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Integral Kannappan-Cosine Addition Law on Semigroups

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ABSTRACT: Let S be a semigroup, $\sigma : S \longrightarrow S$ be an involutive automorphism, μ be a complex measure that is a linear combination of Dirac measures and $\alpha \in \mathbb{C}$. We determine the complex-valued solutions of the following integral Kannappan-cosine addition law with an additional term

$$\int_S g(x\sigma(y)t)d\mu(t) = g(x)g(y) - f(x)f(y) + \alpha \int_S f(x\sigma(y)t)d\mu(t),\ x, y \in S.$$

As application we solve two functional equations that have not been studied until now. The continuous solutions on topological semigroups are found.

Keywords: Semigroup, exponential, integral equation, cosine addition law.

Contents

1. Introduction

In [15] Stetkær introduced and solved the cosine addition law with an additional term

$$g(xy) = g(x)g(y) - f(x)f(y) + \alpha f(xy),\quad x, y \in S. \tag{1.1}$$

He expressed the solutions of (1.1) in terms of exponentials and the solutions of the following special sine addition law

$$\phi_\chi(xy) = \phi_\chi(x)\chi(y) + \chi(x)\phi_\chi(y),\quad x, y \in S, \tag{1.2}$$

where $\chi : S \longrightarrow \mathbb{C}$ is an exponential. The work in [15] about (1.1) was generalized by Asserar and Elqorachi [4], by solving the functional equation

$$g(x\sigma(y)) = g(x)g(y) - f(x)f(y) + \alpha f(x\sigma(y)),\quad x, y \in S. \tag{1.3}$$

The special case of (1.3) in which $\alpha = 0$ and $f = ih$ is the cosine subtraction law

$$g(x\sigma(y)) = g(x)g(y) + h(x)h(y),\quad x, y \in S, \tag{1.4}$$

of wich the most current results about on semigroups were given in [5, Theorem 3.3] and [8, Theorem 4.2]. For additional discussions about (1.4) see [1, Theorem 4.3], [8, Theorem 4.1], [14, Theorem 4.16] and their references.

In [10, Theorem 3.3], Elqorachi and Redouani determined the continuous and bounded solutions of the functional equation

$$\int_G f(xt\sigma(y))d\mu(t) = g(x)g(y) + f(x)f(y),\quad x, y \in G, \tag{1.5}$$

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where G is a locally compact group. They expressed the solutions by means of μ-spherical functions: $\int_G \psi(xty)d\mu(t) = \psi(x)\psi(y)$ and solutions of the functional equation $\int_G f(xty)d\mu(t) = f(x)\psi(y) + \psi(x)f(y),\ \ x, y \in G$.

In [13, Theorem 3.1], Kabbaj et al. determined the solutions of the integral cosine addition law

$$\int_G f(xyt)d\mu(t) = g(x)g(y) - f(x)f(y),\ \ x, y \in G, \tag{1.6}$$

where G is a locally compact Hausdorff group.

In [2] Ajebbar et al. solved the integral Kannappan-sine addition and subtraction laws

$$\int_S f(x\sigma(y)t)d\mu(t) = f(x)g(y) + f(y)g(x),\ \ x, y \in S, \tag{1.7}$$

$$\int_S f(x\sigma(y)t)d\mu(t) = f(x)g(y) - f(y)g(x),\ \ x, y \in S, \tag{1.8}$$

where S is a semigroup. They expressed the solutions in terms of exponentials, the solution of (1.2) and the solutions of the following special integral Kannappan-sine addition law

$$\int_S f(x\sigma(y)t)d\mu(t) = f(x)\chi(y)\int_S \chi(t)d\mu(t) + f(y)\chi(x)\int_S \chi(t)d\mu(t),\ x, y \in S, \tag{1.9}$$

where $\chi : S \to \mathbb{C}$ is an exponential such that $\int_S \chi(t)d\mu(t) \neq 0$.
For further details we refer to [3] and [9].

The aim of this paper is to solve the following integral Kannappan-cosine addition law with an additional term

$$\int_S g(x\sigma(y)t)d\mu(t) = g(x)g(y) - f(x)f(y) + \alpha\int_S f(x\sigma(y)t)d\mu(t),\ \ x, y \in S, \tag{1.10}$$

where S is a semigroup, for unknown functions $f, g : S \longrightarrow \mathbb{C}$.

For $\alpha = 0$, $\sigma = Id$ and $\mu = \delta_{z_0}$ (1.10) reduces to the Kannappan-cosine functional equation

$$g(xyz_0) = g(x)g(y) - f(x)f(y),\ \ x, y \in S, \tag{1.11}$$

which has been solved recently by Jafar et al. [11, Theorem 4.1].

For $\alpha = 0$ and $f = 0$ we get the integral Kannappan-Cauchy multiplicative equation

$$\int_S g(x\sigma(y)t)d\mu(t) = g(x)g(y),\ \ x, y \in S, \tag{1.12}$$

which has been solved in [2, Proposition 3.1].

To solve the functional equation (1.10), we relate it to the functional equations (1.3), (1.4) and (1.12). We express the solutions of (1.10) in terms of exponentials, the solution of (1.2) and the solutions of (1.9).

As an application of our main result we solve the integral Kannappan-cosine functional equation

$$\int_S g(x\sigma(y)t)d\mu(t) = g(x)g(y) - f(x)f(y)\ \ x, y \in S, \tag{1.13}$$

and the following Kannappan-cosine addition law with an additional term

$$g(x\sigma(y)z_0) = g(x)g(y) - f(x)f(y) + \alpha f(x\sigma(y)z_0),\ x, y \in S. \tag{1.14}$$

2. Setup, notation and terminology

Throughout this paper S is a semigroup (i.e., a set with an associative composition rule), z_0 is a fixed element in S, $\alpha \in \mathbb{C}$, δ_{z_0} is a Dirac measure concentrated at z_0, μ is complex measure that is a linear combinations of Dirac measures and $\sigma : S \longrightarrow S$ is an involutive automorphism. That is σ is involutive means $\sigma \circ \sigma(x) = x$ for all $x \in S$.

A map $A : S \longrightarrow \mathbb{C}$ is said to be additive if $A(xy) = A(x) + A(y)$ for all $x, y \in S$, and a function $\chi : S \longrightarrow \mathbb{C}$ is multiplicative if $\chi(xy) = \chi(x)\chi(y)$ for all $x, y \in S$. If χ is multiplicative and $\chi \neq 0$ then we call χ an exponential. For an exponential χ we define the null space I_χ by $I_\chi := \{x \in S \mid \chi(x) = 0\}$. Then I_χ is either empty or a proper subset of S and I_χ is a two sided ideal in S if not empty and $S \setminus I_\chi$ is a subsemigroup of S.

If X is a topological space we denote by $C(X)$ the algebra of continuous functions from X to the field of complex numbers $\mathbb{C}$. Let $\mathbb{C}^* := \mathbb{C} \setminus \{0\}$.

The following lemma will be used in the proof of our main result.

Lemma 2.1 *Let $n \in \mathbb{N}$, and let $\chi_1, \chi_2, \cdots, \chi_n : S \longrightarrow \mathbb{C}$ be different exponentials. Then*
(a) $\{\chi_1, \chi_2, \cdots, \chi_n\}$ is linearly independent.
(b) If $\chi : S \longrightarrow \mathbb{C}$ is an exponential and ϕ_χ is a solution of (1.2), then the set $\{\phi_\chi, \chi\}$ is linearly independent.
(c) Let $a_1, a_2, ..., a_n \in \mathbb{C}$. If S is a topological semigroup and the function $a_1\chi_1 + a_2\chi_2 + \cdots + a_n\chi_n$ is continuous on S then each of the functions $a_1\chi_1, a_2\chi_2, \cdots, a_n\chi_n$ is continuous.

Proof: (a) and (c): See [14, Theorem 3.18]. (b): See [7, Lemma 5.1]. □

For convenience we introduce the following notations:
Let $\phi_\chi : S \longrightarrow \mathbb{C}$ denote a function of the form of [7, Theorem 3.1 (B)]. Notice that the function ϕ_χ defined above is a solution of (1.2).
Let $\Phi_\chi : S \longrightarrow \mathbb{C}$ denote the solution of the special case of the integral Kannappan-sine addition law (1.9), where $\chi : S \longrightarrow \mathbb{C}$ is an exponential such that $\int_S \chi(t)d\mu(t) \neq 0$.

3. Preparatory work

This section is devoted to prove some useful results to solve the functional equation (1.10). Throughout this paper, for a solution $f, g : S \longrightarrow \mathbb{C}$ of (1.10) we define $h := f - \alpha g$.

