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Some Special Vector Fields in Mixed Quasi-Einstein Weyl Manifolds

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ABSTRACT: In this paper, we introduce the notion of a mixed quasi-Einstein Weyl manifold, which extends the classical concept of quasi-Einstein Weyl manifolds. We provide an explicit example to demonstrate their existence. Moreover, we investigate mixed quasi-Einstein Weyl manifolds that admit certain special vector fields and derive several related structural results.

Keywords: Special vector fields, mixed quasi-Einstein Weyl manifold.

Contents

1. Introduction

A Riemannian manifold M_n $(n > 2)$ with metric g is an Einstein manifold if its Ricci tensor Ric is of the form

$$Ric(X, Y) = \frac{r}{n} g(X, Y) \tag{1.1}$$

where r is the scalar curvature of M_n [1].

A non-Einstein Riemannian manifold M_n $(n > 2)$ is called a quasi-Einstein manifold if its Ricci tensor Ric of type $(0, 2)$ is not identically zero and is of the form

$$Ric(X, Y) = ag(X, Y) + bA(X)A(Y), \tag{1.2}$$

where a, b are functions and A is a non-zero 1-form defined by $A(X) = g(X, U)$ for all vector fields X and a unit vector field U. A is called the associated 1-form and U is called the generator of the manifold [2].

The notion of quasi-Einstein manifolds was generalized in different ways such as generalized quasi-Einstein manifolds ([3],[4], [5]), nearly quasi-Einstein manifolds [6], generalized Einstein manifolds [7], super quasi-Einstein manifolds [8], pseudo quasi-Einstein manifolds [9], extended quasi-Einstein manifolds [10], N(k)-mixed quasi-Einstein manifolds [11], mixed generalized quasi-Einstein manifolds [12], etc.

A non-flat Riemannian manifold M_n $(n > 2)$ is called a mixed quasi-Einstein manifold if its Ricci tensor Ric of type $(0, 2)$ is not identically zero and is of the form

$$Ric(X, Y) = ag(X, Y) + b(A(X)B(Y) + B(X)A(Y)), \tag{1.3}$$

where a, b are certain nonzero functions and A, B are two non-zero 1-forms. The unit vector fields U and V corresponding to 1-forms A and B respectively defined by

$$g(X, U) = A(X), \quad g(X, V) = B(X), \quad g(U, V) = 0.$$

The vector fields U and V are called the generators of the manifold [13]. If $b = 0$, then the manifold becomes an Einstein manifold. If $A = B$, then the manifold reduces to a quasi-Einstein manifold.

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The aim of this paper is to introduce a mixed quasi-Einstein Weyl manifold, extending the concept of a quasi-Einstein Weyl manifold.

A Weyl manifold is a conformal manifold endowed with a torsion-free connection that preserves its conformal structure. It is called an Einstein-Weyl manifold if the symmetric part of the Ricci tensor of the Weyl connection is proportional to the conformal metric. This condition represents a generalization of classical Einstein-Weyl manifolds.

Einstein-Weyl manifolds have been studied by Folland [14], Tod [15], Pedersen and Tod [16], and many others. Quasi-Einstein Weyl manifolds were defined and studied by Gül and Canfes [17].

In the present work, we define mixed quasi-Einstein Weyl manifolds and provide an example of their existence. Then, we examine some specific properties of these manifolds and special vector fields in mixed quasi-Einstein Weyl manifolds.

2. Preliminaries

A differentiable manifold of dimension n having a conformal class C of metrics and a torsion-free connection D preserving the conformal class C is called a Weyl manifold, denoted by $W_n(g,\omega)$, where $g \in C$ and ω is a 1-form satisfying the so-called compatibility condition (see [18,19,20])

$$Dg = 2(g \otimes \omega). \tag{2.1}$$

Under the conformal change

$$\bar{g} = \lambda^2 g, \quad \lambda > 0 \tag{2.2}$$

of the representative metric tensor g, the 1-form ω changes by the law

$$\bar{\omega} = \omega + d \ln \lambda. \tag{2.3}$$

Assume that $M_n(g,\omega)$ is a Weyl manifold of class C^∞, covered by a system of coordinate neighborhoods (U, x^h). Then, Eq. (2.1) can be written in local coordinates by

$$D_k g_{ij} = 2\omega_k g_{ij}. \tag{2.4}$$

We note that, throughout the paper, we will use Einstein summation convention over the repeated indices.

