both the principles of GR which are covariance and equivalence, also it removes the singularities and satisfies with the observations given by General Relativity. This theory is depends on two matrices, one of them is flat metric tensor γ_{ij} and other is fundamental metric tensor g_{ij}

$$ds^2 = g_{ij}\, dx^i dx^j, \tag{1.1}$$

$$d\sigma^2 = \gamma_{ij}\, dx^i dx^j. \tag{1.2}$$

The field equations of BTG given by Rosen are:

$$N_i^j - \frac{1}{2} N \delta_i^j + \Lambda = -8\pi k\, T_i^j, \tag{1.3}$$

where

$$N_i^j = \frac{1}{2} \gamma^{pr} \left(g^{sj} g_{si|p} \right)_{|r},$$

the symbol $|$ denotes $\gamma-$ covariant differentiation, and T_i^j is the energy-momentum tensor. Also,

$$N = N_i^i, \qquad k = \sqrt{\frac{g}{\gamma}},$$

with $g = \det(g_{ij})$ and $\gamma = \det(\gamma_{ij})$.

Starting from Rosen many researchers such as Isrelit, Katore and Rane, Gaikwad and Borkar, Karade, Bali et al., Reddy and Rao, Khadekar and Tade [26,27,28,29,30,31,32,33,34] have studied various cosmological models in GR and BTG of the universe.

The standard Λ- Cold Dark Matter and the Early Dark Energy model studied and they agree with the present observations, but the DE Spectroscopic Instruments and Euclid measurements will provoid stringent new test [35]. The problem related to cosmological constant is very interesting. Recent observational data shows that cosmological term Λ has approximate value $10^{-55}\,\text{cm}^{-2}$, In contrast, predictions from particle physics shows value of Λ is greater than observed value by a term 10^{120} approximately. therefore inconsistency mentioned above is known as the cosmological constant problem. Without a doubt, we can resolve this dilemma by considering an alternative cosmological term, which has large value at initial time and decreases to the small value in a universe which is expanding [34,35,36,37].

Numerous phenomenological models that take into account that Λ is a function of time have been put forth. Lorenz gives the specific answers for Locally Rotationaly Symmetric Bianchi II spacetime [38]. Boutros inspected Bianchi type-II spacetime with perfect fluid by a making strategy and furthermore fostered a locally rotationaly symmetric bianchi type-II perfect fluid having a time-dependent equation of state parameter [39]. Chakraborty solved locally rotationaly symmetric bianchi type-II which will have variables G and Λ [40]. Mahanta and Das study the correspondence between the quintessence scalar field and the new holographic dark energy, also reconstruct the quintessence scalar potential [41]. Paul proposed a new mechanism for the origin of bulk viscosity in cosmological context [42]. Koussoul et al. study the bulk viscous fluid in extended symmetric teleparallel gravity [43]. Brizuela et al. examined perturbation in a general static, spherically symmetric and asymptotically flat vaccume spacetime where the two metrices are non-bidiagonal yet share identical isometries [44].

In this work, Locally Rotationally Symmetric Bianchi model of type-II DE cosmological model in BTG with the cosmological term Λ by solving equation (1.3) and investigated the power as well as exponential law expansions of this model.

2. Line Element and Field Equation

Let us take into account the LRS Bianchi Type-II spacetime expressed in the form

$$ds^2 = -dt^2 + A^2(dx^2 + dz^2) + B^2(dy - xdz)^2, \tag{2.1}$$

where the scale factors A and B depend solely on the time variable t.

Corresponding to metric (2.1), the flat metric is

$$d\sigma^2 = -dt^2 + (dx^2 + dz^2) + (dy - dz)^2. \tag{2.2}$$

For string dust, the energy-momentum tensor T_i^j can be written as

$$T_i^j = \rho \upsilon_i \upsilon^j - \lambda x_i x^j - \xi\phi(g_i^j + \upsilon_i \upsilon^j) + E_i^j, \tag{2.3}$$

with

$$\nu_i \nu^i = -x_i x^i = -1, \text{and} \quad \nu^i x_i = 0 \tag{2.4}$$

In equation (2.3) the String tension density of the system of strings and the rest energy density of a cloud of strings are shown by λ and ρ respectively, also the measure of internal resistance to compression, known as the bulk viscosity coefficient ξ and direction of strings is x^i.

