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Local Criteria for u-S-flat Modules

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ABSTRACT: Let R be a commutative ring and S a multiplicative subset of R. Following Zhang's notion of uniformly S-flat modules, we investigate local and homological properties of flatness in this relative setting. We first establish finite local criteria for uniformly S-flat modules under the assumption that a finite family $(f_1, \dots, f_p)$ of elements of R meets S. In particular, we show that an R-module M is uniformly S-flat if and only if its localizations M_{f_i} are uniformly S_{f_i}-flat. We also obtain characterizations in terms of prime ideals disjoint from S. In the second part, we study the uniformly S-flat dimension of modules and the uniformly S-weak global dimension of rings. Using the above finite local techniques, we establish prime ideal formulas and finite local characterizations for these dimensions, extending classical results on weak global dimension to the uniformly S-flat framework.

Keywords: Commutative ring, modules, u-S-flat modules, u-S-flat dimension, u-S-weak global dimension.

Contents

1. Introduction

Throughout this paper, R denotes a commutative ring with identity and $S \subseteq R$ denotes a fixed multiplicative subset, that is, $1 \in S$ and $s_1 s_2 \in S$ for all $s_1, s_2 \in S$. Homological and finiteness conditions relative to such a subset S have played an important role in commutative algebra, especially in the study of localization, torsion theories and various S-analogues of classical notions.

Recall that an R-module M is called an *S-torsion module* if for every $m \in M$ there exists $s \in S$ such that $sm = 0$. This notion naturally leads to the hereditary torsion theory generated by S-torsion modules. Several finiteness conditions relative to S have been introduced and studied over the years. In particular, Anderson and Dumitrescu in [3] introduced the notion of *S-finite modules*, namely R-modules M such that sM is contained in a finitely generated submodule of M for some $s \in S$. Although this definition does not explicitly involve uniformity, it already suggests that a single element of S may control global properties of a module.

Motivated by this observation, Zhang introduced in [1] the notion of *uniformly S-torsion* modules. An R-module T is said to be uniformly S-torsion (or u-S-torsion) if there exists a single element $s \in S$ such that $sT = 0$. This condition is strictly stronger than ordinary S-torsion and captures a genuine uniformity phenomenon. Based on this idea, Zhang further defined *u-S-exact sequences* and *uniformly S-flat modules*. An R-module M is called *u-S-flat* if for every R-module N, the module $Tor_1^R(N, M)$ is uniformly S-torsion. Equivalently, there exists an element $s \in S$ (depending on N) such that $sTor_1^R(N, M) = 0$.

The notion of u-S-flatness provides a natural weakening of classical flatness, allowing controlled torsion governed by elements of S. It has been shown in [2] that this class of modules exhibits behaviors that are markedly different from those of flat modules: for instance, u-S-flat modules need not be closed under direct sums or direct limits, and the flatness of the localization M_S does not imply that M is u-S-flat.

The first aim of this paper is to develop new local and homological characterizations of u-S-flat modules. Our approach relies on localization techniques and on the behavior of the functor $Tor_1^R(-, -)$. In particular, we show that if $(f_1, \dots, f_p)$ is a finite family of elements of R whose generated ideal meets

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S, then the u-S-flatness of a module can be detected either through suitable uniform annihilators of the form $s_i f_i$ or through localizations at the elements f_i. This leads to characterizations of u-S-flatness in terms of prime ideals disjoint from S.

The second aim of the paper is to investigate homological dimensions adapted to this uniform setting. Following the ideas developed in [2], we consider the *u-S-flat dimension* of a module, defined as the least integer n such that $Tor^R_{n+1}(N, M)$ is uniformly S-torsion for all R-modules N. This naturally gives rise to the notion of *u-S-weak global dimension* of a ring. We establish prime ideal formulas for these invariants and show that they can be computed as suprema of classical flat or weak global dimensions of localizations R_P with $P \cap S = \varnothing$.

The paper is organized as follows. In Section 2, we study uniformly S-flat modules and provide several equivalent characterizations based on localization and annihilation properties. Section 3 is devoted to the study of u-S-flat dimensions and the u-S-weak global dimension, together with their fundamental properties and local descriptions.