The curvature tensor, the covariant curvature tensor, the Ricci tensor, and the scalar curvature of $M_n(g,\omega)$ are defined, respectively, as follows (see [21]):

$$v^j W^p_{jkl} = (D_k D_l - D_l D_k) v^p, \tag{2.5}$$

$$W_{hjkl} = g_{hp} W^p_{jkl}, \tag{2.6}$$

$$W_{ij} = W^p_{ijp} = g^{hk} W_{hijk}, \tag{2.7}$$

$$s = g^{ij} W_{ij}. \tag{2.8}$$

From (2.5) it follows that

$$W^p_{jkl} = \partial_k \Gamma^p_{jl} - \partial_l \Gamma^p_{jk} + \Gamma^p_{hk}\Gamma^h_{jl} - \Gamma^p_{hl}\Gamma^h_{jk}, \tag{2.9}$$

where $\partial_k = \dfrac{\partial}{\partial x^k}$ and Γ^i_{kl} are the coefficients of the Weyl connection D given by

$$\Gamma^i_{kl} = \left\{ \begin{matrix} i \\ kl \end{matrix} \right\} - g^{im}(g_{mk}\omega_l + g_{ml}\omega_k - g_{kl}\omega_m), \tag{2.10}$$

in which $\left\{ \begin{matrix} i \\ kl \end{matrix} \right\}$ are the coefficients of the Levi-Civita connection.

By straightforward calculations it is easy to see that the antisymmetric part of W_{ij} has the property

$$W_{[ij]} = n \nabla_{[i}\omega_{j]}, \tag{2.11}$$

where brackets indicate antisymmetrization [21].

Definition 2.1 *A tensor field A is called a satellite of g with weight p if it admits a transformation of the form*

$$\bar{A} = \mu^p A, \tag{2.12}$$

under the change (2.2) of the metric tensor g, following [18].

From (2.7), (2.8), (2.9) and (2.10), it is easy to see that the Ricci tensor, the curvature tensor and Weyl connection coefficients are satellites of g with weight 0, and the scalar curvature s is a satellite of g with weight -2.

Definition 2.2 *The prolonged derivative of a satellite A of g with weight p is defined as in [18] by*

$$\dot{\partial}_k A = \partial_k A - p\omega_k A. \tag{2.13}$$

Definition 2.3 *The prolonged covariant derivative of a satellite A of g with weight p is introduced in [18] as*

$$\dot{D}_k A = D_k A - p\omega_k A. \tag{2.14}$$

We note that the prolonged covariant derivative and the prolonged derivative preserve the weights of the tensors.

From (2.2), (2.4) and (2.14) it follows that

$$\dot{D}_k g_{ij} = 0. \tag{2.15}$$

Moreover, since $\dot{\partial}_k g_{ij} = \partial_k g_{ij} - 2\omega_k g_{ij}$, it follows from (2.10) that

$$\Gamma^i_{kl} = \frac{1}{2} g^{im} \left(\dot{\partial}_k g_{lm} + \dot{\partial}_l g_{km} - \dot{\partial}_m g_{kl} \right). \tag{2.16}$$

A satellite of g is said to be prolonged covariantly constant if its prolonged covariant derivative is zero.