The electromagnetic field E_{ij} is

$$E_{ij} = \overline{\mu}\left[|h|^2\left(\upsilon_i\upsilon_j + \frac{1}{2}g_{ij}\right) - h_i h_j\right], \tag{2.5}$$

here $\bar{\mu}$ is the magnetic permeability and the four-velocity vector ν_i has the form

$$g_{ij}\upsilon^i\upsilon^j = -1 \tag{2.6}$$

The magnetic flux vector h_i is defined by

$$h_i = \frac{\sqrt{-g}}{2\overline{\mu}}\,\epsilon_{ijkl}F^{kl}\upsilon^j, \tag{2.7}$$

where F^{kl} is the electromagnetic field tensor and ϵ_{ijkl} is the Levi-Civita tensor density.

We consider the comoving coordinate system as

$$\upsilon^4 = 1, \qquad \upsilon^1 = \upsilon^2 = \upsilon^3 = 0.$$

Additionally, we consider the magnetic field incident along the x-axis, so that

$$h_4 = h_3 = h_2 = 0 \quad \text{and} \quad h_1 \neq 0$$

Maxwell's equation can be written as

$$F_{[ij,k]} = 0, \tag{2.8}$$

which yields

$$F_{23} = \text{ constant } \quad (\text{say } H) \tag{2.9}$$

Let us consider that electrical conductivity has no limits

we have, The sole non-disappearing expert of F_{ij} is F_{23} and $F_{14} = F_{34} = F_{24} = 0$. so that

$$|h|^2 = \frac{H^2}{\overline{\mu}^2 B^4} \text{ and } h_1 = \frac{AH}{\overline{\mu}B^2}, \tag{2.10}$$

form equation (2.9) we obtain

$$-E_1^1 = E_2^2 = E_3^3 = -E_4^4 = \frac{H^2}{2\overline{\mu}B^2A^2}, \tag{2.11}$$

from equation (2.6) we obtain

$$T_1^1 = \left(\lambda + \xi\varphi + \frac{H^2}{2\overline{\mu}B^2A^2}\right), \qquad T_2^2 = T_3^3 = \left(\xi\varphi - \frac{H^2}{2\overline{\mu}B^2A^2}\right), \qquad T_4^4 = \left(\rho + \frac{H^2}{2\overline{\mu}B^2A^2}\right), \tag{2.12}$$

Using equation (2.12) and Rosen's field equation (1.3) gives

$$\frac{B_{44}}{B} - \frac{{B_4}^2}{B^2} = 16\pi A^2 B\left(\lambda + \xi\phi + \phi + \frac{H^2}{2\overline{\mu}B^2A^2}\right) - 2\Lambda, \tag{2.13}$$

$$\frac{2\ A_{44}}{A} - \frac{2\ A_4^2}{A^2} - \frac{B_{44}}{B} - \frac{B_4^2}{B^2} - \frac{B^2}{A^2} = 16\pi A^2 B\left(\xi\phi - \frac{H^2}{2\overline{\mu}B^2A^2}\right) - 2\Lambda, \tag{2.14}$$

$$\frac{B_{44}}{B} - \frac{B_4^2}{B^2} + \frac{B^2}{A^2} = 16\pi A^2 B\left(\xi\phi - \frac{H^2}{2\overline{\mu}B^2A^2}\right) - 2\Lambda, \tag{2.15}$$

$$\frac{2\ A_{44}}{A} - \frac{2\ A_4^2}{A^2} + \frac{B_{44}}{B} - \frac{B_4^2}{B^2} = 16\pi A^2 B\left(\rho + \frac{H^2}{2\overline{\mu}B^2A^2}\right) - 2\Lambda, \tag{2.16}$$

3. Solution of the Field Equations

Equation (2.13)-(2.16) are the system consisting of four equations and seven unknown variables $A, B, \rho, \lambda, \varphi, \xi$ and Λ. Thus the system is initially indeterminate and therefore, three additional conditions are required to obtain its complete solution.

We begin by assuming that the shear tensor component σ is directly proportional to the scalar expansion ϕ. This assumption leads to a specific relation between the metric potentials:

$$A = B^m, \tag{3.1}$$

The second condition is the Zel'dovich condition:

$$\rho = \lambda. \tag{3.2}$$

Thirdly, we take

$$H = la^{-n} \tag{3.3}$$

The spatial volume for the model is given by

$$V^3 = AB^2. \tag{3.4}$$

Let us define the average scale factor a as

$$a = (AB^2)^{1/3}. \tag{3.5}$$

The generalized mean Hubble parameter H can be demined as

$$H = \frac{1}{3}\left(H_x + H_y + H_z\right), \tag{3.6}$$

where the directional Hubble parameters in the x, y, and z directions are

$$H_x = \frac{A_4}{A} \qquad H_y = \frac{B_4}{B}, \qquad H_z = \frac{B_4}{B}.$$

The deceleration parameter q in mathematical form is

$$q = \left(-\frac{aa_{44}}{a_4^2}\right) = (n-1), \qquad n \neq 0 \tag{3.7}$$

The deceleration parameter takes a constant value. The model speeds up or not depends on sign of q. For n is greater than 1 the universe indicate non accelerating behavior, where as for n is less than 1 we have a speeding-up of a cosmos.