Lemma 3.1 *Let $f, g : S \longrightarrow \mathbb{C}$ be a solution of (1.10). Then we have the following.*
(a) If $\int_S h(t)d\mu(t) = 0$ then we have for all $x, y \in S$

$$g(x\sigma(y)) \int_S \int_S g(\sigma(t)s)d\mu(s)d\mu(t) \tag{3.1}$$

$$= [g(x)g(y) - f(x)f(y)] \int_S g(t)d\mu(t) + f(x\sigma(y)) \int_S \int_S f(\sigma(t)s)d\mu(s)d\mu(t).$$

(b) If $\int_S h(t)d\mu(t) \neq 0$ then there exists a constant $\omega \in \mathbb{C}$ such that for all $x, y \in S$ we have

$$\int_S h(x\sigma(y)t)d\mu(t) = g(x)h(y) + g(y)h(x) + [(1 - \alpha^2)\omega + \alpha]h(x)h(y). \tag{3.2}$$

Proof: Let $x, y, z, s \in S$ be arbitrary. (a) Assume that $\int_S h(t)d\mu(t) = 0$. Making the substitutions $(x\sigma(yz), s)$ and $(x\sigma(y), zs)$ in (1.10) and then integrating with respect to s we get receptively

$$\int_S \int_S g(x\sigma(yz)\sigma(s)t)d\mu(t)d\mu(s) = g(x\sigma(yz)) \int_S g(s)d\mu(s)$$

$$-f(x\sigma(yz)) \int_S f(s)d\mu(s) + \alpha \int_S \int_S f(x\sigma(yz)\sigma(s)t)d\mu(t)d\mu(s),$$

and

$$\int_S\int_S g(x\sigma(y)\sigma(zs)t)d\mu(t)d\mu(s) = g(x\sigma(y))\int_S g(zs)d\mu(s)$$
$$-f(x\sigma(y))\int_S f(zs)d\mu(s) + \alpha\int_S\int_S f(x\sigma(y)\sigma(zs)t)d\mu(t)d\mu(s).$$

Since $g(x\sigma(y)\sigma(zs)t) = g(x\sigma(yz)\sigma(s)t)$ we get from the previous identities that

$$g(x\sigma(yz))\int_S g(s)d\mu(s) - f(x\sigma(yz))\int_S f(s)d\mu(s)$$
$$= g(x\sigma(y))\int_S g(zs)d\mu(s) - f(x\sigma(y))\int_S f(zs)d\mu(s).$$

Now, by putting $z = \sigma(t)$, integrating the result obtained with respect to t, using (1.10) and taking into account that $\int_S h(s)d\mu(s) = 0$, we obtain

$$g(x\sigma(y))\int_S\int_S g(\sigma(t)s)d\mu(s)d\mu(t) - f(x\sigma(y))\int_S\int_S f(\sigma(t)s)d\mu(s)d\mu(t)$$

$$\begin{aligned}
&= \int_S g(s)d\mu(s)\int_S g(x\sigma(y)t)d\mu(t) - \int_S f(s)d\mu(s)\int_S f(x\sigma(y)t)d\mu(t)\\
&= \left[g(x)g(y) - f(x)f(y) + \alpha\int_S f(x\sigma(y)t)d\mu(t)\right]\int_S g(s)d\mu(s)\\
&- \int_S f(x\sigma(y)t)d\mu(t)\int_S f(s)d\mu(s)\\
&= [g(x)g(y) - f(x)f(y)]\int_S g(t)d\mu(t)\\
&- \int_S h(s)d\mu(s)\int_S f(x\sigma(y)t)d\mu(t)\\
&= [g(x)g(y) - f(x)f(y)]\int_S g(t)d\mu(t),
\end{aligned}$$

which proves (3.1).
(b) Assume that $\int_S h(t)d\mu(t) \neq 0$. By applying (1.10) to the pairs (x, yzs) and $(x\sigma(yz), s)$ we obtain

$$\int_S\int_S g(x\sigma(yzs)t)d\mu(t)d\mu(s) = g(x)\int_S g(yzs)d\mu(s) - f(x)\int_S f(yzs)d\mu(s)$$
$$+\alpha\int_S\int_S f(x\sigma(yzs)t)d\mu(t)d\mu(s)$$

and

$$\int_S\int_S g(x\sigma(yz)\sigma(s)t)d\mu(t)d\mu(s) = g(x\sigma(yz))\int_S g(t)d\mu(t)$$
$$- f(x\sigma(yz))\int_S f(s)d\mu(s) + \alpha\int_S\int_S f(x\sigma(yz)\sigma(s)t)d\mu(t)d\mu(s).$$

So that

$$g(x)\int_S g(yzs)d\mu(s) - f(x)\int_S f(yzs)d\mu(s)$$
$$= g(x\sigma(yz))\int_S g(t)d\mu(t) - f(x\sigma(yz))\int_S f(t)d\mu(t).$$

Setting $z=\sigma(t)$ in the last identity and then integrating the result obtained with respect to t we get, by using (1.10), that

$$\begin{aligned}
&g(x)g(y)\int_S g(t)d\mu(t) - g(x)f(y)\int_S f(t)d\mu(t) + \alpha g(x)\int_S f(y\sigma(t)s)d\mu(s)d\mu(t)\\
&\quad - f(x)\int_S\int_S f(y\sigma(t)s)d\mu(s)d\mu(t)\\
&= g(x)g(y)\int_S g(t)d\mu(t) - f(x)f(y)\int_S g(t)d\mu(t)\\
&\quad + \alpha\int_S g(t)d\mu(t)\int_S f(x\sigma(y)t)d\mu(t) - \int_S f(x\sigma(y)t)d\mu(t)\int_S f(t)d\mu(t),
\end{aligned}$$

from which we derive, by using $h=f-\alpha g$, that

$$\begin{aligned}
&\int_S h(t)d\mu(t)\left[\int_S f(x\sigma(y)t)d\mu(t) - g(x)f(y)\right] \qquad (3.3)\\
&= h(x)\left[\int_S\int_S f(y\sigma(t)s)d\mu(s)d\mu(t) - f(y)\int_S g(t)d\mu(t)\right].
\end{aligned}$$

Since $\int_S h(t)d\mu(t)\neq 0$, we infer from the last identity that

$$\int_S f(x\sigma(y)t)d\mu(t) = g(x)f(y) + h(x)\delta(y), \qquad (3.4)$$

where

$$\delta(y) := \frac{\int_S\int_S f(y\sigma(t)s)d\mu(s)d\mu(t) - f(y)\int_S g(t)d\mu(t)}{\int_S h(t)d\mu(t)}.$$

Substituting (3.4) in (3.3) and dividing the result obtained by $\int_S h(t)d\mu(t)\neq 0$, and using that $h=f-\alpha g$, we obtain

$$\begin{aligned}
\delta(y)\int_S h(t)d\mu(t) &= g(y)\int_S f(t)d\mu(t) + h(y)\int_S \delta(t)d\mu(t) - f(y)\int_S g(t)d\mu(t)\\
&= g(y)\int_S h(t)d\mu(t) + h(y)\left[\int_S \delta(t)d\mu(t) - \int_S g(t)d\mu(t)\right].
\end{aligned}$$

Thus, we conclude that

$$\delta(y) = g(y) + \omega h(y), \qquad (3.5)$$

where

$$\omega := \frac{\int_S \delta(t)d\mu(t) - \int_S g(t)d\mu(t)}{\int_S h(t)d\mu(t)}. \qquad (3.6)$$

From (3.5) and (3.4) we get that

$$\int_S f(x\sigma(y)t)d\mu(t) = g(x)f(y) + [g(y)+\omega h(y)]h(x). \qquad (3.7)$$

Next, using (3.7) we get

$$\begin{aligned}
\int_S g(x\sigma(y)t)d\mu(t) &= g(x)g(y) - f(x)f(y) + \alpha\int_S f(x\sigma(y)t)d\mu(t)\\
&= g(x)g(y) - f(x)f(y) + \alpha[g(x)f(y) + (g(y)+\omega h(y))h(x)] \qquad (3.8)\\
&= g(x)g(y) + (\alpha\omega - 1)h(x)h(y).
\end{aligned}$$

So, from this and (3.7) we derive that

$$\begin{aligned}\int_S h(x\sigma(y)t)d\mu(t) &= \int_S f(x\sigma(y)t)d\mu(t) - \alpha\int_S g(x\sigma(y)t)d\mu(t)\\ &= g(x)f(y) + [g(y+\omega h(y)]h(x) - \alpha[g(x)g(y) + (\alpha\omega - 1)h(x)h(y)]\\ &= g(x)h(y) + g(y)h(x) + [(1-\alpha^2)\omega + \alpha]h(x)h(y).\end{aligned}$$

This proves (3.2). □

Lemma 3.2 *Let $f, g : S \longrightarrow \mathbb{C}$ be a solution of (1.10) such that $\{f, g\}$ is linearly independent and $\int_S h(t)d\mu(t) = 0$. Then we have the following:*
(a)

$$\int_S g(t)d\mu(t) \neq 0.$$

(b)

$$\int_S\int_S g(\sigma(t)s)d\mu(t)d\mu(s) = 0 \implies \int_S\int_S f(\sigma(t)s)d\mu(t)d\mu(s) \neq 0.$$

Proof: Let $x, y, s \in S$ be arbitrary. (a) Assume that $\int_S g(t)d\mu(t) = 0$. By applying (1.10) to the pairs $(x\sigma(y), s)$ and (x, ys), and integrating with respect to s we obtain

$$\begin{aligned}&\int_S\int_S g(x\sigma(ys)t)d\mu(t)d\mu(s)\\ &= g(x\sigma(y))\int_S g(s)d\mu(s) - f(x\sigma(y))\int_S f(s)d\mu(s)\\ &\quad + \alpha\int_S\int_S f(x\sigma(ys)t)d\mu(t)d\mu(s)\\ &= g(x)\int_S g(ys)d\mu(s) - f(x)\int_S f(ys)d\mu(s) + \alpha\int_S\int_S f(x\sigma(ys)t)d\mu(t)d\mu(s),\end{aligned}$$

from which we derive that

$$g(x)\int_S g(ys)d\mu(s) - f(x)\int_S f(ys)d\mu(s) = 0, \tag{3.9}$$

because $\int_S f(s)d\mu(s) = \int_S h(s)d\mu(s) + \alpha\int_S g(s)d\mu(s) = 0$. Since $\{f, g\}$ is independent we get from (3.9) that $\int_S g(ys)d\mu(s) = \int_S f(ys)d\mu(s) = 0$ for all $y \in S$. Then by using this and (1.10) we obtain $0 = \int_S g(x\sigma(y)t)d\mu(t) - \alpha\int_S f(x\sigma(y)t)d\mu(t) = g(x)g(y) - f(x)f(y)$, which contradicts that $\{f, g\}$ is linearly independent. Therefore $\int_S\int_S f(\sigma(t)s)d\mu(t)d\mu(s) \neq 0$. □

4. Main result

Now we are ready to describe the solutions of the functional equation (1.10) on semigroups. The following theorem is the aim of this paper.