3. Mixed Quasi-Einstein Weyl Manifolds

A Weyl manifold $M_n(g,\omega)$ is called a mixed quasi-Einstein Weyl manifold if

$$S_{ij} = a g_{ij} + b(A_i B_j + A_j B_i), \tag{3.1}$$

where a, b are functions of weight -2, S_{ij} is not identically zero and denotes the symmetric part of the Ricci tensor W_{ij} of weight 0, and A_i, B_i are non-zero 1-forms of weight 1 satisfying

$$g^{ij} A_i A_j = 1, \quad g^{ij} B_i B_j = 1, \quad g^{ij} A_i B_j = 0. \tag{3.2}$$

In this case, A_i and B_i are called associated 1-forms, and a, b are called associated scalar functions. If $b = 0$, then $M_n(g,\omega)$ is called Einstein-Weyl manifold. If $A = B$, then $M_n(g,\omega)$ is called quasi-Einstein Weyl manifold [17].

Multiplying (3.1) by g^{ij}, we get

$$s = an, \tag{3.3}$$

which is the scalar curvature of a mixed quasi-Einstein Weyl manifold.

Theorem 3.1 *The vector fields dual to the associated 1-forms for a a mixed quasi-Einstein Weyl manifold $M_n(g,\omega)$ can not be parallel vector fields.*

Proof: Suppose that the vector field A^i dual to the associated 1-form A_i is parallel for a mixed quasi-Einstein Weyl manifold then, we have $\nabla_i A^j = 0$. Using (2.5) and (2.7), we get

$$A^i W_{ij} = 0. \tag{3.4}$$

Using (3.1) and (3.4) we obtain

$$\begin{aligned} 0 &= A^i W_{ij} = A^i (S_{ij} + W_{[ij]}) \\ &= A^i (a g_{ij} + b\,(A_i B_j + A_j B_i)) + A^i W_{[ij]} \\ &= a A_j + b B_j + A^i W_{[ij]}. \end{aligned} \tag{3.5}$$

Contracting (3.5) with A^j and then with B^j, we get

$$a = 0 \tag{3.6}$$

and

$$b = -A^i B^j W_{[ij]}, \tag{3.7}$$

respectively. Using (3.6) and (3.7) in (3.1), we get

$$S_{ij} = -W_{[ij]}, \tag{3.8}$$

which is not possible, since S_{ij} is not identically zero. Hence the vector field A^i dual to the associated 1-form A_i can not be parallel. The proof is similar for the case B^i. □

Theorem 3.2 *If the associated scalar function a of mixed quasi-Einstein Weyl manifold $M_n(g,\omega)$ is prolonged covariantly constant, then $M_n(g,\omega)$ is conformal to a mixed quasi-Einstein manifold.*

Proof: Assume that $M_n(g,\omega)$ is a mixed quasi-Einstein Weyl manifold. The prolonged covariant derivative of (3.3) is

$$\dot{D}_k s = \dot{D}_k (an) = n \dot{D}_k a. \tag{3.9}$$

If the associated scalar function a is prolonged covariantly constant, and since the weight of s is -2, then from (3.9) we find

$$\dot{D}_k s = D_k s + 2\omega_k s = 0. \tag{3.10}$$

Hence, we have

$$\omega_k = -\frac{D_k s}{2s}, \tag{3.11}$$

from which it follows that ω_k is locally a gradient which completes the proof. □

It is easy to see that contracting (3.1) with $A^i A^j$, $B^i B^j$, and $A^i B^j$ yields

$$A^i A^j S_{ij} = a, \tag{3.12}$$

$$B^i B^j S_{ij} = a, \tag{3.13}$$

$$A^i B^j S_{ij} = b. \tag{3.14}$$

Example 3.1 *We consider a $3-$dimensional Weyl manifold $M_3(g,\omega)$ endowed with a metric by $ds^2 = g_{ij}dx^i dx^j = e^{-x^1}(dx^1)^2 + e^{-x^1}(dx^2)^2 + (dx^3)^2$ and a 1-form $\omega = \omega_i dx^i = -dx^1 + \frac{e^{x^1/2}}{\sqrt{2}}dx^3$. The nonzero Weyl connection coefficients are*