The value of average scale factor (a) can be obtained from equation (3.5) and is given by

$$a = (n\ell t + c_1)^{\frac{1}{n}}, \qquad n \neq 0 \tag{3.8}$$

here, c_1 represents a constant of integration.

$$a = c_2 e^{lt} \quad \text{for } n = 0, \tag{3.9}$$

here, c_2 represents a constant of integration.

Thus, the variation law of the Hubble parameter gives rise to two kinds of cosmic expansion: the power-law expansion described by equation (3.8), and the exponential-law expansion described by equation (3.9).

3.1. Power Law Model

In the power law model, the average scale factor a is given by

$$a = (nlt + c_1)^{\frac{1}{n}}, \qquad n \neq 0.$$

Using the first and third conditions along with equation (3.8), we get the values of the scale factors $A(t)$ and $B(t)$ as

$$A(t) = (nlt + c_1)^{\frac{3m}{n(m+2)}} \tag{3.10}$$

$$B(t) = (nlt + c_1)^{\frac{3}{n(m+2)}} \tag{3.11}$$

where c_1, $n \neq 0$, $m > 0$ are constants.

Therefore, in the power–law model, the metric (2.1) is given by

$$ds^2 = -dt^2 + (nlt + c_1)^{\frac{6m}{n(m+2)}}(dx^2 + dz^2) + (nlt + c_1)^{\frac{6}{n(m+2)}}(dy - x\,dz)^2. \tag{3.12}$$

This is the required LRS Bianchi Type II cosmological model of dark energy in power law expansion with steady deceleration parameter in bimetric theory.

3.2. Exponential Law Model

The average scale factor a can be written as

$$a = c_2 e^{lt}, \qquad \text{for } n = 0.$$

We can determine values of the scale factors A and B, using the first and third conditions along with equation (3.9), as follows:

$$A(t) = \left(c_2 e^{lt}\right)^{\frac{3m}{m+2}} \tag{3.13}$$

$$B(t) = \left(c_2 e^{lt}\right)^{\frac{3}{m+2}} \tag{3.14}$$

where $n > 0$, c_2 are constants.

Therefore, the metric represented by equation (2.1) is given by

$$ds^2 = -dt^2 + \left(c_2 e^{lt}\right)^{\frac{6m}{m+2}}(dx^2 + dz^2) + \left(c_2 e^{lt}\right)^{\frac{6}{m+2}}(dy - x\,dz)^2. \tag{3.15}$$

This is the required LRS Bianchi Type II cosmological model of dark energy in exponential law expansion with steady deceleration parameter in bimetric theory.

4. Geometrical and Physical Significance

4.1. Power Law Model

Equation (3.12) represents the line element for the power-law model having scale factors A and B, which eventually become infinite after continuously growing over time. Subsequently, around the beginning phase, the model turns out to be flat, while at later stages the scale factors diverge to infinity.

Various physical quantities hold significant importance in cosmology; some of them are the string tension per unit length (λ), the distribution mass-energy (ρ), and the measure of internal resistance to compression, known as the bulk viscosity coefficient (ξ), the scalar expansion (φ), the anisotropy parameter (A_m), the mean Hubble parameter (H), and the shear (σ). These quantities have been evaluated for the power-law model (3.12) as follows:

$$\rho = \lambda = -\left(nlt + c_1\right)^{-\frac{6(m-1)}{n(m+2)}} - \frac{H^2}{\bar{\mu}}\left(nlt + c_1\right)^{-\frac{6(m+1)}{n(m+2)}}, \tag{4.1}$$

$$\varphi = \frac{3l(2m+1)}{(m+2)}\left(nlt + c_1\right)^{-1}, \tag{4.2}$$

$$\xi = \frac{nml}{8\pi(2m+1)}\left(nlt + c_1\right)^{-\left(\frac{3(2m+1)}{n(m+2)}+1\right)}. \tag{4.3}$$