Theorem 4.1 *The solutions $f, g : S \longrightarrow \mathbb{C}$ of the functional equation (1.10) can be listed as follows:*
(1) $\alpha = \pm 1$, $f = \pm g$ and g is an arbitrary non-zero function.
(2) $\alpha \neq \pm 1$, $f = \pm g$ and g is an arbitrary non-zero function such that $\int_S g(xyt)d\mu(t) = 0$ for all $x, y \in S$.
(3) There exist constants $q, \delta \in \mathbb{C}$ and an exponential $\chi = \chi \circ \sigma$ with $\delta := \pm\sqrt{1 + q^2 - \alpha^2}$ and $\int_S \chi(t)d\mu(t) \neq 0$ such that

$$f = \frac{q+\alpha}{2}\chi\int_S \chi(t)d\mu(t) \quad \text{and} \quad g = \frac{1+\delta}{2}\chi\int_S \chi(t)d\mu(t).$$

(4) $\alpha \notin \{-ic, ic^{-1}\}$*, and there exist constants* $\beta \in \mathbb{C}^*, c \in \mathbb{C}^* \setminus \{\pm i\}$ *and two different exponentials* $\chi_1 = \chi_1 \circ \sigma$ *and* $\chi_2 = \chi_2 \circ \sigma$ *with* $\int_S \chi_1(t)d\mu(t) = \dfrac{1}{\beta(\alpha + ic)}$ *and* $\int_S \chi_2(t)d\mu(t) = \dfrac{1}{\beta(\alpha - ic^{-1})}$ *such that*

$$f = \frac{c^{-1}\chi_1 + c\chi_2}{\beta(c^{-1} + c)} \quad and \quad g = \frac{\chi_1 - \chi_2}{i\beta(c^{-1} + c)}.$$

(5) There exist constants $\lambda, \delta \in \mathbb{C}$, $q \in \mathbb{C} \setminus \{\lambda\delta\}$, $\gamma \in \mathbb{C} \setminus \{0\}$, *with* $\delta = \pm\sqrt{1 + q^2 - \lambda^2}$, $(1 + \delta - \alpha(\lambda + q))(1 - \delta - \alpha(\lambda - q)) \neq 0$, *and two different exponentials* $\chi_1 = \chi_1 \circ \sigma$ *and* $\chi_2 = \chi_2 \circ \sigma$ *satisfying* $\int_S \chi_1(t)d\mu(t) = \dfrac{1 + \delta - \lambda(\lambda + q)}{\gamma(1 + \delta - \alpha(\lambda + q))}$ *and* $\int_S \chi_2(t)d\mu(t) = \dfrac{1 - \delta - \lambda(\lambda - q)}{\gamma(1 - \delta - \alpha(\lambda - q))}$ *such that*

$$f = \lambda\frac{\chi_1 + \chi_2}{2\gamma} + q\frac{\chi_1 - \chi_2}{2\gamma} \quad and \quad g = \frac{\chi_1 + \chi_2}{2\gamma} + \delta\frac{\chi_1 - \chi_2}{2\gamma}.$$

(6) $\alpha \neq \pm 1$*, and there exist constants* $\lambda \in \mathbb{C} \setminus \{\pm 1\}$, $\gamma \in \mathbb{C} \setminus \{0\}$*, and an exponential* $\chi \neq \chi \circ \sigma$ *with* $\int_S \chi(t)d\mu(t) = \dfrac{\lambda - 1}{\gamma(\alpha - 1)}$ *and* $\int_S \chi \circ \sigma(t)d\mu(t) = \dfrac{\lambda + 1}{\gamma(\alpha + 1)}$ *such that*

$$f = \frac{1 + \lambda}{2\gamma}\chi - \frac{1 - \lambda}{2\gamma}\chi \circ \sigma \quad and \quad g = \frac{1 + \lambda}{2\gamma}\chi + \frac{1 - \lambda}{2\gamma}\chi \circ \sigma.$$

(7) $\alpha \neq \pm 1$*, and there exist a constant* $\beta \in \mathbb{C}^*$ *and an exponential* $\chi \neq \chi \circ \sigma$ *with* $\int_S \chi(t)d\mu(t) = \dfrac{1}{\beta(\alpha - 1)}$ *and* $\int_S \chi \circ \sigma(t)d\mu(t) = \dfrac{1}{\beta(\alpha + 1)}$ *such that*

$$f = \frac{\chi + \chi \circ \sigma}{2\beta} \quad and \quad g = \frac{\chi - \chi \circ \sigma}{2\beta}.$$

(8) $\alpha \neq \pm 1$*, and there exist a constant* $\beta \in \mathbb{C}^*$*, an exponential* $\chi = \chi \circ \sigma$ *and a non-zero even function* ϕ_χ *satisfying (1.2) with* $\int_S \chi(t)d\mu(t) = \dfrac{\varepsilon}{\beta(\varepsilon\alpha - 1)}$ *and* $\int_S \phi_\chi(t)d\mu(t) = -\dfrac{1}{\beta(\varepsilon\alpha - 1)^2}$ *such that*

$$f = (\chi + \varepsilon\phi_\chi)/\beta \quad and \quad g = \phi_\chi/\beta.$$

(9) $\alpha \neq \pm 1$*, and there exist a constant* $\lambda \neq -\varepsilon$*, an exponential* $\chi = \chi \circ \sigma$ *and a non-zero even function* ϕ_χ *satisfying (1.2) with*

$$\int_S \chi(t)d\mu(t) = \frac{\lambda + \varepsilon}{\gamma(\alpha + \varepsilon)} \quad and \quad \int_S \phi_\chi(t)d\mu(t) = \frac{(\lambda - \alpha)(1 + \varepsilon\lambda)}{\gamma(\alpha + \varepsilon)^2},$$

where $\varepsilon = \pm 1$*, such that*

$$f = (\lambda\chi + \phi_\chi)/\gamma \quad and \quad g = (\chi - \varepsilon\phi_\chi)/\gamma.$$

(10) $\alpha \neq \pm 1$*, and there exist an exponential* $\chi = \chi \circ \sigma$ *with* $\int_S \chi(t)d\mu(t) \neq 0$*, and non-zero function* Φ_χ *satisfying (1.9) such that*

$$f = \frac{1}{1 + \varepsilon\alpha}\Phi_\chi + \alpha\chi\int_S \chi(t)d\mu(t) \quad and \quad g = \frac{-\varepsilon}{1 + \varepsilon\alpha}\Phi_\chi + \chi\int_S \chi(t)d\mu(t),$$

where $\varepsilon = \pm 1$.

If S *is a topological semigroup and* $f \in C(S)$ *then:*
(i) $g \in C(S)$ *in (1), (2), (4), (6)-(9); (10) if* $\alpha = 0$.
(ii) $g \in C(S)$ *in (10) if* $\sigma \in C(S)$ *for* $\alpha \neq 0$.
(iii) $g \in C(S)$ *in (3) if* $q \neq -\alpha$.
(iv) $g \in C(S)$ *in (5) if* $q^2 \neq \delta^2$.

Proof: If $g=0$ then (1.10) becomes

$$\alpha\int_S f(x\sigma(y)t)d\mu(t)=f(x)f(y),\ \ x,y\in S. \tag{4.1}$$

For the case $f=0$ we get a special case of part (3) corresponding to $q=-\alpha$ and $\delta=-1$. Otherwise, from (4.1) we read that $\alpha\neq 0$ and the function f/α satisfies the functional equation (1.12). So, according to [2, Proposition 3.1] we get $f/\alpha=:\chi\int_S\chi(t)d\mu(t)$ where χ is an exponential on S such that $\chi\circ\sigma=\chi$ and $\int_S\chi(t)d\mu(t)\neq 0$. Then we are in part (3) with $q=\alpha\neq 0$ and $\delta=-1$.
Next, we assume that $g\neq 0$. Here we split the proof in two cases according to $\{f,g\}$ is linearly independent or not.
Case 1: $\{f,g\}$ is linearly dependent. Then there exists $c\in\mathbb{C}$ such that $f=cg$ and therefore the functional equation (1.10) reduces to

$$(1-\alpha c)\int_S g(x\sigma(y)t)d\mu(t)=(1-c^2)g(x)g(y),\ \ x,y\in S. \tag{4.2}$$

If $c=1$, then $f=g$ and Eq.(4.2) reduces to $(1-\alpha)\int_S g(x\sigma(y)t)d\mu(t)=0$ for all $x,y\in S$. For $\alpha=1$ we get part (1). For $\alpha\neq 1$, then we get $\int_S g(xyt)d\mu(t)=0$ for all $x,y\in S$ with $g\neq 0$. This gives the solution part (2).

If $c=-1$ we proceed as above and then we obtain solution part (1) for $\alpha=-1$, and solution part (2) for $\alpha\neq -1$.