$$\begin{aligned} \Gamma^1_{11} &= \frac{1}{2}, \quad \Gamma^1_{13} = -\frac{e^{x^1/2}}{\sqrt{2}}, \quad \Gamma^1_{22} = -\frac{1}{2}, \quad \Gamma^1_{33} = -e^{x^1}, \\ \Gamma^2_{12} &= \frac{1}{2}, \quad \Gamma^2_{23} = -\frac{e^{x^1/2}}{\sqrt{2}}, \\ \Gamma^3_{11} &= \frac{e^{-x^1/2}}{\sqrt{2}}, \quad \Gamma^3_{13} = 1, \quad \Gamma^3_{22} = \frac{e^{-x^1/2}}{\sqrt{2}}, \quad \Gamma^3_{33} = -\frac{e^{x^1/2}}{\sqrt{2}}. \end{aligned} \tag{3.15}$$

A direct computation yields the following nonzero components of the Ricci tensor:

$$W_{11} = 1, \quad W_{13} = \frac{3e^{x^1/2}}{2\sqrt{2}}, \quad W_{22} = 1, \quad W_{33} = e^{x^1}. \tag{3.16}$$

Moreover, the nonzero components of symmetric parts of the Ricci tensor S_{ij} and the scalar curvature s are

$$S_{11} = 1, \quad S_{13} = \frac{3e^{x^1/2}}{4\sqrt{2}}, \quad S_{22} = 1, \quad S_{33} = e^{x^1}, \quad s = 3e^{x^1}. \tag{3.17}$$

If the associated scalar functions are given by

$$a = e^{x^1}, \quad b = \frac{3e^{x^1}}{4\sqrt{2}}, \tag{3.18}$$

and the associated 1-forms are $A = A_i dx^i = e^{-x^1/2}dx^1, B = B_i dx^i = dx^3$ then, (3.1) and (3.2) are satisfied. Therefore, $M_3(g,\omega)$ is a mixed quasi-Einstein Weyl manifold.

4. Some Special Vector Fields in Mixed Quasi-Einstein Weyl Manifolds

In this section, we define some special vector fields on a Weyl manifold and then present the results related to a mixed quasi-Einstein Weyl manifold.

A vector field ϕ in a Riemannian manifold M is called torse-forming if it satisfies the condition

$$\nabla_i \phi^h = \alpha \delta_i^h + \phi^h \gamma_i,$$

where α is a smooth function, ϕ^h and γ_i are the components of the vector field ϕ and 1-form γ, respectively, and δ_i^h is the Kronecker symbol [22]. For details, see [23, p. 168].

If $\alpha = 0$, then the torse-forming vector field is called recurrent vector field, that is, the vector field ϕ satisfies

$$\nabla_i \phi^h = \phi^h \gamma_i.$$

If $\gamma = 0$, then the torse-forming vector field is called concircular vector field, that is, the vector field ϕ satisfies

$$\nabla_i \phi^h = \alpha \delta_i^h.$$

A $\varphi(Ric)$-vector field is a vector field φ on a Riemannian manifold M satisfying

$$\nabla_i \varphi^h = \beta R_i^h, \tag{4.1}$$

where φ^h and R_i^h are the components of the vector field φ and the Ricci tensor of the Riemannian manifold M, respectively, and β is a constant [24]. Equation (4.1) can also be expressed in index-lowered form as

$$\nabla_i \varphi_j = \beta R_{ij},$$

where R_{ij} is the Ricci tensor of the Riemannian manifold. We note that generalized $\varphi(Ric)$-vector fields are also defined by taking β as a function [25]. A comprehensive study of these equations on (pseudo)-Riemannian spaces can be found in [26]. The present work follows the method introduced therein.

Now, we define these vector fields by using prolonged covariant derivative on a Weyl manifold.

A vector field ϕ of weight p in a Weyl manifold $M_n(g,\omega)$ is called generalized torse-forming vector field if it satisfies the condition

$$\dot{D}_i \phi^h = \alpha \delta_i^h + \phi^h \gamma_i, \tag{4.2}$$

where α is a smooth function of weight p, ϕ^h and γ_i are the components of the vector field ϕ and 1-form γ of weights p and 0, respectively.