$$\Lambda = \frac{3l^2\left(2m^2+5m-1\right)}{2\left(m+2\right)^2}\left(nlt + c_1\right)^{-2} - \frac{9l^2}{2\left(m+2\right)}\left(nlt + c_1\right)^{-\frac{3}{2}} - \frac{1}{2}\left(nlt + c_1\right)^{-\frac{6(1-m)}{n(m+2)}} - \frac{4\pi H^2}{\bar{\mu}}\left(nlt + c_1\right)^{-\frac{3}{n(m+2)}} \tag{4.4}$$

$$H = l\left(nlt + c_1\right)^{-1} \tag{4.5}$$

$$\sigma^2 = \frac{3l^2(m-1)^2}{(m+2)^2}\left(nlt + c_1\right)^{-2} \tag{4.6}$$

$$A_m = \frac{2(m-1)^2}{(m+2)^2} \tag{4.7}$$

$$q = n - 1 \tag{4.8}$$

$$V = \left(nlt + c_1\right)^{\frac{1}{n}} \tag{4.9}$$

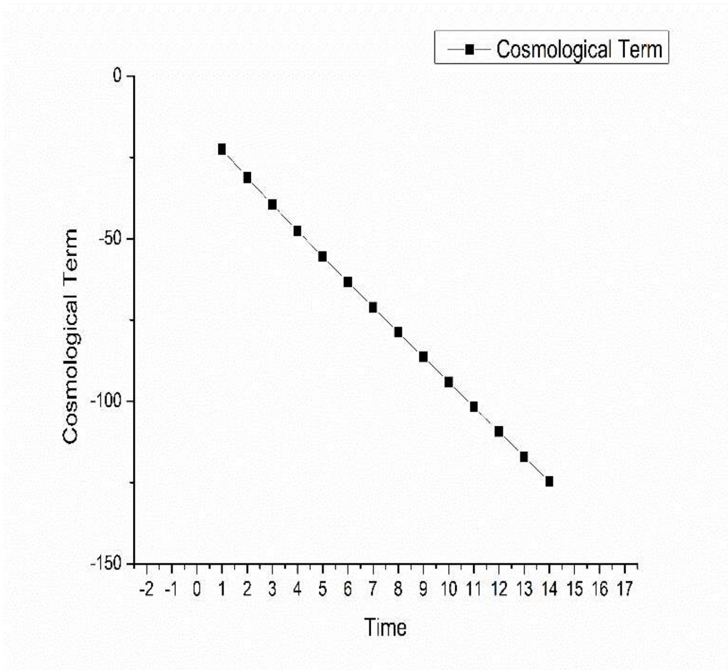

Figure 1: Graph of Cosmological Term and Time

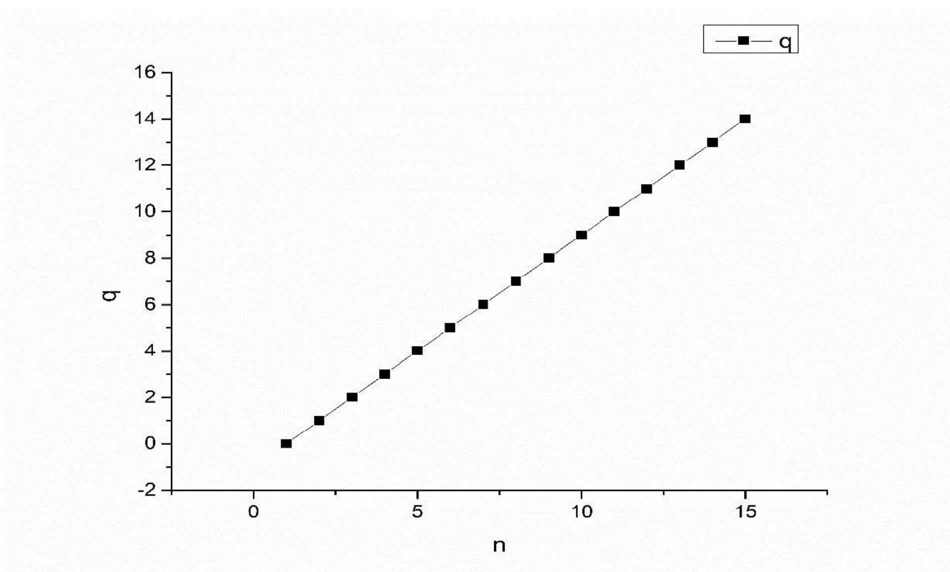

Figure 2: Graph of q and constant n

A maximum value of Λ followed by a decrease into negative values could correspond to different stages of the expansion of the universe. Initially, a maximum might correspond to a phase of rapid acceleration, while the subsequent decrease into negative values might imply a transition to deceleration. If $n < 1$, then q becomes negative, which shows that the universe is expanding at an accelerating rate, and if $n > 1$, q is positive, indicating that slow down the expansion of the universe. Since σ/θ remains a constant, the model generally does not tend to become isotropic [20,37].