If $c\neq\pm1$, then from (4.2) we read that $1-\alpha c\neq 0$, because $g\neq 0$. Therefore, by using [2, Proposition 3.1], we get that $(1-c^2)g/(1-\alpha c)=\chi\int_S\chi(t)d\mu(t)$, where χ is an exponential such that $\chi=\chi\circ\sigma$ and $\int_S\chi(t)d\mu(t)\neq 0$. So,

$$g=\frac{1-\alpha c}{1-c^2}\chi\int_S\chi(t)d\mu(t)\ \ \textit{and}\ \ f=cg=\frac{c-\alpha c^2}{1-c^2}\chi\int_S\chi(t)d\mu(t).$$

Introducing the constants $d,q\in\mathbb{C}$ such that $d:=\dfrac{1-\alpha c}{1-c^2}$ and $\dfrac{q+\alpha}{2}:=\dfrac{c-\alpha c^2}{1-c^2}$, and using the same calculations as those in the proof of [15, Lemma 4.3] we get that

$$f=\frac{q+\alpha}{2}\chi\int_S\chi(t)d\mu(t)\ \ \textit{and}\ \ g=\frac{1\pm\sqrt{1+q^2-\alpha^2}}{2}\chi\int_S\chi(t)d\mu(t).$$

This solution occurs in part (3).
Case 2: $\{f,g\}$ is independent. Here we discuss according to $\int_S h(t)d\mu(t)=0$ or $\int_S h(t)d\mu(t)\neq 0$.
<u>Case 2.A</u>: $\int_S h(t)d\mu(t)=0$. Then, from Lemma 3.1 we get the functional equation (3.1) and from Lemma 3.2 (a) we read that $\int_S g(t)d\mu(t)\neq 0$.
<u>Case 2.A.1</u>: $\int_S\int_S g(\sigma(t)s)d\mu(s)d\mu(t)=0$. We infer from Lemma 3.2 (b) that $\int_S\int_S f(\sigma(t)s)d\mu(s)d\mu(t)\neq 0$, so Eq. (3.1) can be reformulated as follows

$$\beta f(x\sigma(y))=\beta^2 f(x)f(y)+(-i\beta)^2g(x)g(y),\ \ x,y\in S, \tag{4.3}$$

with $\beta:=\int_S g(t)d\mu(t)/\int_S\int_S f(\sigma(t)s)d\mu(s)d\mu(t)$ is a non-zero complex constant. Now, from (4.3) we read that the pair $(\beta f,-i\beta g)$ satisfies the cosine subtraction law (1.4). So, according to [8, Theorem 4.2] and using the fact that $\{f,g\}$ is linearly independent, we discuss the following cases:
(i)

$$\beta f=\frac{\chi+\chi\circ\sigma}{2}\ \ \textit{and}\ \ -i\beta g=\frac{\chi-\chi\circ\sigma}{2i},$$

where χ is an exponential on S such that $\chi\neq\chi\circ\sigma$. Then we obtain

$$f=\frac{\chi+\chi\circ\sigma}{2\beta}\ \ \textit{and}\ \ g=\frac{\chi-\chi\circ\sigma}{2\beta}.$$

By using (1.10) we get after some calculations that

$$[1+\beta(1-\alpha)\int_S \chi(t)d\mu(t)]\chi(x\sigma(y))+[1-\beta(1+\alpha)\int_S \chi\circ\sigma(t)d\mu(t)]\chi(\sigma(x)y)=0,$$

for all $x,y\in S$, and then by using Lemma 2.1 (a) we get that

$$\beta(1-\alpha)\int_S \chi\circ\sigma(t)d\mu(t)+1=0 \;\; and \;\; 1-\beta(\alpha+1)\int_S \chi(t)d\mu(t)=0,$$

from which we read $\alpha\neq\pm1$ (because otherwise we get $1=0$ which is impossible). So we conclude that

$$\int_S \chi(t)d\mu(t)=\frac{1}{\beta(\alpha-1)} \;\; and \;\; \int_S \chi\circ\sigma(t)d\mu(t)=\frac{1}{\beta(\alpha+1)}.$$

So we are in part (7) of our statement.
(ii)

$$\beta f=\frac{c^{-1}\chi_1+c\chi_2}{c^{-1}+c} \;\; and \;\; -i\beta g=\frac{\chi_2-\chi_1}{c^{-1}+c},$$

where $c\in\mathbb{C}^*\setminus\{\pm i\}$ and χ_1 and χ_2 are two different exponentials on S such that $\chi_1\circ\sigma=\chi_1$ and $\chi_2\circ\sigma=\chi_2$. So,

$$f=\frac{c^{-1}\chi_1+c\chi_2}{\beta(c^{-1}+c)} \;\; and \;\; g=\frac{\chi_1-\chi_2}{i\beta(c^{-1}+c)}.$$

Using (1.10) we get after some simplifications that

$$[\beta(c^{-1}+c)(1-i\alpha c^{-1})\textstyle\int_S \chi_1(t)d\mu(t)+i(1+c^{-2})]\chi_1(xy)$$

$$+[i(1+c^2)-\beta(c^{-1}+c)(1+i\alpha c)\int_S \chi_2(t)d\mu(t)]\chi_2(xy)=0,$$

which implies, by using Lemma 2.1(a), that

$$\beta(c^{-1}+c)(1-i\alpha c^{-1})\int_S \chi_1(t)d\mu(t)+i(1+c^{-2})=0$$

and

$$i(1+c^2)-\beta(c^{-1}+c)(1+i\alpha c)\int_S \chi_2(t)d\mu(t)]=0.$$

From the two last identities we read that $1-i\alpha c^{-1}\neq0$ and $1+i\alpha c\neq0$, i.e. $\alpha\notin\{-ic,ic^{-1}\}$, because otherwise we get $c=\pm i$ which contradicts the assumption on c. Thus we conclude that

$$\int_S \chi_1(t)d\mu(t)=\frac{1}{\beta(\alpha+ic)} \;\; and \;\; \int_S \chi_2(t)d\mu(t)=\frac{1}{\beta(\alpha-ic^{-1})}.$$

The solution occurs in part (4).
(iii) $\beta f=\chi\pm i\phi_\chi$ and $-i\beta g=\phi_\chi$, where χ is an exponential on S such that $\chi\circ\sigma=\chi$ and ϕ_χ is a non-zero solution of (1.2) and $\phi_\chi\circ\sigma=\phi_\chi$. Seeing that $i\phi_\chi$ is also a non-zero solution of (1.2) and $i\phi_\chi\circ\sigma=i\phi_\chi$ we get, by writing ϕ_χ instead of $i\phi_\chi$, that

$$f=(\chi+c\phi_\chi)/\beta \text{ and } g=\phi_\chi/\beta,$$

where $c=\pm1$.
When we substitute this in (1.10), we obtain, after some computation, that

$$\begin{aligned}0=&[(1-\alpha\beta\int_S \chi(t)d\mu(t)+\beta(1-c\alpha)\int_S \phi_\chi(t)d\mu(t))\chi(y)\\&+(c+\beta(1-c\alpha)\int_S \chi(t)d\mu(t))\phi_\chi(y)]\chi\\&+[(c+\beta(1-c\alpha)\int_S \chi(t)d\mu(t))\chi(y)]\phi_\chi\end{aligned}$$

for all $y \in S$, which implies, by using Lemma 2.1 (b), the following identities

$$1-\alpha\beta\int_S \chi(t)d\mu(t)+\beta(1-c\alpha)\int_S \phi_\chi(t)d\mu(t)=0$$

and

$$c+\beta(1-c\alpha)\int_S \chi(t)d\mu(t)=0.$$

From the identities above we read that $1-c\alpha\neq 0$ (because otherwise we get $c=0$, contradicting that $c=\pm 1$). Then we get that

$$\int_S \chi(t)d\mu(t)=\frac{\pm 1}{\beta(\pm\alpha-1)} \quad \textit{and} \quad \int_S \phi_\chi(t)d\mu(t)=-\frac{1}{\beta(\pm\alpha-1)^2}.$$

The solution occurs in part (8).
Case 2.A.2 : Suppose $\int_S\int_S g(\sigma(t)s)d\mu(s)d\mu(t)\neq 0$. Then the functional equation (3.1) can be rewritten as follows

$$\gamma g(x\sigma(y))=\gamma^2 g(x)g(y)-\gamma^2 f(x)f(y)+\lambda\gamma f(x\sigma(y)),\ \ x,y\in S, \tag{4.4}$$

where

$$\gamma:=\frac{\int_S g(t)d\mu(t)}{\int_S\int_S g(\sigma(t)s)d\mu(s)d\mu(t)}\neq 0 \quad \textit{and} \quad \lambda:=\frac{\int_S\int_S f(\sigma(t)s)d\mu(s)d\mu(t))}{\int_S\int_S g(\sigma(t)s)d\mu(s)d\mu(t)}.$$

So, the pair $(\gamma g,\gamma f)$ satisfies (1.3). Now, according to [4, Theorem 3.4] and taking into account that $\{f,g\}$ is linearly independent we have only the following possibilities:
(i)

$$\gamma f=\lambda\frac{\chi_1+\chi_2}{2}+q\frac{\chi_1-\chi_2}{2} \quad \textit{and} \quad \gamma g=\frac{\chi_1+\chi_2}{2}\pm\sqrt{1+q^2-\lambda^2}\,\frac{\chi_1-\chi_2}{2},$$

where $q\in\mathbb{C}$ and χ_1 and χ_2 are two different exponentials such that $\chi_1\circ\sigma=\chi_1$ and $\chi_2\circ\sigma=\chi_2$. Introducing $\delta:=\pm\sqrt{1+q^2-\lambda^2}$, we get that

$$f=\lambda\frac{\chi_1+\chi_2}{2\gamma}+q\frac{\chi_1-\chi_2}{2\gamma} \quad \textit{and} \quad g=\frac{\chi_1+\chi_2}{2\gamma}+\delta\frac{\chi_1-\chi_2}{2\gamma}. \tag{4.5}$$