If $\alpha = 0$, then the generalized torse-forming vector field is called generalized recurrent vector field, that is, the vector field ϕ satisfies

$$\dot{D}_i \phi^h = \phi^h \gamma_i. \tag{4.3}$$

If $\gamma = 0$, then the generalized torse-forming vector field is called generalized concircular vector field, that is, the vector field ϕ satisfies

$$\dot{D}_i \phi^h = \alpha \delta_i^h. \tag{4.4}$$

Since, the Ricci tensor W_{ij} of a Weyl manifold $M_n(g, \omega)$ is not symmetric, we define generalized $\varphi(Ric)$-vector fields by taking symmetric part of the Ricci tensor S_{ij}.

Definition 4.1 *A generalized $\varphi(Ric)$-vector field of weight p is a vector field φ on a Weyl manifold $M_n(g, \omega)$ satisfying*

$$\dot{D}_i \varphi^h = \beta S_i^h, \tag{4.5}$$

where φ^h are the components of the vector field φ, S_i^h is defined $S_i^h = g^{hj} S_{ij}$ with weight -2, and β is a smooth function of weight $p + 2$.

Equation (4.5) can also be written index-lowered form as

$$\dot{D}_i \varphi_j = \beta S_{ij}, \tag{4.6}$$

where $\varphi_j = \varphi^h g_{hj}$, which is of weight $p + 2$.

Theorem 4.1 *The vector fields dual to the associated 1-forms cannot be generalized $A(Ric)$ and $B(Ric)$ vector fields in a mixed quasi-Einstein Weyl manifold.*

Proof: Suppose that the vector fields dual to the associated 1-forms are generalized $A(Ric)$ and $B(Ric)$ vector fields on a mixed quasi-Einstein Weyl manifold. Then, by (4.6), we have

$$\dot{D}_i A_j = \beta S_{ij}, \tag{4.7}$$
$$\dot{D}_i B_j = \breve{\beta} S_{ij}, \tag{4.8}$$

where A_j and B_j are the components of the associated 1-forms of weight 1, β and $\breve{\beta}$ are nonzero scalar functions of weight 1, and S_{ij} denotes the components of the symmetric part of the Ricci tensor of the Weyl manifold, which has weight 0.

Multiplying (4.7) and (4.8) by A^j and B^j, respectively, we get

$$0 = \beta A^j S_{ij}, \tag{4.9}$$
$$0 = \breve{\beta} B^j S_{ij}, \tag{4.10}$$

where we used the facts $A^j \dot{D}_i A_j = 0$ and $B^j \dot{D}_i B_j = 0$.

Multiplying (4.9) and (4.10) again by A^i and using Equations (3.12) and (3.13), we get

$$0 = a\beta, \tag{4.11}$$
$$0 = b\breve{\beta}. \tag{4.12}$$

From (4.11) and (4.12), since β and $\breve{\beta}$ are nonzero scalar functions, we obtain $a = b = 0$ which is not possible since we assume that S_{ij} is not identically zero. Hence the theorem is proved. □

Theorem 4.2 *If $M_n(g, \omega)$ is a mixed quasi-Einstein Weyl manifold, then both of the vector fields dual to the associated 1-forms cannot be generalized torse-forming vector fields.*

Proof: Assume that the vector fields dual to the associated 1-forms are both generalized torse-forming vector fields on a mixed quasi-Einstein Weyl manifold. Then, from (4.2) we have:

$$\dot{D}_i A^h = \alpha \delta_i^h + A^h \gamma_i, \tag{4.13}$$
$$\dot{D}_i B^h = \breve{\alpha} \delta_i^h + B^h \breve{\gamma}_i. \tag{4.14}$$