The properties of a material can be characterized by various physical parameters, including string tension per unit length (λ), the distribution of mass-energy (ρ), and the measure of internal resistance to compression, known as the bulk viscosity coefficient (ξ), the scalar expansion (φ), the mean Hubble parameter (H), and the shear (σ). All the physical quantities initially attain their maximum values and goes on declining with increment in time.

4.2. Exponential Law Model

The model of exponential law given in equation (3.15) has scale factors A and B which increase exponentially with time and finally become infinite. Subsequently, toward the early phase, the model turns out to be flat, while at a later stage the scale factor of the model diverges to infinity.

Various physical quantities hold significant importance in cosmology. Some of them are the string tension per unit length (λ), the distribution of mass-energy (ρ), and the measure of internal resistance to compression, known as the bulk viscosity coefficient (ξ), the scalar expansion (φ), the anisotropy parameter (A_m), the mean Hubble parameter (H), and the shear (σ). These quantities have been calculated for the exponential law model (3.15) as follows:

$$\rho = \lambda = -C_5 e^{-\frac{6lt(m-1)}{(m+2)}} - \frac{H^2}{\bar{\mu} C_6} e^{-\frac{6lt(m+1)}{(m+2)}} \tag{4.10}$$

$$\varphi = \frac{3l(2m+1)}{m+2} \tag{4.11}$$

$$\xi = 0 \tag{4.12}$$

$$\Lambda = -\frac{C_5}{2} e^{-\frac{6lt(m-1)}{m+2}} - \frac{4\pi H^2}{\bar{\mu} C_4} e^{-\frac{3lt}{m+2}} \tag{4.13}$$

