Bol. Soc. Paran. Mat.
©SPM – E-ISSN-2175-1188
SPM: www.spm.uem.br/bspm

(3s.) **v. 2026 (44) 5** : 1–7.
ISSN-0037-8712
doi:10.5269/bspm.81119

Ideal (λ, μ) −Statistical Convergence of Order α with Respect to Seminorm q

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ABSTRACT: In this paper, we introduce a new convergence concept of Ideal (λ, μ) -Statistical Convergence of order α with respect to seminorm q. Our main objective is to improve and generalize several existing results related to sequence spaces. Furthermore, we establish detailed inclusion relations among the newly defined sequence spaces and the existing ones. Finally, a fruitful open problem is presented for future research.

Keywords: Seminorm q, double statistical convergence, pringsheim limit.

Contents

1. Introduction

One of the most important research areas of Summability Theory is examining the notion of statistical convergent sequences. The important of this area came to prominent after the examination of Fridy and Šalát in [12] and [23], respectively. Additionally, Statistical convergence has important applications in several areas of mathematical analysis. It is widely used in summability theory to study sequences and series that fail to converge in the classical sense but exhibit regular behavior on average. The relationship between the summability theory and statistical convergence has been introduced by Schoenberg [37]. The natural density of the set U, where $U \subset \mathbb{N}$ has been given by

$$\delta(U) = \lim_n \frac{1}{n} |\{k < n : |k \in U|\}|$$

where $|.|$ used for the order of an enclosed set. Moreover, sequence $x = (x_k)$ statistically convergent to the number x_0, if for arbitrary $\varepsilon > 0$,

$$U(\varepsilon) = \{k \in \mathbb{N} : |x_k - x_0| > \varepsilon\}$$

has zero natural density in [11]. In this case we write if $U \subset \mathbb{N}$ is finite set, then $\delta(U) = 0$ and for any set $U \subset \mathbb{N}$, $\delta(U^c) = 1 - \delta(U)$. It is represented symbolically S, the set of all statistically convergent sequences.

The concept has been extended to functional analysis, particularly in normed and Banach spaces, to analyze sequences of functions and operators under generalized convergence notions in [4]. Moreover, statistical convergence is related to probability theory in [7], especially to convergence in probability, and has been further generalized to settings such as ideal convergence and fuzzy normed spaces in [32].

Afterward, the concept of $\lambda-$statistically convergent sequences was defined by Mursaleen [17]. Moreover, the definition of $[V, \lambda]-$summability was extended by Leindler [15]. On the other hand, in [5] and [6] a different direction were given to the study of statistical convergence where the notion of statistical convergence of order α, $0 < \alpha < 1$ was introduced by replacing n by n^α in the denominator in the definition of statistical convergence, also [1] for related works.

In other respects, in numerous fields of science and engineering, we often deal with double sequences, such as sequences of matrices. There are certainly cases where the concept of classical convergence is inadequate, or the structure of the underlying space does not meet our requirements. To address these

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2020 *Mathematics Subject Classification:* 40A35.
Submitted December 20, 2025. Published February 14, 2026

challenges, it becomes necessary to develop alternative notions of convergence or new measurement tools that offer a more effective and suitable framework for analysis.

Now let us present the definition of convergent in Prinsgheim sense [22] which is as follows: A double sequence $x = (x_{k,l})$ of real numbers is said to be convergent to $L \in \mathbb{R}$ in Pringsheim sense if for $\varepsilon > 0$, there exists $N_\varepsilon \in \mathbb{N}$ such that $|x_{k,l} - L| < \varepsilon$, whenever $k, l > N_\varepsilon$. In this case, we denote such limit as follow:

$$P - \lim_{k,l\to\infty} x_{k,l} = L.$$

The Pringsheim limit plays a central role in the theory of double sequences and serves as a basis for defining generalized convergence concepts, including statistical convergence and ideal convergence for double sequences.