As $\chi_1+\chi_2,\chi_1-\chi_2$ is linearly independent we get from (4.5), by using the linearl independence of f,g, that $q\neq\alpha\lambda$. On the other hand, by using (1.10) a small computation shows that

$$\begin{aligned}&\left(2\gamma[(1+\delta)-\alpha(\lambda+q)]\int_S \chi_1(t)d\mu(t)+(\lambda+q)^2-(1+\delta)^2\right)\chi_1(xy)\\ &\quad+\left(2\gamma[(1-\delta)-\alpha(\lambda-q)]\int_S \chi_2(t)d\mu(t)+(\lambda-q)^2-(1-\delta)^2\right)\chi_2(xy)\\ &=0,\end{aligned}$$

and then by Lemma 2.1 (a) we obtain

$$2\gamma[(1+\delta)-\alpha(\lambda+q)]\int_S \chi_1(t)d\mu(t)+(\lambda+q)^2-(1+\delta)^2=0 \tag{4.6}$$

and

$$2\gamma[(1-\delta)-\alpha(\lambda-q)]\int_S \chi_2(t)d\mu(t)+(\lambda-q)^2-(1-\delta)^2=0. \tag{4.7}$$

Moreover, we have

$$1+\delta-\alpha(\lambda+q)\neq 0 \quad \textit{and} \quad 1-\delta-\alpha(\lambda-q)\neq 0,$$

Indeed, if $1+\delta-\alpha(\lambda+q)=0$ then $1+\delta=\alpha(\lambda+q)$, and (4.6) implies that $(\lambda+q)^2=(1+\delta)^2$. Hence $(\lambda+q)^2(\alpha^2-1)=0$. Then $q=-\lambda$ or $\alpha^2=1$.
If $q=-\lambda$ then $\delta=-1$. Substituting this in (4.5) we get that $f=\lambda g$, which contradicts the fact that f

and g are linearly independent.
Hence $q \neq -\lambda$ and $\alpha^2 = 1$. So that $(1+\delta)^2 = (\lambda+q)^2$, which implies that $1 + 2\delta + \delta^2 = \lambda^2 + 2\lambda q + q^2$. As $\delta^2 = 1 + q^2 - \lambda^2$ we derive that $\alpha(\lambda + q) = 1 + \delta = \lambda(\lambda + q)$. Then $\lambda = \alpha$ because $\lambda + q \neq 0$. As $1+\delta = \alpha(\lambda+q)$ we get that $\delta = \alpha q = \lambda q$. Now, multiplying the expression of f in (4.5) by λ we get that $g = \lambda f$ which contradicts the linear independence of f and g. Similarly, we prove that $1-\delta-\alpha(\lambda-q) \neq 0$.

Therefore from (4.6) and (4.7) we derive

$$\int_S \chi_1(t)d\mu(t) = \frac{(1+\delta)^2 - (\lambda+q)^2}{2\gamma(1+\delta-\alpha(\lambda+q))} = \frac{1+\delta-\lambda(\lambda+q)}{\gamma(1+\delta-\alpha(\lambda+q))}$$

and

$$\int_S \chi_2(t)d\mu(t) = \frac{(1-\delta)^2 - (\lambda-q)^2}{2\gamma(1-\delta-\alpha(\lambda-q))} = \frac{1-\delta-\lambda(\lambda-q)}{\gamma(1-\delta-\alpha(\lambda-q))}.$$

The solution occurs in part (5).
(ii) $\lambda \neq 0$, and we have

$$\gamma f = \lambda\chi_1 \ \ \textit{and} \ \ \gamma g = \chi_2,$$

where χ_1 and χ_2 are two different exponentials such that $\chi_1 \circ \sigma = \chi_1$ and $\chi_2 \circ \sigma = \chi_2$. Then we get

$$f = \frac{\alpha}{\gamma}\chi_1 \quad \textit{and} \quad g = \frac{1}{\gamma}\chi_2. \tag{4.8}$$

A small computation based on (1.10) shows that

$$\left(\lambda^2 - \lambda\alpha\gamma\int_S \chi_1(t)d\mu(t)\right)\chi_1(xy) + \left(\gamma\int_S \chi_2(t)d\mu(t) - 1\right)\chi_2(xy) = 0$$

for all $x, y \in S$ because $\gamma \neq 0$. Then we get, by using Lemma 2.1 (a), that

$$\int_S \chi_1(t)d\mu(t) = \frac{\lambda}{\alpha\gamma} \quad \text{and} \quad \int_S \chi_2(t)d\mu(t) = \frac{1}{\gamma}. \tag{4.9}$$

From (4.8) and (4.9) we deduce that

$$f = \alpha\int_S \chi_1(t)d\mu(t)\chi_1 \text{ and } g = \int_S \chi_2(t)d\mu(t)\chi_2.$$

So, we get a special case of part (5) corresponding to $\lambda = \alpha$, $(q, \delta) = (\alpha, -1)$ and $\alpha \neq 0$.
(iii) $\lambda \neq \pm 1$, and

$$\gamma f = \frac{1+\lambda}{2}\chi + \frac{\lambda-1}{2}\chi\circ\sigma \quad \textit{and} \quad \gamma g = \frac{1+\lambda}{2}\chi + \frac{1-\lambda}{2}\chi\circ\sigma,$$

where χ is an exponential such that $\chi \neq \chi\circ\sigma$. Then we get

$$f = \frac{1+\lambda}{2\gamma}\chi - \frac{1-\lambda}{2\gamma}\chi\circ\sigma \quad \textit{and} \quad g = \frac{1+\lambda}{2\gamma}\chi + \frac{1-\lambda}{2\gamma}\chi\circ\sigma.$$

By using (1.10) we get that

$$\left(\gamma(\lambda+1)(1-\alpha)\int_S \chi(t)d\mu(t) + \lambda^2 - 1\right)\chi(xy)$$
$$+\left(\gamma(1-\lambda)(1+\alpha)\int_S \chi\circ\sigma(t)d\mu(t) + \lambda^2 - 1\right)\chi\circ\sigma(xy) = 0,$$

for all $x, y \in S$. This implies by using Lemma 2.1 (a) that

$$\gamma(\lambda+1)(1-\alpha)\int_S \chi(t)d\mu(t) + \lambda^2 - 1 = 0$$

and

$$\gamma(1-\lambda)(1+\alpha)\int_S \chi\circ\sigma(t)d\mu(t)+\lambda^2-1=0.$$

If $\alpha=\pm1$ then $\lambda=\pm1$ which a contradiction. So, $\alpha\neq\pm1$ and therefore

$$\int_S \chi(t)d\mu(t)=\frac{\lambda-1}{\gamma(\alpha-1)}\quad \textit{and}\quad \int_S \chi\circ\sigma(t)d\mu(t)=\frac{\lambda+1}{\gamma(\alpha+1)}.$$

The solution occurs in part (6).
(iv)$\gamma f=\lambda\chi+\phi_\chi$ and $\gamma g=\chi\pm\phi_\chi$ such that $\chi\circ\sigma=\chi$ and $\phi_\chi\circ\sigma=\phi_\chi$, which gives $f=(\lambda\chi+\phi_\chi)/\gamma$ and $g=(\chi\pm\phi_\chi)/\gamma$. For the case $g=(\chi+\phi_\chi)/\gamma$ we get by using (1.10) that
$\chi(x)[\chi(y)\left(\gamma(1-\alpha\lambda)\int_S\chi(t)d\mu(t)+\gamma(1-\alpha)\int_S\phi_\chi(t)d\mu(t)+\lambda^2-1\right)$
$+\phi_\chi(y)\left(\gamma(1-\alpha)\int_S\chi(t)d\mu(t)+\lambda-1\right)]$
$+\phi_\chi(x)\left[\chi(y)\left(\gamma(1-\alpha)\int_S\chi(t)d\mu(t)+\lambda-1\right)\right]=0$, for all $x,y\in S$, which gives by using Lemma 2.1 (b) that
$\gamma(1-\alpha\lambda)\int_S\chi(t)d\mu(t)+\gamma(1-\alpha)\int_S\phi_\chi(t)d\mu(t)+\lambda^2-1=0$
and $\gamma(1-\alpha)\int_S\chi(t)d\mu(t)+\lambda-1=0$.