Multiplying (4.13) and (4.14) by g_{hj}, we obtain:

$$\dot{D}_i A_j = \alpha g_{ij} + A_j \gamma_i, \tag{4.15}$$
$$\dot{D}_i B_j = \breve{\alpha} g_{ij} + B_j \breve{\gamma}_i, \tag{4.16}$$

where A_j and B_j are the components of the associated 1-forms of weight 1, α and $\breve{\alpha}$ are nonzero scalar functions of weight -1, γ_i and $\breve{\gamma}_i$ are 1-forms of weight 0. Contracting (4.15) and (4.16) with A^j and B^j, respectively, we get

$$0 = \alpha A_i + \gamma_i, \tag{4.17}$$
$$0 = \breve{\alpha} B_i + \breve{\gamma}_i, \tag{4.18}$$

since $A^j \dot{D}_i A_j = B^j \dot{D}_i B_j = 0$ and $A^j A_j = B^j B_j = 1$. Hence we obtain

$$\gamma_i = -\alpha A_i, \tag{4.19}$$
$$\breve{\gamma}_i = -\breve{\alpha} B_i. \tag{4.20}$$

Using Eqs. (4.19) and (4.20) in (4.15) and (4.16), we get

$$\dot{D}_i A_j = \alpha \left(g_{ij} - A_i A_j \right), \tag{4.21}$$
$$\dot{D}_i B_j = \breve{\alpha} \left(g_{ij} - B_i B_j \right). \tag{4.22}$$

From (3.2), we have $A^i B_i = 0$. Taking prolonged covariant derivative this equation and using (4.21) and (4.22), we obtain

$$0 = \dot{D}_k \left(A^i B_i \right) = A^i \dot{D}_k B_i + B_i \dot{D}_k A^i = \breve{\alpha} A_k + \alpha B_k. \tag{4.23}$$

Contracting (4.23) with A^k and B^k, respectively, we get $\alpha = \breve{\alpha} = 0$, which contradicts the assumption that α and $\breve{\alpha}$ are nonzero scalar functions. Therefore, both of the vector fields dual to the associated 1-forms cannot be generalized torse-forming vector fields in a mixed quasi-Einstein Weyl manifold. □

Theorem 4.3 *If the vector fields dual to the associated 1-forms are generalized concircular vector fields on a mixed quasi-Einstein Weyl manifold, then the associated 1-forms are prolonged covariantly constant.*

Proof: Assume that the vector fields dual to the associated 1-forms are generalized concircular vector fields on a mixed quasi-Einstein Weyl manifold. Then, by (4.4), we have

$$\dot{D}_i A^h = \alpha \delta_i^h, \tag{4.24}$$
$$\dot{D}_i B^h = \breve{\alpha} \delta_i^h, \tag{4.25}$$

from which it follows that

$$\dot{D}_i A_j = \alpha g_{ij}, \tag{4.26}$$
$$\dot{D}_i B_j = \breve{\alpha} g_{ij}, \tag{4.27}$$

where A_j and B_j are the components of the associated 1-forms of weight 1, α and $\breve{\alpha}$ are scalar functions of weight -1. Multiplying (4.26) and (4.27) by A^j and B^j, respectively, we get

$$0 = \alpha A_i, \tag{4.28}$$
$$0 = \breve{\alpha} B_i, \tag{4.29}$$

since $A^j \dot{D}_i A_j = 0$ and $B^j \dot{D}_i B_j = 0$. Multiplying (4.28) and (4.29) by A^j and B^j, respectively, we obtain $\alpha = \breve{\alpha} = 0$. Therefore, by (4.26) and (4.27), the result follows. □

Theorem 4.4 *If the vector fields dual to the associated 1-forms are generalized recurrent vector fields on a mixed quasi-Einstein Weyl manifold, then the associated 1-forms are prolonged covariantly constant.*

Proof: Assume that the vector fields dual to the associated 1-forms are generalized recurrent vector fields on a mixed quasi-Einstein Weyl manifold. Then, by (4.3), we have

$$\dot{D}_i A^h = A^h \gamma_i, \tag{4.30}$$
$$\dot{D}_i B^h = B^h \breve{\gamma}_i, \tag{4.31}$$

from which it follows that

$$\dot{D}_i A_j = A_j \gamma_i, \tag{4.32}$$
$$\dot{D}_i B_j = B_j \breve{\gamma}_i, \tag{4.33}$$

where A_j and B_j are the components of the associated 1-forms of weight 1, γ_i and $\breve{\gamma}_i$ are components of the 1-forms γ and $\breve{\gamma}$ of weight 0.