Definition 1.1 *Let $K \subseteq \mathbb{N} \times \mathbb{N}$ be a two-dimensional set of positive integers. The double natural density of K is defined by*

$$\delta_2(K) := P - \lim_{m,n\to\infty} \frac{1}{mn} \left|\{(m,n) \in \mathbb{N} \times \mathbb{N} : (m,n) \in K\}\right|.$$

A double sequence $x = (x_{k,l})$ of real numbers is statistically convergent to L if for each $\varepsilon > 0$,

$$P - \lim_{m,n\to\infty} \frac{1}{mn} \left|\{(k,l) \in \mathbb{N} \times \mathbb{N} : k \leq m,\ l \leq n : |x_{k,l} - L| \geq \varepsilon\}\right| = 0.$$

In this case, we write $S_2 - \lim x = L$ and denote the set of all statistically convergent double sequences with S_2 [16].

Recently, Mursaleen et al. in [18] studied generalized statistical convergence and statistical core of double sequences. Let $\lambda = (\lambda_m)$ and $\mu = (\mu_n)$ are two non-decreasing sequences of positive real numbers such that each tending to ∞ with $\lambda_{m+1} \leq \lambda_m + 1, \lambda_1 = 1$ and $\mu_{n+1} \leq \mu_n + 1$, $\mu_1 = 1$. The collection of such sequences (λ, μ) will be denoted by Δ.
Let $K \subseteq \mathbb{N} \times \mathbb{N}$ be a two-dimensional set of positive integers then the (λ, μ) density of K is defined as

$$\delta_{r,s}(K) = P - \lim_{m,n\to\infty} \frac{1}{\lambda_m \mu_n} \left| \left\{ \begin{array}{c} m - \lambda_m + 1 \leq k \leq m; \\ n - \mu_n + 1 \leq l \leq n : (k,l) \in K \end{array} \right\} \right|$$

provided that the limit exists, where $I_m = [m - \lambda_m + 1, m]$ and $J_n = [n - \mu_n + 1, n]$ and $\lambda_{m,n} = \lambda_m \mu_n$ in [18].

Throughout this paper, we shall denote $(k \in I_m, l \in J_n)$ by $(k,l) \in I_{m,n}$. We remark that for $I_m = m$ and $I_n = n$, the above density reduces to the double natural density. We say that $x = (x_{k,l})$ is $(\lambda\mu)$ $-$statistically convergent to the number L if, for every $\varepsilon > 0$,

$$P - \lim_{m,n\to\infty} \frac{1}{\lambda_{m,n}} \left|\{(k,l) \in I_{m,n} : |x_{k,l} - L| \geq \varepsilon\}\right| = 0.$$

We denote this by $S_{\lambda,\mu} - \lim x = L$.

Before establishing to the main results of this paper, one way to generalize the original definition of statistical convergence is to define statistical convergence with respect to an ideal of subsets. Sequential convergence with respect to an ideal was introduced by Kostyrko, Šalát and Wilczyński [14]. Later it was further studied by Dem [10], Das, Savas, and Ghosal in [8], Savas ([26], [27], [28], [29], [30], [31]), Savas and Gumus [33], and many others. More applications of ideals can be found in [9]. An admissible ideal of $\mathbb{N}\times\mathbb{N}$ is called strongly admissible if $\{i\}\times\mathbb{N}$ and $\mathbb{N}\times\{i\}$ belong to for each $i \in \mathbb{N}$. It is clear to see that a strongly admissible ideal is also admissible. Let $\mathcal{I}_0 = \{A \subset \mathbb{N} \times \mathbb{N} : (\exists m(A) \in \mathbb{N})\,(i,j \geq m\,(A) \Rightarrow (i,j) \notin A)\}$. Then $\mathcal{I}_0$ is a proper strongly admissible ideal and $\mathcal{I}_2$ is admissible if and only if $\mathcal{I}_0 \subset \mathcal{I}_2$.

Now let $\mathcal{I}_2$ be a proper admissible ideal in $\mathbb{N}\times\mathbb{N}$. A double sequence $x = (x_{k,l})$ of real numbers is said to be convergent to L with respect to the ideal $\mathcal{I}_2$, if for every $\varepsilon > 0$, $A(\varepsilon) = \{(k,l) \in \mathbb{N} \times \mathbb{N} : |x_{k,l} - L| \geq \varepsilon\} \in \mathcal{I}_2$. In this case, we write $\mathcal{I}_2 - \lim_{k,l\to\infty} x_{k,l} = L$.