If $\alpha=1$ then $\lambda=1$ and we get $f=g$ and this contradicts the assumption that $\{f,g\}$ is linearly independent. So $\alpha\neq1$ and then we obtain

$$\int_S \chi(t)d\mu(t)=\frac{\lambda-1}{\gamma(\alpha-1)}\quad \textit{and}\quad \int_S \phi_\chi(t)d\mu(t)=\frac{(\lambda-\alpha)(1-\lambda)}{\gamma(\alpha-1)^2}.$$

So we get part (9) for $\varepsilon=-1$. The case $g=(\chi-\phi_\chi)/\gamma$ can be treated similarly, and we obtain

$$\int_S \chi(t)d\mu(t)=\frac{\lambda+1}{\gamma(\alpha+1)}\quad \textit{and}\quad \int_S \phi_\chi(t)d\mu(t)=\frac{(\lambda-\alpha)(1+\lambda)}{\gamma(\alpha+1)^2}.$$

So we get part (9) for $\varepsilon=1$.
<u>Case 2.B</u>: Suppose $\int_S h(t)d\mu(t)\neq0$. By using the system (3.2) and (3.8) we get for any $\lambda\in\mathbb{C}$ that

$$\int_S(g-\lambda h)(x\sigma(y)t)d\mu(t)=(g-\lambda h)(x)(g-\lambda h)(y)+p(\lambda)h(x)h(y),\ \ x,y\in S, \tag{4.10}$$

where p denotes the second order polynomial

$$p(\lambda)=-\lambda^2-[(1-\alpha^2)\omega+\alpha]\lambda+\omega\alpha-1,\ \ \lambda\in\mathbb{C}, \tag{4.11}$$

where ω is defined in (3.6).
Let λ_1 and λ_2 be the roots of the polynomial (4.11). By using [2, Proposition 3.1] we get from (4.10) that

$$g-\lambda_2 h=\chi_1\int_S\chi_1(t)d\mu(t)\quad \textit{and}\quad g-\lambda_1 h=\chi_2\int_S\chi_2(t)d\mu(t), \tag{4.12}$$

where χ_1 and χ_2 are two exponentials such that $\chi_1\circ\sigma=\chi_1$, $\chi_2\circ\sigma=\chi_2$, $\int_S\chi_1(t)d\mu(t)\neq0$ and $\int_S\chi_2(t)d\mu(t)\neq0$, because $g-\lambda_1h\neq0$ and $g-\lambda_2h\neq0$ since h and g are linearly independent.
If $\lambda_1\neq\lambda_2$ then from (4.12) we derive that $\chi_1\int_S\chi_1(t)d\mu(t)\neq\ \chi_2\int_S\chi_2(t)d\mu(t)$ and therefore

$$h=\frac{\chi_1\int_S\chi_1(t)d\mu(t)-\chi_2\int_S\chi_2(t)d\mu(t)}{\lambda_1-\lambda_2}$$

and

$$g=\frac{\lambda_1\int_S\chi_1(t)d\mu(t)\chi_1-\lambda_2\int_S\chi_2(t)d\mu(t)\chi_2}{\lambda_1-\lambda_2},$$

and then we deduce that

$$f=h+\alpha g=\frac{(1+\alpha\lambda_1)\chi_1\int_S\chi_1(t)d\mu(t)-(1+\alpha\lambda_2)\chi_2\int_S\chi_2(t)d\mu(t)}{\lambda_1-\lambda_2}..$$

Using this and introducing the constant $q := \dfrac{2+\alpha(\lambda_1+\lambda_2)}{\lambda_1-\lambda_2} \in \mathbb{C}$ we proceed as in the proof of [15, Lemma 4.6] we show that f and g depend on q as follows

$$f = \frac{\alpha+q}{2}\int_S \chi_1(t)d\mu(t)\chi_1 + \frac{\alpha-q}{2}\int_S \chi_2(t)d\mu(t)\chi_2$$

and

$$g = \frac{1+\delta}{2}\int_S \chi_1(t)d\mu(t)\chi_1 + \frac{1-\delta}{2}\int_S \chi_2(t)d\mu(t)\chi_2,$$

with $\delta := \pm\sqrt{1+q^2-\alpha^2}$. We omit the details. As f and g are linearly independent, $\int_S \chi_1(t)d\mu(t) \neq 0$ and $\int_S \chi_2(t)d\mu(t) \neq 0$, we get that $(1-\delta)(\alpha+q)-(1+\delta)(\alpha-q) \neq 0$ which reduces to $q \neq \alpha\delta$. The solution obtained is a special case of part (5) corresponding to $\lambda = \alpha$.
If $\lambda_1 = \lambda_2 =: \lambda$ then we have $\chi_1 \int_S \chi_1(t)d\mu(t) = \chi_2 \int_S \chi_2(t)d\mu(t)$ which gives by Lemma 2.1 (a) that $\chi_1 = \chi_2$. This allows us to define $\chi := \chi_1 = \chi_2$. Notice that χ is also an exponential such that $\int_S \chi(t)d\mu(t) \neq 0$, and then $g - \lambda h =: \chi \int_S \chi(t)d\mu(t)$ from which we derive $g = \lambda h + \chi \int_S \chi(t)d\mu(t)$. We get also from (4.11) and by elementary algebra that $2\lambda = -[(1-\alpha^2)\omega+\alpha]$. Using this and substituting the form of g into (3.2) we get

$$\begin{aligned}
\int_S h(x\sigma(y)t)d\mu(t) &= g(x)h(y)+g(y)h(x)+[(1-\alpha^2)\omega+\alpha]h(x)h(y)\\
&= \left(\lambda h(x)+\chi(x)\int_S \chi(t)d\mu(t)\right)h(y)\\
&\quad + \left(\lambda h(y)+\chi(y)\int_S \chi(t)d\mu(t)\right)h(x)\\
&\quad + [(1-\alpha^2)\omega+\alpha]h(x)h(y)\\
&= [h(x)\chi(y)+h(y)\chi(x)]\int_S \chi(t)d\mu(t)\\
&\quad + [(1-\alpha^2)\omega+\alpha+2\lambda]h(x)h(y)\\
&= [h(x)\chi(y)+h(y)\chi(x)]\int_S \chi(t)d\mu(t).
\end{aligned}$$

This shows that the function h is a non-zero solution of the special integral Kannappan-sine addition law (1.9). Then $h =: \Phi_\chi \neq 0$, so we get that

$$g = \lambda h + \chi\int_S \chi(t)d\mu(t) = \lambda\Phi_\chi + \chi\int_S \chi(t)d\mu(t) \tag{4.13}$$

and

$$f = \alpha g + h = (\alpha\lambda+1)\Phi_\chi + \alpha\chi\int_S \chi(t)d\mu(t). \tag{4.14}$$

A short computation based on (1.10) shows that

$$0 = [\lambda^2-(\alpha\lambda+1)^2]\Phi_\chi(x)\Phi_\chi(y) = (\lambda(1+\alpha)+1)(\lambda(1-\alpha)-1)\Phi_\chi(x)\Phi_\chi(y)$$

for all $x, y \in S$, from which we drive $\lambda(1+\alpha)+1 = 0$ or $\lambda(1-\alpha)-1 = 0$ because $\Phi_\chi \neq 0$. If $\lambda(1+\alpha)+1 = 0$ we get $\lambda = -1/(1+\alpha)$ and if $\lambda(1-\alpha)-1 = 0$ we obtain $\lambda = 1/(1-\alpha)$. Combining this with (4.13) and (4.14) we obtain $f = \dfrac{1}{1+\varepsilon\alpha}\Phi_\chi + \alpha\chi\int_S \chi(t)d\mu(t)$ *and* $g = \dfrac{-\varepsilon}{1+\varepsilon\alpha}\Phi_\chi + \chi\int_S \chi(t)d\mu(t)$, where $\varepsilon = \pm 1$. The solution occurs in part (10).

Conversely, if f and g are of the forms (1)-(10) in Theorem 4.1 we check that the pair (g, f) is a solution of equation (1.10).

Now, suppose that S is a topological semigroup and $f \in C(S)$.

In parts (1)-(2) the continuity of g is evident, because g is proportional to f.

In part (3) the continuity of g is also trivial in each cases $q \neq -\alpha$, and $q = -\alpha$ and $\delta = -1$; but g is not necessarily continuous if $q = -\alpha$ and $\delta = 1$.

In parts (4), (6), (7) and (8) the continuity of g is derived by using Lemma 2.1(c) since $c \neq 0$ and $\lambda \neq \pm 1$.

In part (5) we rewrite $f = \dfrac{\lambda+q}{2}\chi_1 + \dfrac{\lambda-q}{2}\chi_2$ and $g = \dfrac{1+\delta}{2}\chi_1 + \dfrac{1-\delta}{2}\chi_2$. Then, if $q^2 \neq \lambda^2$ we get, according to Lemma 2.1(c), that the exponentials χ_1 and χ_2 are continuous. So g is. If $q = \lambda$ the $\delta = \pm 1$, then $\delta = 1$ because $q \neq \lambda\delta$, hence $f = \dfrac{\lambda}{\gamma}\chi_1$ and $g = \dfrac{1}{\gamma}\chi_2$, so g is not necessarily continuous. Similarly we check that if $q = -\lambda$ g is not necessarily continuous.

In part (9) we have $f = (\lambda\chi + \phi_\chi)/\gamma$, then we get for all $x, y \in S$ that

$$\begin{aligned}\phi_\chi(x)\chi(y) &= \phi_\chi(xy) - \phi_\chi(y)\chi(x)\\ &= \gamma f(xy) - \lambda\chi(xy) - \phi_\chi(y)\chi(x)\\ &= \gamma f(xy) - \lambda\chi(xy) - [\gamma f(y) - \lambda\chi(y)]\chi(x)\\ &= \gamma f(xy) - \gamma f(y)\chi(x).\end{aligned}$$

It follows that $\chi \in C(S)$, because $\phi_\chi \neq 0$. Then we get that $\phi_\chi \in C(S)$, because $\chi \neq 0$. Thus $g = (\chi \pm \phi_\chi)/\gamma \in C(S)$.

The part (8) can be treated similarly to (9).