$$H = l \tag{4.14}$$

$$\sigma^2 = \frac{3l^2(m-1)^2}{(m+2)^2} \tag{4.15}$$

$$A_m = \frac{2(m-1)^2}{(m+2)^2} \tag{4.16}$$

$$q = -1 \tag{4.17}$$

$$V = \left(c_2 e^{lt}\right)^2 \tag{4.18}$$

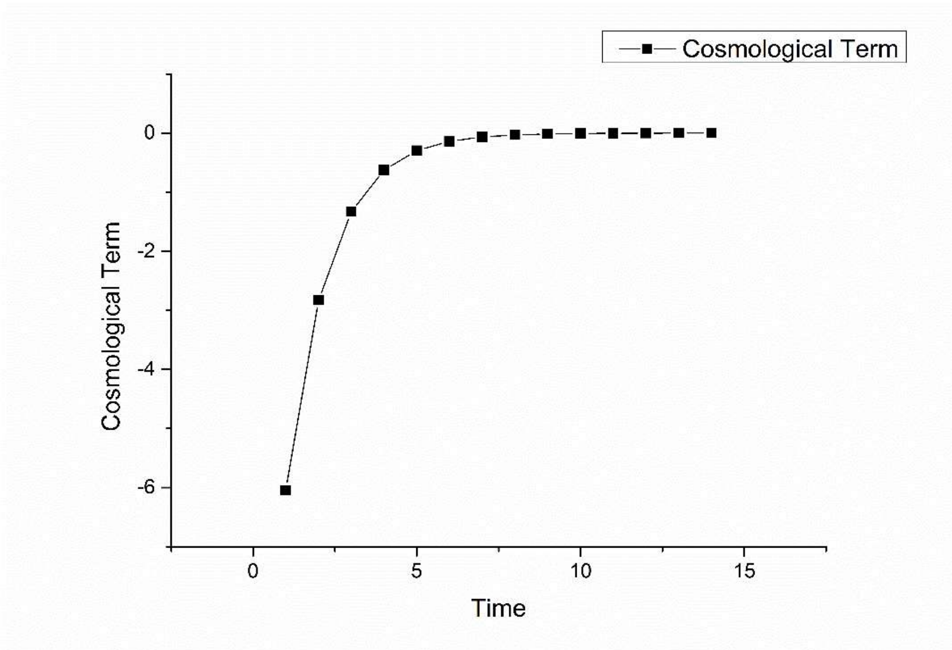

Figure 3: Graph of Cosmological Term And Time

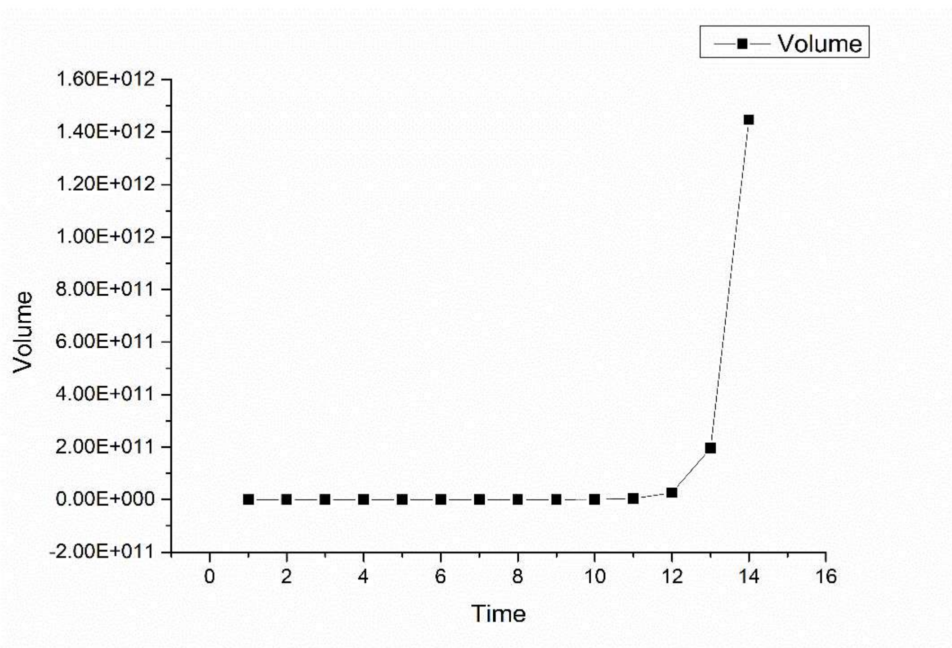

Figure 4: Graph of Volume And Time

The cosmological term Λ initially attains its minimum and goes on increasing and aproaches to zero with increase in time indicate that the presence of alternative theory to the general theory of relativity. If Λ starts negative and approaches zero, it reflects a transitional phase in the universe's expansion. Initially, a negative Λ causes deceleration in the universe's expansion, while its approach to zero leads to a transition to a more standard expansion behavior (acceleration) dominated by other components like dark energy or matter.

At $t = 0$, the model's volume reaches its lowest point and then progressively grows as t increases, ultimately nearing its peak as t approaches infinity. This indicates that the model begins with minimum volume and approaches a maximum value at a later stage. It has been noted that when q is less than zero (i.e., $q = -1$), the universe is at an accelerating phase always. Given that the ratio of σ to φ remains unchanged, the model generally does not exhibit isotropic behavior [37].

We notice that the scalar expansion stays constant while the scale factor rises with increasing t. This cosmic state is consistent with astronomical findings of SNe I_0, Spergel [10], Riess [6,11], and Knop [40]. The physical quantities $\rho = \lambda$ and ξ tend to infinity at $t = -\frac{c_1}{nl}$ and become 0 when $t = \infty$.

5. Component of Dark Energy

5.1. Power Law Model

An alternative model for dark energy (DE) is the Chaplygin gas, a theoretical substance found in some cosmological frameworks. It adheres to the standard Chaplygin gas equation, expressed as

$$p = -\frac{\mu}{\rho}, \tag{5.1}$$

where ρ and p denote the density and pressure of matter, and μ is a positive constant.

The DE is traditionally defined through the equation of state (EOS) parameter ω. For the power-law model, the EOS parameter ω takes the form

$$\omega = \left\{\mu\left[(nlt + c_1)^{\frac{-6(m-1)}{n(m+2)}} + \frac{H^2}{\bar{\mu}}(nlt + c_1)^{\frac{-6(m+1)}{n(m+2)}}\right]\right\}^{-2}. \tag{5.2}$$