Note that if $\mathcal{I}_2$ is the ideal $\mathcal{I}_0$, then $\mathcal{I}_2-$convergence coincides with the convergence in Prinsheim sense and if we take $\mathcal{I}_d=\{A\subset\mathbb{N}\times\mathbb{N}:\delta_2(A)=0\}$, then $\mathcal{I}_d-$convergence becomes statistical convergence for double sequences.

Recently, the ideas of $\mathcal{I}-$statistical convergence and $\mathcal{I}-\lambda-$statistical convergence, which are more general than statistical convergence and $\lambda-$statistical convergence. Following these works, Balcerzak and Dems in [2] introduced the notion of $\mathcal{I}-$convergence in measure which is more general than convergence. In [24] Savas and Das presented the notions of $\mathcal{I}-$statistical convergence and $\mathcal{I}_\lambda-$statistically, and many researchers studied this concept. (see [13], [19], [20], [25]). Additionally, Belen and Yildirim in [3] defined some methods of summability by unifying the notions of statistical convergence, $(\lambda\mu)-$statistical convergence, and ideal convergence for double sequences.

In this paper, we shall consider the following definitions to get some new summability methods and examine the relationship betwen them.

As discussed by Schaefer and Wolff [36], let X be a vector space over the field $K=\mathbb{R}$ or $\mathbb{C}$. A function $q:X\to[0,\infty)$ is called a seminorm if it satisfies the following properties for all $x,y\in X$ and all scalars $\alpha\in K$; (i) $q(\alpha x)=|\alpha|\,q(x)$ and (ii) $q(x,y)\leq q(x)+q(y)$. A topological vector space X is locally convex if and only if its topology can be generated by a family of continuous seminorms q. In this case, (x_k) converges $x_0\in X$ if and only if $q(x_k-x_0)\to0$.

2. Main Results

We begin this section with the following definitions which will be used in establishing the main results of this paper.

Definition 2.1 *Let X be a locally convex Hausdorff topological linear space whose topology is determined by a set U of continuous seminorms q and $(x_{k,l})\in U$. Let a double sequence $x=(x_{k,l})$ is said to be double $\mathcal{I}_2-$statistically convergent of order α to x_0, where $0<\alpha<1$, if for every given $\varepsilon>0$ and $\delta>0$*

$$\left\{(m,n)\in\mathbb{N}\times\mathbb{N}:\frac{1}{(mn)^\alpha}\left|k\leq m,l\leq n:q\left(x_{k,l}-x_0\right)\geq\varepsilon\right|\geq\delta\right\}\in\mathcal{I}_2.$$

We say that a sequence $x=(x_{k,l})$ is double $\mathcal{I}_2-$statistically convergent of order α to x_0 or $S^\alpha(\mathcal{I}_2)(q)-$convergent to x_0, where $0<\alpha\leq1$, it is represented symbolically by $x_{k,l}\overset{P}{\to}S^\alpha(\mathcal{I}_2)(q)$. The set of all double$-I_2-$statistically convergent of order α sequences will be presented by $S^\alpha(\mathcal{I}_2)(q)$.

Remark 2.1 *If we consider $\mathcal{I}_2=\mathcal{I}_2(fin)$, then $S^\alpha(\mathcal{I}_2)(q)-$convergence coincides with double statistical convergence of order α with respect to seminorm q. For an arbitrary ideal $\mathcal{I}_2$ and for $\alpha=1$, it coincides with double $\mathcal{I}_2-$statistical convergence. When $\mathcal{I}_2=\mathcal{I}_2(fin)$, and $\alpha=1$ it is only double statistical convergence with respect to seminorm q.*

Definition 2.2 *Let the sequence $x=(x_{k,l})$. If for a given $\varepsilon>0$ and $\delta>0$,*

$$\left\{(m,n)\in\mathbb{N}\times\mathbb{N}:\frac{1}{\lambda^\alpha_{m,n}}\left|\{(k,l)\in I_{m,n}:q\left(x_{k,l}-x_0\right)\geq\varepsilon\}\right|\geq\delta\right\}\in\mathcal{I}_2$$

We say that all these types of sequences will represent by $S^\alpha_{\lambda,\mu}(\mathcal{I}_2)(q)-$statistically convergent of order α or $x_{k,l}\overset{P}{\to}x_0\left(S^\alpha_{\lambda,\mu}(\mathcal{I}_2)(q)\right)$.