In part (10) we have by (4.14) that $\Phi_\chi = (1+\varepsilon\alpha)(f - \alpha\chi\int_S \chi(t)d\mu(t))$ and for any $x, y \in S$ we get

$$\begin{aligned}&\Phi_\chi(x)\chi(y)\int_S \chi(t)d\mu(t)\\ &= \int_S \Phi_\chi(x\sigma(y)t)d\mu(t) - \Phi_\chi(y)\chi(x)\int_S \chi(t)d\mu(t)\\ &= (1+\varepsilon\alpha)\int_S f(x\sigma(y)t)d\mu(t) - \alpha(1+\varepsilon\alpha)\int_S \chi(x\sigma(y)t)d\mu(t)\int_S \chi(t)d\mu(t)\\ &\quad - (1+\varepsilon\alpha)[f(y) - \alpha\chi(y)\int_S \chi(t)d\mu(t)]\chi(x)\int_S \chi(t)d\mu(t).\end{aligned}$$

So that

$$\Phi_\chi(x)\chi(y)\int_S \chi(t)d\mu(t) = (1+\varepsilon\alpha)\left(\int_S f(x\sigma(y)t)d\mu(t) - \chi(x)f(y)\int_S \chi(t)d\mu(t)\right), \tag{4.15}$$

for all $x, y \in S$.

If $\alpha = 0$ then $f = \Phi_\chi$ so Φ_χ is continuous. Moreover (4.15) reduces to $f(y)\chi(x)\int_S \chi(t)d\mu(t) = \int_S f(x\sigma(y)t)d\mu(t) - f(x)\chi(y)\int_S \chi(t)d\mu(t)$. As $f \neq 0$ there exists $y_0 \in S$ such that $f(y_0) \neq 0$. So, by putting $y = y_0$ in the last identity and dividing the identity obtained by $f(y_0)\int_S \chi(t)d\mu(t)$, since $\int_S \chi(t)d\mu(t) \neq 0$, and seeing that f is continuous and μ is a linear combination of Dirac measures we derive that χ is continuous. So, from (4.13) we deduce that g is continuous.

Now, assume that $\alpha \neq 0$. As $\Phi_\chi \neq 0$ there exists $a \in S$ such that $\Phi_\chi(a) \neq 0$. Moreover $\int_S \chi(t)d\mu(t) \neq 0$. By putting $x = a$ in the identity above and diving the identity obtained by $\int_S \chi(t)d\mu(t)\Phi_\chi(a)$ we get that

$$\chi(y) = (1+\varepsilon\alpha)\left(\int_S f(a\sigma(y)t)d\mu(t) - \chi(a)f(y)\int_S \chi(t)d\mu(t)\right) / \int_S \chi(t)d\mu(t)\Phi_\chi(a)$$

from which we deduce that $\chi \in C(S)$ because $\sigma, f \in C(S)$ and μ is a linear combination of Dirac measures. Therefore Φ_χ is continuous, because $\chi \neq 0$. So, in part (10) $g \in C(S)$, because g is a linear combination of χ and Φ_χ. This completes the proof of Theorem 4.1. □

5. Application

In this section we present two applications of our main result.

If we take $\alpha = 0$ in (1.10) we get the integral Kannappan-cosine functional equation (1.13) which has not been studied until now except for the case $\sigma = Id$ and S is a compact group. The solutions of (1.13) on semigroups are given in Theorem 5.1, by taking $\alpha = 0$ in Theorem 4.1.

Theorem 5.1 *The solutions $f, g : S \longrightarrow \mathbb{C}$ of the functional equation (1.13) can be listed as follows*
(1) $f = \pm g$ and g is an arbitrary non-zero function such that $\int_S g(xyt)d\mu(t) = 0$ for all $x, y \in S$.
(2) There exist constants $q, \delta \in \mathbb{C}$ and an exponential $\chi = \chi \circ \sigma$ with $\delta := \pm\sqrt{1+q^2}$ and $\int_S \chi(t)d\mu(t) \neq 0$ such that

$$f = \frac{q}{2}\chi \int_S \chi(t)d\mu(t) \quad \text{and} \quad g = \frac{1+\delta}{2}\chi \int_S \chi(t)d\mu(t).$$

(3) There exist constants $\beta \in \mathbb{C}^*, c \in \mathbb{C}^* \setminus \{\pm i\}$ and two different exponentials $\chi_1 = \chi_1 \circ \sigma$ and $\chi_2 = \chi_2 \circ \sigma$ with $\int_S \chi_1(t)d\mu(t) = \dfrac{1}{i\beta c}$ and $\int_S \chi_2(t)d\mu(t) = \dfrac{1}{-i\beta c^{-1}}$ such that

$$f = \frac{c^{-1}\chi_1 + c\chi_2}{\beta(c^{-1}+c)} \quad \text{and} \quad g = \frac{\chi_1 - \chi_2}{i\beta(c^{-1}+c)}.$$

(4) There exist constants $\lambda, \delta \in \mathbb{C}$, $q \in \mathbb{C} \setminus \{\lambda\delta\}$, $\gamma \in \mathbb{C} \setminus \{0\}$, with $\delta = \pm\sqrt{1+q^2-\lambda^2} \neq \pm 1$ and two different exponentials $\chi_1 = \chi_1 \circ \sigma$ and $\chi_2 = \chi_2 \circ \sigma$ satisfying $\int_S \chi_1(t)d\mu(t) = \dfrac{1+\delta-\lambda(\lambda+q)}{\gamma(1+\delta)}$ and $\int_S \chi_2(t)d\mu(t) = \dfrac{1-\delta-\lambda(\lambda-q)}{\gamma(1-\delta)}$ such that

$$f = \lambda\frac{\chi_1+\chi_2}{2\gamma} + q\frac{\chi_1-\chi_2}{2\gamma} \quad \text{and} \quad g = \frac{\chi_1+\chi_2}{2\gamma} + \delta\frac{\chi_1-\chi_2}{2\gamma}.$$

(5) There exist constants $\lambda \in \mathbb{C}\setminus\{\pm 1\}$, $\gamma \in \mathbb{C}\setminus\{0\}$ and an exponential $\chi \neq \chi \circ \sigma$ satisfying $\int_S \chi(t)d\mu(t) = \dfrac{1-\lambda}{\gamma}$ and $\int_S \chi(\sigma(t))d\mu(t) = \dfrac{\lambda+1}{\gamma}$ such that

$$f = \frac{1+\lambda}{2\gamma}\chi - \frac{1-\lambda}{2\gamma}\chi\circ\sigma \quad \text{and} \quad g = \frac{1+\lambda}{2\gamma}\chi + \frac{1-\lambda}{2\gamma}\chi\circ\sigma.$$

(6) There exist a constant $\beta \in \mathbb{C}^*$ and an exponential $\chi \neq \chi \circ \sigma$ with $\int_S \chi(t)d\mu(t) = -\int_S \chi(\sigma(t))d\mu(t) = -\dfrac{1}{\beta}$ such that

$$f = \frac{\chi + \chi\circ\sigma}{2\beta} \quad \text{and} \quad g = \frac{\chi - \chi\circ\sigma}{2\beta}.$$

(7) There exist a constant $\beta \in \mathbb{C}^*$, an exponential $\chi = \chi \circ \sigma$ and a non-zero function ϕ_χ satisfying (1.2) with $\int_S \chi(t)d\mu(t) = -\dfrac{\varepsilon}{\beta}$ and $\int_S \phi_\chi(t)d\mu(t) = -\dfrac{1}{\beta}$, where $\varepsilon = \pm 1$, such that

$$f = (\chi + \varepsilon\phi_\chi)/\beta \quad \text{and} \quad g = \phi_\chi/\beta.$$

(8) There exist a constant $\lambda \neq -\varepsilon$, an exponential $\chi = \chi \circ \sigma$ and a non-zero even function ϕ_χ satisfying (1.2) with

$$\int_S \chi(t)d\mu(t) = \frac{1+\varepsilon\lambda}{\gamma} \quad \text{and} \quad \int_S \phi_\chi(t)d\mu(t) = \frac{\lambda(1+\varepsilon\lambda)}{\gamma},$$

where $\varepsilon = \pm 1$, such that

$$f = (\lambda\chi + \phi_\chi)/\gamma \quad \text{and} \quad g = (\chi - \varepsilon\phi_\chi)/\gamma.$$

(9) There exist an exponential $\chi = \chi \circ \sigma$ with $\int_S \chi(t)d\mu(t) \neq 0$, and non-zero function Φ_χ satisfying (1.9) such that

$$f = \Phi_\chi \quad \text{and} \quad g = \varepsilon\Phi_\chi + \chi\int_S \chi(t)d\mu(t),$$

where $\varepsilon = \pm 1$.

If S *is a topological semigroup and* $f \in C(S)$ *then:*
(i) $g \in C(S)$ *in (1), (3), (5)-(9).*
(ii) $g \in C(S)$ *in (2) if* $q \neq 0$.
(iii) $g \in C(S)$ *in (4) if* $q^2 \neq \delta^2$.

The following proposition gives the solutions of the functional equation

$$f(x\sigma(y)z_0) = \chi(z_0)f(x)\chi(y) + \chi(z_0)f(y)\chi(x), \quad x, y \in S, \tag{5.1}$$

where χ is an exponential such that $\chi(z_0) \neq 0$.