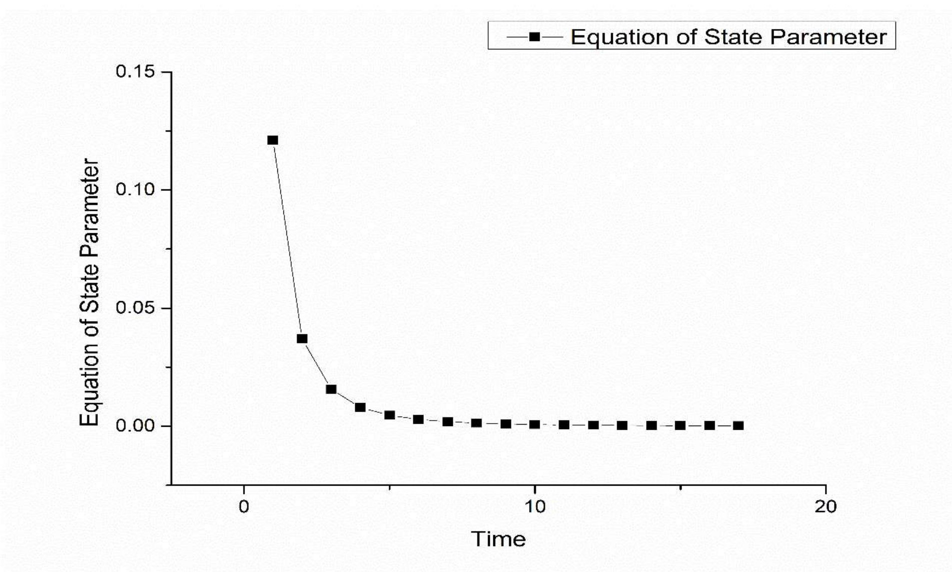

Figure 5: Equation of State Parameter and Time

Initially, ω is at its maximum, which corresponds to an initial condition of the universe. As time progresses, ω decreases, implying changes in the energy density or pressure of this component. When the EoS parameter ω attains a maximum value and then decreases with time, it suggests that the universe's dark energy component is dynamically evolving. This evolution affects the expansion rate and the transition from accelerated to potentially decelerated expansion phases.

During the early evolution of the universe, the EOS parameter (ω) exhibited a positive value, indicating a matter-dominated phase. Over time, this parameter transitioned to a negative value, shaping the current state of the universe. Initially composed of real matter, the universe gradually shifted into a dark-energy-dominated phase, experiencing both acceleration and deceleration throughout its evolution.

5.2. Exponential Law Model

For the exponential law model, ω takes the value

$$\omega = \left\{ \mu \left[c_5 e^{\frac{-6lt(m-1)}{(m+2)}} + \frac{H^2}{\bar{\mu} C_6} e^{-\left(\frac{6lt(m+1)}{(m+2)}\right)} \right] \right\}^{-2}. \tag{5.3}$$

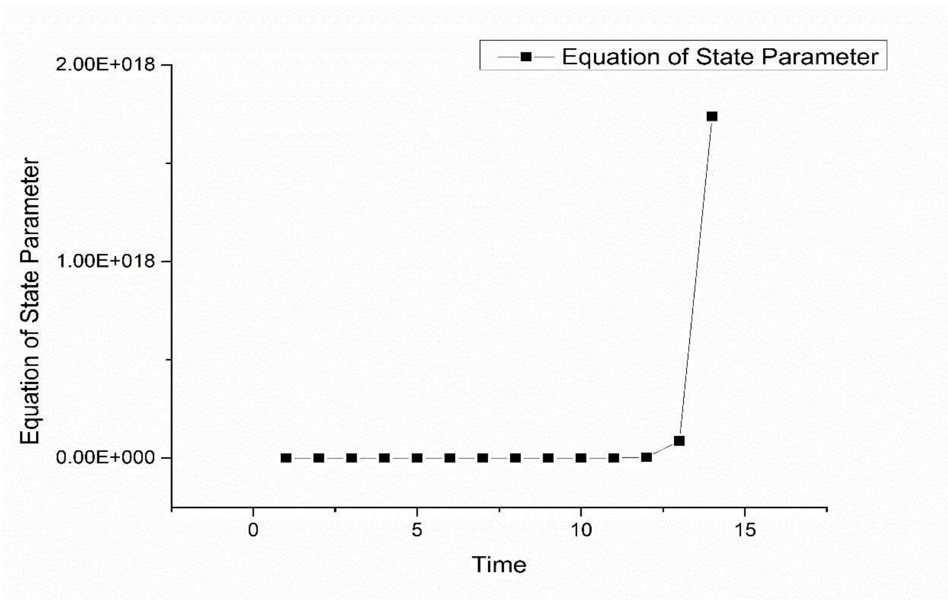

Figure 6: Equation of State Parameter and Time

The EoS parameter $\omega = 0$ corresponds to non-relativistic matter (dust). If ω is at 0 and then increases with time, it suggests that the component initially behaves like matter but transitions to a state with a different pressure-to-density ratio. As ω increases from 0, it moves towards positive values.