Remark 2.2 *If we consider $\mathcal{I}_2=\mathcal{I}_2(fin)$, $S^\alpha_{\lambda,\mu}(\mathcal{I}_2)(q)-$convergence reduces $S^\alpha(\mathcal{I}_2)(q)-$statistical convergence of order α. For an arbitrary ideal $\mathcal{I}_2$ and for $\alpha=1$, it coincides with $S_{\lambda,\mu}(\mathcal{I}_2)(q)-$statistically convergence. Finally for $\mathcal{I}_2=\mathcal{I}_2(fin)$ and $\alpha=1$ it becomes $S_{\lambda,\mu}(q)-$statistical convergence. Also, if we consider $\lambda_n=n$ and $\mu_m=m$, we have definition 2.1 from definition 2.3.*

Definition 2.3 *Let X be a locally convex Hausdorff topological linear space whose topology is determined by a set U of continuous seminorms q and $(x_{k,l}) \in U$. Let $\lambda, \mu \in \Delta, \alpha \in (0,1]$ be any real number. If there is a real number x_0 such that*

$$\left\{(m,n) \in \mathbb{N} : \frac{1}{\lambda_{m,n}^{\alpha}} \sum_{(k,l)\in I_{m,n}} q\left(x_{k,l} - x_0\right) \geq \varepsilon\right\} \in \mathcal{I}_2$$

then we call that a double sequence $x = (x_{k,l})$ is $[V, \lambda\mu, q]^{\alpha}(\mathcal{I}_2)$ $-$summable to x_0.

Remark 2.3 *If we take $\mathcal{I}_2 = \mathcal{I}_2(fin)$, then $[V, \lambda\mu, q]^{\alpha}(\mathcal{I}_2)$ $-$summability reduce to $[V, \lambda\mu, q]^{\alpha}$ $-$summability.*

Theorem 2.1 *Suppose $0 < \alpha \leq \beta \leq 1$. Then we obtain $S_{\lambda,\mu}^{\alpha}(\mathcal{I}_2)(q) \subset S_{\lambda,\mu}^{\beta}(\mathcal{I}_2)(q)$.*

Proof: If $0 < \alpha \leq \beta \leq 1$, then

$$\frac{\left|\left\{(k,l) \in I_{m,n} : q\left(x_{k,l} - x_0\right) \geq \varepsilon\right\}\right|}{\lambda_{m,n}^{\beta}} \leq \frac{\left|\left\{(k,l) \in I_{m,n} : q\left(x_{k,l} - x_0\right) \geq \varepsilon\right\}\right|}{\lambda_{m,n}^{\alpha}}$$

and for any $\delta > 0$,

$$\begin{aligned} & \left\{(m,n) \in \mathbb{N} \times \mathbb{N} : \frac{\left|\left\{(k,l) \in I_{m,n} : q\left(x_{k,l} - x_0\right) \geq \varepsilon\right\}\right|}{\lambda_{m,n}^{\beta}} \geq \delta\right\} \\ \subset & \left\{(m,n) \in \mathbb{N} \times \mathbb{N} : \frac{\left|\left\{(k,l) \in I_{m,n} : q\left(x_{k,l} - x_0\right) \geq \varepsilon\right\}\right|}{\lambda_{m,n}^{\alpha}} \geq \delta\right\}. \end{aligned}$$

Thus, if the set on the right-hand side belongs to the ideal $\mathcal{I}_2$ then the set on the left-hand side also belongs to $\mathcal{I}_2$. This shows that

$$S_{\lambda,\mu}^{\alpha}(\mathcal{I}_2)(q) \subset S_{\lambda,\mu}^{\beta}(\mathcal{I}_2)(q).$$