2. u-S-Flat Modules and Localization

Lemma 2.1 *Let R be a commutative ring, $S \subseteq R$ a multiplicative subset, and $f_1, \dots, f_p \in R$ such that $(f_1, \dots, f_p) \cap S \neq \varnothing$. Let $s_1, \dots, s_p \in S$ and $n_1, \dots, n_p \in \mathbb{N}$. Then $S \cap (s_1 f_1^{n_1}, \dots, s_p f_p^{n_p}) \neq \varnothing$.*

Proof: Assume, by contradiction, that $S \cap (s_1 f_1^{n_1}, \dots, s_p f_p^{n_p}) = \varnothing$. Then there exists a prime ideal P of R such that $(s_1 f_1^{n_1}, \dots, s_p f_p^{n_p}) \subseteq P$ and $P \cap S = \varnothing$. Since $s_i f_i^{n_i} \in P$ and $s_i \in S$ while $P \cap S = \varnothing$, it follows that $f_i^{n_i} \in P$, hence $f_i \in P$ for all $i = 1, \dots, p$. Therefore $(f_1, \dots, f_p) \subseteq P$, which implies $(f_1, \dots, f_p) \cap S \subseteq P \cap S = \varnothing$, a contradiction with the hypothesis. □

Proposition 2.1 *Let M be an R-module and let $f_1, \dots, f_p \in R$ be such that $(f_1, \dots, f_p) = R$ Then M is flat if and only if M is u-f_i-flat for every $i \in \{1, \dots, p\}$.*

Proof: If M is flat, then $Tor^R_1(N, M) = 0$ for every R-module N. Hence $Tor^R_1(N, M)$ is annihilated by f_i for each i, and M is u-f_i-flat.

Conversely, assume that M is u-f_i-flat for every $i \in \{1, \dots, p\}$. Let N be an R-module and set $T := Tor^R_1(N, M)$. For each i, since M is u-f_i-flat, there exists an integer $n_i \geq 1$ such that $f_i^{n_i} T = 0$.

Let S be a multiplicative subset of R containing all invertible elements of R. Since $(f_1, \dots, f_p) = R$, we have $(f_1, \dots, f_p) \cap S \neq \varnothing$. By Lemma 2.1, there exists an element $s \in S$ such that $sT = 0$. As s is invertible in R, this implies that $T = 0$. Hence $Tor^R_1(N, M) = 0$ for all R-modules N, and therefore M is flat.

□

Theorem 2.1 *Let R be a commutative ring, $S \subseteq R$ a multiplicative subset, and $f_1, \dots, f_p \in R$ such that $(f_1, \dots, f_p) \cap S \neq \varnothing$. Let M be an R-module and, for each i, let S_{f_i} denote the image of S in R_{f_i}. Then the following conditions are equivalent:*

- *(i) M is u-S-flat as an R-module;*
- *(ii) for each $i = 1, \dots, p$, the localization M_{f_i} is u-S_{f_i}-flat as an R_{f_i}-module.*

Proof:

$(i) \Rightarrow (ii)$. Assume that M is u-S-flat. Fix $i \in \{1, \dots, p\}$ and let N' be any R_{f_i}-module. By restriction of scalars, N' is also an R-module. By Zhang's criterion, there exists $s \in S$ such that $s \cdot Tor^R_1(N', M) = 0$. Localizing at f_i gives $s \cdot (Tor^R_1(N', M))_{f_i} = 0.\big(Tor^R_1(N', M)\big)_{f_i} \cong Tor^{R_{f_i}}_1(N', M_{f_i})$. Thus the image of s in $S_{f_i} \subseteq R_{f_i}$ annihilates $Tor^{R_{f_i}}_1(N', M_{f_i})$. Since N' was arbitrary, it follows again from Zhang's characterization (applied over R_{f_i} with the multiplicative set S_{f_i}) that M_{f_i} is u-S_{f_i}-flat as an R_{f_i}-module. This holds for every i, hence (ii).

$(ii) \Rightarrow (i)$. Assume now that M_{f_i} is u-S_{f_i}-flat as an R_{f_i}-module for each $i = 1, \ldots, p$. We want to show that M is u-S-flat. By Zhang's criterion, it suffices to prove that for every R-module N, the module $T = Tor_1^R(N, M)$ is u-S-torsion. Fix an R-module N. For each i, consider the R_{f_i}-module N_{f_i}. By the u-S_{f_i}-flatness of M_{f_i} over R_{f_i}, there exists $s_i \in S$ whose image in S_{f_i} annihilates $Tor_1^{R_{f_i}}(N_{f_i}, M_{f_i})$. or $Tor_1^{R_{f_i}}(N_{f_i}, M_{f_i}) \cong \big(Tor_1^R(N, M)\big)_{f_i} = T_{f_i}$. Thus s_i annihilates T_{f_i} for each i. Concretely, this means that for every i, there exists an integer $n_i \geq 0$ such that $f_i^{n_i} s_i T = 0$. Since $(f_1, \ldots, f_p) \cap S \neq \varnothing$, the lemma 2.1 proved above implies $S \cap (s_1 f_1^{n_1}, \ldots, s_p f_p^{n_p}) \neq \varnothing$. Hence there exists an element $s \in S$ such that $s \in (s_1 f_1^{n_1}, \ldots, s_p f_p^{n_p})$. Therefore $sT = 0$ Since the same argument works for every $x \in T$, we see that T is u-S-torsion. By Zhang's criterion, M is u-S-flat.