If ω starts at 0 and increases, it suggests a shift from a matter-like component to something with positive pressure relative to its energy density. If ω increases towards $\frac{1}{3}$, the component would start behaving more like radiation. If ω increases towards a value greater than $\frac{1}{3}$, it might indicate a transition to a form of dark energy with a higher EoS parameter, such as quintessence models where ω is positive but less than 1.

When the EoS parameter ω attains a minimum value of 0 and then increases with time, it indicates a transition from a matter-like component ($\omega = 0$) to a state with positive pressure. This shift can affect the dynamics of cosmic expansion, potentially leading to changes from deceleration to acceleration in the expansion behavior.

6. Key Findings

6.1. Power Law Model

- The value of Λ indicates that the expansion of the universe transitions from an accelerated phase to a decelerated phase.

- The energy density ρ, the mean Hubble parameter H, the coefficient of bulk viscosity ξ, the shear σ, the string tension density λ, and the scalar expansion φ all the physical quantities initially attains its maximum and goes on diminishing with increment in time.

- The value of q is $n-1$ shows that for $n<1$, q becomes negative, shows that the universe is expanding at an accelerating rate; and for $n>1$, q becomes positive, indicating a slowing down of the universe's expansion.

- If the Hubble parameter H, which measures the cosmos expansion rate, declines over time, it indicates a deceleration in cosmic expansion.

- Since the ratio $\frac{\sigma}{\varphi}$ remains constant, the model generally does not exhibit isotropic behavior.

- We infer that during the initial stages of cosmic evolution, the equation-of-state parameter ω had a positive value. Over time, it transitioned to a negative value. Initially, the universe was dominated by real matter, which later evolved into a dark-energy-dominated phase, exhibiting both accelerating and decelerating behavior.

- At $t=0$, the model's volume reaches its lowest point and then gradually expands as t progresses, eventually nearing its peak. This shows that the model starts with minimal volume and later stabilizes at a higher value.

6.2. Exponential Law Model

- The cosmological term Λ initially attains its minimum and goes on increasing and approaches zero with increase in time, indicating the presence of an alternative theory to the general theory of relativity.

- Initially, the value of Λ is negative, causing deceleration in the universe's expansion. Later, it approaches zero, leading to a transition to a more standard expansion behavior (acceleration) dominated by other components like dark energy or matter.

- A deceleration parameter $q=-1$ indicates that the universe undergoes accelerated expansion always.

- The EoS parameter $\omega=0$ corresponds to non-relativistic matter. As ω increases towards $\frac{1}{3}$, the component would start behaving more like radiation and quintessence models, where ω is positive but less than 1.

- The value of ω increasing with time indicates accelerated expansion of the universe.

- The ratio of σ to φ remains unchanged, implying that the model generally does not exhibit isotropic behavior.

- At $t=0$, the model's volume reaches its lowest point and then steadily grows as t progresses, eventually nearing its peak. This shows that the model starts with a minimal volume and later expands to its maximum value.

7. Conclusion

The comparative analysis of power-law and exponential-law cosmological models reveals that both frameworks describe a universe transitioning from an initial decelerating stage to a later period of accelerated expansion, dominated by dark energy. In the power-law model, key physical quantities such as the mean Hubble parameter, matter-energy density, bulk viscosity, and string tension density initially reach their maximum values and then decrease with time, indicating a transition from acceleration to deceleration or vice versa, depending on the sign of the deceleration parameter (q). The behavior of the equation of state (EoS) parameter supports this transition, shifting from positive (real matter-dominated era) to negative (dark energy-dominated era), suggesting a dynamic cosmic evolution. The model's volume also expands over time, starting from a minimum and stabilizing at a larger value.

Similarly, the exponential-law model describes an initially negative cosmological term that increases with time, aligning with alternative gravity theories and eventually supporting accelerated expansion. Here, the deceleration parameter remains negative, confirming continuous acceleration, and the EoS parameter evolves toward values resembling quintessence or radiation. Despite the evolution, both models retain anisotropic characteristics due to the constant ratio of shear to expansion.

Overall, the study suggests that the universe evolves from a matter-dominated era characterized by deceleration to a dark energy-dominated era marked by accelerated expansion, reduction in physical quantities, and anisotropy.

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