□

Corollary 2.1 *If a double sequence is $\mathcal{I}_2^{\lambda\mu}$ $-$double statistically convergent of order α to x_0 for some $0 < \alpha \leq 1$, then it is $\mathcal{I}_2^{\lambda\mu}$ $-$double statistically convergent to x_0, i.e., $S_{\lambda,\mu}^{\alpha}(\mathcal{I}_2)(q) \subset S_{\lambda,\mu}(\mathcal{I}_2)(q)$.*

Similaary, we can prove the following theorem

Theorem 2.2 *Suppose $0 < \alpha \leq \beta \leq 1$. Then*

(i) $S^{\alpha}(\mathcal{I}_2)(q) \subset S^{\beta}(\mathcal{I}_2)(q)$

(iii) In particular $S^{\alpha}(\mathcal{I}_2)(q) \subset S(\mathcal{I}_2)(q)$.

Theorem 2.3 $S^{\alpha}(\mathcal{I}_2)(q) \subset S_{\lambda,\mu}^{\alpha}(\mathcal{I}_2)(q)$ *if* $P - \liminf_{m,n\to\infty} \frac{\lambda_{mn}^{\alpha}}{(mn)^{\alpha}} > 0$.

Proof: For any $\varepsilon > 0$,

$$\begin{aligned} \frac{1}{(mn)^{\alpha}} \left|\left\{k \leq m,\, l \leq n : q\left(x_{k,l} - x_0\right) \geq \varepsilon\right\}\right| & \geq \frac{1}{(mn)^{\alpha}} \left|\left\{(k,l) \in I_{m,n} : q\left(x_{k,l} - x_0\right) \geq \varepsilon\right\}\right| \\ & \geq \frac{\lambda_{mn}^{\alpha}}{(mn)^{\alpha}} \frac{1}{\lambda_{mn}^{\alpha}} \left|\left\{(k,l) \in I_{m,n} : q\left(x_{k,l} - x_0\right) \geq \varepsilon\right\}\right|. \end{aligned}$$

If

$$\liminf_{m,n\to\infty} \frac{\lambda_{mn}^{\alpha}}{(mn)^{\alpha}} = u,$$

then we obtain $\left\{(m,n)\in\mathbb{N}\times\mathbb{N}:\frac{\lambda_{mn}^{\alpha}}{(mn)^{\alpha}}<\frac{u}{2}\right\}$ is finite. For $\delta>0$,

$$\begin{aligned}
&\left\{(m,n)\in\mathbb{N}\times\mathbb{N}:\frac{1}{\lambda_{m,n}^{\alpha}}\left|\left\{(k,l)\in I_{m,n}:q\left(x_{k,l}-x_{0}\right)\geq\varepsilon\right\}\right|\geq\delta\right\}\\
\subset\ &\left\{(m,n)\in\mathbb{N}\times\mathbb{N}:\frac{1}{(mn)^{\alpha}}\left|\left\{k\leq m,\ l\leq n:q\left(x_{k,l}-x_{0}\right)\geq\varepsilon\right\}\right|\geq\frac{u}{2}\delta\right\}\\
&\cup\left\{(m,n)\in\mathbb{N}\times\mathbb{N}:\frac{\lambda_{m,n}^{\alpha}}{(mn)^{\alpha}}<\frac{u}{2}\right\}.
\end{aligned}$$

The set on the right-hand side belongs to $\mathcal{I}_2$. □

Theorem 2.4 *Suppose* α, $0<\alpha<1$ *and if* $(\lambda_{m,n})\in\Delta$ *be such that* $P-\lim\limits_{mn}\frac{(m-\lambda_m)(n-\mu_n)}{(mn)^{\alpha}}=0$, *then we obtain* $S_{\lambda,\mu}^{\alpha}\left(\mathcal{I}_2\right)(q)\subset S^{\alpha}\left(\mathcal{I}_2\right)(q)$.