Multiplying (4.32) and (4.33) by A^j and B^j, respectively, we get $\gamma_i = \breve{\gamma}_i = 0$, since $A^j \dot{D}_i A_j = 0$ and $B^j \dot{D}_i B_j = 0$. Hence, the theorem is proved. □

Theorem 4.5 *If a mixed quasi-Einstein Weyl manifold satisfies the Codazzi type condition on the symmetric part of the Ricci tensor and one of the vector fields dual to the associated 1-forms is concircular, then the manifold reduces to an Einstein–Weyl manifold.*

Proof: If a mixed quasi-Einstein Weyl manifold satisfies the Codazzi type condition on the symmetric part of the Ricci tensor, then we have

$$\dot{\nabla}_k S_{ij} = \dot{\nabla}_i S_{kj}. \tag{4.34}$$

Using (3.1), we get

$$\begin{aligned} \dot{\nabla}_k S_{ij} &= \dot{\nabla}_k (a g_{ij} + b(A_i B_j + A_j B_i)) \\ &= g_{ij} \dot{\nabla}_k a + (A_i B_j + A_j B_i) \dot{\nabla}_k b + b \dot{\nabla}_k (A_i B_j + A_j B_i). \end{aligned} \tag{4.35}$$

Multiplying (4.35) by $A^i B^j$, we obtain

$$A^i B^j \dot{\nabla}_k S_{ij} = \dot{\nabla}_k b. \tag{4.36}$$

Similarly, we have

$$\begin{aligned} \dot{\nabla}_i S_{kj} &= \dot{\nabla}_i (a g_{kj} + b(A_k B_j + A_j B_k)) \\ &= g_{kj} \dot{\nabla}_i a + (A_k B_j + A_j B_k) \dot{\nabla}_i b + b \dot{\nabla}_i (A_k B_j + A_j B_k) \end{aligned} \tag{4.37}$$

and multiplying (4.37) by $A^i B^j$, we obtain

$$A^i B^j \dot{\nabla}_i S_{kj} = B_k A^i \dot{\nabla}_i a + A_k A^i \dot{\nabla}_i b + b(A^i \dot{\nabla}_i A_k + B_k A^i B^j \dot{\nabla}_i A_j). \tag{4.38}$$

Using (4.34), (4.36) and (4.38), we obtain

$$\dot{\nabla}_k b = B_k A^i \dot{\nabla}_i a + A_k A^i \dot{\nabla}_i b + b(A^i \dot{\nabla}_i A_k + B_k A^i B^j \dot{\nabla}_i A_j). \tag{4.39}$$

Suppose that the vector field dual to the associated the 1-form A_i is generalized concircular. Then, using (4.3), we have

$$\dot{\nabla}_i A_j = \gamma_i A_j, \tag{4.40}$$

where γ_i is the component of a 1-form γ of weight 0. Using (4.40) in (4.39) and then contracting (4.39) by A^k, we get

$$A^k \dot{\nabla}_k b = 0 + A^i \dot{\nabla}_i b + b(A^i \gamma_i + 0), \tag{4.41}$$

from which we conclude that $b = 0$. Hence the manifold reduces to an Einstein-Weyl manifold.

Similarly, multipliying (4.35) by $A^j B^i$ and contracting the result by B^k under the assumption the vector field dual to the associated the 1-form B_i is generalized concircular, we obtain $b = 0$, which proves the theorem. □

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