Proposition 5.1 *Let* $\chi : S \longmapsto \mathbb{C}$ *be an exponential such that* $\chi \circ \sigma = \chi$ *and* $\chi(z_0) \neq 0$. *If* $f : S \longmapsto \mathbb{C}$ *is a solution of (5.1) with* $f \neq 0$, *then*

$$f = \phi_\chi + \chi(z_0)A(z_0),$$

where $A : S \setminus I_\chi \longmapsto \mathbb{C}$ *is an additive function.*

Proof: See [12, Proposition 4.3]. □

In what follows we describe the solutions of the functional equation (1.14) on semigroups.

Theorem 5.2 *The solutions* $f, g : S \longrightarrow \mathbb{C}$ *of the functional equation (1.14) can be listed as follows*
(1) $\alpha = \pm 1$, $f = \pm g$ *and* g *is an arbitrary non-zero function.*
(2) $\alpha \neq \pm 1$, $f = \pm g$ *and* g *is an arbitrary non-zero function such that* $g(S^2 z_0) = \{0\}$.
(3) There exist constants $q, \delta \in \mathbb{C}$ *and an exponential* $\chi = \chi \circ \sigma$ *with* $\delta := \pm\sqrt{1 + q^2 - \alpha^2}$ *and* $\chi(z_0) \neq 0$ *such that*

$$f = \frac{q + \alpha}{2}\chi(z_0)\chi \quad \textit{and} \quad g = \frac{1 + \delta}{2}\chi(z_0)\chi.$$

(4) $\alpha \notin \{-ic, ic^{-1}\}$, *and there exist constants* $\beta \in \mathbb{C}^*, c \in \mathbb{C}^* \setminus \{\pm i\}$ *and two different exponentials* $\chi_1 = \chi_1 \circ \sigma$ *and* $\chi_2 = \chi_2 \circ \sigma$ *with* $\chi_1(z_0) = \dfrac{1}{\beta(\alpha + ic)}$ *and* $\chi_2(z_0) = \dfrac{1}{\beta(\alpha - ic^{-1})}$ *such that*

$$f = \frac{c^{-1}\chi_1 + c\chi_2}{\beta(c^{-1} + c)} \quad \textit{and} \quad g = \frac{\chi_1 - \chi_2}{i\beta(c^{-1} + c)}.$$

(5) There exist constants $\lambda, \delta \in \mathbb{C}$, $q \in \mathbb{C} \setminus \{\lambda\delta\}$, $\gamma \in \mathbb{C} \setminus \{0\}$, *with* $\delta = \pm\sqrt{1 + q^2 - \lambda^2}$, $(1 + \delta - \alpha(\lambda + q))(1 - \delta - \alpha(\lambda - q)) \neq 0$, *and two different exponentials* $\chi_1 = \chi_1 \circ \sigma$ *and* $\chi_2 = \chi_2 \circ \sigma$ *satisfying* $\chi_1(z_0) = \dfrac{1 + \delta - \lambda(\lambda + q)^2}{\gamma(1 + \delta - \alpha(\lambda + q))}$ *and* $\chi_2(z_0) = \dfrac{1 - \delta - \lambda(\lambda - q)}{\gamma(1 - \delta - \alpha(\lambda - q))}$ *such that*

$$f = \lambda\frac{\chi_1 + \chi_2}{2\gamma} + q\frac{\chi_1 - \chi_2}{2\gamma} \quad \textit{and} \quad g = \frac{\chi_1 + \chi_2}{2\gamma} + \delta\frac{\chi_1 - \chi_2}{2\gamma}.$$

(6) $\alpha \neq \pm 1$, *and there exist constants* $\lambda \in \mathbb{C} \setminus \{\pm 1\}$, $\gamma \in \mathbb{C} \setminus \{0\}$, *and an exponential* $\chi \neq \chi \circ \sigma$ *with* $\chi(z_0) = \dfrac{\lambda - 1}{\gamma(\alpha - 1)}$ *and* $\chi \circ \sigma(z_0) = \dfrac{\lambda + 1}{\gamma(\alpha + 1)}$ *such that*

$$f = \frac{1 + \lambda}{2\gamma}\chi - \frac{1 - \lambda}{2\gamma}\chi \circ \sigma \quad \textit{and} \quad g = \frac{1 + \lambda}{2\gamma}\chi + \frac{1 - \lambda}{2\gamma}\chi \circ \sigma.$$

(7) $\alpha \neq \pm 1$, *and there exist a constant* $\beta \in \mathbb{C}^*$ *and an exponential* $\chi \neq \chi \circ \sigma$ *with* $\chi(z_0) = \dfrac{1}{\beta(\alpha - 1)}$ *and* $\chi \circ \sigma(z_0) = \dfrac{1}{\beta(\alpha + 1)}$ *such that*

$$f = \frac{\chi + \chi \circ \sigma}{2\beta} \quad \textit{and} \quad g = \frac{\chi - \chi \circ \sigma}{2\beta}.$$

(8) $\alpha \neq \pm 1$, *and there exist a constant* $\beta \in \mathbb{C}^*$, *an exponential* $\chi = \chi \circ \sigma$ *and a non-zero function* ϕ_χ *satisfying (1.2) with* $\chi(z_0) = \dfrac{\varepsilon}{\beta(\varepsilon\alpha - 1)}$ *and* $\phi_\chi(z_0) = -\dfrac{1}{\beta(\varepsilon\alpha - 1)^2}$, *where* $\varepsilon = \pm 1$, *such that*

$$f = (\chi + \varepsilon\phi_\chi)/\beta \quad \text{and} \quad g = \phi_\chi/\beta.$$

(9) $\alpha \neq \pm 1$, *and there exist a constant* $\lambda \neq -\varepsilon$, *an exponential* $\chi = \chi \circ \sigma$ *and a non-zero even function* ϕ_χ *satisfying (1.2) with*

$$\chi(z_0) = \frac{\lambda + \varepsilon}{\gamma(\alpha + \varepsilon)} \quad \text{and} \quad \phi_\chi(z_0) = \frac{(\lambda - \alpha)(1 + \varepsilon\lambda)}{\gamma(\alpha + \varepsilon)^2},$$

where $\varepsilon = \pm 1$, *such that*

$$f = (\lambda\chi + \phi_\chi)/\gamma \quad \text{and} \quad g = (\chi - \varepsilon\phi_\chi)/\gamma.$$

(10) $\alpha \neq \pm 1$, *and there exist an exponential* $\chi = \chi \circ \sigma$ *with* $\chi(z_0) \neq 0$, *and non-zero function* Φ_χ *satisfying (1.9) such that*

$$f = \frac{1}{1 + \varepsilon\alpha}\Phi_\chi + \alpha\chi(z_0)\chi \quad \text{and} \quad g = \frac{-\varepsilon}{1 + \varepsilon\alpha}\Phi_\chi + \chi(z_0)\chi,$$

where $\varepsilon = \pm 1$.

If S *is a topological semigroup and* $f \in C(S)$ *then:*
(i) $g \in C(S)$ *in (1), (2), (4), (6)-(9); (10) if* $\alpha = 0$.
(ii) $g \in C(S)$ *in (10) if* $\sigma \in C(S)$ *for* $\alpha \neq 0$.
(iii) $g \in C(S)$ *in (3) if* $q \neq -\alpha$.
(iv) $g \in C(S)$ *in (5) if* $q^2 \neq \delta^2$.

Proof: By taking μ equal to the Dirac measure δ_{z_0} in Theorem 4.1.

□

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References

1. Ajebbar, O., Elqorachi, E., *The Cosine-Sine functional equation on a semigroup with an involutive automorphism.* Aequat. Math. 91(6), 1115-1146, (2017).
2. Ajebbar, O., Elqorachi, E., Jafar, A., *Integral Kannappan-sine subtraction and addition law on semigroups.* arXiv:2504.21057v1
3. Akkouchi, M., Bouikhalene, B., Elqorachi, E., Functional Equations and μ-Spherical Functions. Georgian Mathematical Journal, vol. 15, no. 1, 1-20, (2008).
4. Aserrar, Y., Elqorachi, E., *A generalization of the cosine addition law on semigroups.* Aequat. Math. 97, 787-804, (2023).
5. Aserrar, Y., Elqorachi, E., *Cosine and Sine addition and subtraction law with an automorphism.* Annales Mathematicae Silesianae 38(2), 155-176, (2024).
6. Ebanks, B., *Around the Sine Addition Law and d'Alembert's Equation on Semigroups.* Results Math. 77(1), Paper no. 11, (2022).
7. Ebanks, B., *Cosine subtraction laws.* Aequat. Math. 97, 995-1010, (2023).
8. Elqorachi, E., Akkouchi, M., Bakali, A., Bouikhalene, B., Badora's Equation on Non-Abelian Locally Compact. Groups Georgian Math. J. Vol. 11, no. 3, 449-466, (2004).
9. Elqorachi, E., Redouani, A., *Trigonometric formulas and μ-spherical functions.* Aequat. math. 72, 60-77, (2006).
10. Jafar, A., Ajebbar, O. and Elqorachi, E., *A Kannappan-cosine functional equation on semigroups.* Annales mathematicae silesianae 39(1), 143-154, (2025) .
11. Jafar, A., Ajebbar, O. and Elqorachi, E., *A generalization of the Kannappan-sine addition law on semigroups.* Aequat. Math. 99, 1403-1420, (2025).

12. Kabbaj, S., Tial, M. and Zeglami, D., *The Integral Cosine Addition and Sine Subtraction Laws*. Results Math 73, 97, (2018).

13. Stetkær, H., *Functional Equations on Groups.* World Scientific Publishing Company, Singapore (2013).

14. Stetkær, H., *The cosine addition law with an additional term.* Aequat. Math. 90(6), 1147–1168, (2016).

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