Proof: Suppose $\delta>0$. We know that $P-\lim\limits_{mn}\frac{(\lambda_{m,n})^{\alpha}}{(mn)^{\alpha}}=0$, so we can select $(r,s)\in\mathbb{N}\times\mathbb{N}$ such that $\frac{(m-\lambda_m)(n-\mu_n)}{(mn)^{\alpha}}<\frac{\delta}{2}$, for all $m\geq r$ and $n\geq s$. We write for given $\varepsilon>0$,

$$\begin{aligned}
&\frac{1}{(mn)^{\alpha}}\left|\left\{k\leq m,\ l\leq n:q\left(x_{k,l}-x_{0}\right)\geq\varepsilon\right\}\right|\\
=\ &\frac{1}{(mn)^{\alpha}}\left|\left\{k\leq m-\lambda_m,\ l\leq n-\mu_n:q\left(x_{k,l}-x_{0}\right)\geq\varepsilon\right\}\right|\\
&+\frac{1}{(mn)^{\alpha}}\left|\left\{(k,l)\in I_{m,n}:q\left(x_{k,l}-x_{0}\right)\geq\varepsilon\right\}\right|\\
\leq\ &\frac{(m-\lambda_m)(n-\mu_n)}{(mn)^{\alpha}}+\frac{1}{(mn)^{\alpha}}\left|\left\{(k,l)\in I_{m,n}:q\left(x_{k,l}-x_{0}\right)\geq\varepsilon\right\}\right|\\
\leq\ &\frac{\delta}{2}+\frac{1}{(mn)^{\alpha}}\left|\left\{(k,l)\in I_{m,n}:q\left(x_{k,l}-x_{0}\right)\geq\varepsilon\right\}\right|,
\end{aligned}$$

for all $m\geq r$ and $n\geq s$. Thus,

$$\begin{aligned}
&\left\{(m,n)\in\mathbb{N}\times\mathbb{N}:\frac{1}{(mn)^{\alpha}}\left|\left\{k\leq m,\ l\leq n:q\left(x_{k,l}-x_{0}\right)\geq\varepsilon\right\}\right|\geq\delta\right\}\\
\subset\ &\left\{(m,n)\in\mathbb{N}\times\mathbb{N}:\frac{1}{\lambda_{m,n}^{\alpha}}\left|\left\{(k,l)\in I_{m,n}:q\left(x_{k,l}-x_{0}\right)\geq\varepsilon\right\}\right|\geq\frac{\delta}{2}\right\}\cup B,
\end{aligned}$$

where B is the union of the first m_0 and the first n_0 columns of the double sequence. If $S_{\lambda,\mu}^{\alpha}\left(\mathcal{I}_2\right)(q)-P-\lim x=x_0$, then the set on the right-hand side belongs to $\mathcal{I}_2$. Hence, the set on the left-hand side also belongs to $\mathcal{I}_2$. This shows that $x=(x_{k,l})$ is $S^{\alpha}\left(\mathcal{I}_2\right)(q)-$statistically convergent to x_0. □

Theorem 2.5 *Suppose* $(\lambda_{m,n})\in\Delta$, *Then* $x_{k,l}\rightarrow x_0\left[V,\lambda\mu,q\right]^{\alpha}\left(\mathcal{I}_2\right)\Rightarrow x_{k.l}\rightarrow x_0\left(S_{\lambda,\mu}^{\alpha}\left(\mathcal{I}_2\right)(q)\right)$.

Proof: Suppose $\varepsilon>0$ and $x_{k,l}\rightarrow x_0\left[V,\lambda\mu,q\right]^{\alpha}\left(\mathcal{I}_2\right)$. We obtain

$$\begin{aligned}
\sum_{(k,l)\in I_{m,n}}q\left(x_{k,l}-x_{0}\right)\ &\geq\sum_{\substack{(k,l)\in I_{m,n}\\ \&q\left(x_{k,l}-x_{0}\right)>\varepsilon}}q\left(x_{k,l}-x_{0}\right)\geq\varepsilon\\
&\geq\ \varepsilon\left|\left\{(k,l)\in I_{m,n}:q\left(x_{k,l}-x_{0}\right)\geq\varepsilon\right\}\right|.
\end{aligned}$$

Then for a given $\delta > 0$,

$$\begin{aligned} \frac{1}{(mn)^{\alpha}} \left|\left\{(k,l) \in I_{m,n} : q\left(x_{k,l} - x_0\right) \geq \varepsilon\right\}\right| &\geq \delta \\ &\Rightarrow \frac{1}{\lambda_{m,n}^{\alpha}} \sum_{(k,l)\in I_{m,n}} q\left(x_{k,l} - x_0\right) \geq \varepsilon\delta \end{aligned}$$

that is

$$\begin{aligned} &\left\{(m,n) \in \mathbb{N} \times \mathbb{N} : \frac{1}{\lambda_{m,n}^{\alpha}} \left|\left\{(k,l) \in I_{m,n} : q\left(x_{k,l} - x_0\right) \geq \varepsilon\right\}\right| \geq \delta\right\} \\ \subset &\left\{(m,n) \in \mathbb{N} \times \mathbb{N} : \frac{1}{\lambda_{m,n}^{\alpha}} \left\{\sum_{(k,l)\in I_{m,n}} q\left(x_{k,l} - x_0\right)\right\} \geq \varepsilon\delta\right\}. \end{aligned}$$

We know that $x_{k,l} \to x_0 \left[V, \lambda\mu, q\right]^{\alpha}\left(\mathcal{I}_2\right)$, so the right hand side belongs to $\mathcal{I}_2$, so $x_{k,l} \to x_0 \left(S_{\lambda,\mu}^{\alpha}\left(\mathcal{I}_2\right)(q)\right)$. $\square$

3. Conclusion

We have developed a comprehensive framework for double $\mathcal{I}_2$-statistical convergence of order α in locally convex Hausdorff topological vector spaces equipped with a family of continuous seminorms. Our results generalize classical notions of statistical convergence and summability, showing how various types of double statistical convergence. This framework provides a unified approach to analyze the statistical behavior of double sequences in topological vector spaces, highlighting the role of seminorms in capturing convergence patterns. In particular, the inclusion relations and parameter-dependent results demonstrate that lower-order double statistical convergence implies higher-order convergence, and that summability methods imply statistical convergence in the double sequence context. Overall, these findings extend the classical theory of statistical convergence to double sequences in locally convex spaces, offering a flexible and robust toolset for future research in functional analysis, sequence spaces, and convergence methods in topological vector spaces. This work opens new avenues for studying statistical convergence phenomena in higher-dimensional or double-indexed sequences, as well as their applications in analysis and applied mathematics.

References

1. Altın, Y., Et, M., *Generalized statistical convergence of order alpha.* Journal of Inequalities and Applications, 2014, 2014:68.
2. Balcerzak, M., Dems, K., Komisarski, A., *Statistical Convergence and Ideal convergence for sequences of functions.* J. Math. Anal. Appl. 328 (2007), 715–729.
3. Beren, C, Yildirim, M., *On generalized statistical convergence of double sequences via ideals*, Ann Univ Ferrara 58 (2012), 11–20.
4. Connor, J. S., *A topological approach to statistical convergence.* Analysis, 10, 1990, 373–382
5. Çolak, R., *Statistical convergence of order alpha*, Modern Methods in Analysis and Its Applications, Anamaya Publishers, 2009, 121–129.
6. Çolak, R., Bektaş, Ç. *A. alpha -statistical convergence of order alpha*, Acta Mathematica Scientia, 31B(3), 2011, 953–959.
7. Das, P., Ghosal, S., Sam, S., *Statistical Convergence of order α in probability*, Arab. J. Math. Sci., 21 (2015), 253-265.
8. Das, P., Savas, E., Ghosal, S. Kr., *On generalizations of certain summability methods using ideals*, Appl. Math. Lett. 24 (9) (2011), 1509–1514.
9. Das, P., Kostyrko, P., Wilczynski, W., Malik, P., *I and I^*- convergence of double sequences*, Math. Slovaca, 58 (5) (2008), 605–620.
10. Dems, K., *On $I-$ Cauchy sequences*, Real Anal. Exhance, 30 (2004-2005), 123–128.
11. Fast, H., *Sur la convergence statistique*, Colloq. Math. 2 (1951), 241–244.

12. Fridy, J., *On statistical convergence*, Analysis 5 (1985), 301–313.
13. Gurdal, M., A ¸Sahiner, A., *Extremal I-limit points of double sequences*, Appl. Math. E-Notes, 8 (2008), 131–137.
14. Kostyrko, P., Šalát, T., Wilcznski, W., *$I-$convergence*, Real Anal. Exchange, 26 (2) (2000/2001), 669–686.
15. Liendler, L., *Über die verallgemeinerte de la Vallee-Poussinsche Summierbarkeit allgemeiner Orthogonalreihen*, (German), Acta Math. Acad.. Sci. Hungar., 16 (1965), 375–387.
16. Mursaleen, M., Edely, O. H., *Statistical convergence of double sequences*, J. Math. Anal. Appl., 288 (2003), 223–231.
17. Mursaleen, M., *$\lambda-$ Statistical Convergence*, Math. Slovaca, 50 (11) (2000), 111–115.
18. Mursaleen, M., Cakan, C., Mohiuddine, S. A., Savas, E., *Generalized Statistical Convergence and Statistical Core of Double Sequences*, Acta Mathematica Sinica, English Series Nov., 26 (11) (2010), 2131–2144.
19. Nabiev, A., Savas, E., Gurdal, M., *I-localized sequences in metric spaces*, Facta Univ. Ser. Math. Inform., 35(2) (2020), 459–469.
20. Nabiev, A., Pehlivan, S., Gurdal, M., *On I-Cauchy sequences*, Taiwanese J. Math., 11(2) (2007), 569-566, 2007.
21. Nuray, F., *$\lambda-$ Strongly Summable and $\lambda-$ Statistically Convergent Functions*, Iranian Journal of Science & Technology, Transaction A, 34 (A4) (2010), 335–339.
22. Pringsheim, A., *Zur theorie der zweifach unendlichen zahlen folgen*, Math. Ann. 53 (1900), 289–321.
23. Šalát, T., *On statistically convergent sequences of real numbers*, Math. Slovaca, 30 (1980), 139–150.
24. Savas, E., Das, P., *A generalized statistical convergence via ideals*, Appl. Math Lett. 24 (2011), 826–830.
25. Savas, E., Gurdal, M., *Ideal convergent function sequences in random 2-normed spaces*, Filomat, 30(3) (2016), 557–567.
26. Savas, E., *I_λ-double statistical convergence of order α in topological groups*, Ukraïn. Mat. Zh. 68 (9) (2016), 1251–1258.
27. Savas, E., *Δ^m-strongly summable sequences spaces in $2-$Normed Spaces defined by Ideal Convergence and a Orlicz Function*, App. Math. Comp., 217 (2010), 271–276.
28. Savas, E. *A- sequence spaces in $2-$normed space defined by ideal convergence and an Orlicz function*, Abst. Appl. Analys., Vol. Article ID 741382, (2011).
29. Savas, E., *On some new sequence spaces in $2-$normed spaces using Ideal convergence and an Orlicz functions*, J. Ineq. Appl. Article Number: 482392 DOI: 10.1155/2010/482392 (2010).
30. Savas, E., *On generalized statistical convergence in random $2-$normed space*, Iran. J. Sci. and Tech. Trans. (A4) (2012), 417–423.
31. Savas, E., *On $I-$ Asymptotically Lacunary Statistical Equivalent Sequences*, Advances in Difference Equations, 2013 (111) (2013).
32. Savas, E., *On statistical convergence in fuzzy normed linear spaces*, International Journal of Mathematics and Mathematical Sciences, 2010, Article ID 987089.
33. Savas, E., Gumus, H., *A Generalization on $I-$ Asymptotically Lacunary Statistical Equivalent Sequences*, J. Ineq. Appl. 2013 (270), 2013.
34. Savas, E., *On Generalized Statistically Convergent Functions via Ideals*, Appl. Math. Sci. 10 (3) (2016), 943–947.
35. Savas, R., *lambda-Double Statistical Convergence of Functions*, Filomat, 33 (2) (2019), 519–524.
36. Schaefer, W., *Topological Vector Spaces*, 2nd ed., Springer, 1999, Chapter II
37. Schoenberg, I. J. *The integrability of certain functions and related summability methods.* Amer. Math. Monthly 66 (1959), 361–375.
38. Steinhaus, H., *Sur la convergence ordinaire et la convergence asymptotique*, Colloq. Math. 2 (1951), 73–74.

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