□

Theorem 2.2 *Let R be a commutative ring and $S \subseteq R$ a multiplicative subset. For an R-module M, the following assertions are equivalent:*

(i) *M is u-S-flat;*

(ii) *M_P is flat over R_P for every prime ideal P such that $P \cap S = \varnothing$;*

Proof: $(\Rightarrow)$ Assume that M is u-S-flat. Let P be a prime ideal such that $P \cap S = \varnothing$. For any R-module N, the u-S-flatness of M implies that there exists $s \in S$ such that $s \cdot Tor_1^R(N, M) = 0$. Localizing at P, and using the flatness of localization, we obtain $Tor_1^{R_P}(N_P, M_P) \cong \big(Tor_1^R(N, M)\big)_P$. Since $s \notin P$, it becomes invertible in R_P, hence $Tor_1^{R_P}(N_P, M_P) = 0$. Thus M_P is flat over R_P.

$(\Leftarrow)$ Conversely, assume that M_P is flat for every prime ideal P such that $P \cap S = \varnothing$. Let N be any R-module and set $T = Tor_1^R(N, M)$. We claim that T is u-S-torsion. Suppose by contradiction that T is not u-S-torsion. Then $Ann(T) \cap S = \varnothing$. Hence there exists a prime ideal P of R such that $Ann(T) \subseteq P$ and $P \cap S = \varnothing$. Localizing at P, we get $T_P = Tor_1^{R_P}(N_P, M_P)$. But M_P is flat by assumption, so $Tor_1^{R_P}(N_P, M_P) = 0$, hence $T_P = 0$. This implies that there exists $r \in R \setminus P$ such that $rT = 0$, i.e. $r \in Ann(T)$, contradicting $Ann(T) \subseteq P$.Therefore $Ann(T) \cap S \neq \varnothing$, and so T is u-S-torsion. Since N was arbitrary, M is u-S-flat. □

3. u-S-Flat Dimension and u-S-Weak Global Dimension

The u-S-flat dimension of an R-module M, introduced in [2], is defined as the smallest integer n (if it exists) such that the higher Tor modules $Tor_{n+1}^R(N, M)$ are uniformly S-torsion for all R-modules N.

In the same spirit, the *u-S-weak global dimension* of the ring R is defined as the supremum of the u-S-flat dimensions taken over all R-modules. That is, u-S-w.gl.dim$(R) = \sup\{\, u\text{-}S\text{-fd}_R(M) \mid M \text{ is an } R\text{-module}\,\}$.

Proposition 3.1 *Let R be a commutative ring, $S \subseteq R$ a multiplicative subset, and let $f \in R$. For every R-module M, u-S_f-$fd_{R_f}(M_f) \leq u$-S-$fd_R(M)$, where S_f denotes the image of S in R_f.*

Proof: Let $n = u$-S-fd$_R(M)$. By definition, for every R-module N we have $Tor_{n+1}^R(N, M)$ is u-S-torsion. Localizing at f and using the flatness of R_f gives $Tor_{n+1}^{R_f}(N_f, M_f) \cong \big(Tor_{n+1}^R(N, M)\big)_f$. Since an element of S annihilates $Tor_{n+1}^R(N, M)$, its image in S_f annihilates $\big(Tor_{n+1}^R(N, M)\big)_f$. Thus $Tor_{n+1}^{R_f}(N_f, M_f)$ is u-S_f-torsion for all N_f, showing that u-S_f-fd$_{R_f}(M_f) \leq n$. □

Theorem 3.1 *Let R be a commutative ring, $S \subseteq R$ a multiplicative subset, and let $f_1, \ldots, f_p \in R$ such that $(f_1, \ldots, f_p) \cap S \neq \varnothing$. Then, for any R-module M, one has u-S-$fd_R(M) = \sup_{1 \leq i \leq p}\big(u\text{-}S_{f_i}\text{-}fd_{R_{f_i}}(M_{f_i})\big)$, where S_{f_i} denotes the image of S in the localization R_{f_i}.*

Proof: We prove the two inequalities separately.

"$\geq$". For each $i \in \{1, \ldots, p\}$, localization at f_i is a flat base change. Hence, for every R-module M, we have u-S_{f_i}-$\mathrm{fd}_{R_{f_i}}(M_{f_i}) \leq u$-$S$-$\mathrm{fd}_R(M)$. Taking the supremum over all i yields $\sup_{1\leq i\leq p} u$-S_{f_i}-$\mathrm{fd}_{R_{f_i}}(M_{f_i}) \leq u$-$S$-$\mathrm{fd}_R(M)$.

"$\leq$". Set $n := \sup_{1\leq i\leq p} u$-$S_{f_i}$-$\mathrm{fd}_{R_{f_i}}(M_{f_i})$, and let N be any R-module. By definition of the u-S_{f_i}-flat dimension, for each i the module $Tor_{n+1}^{R_{f_i}}(N_{f_i}, M_{f_i})$ is uniformly S_{f_i}-torsion. Using the localization isomorphism for Tor, we obtain $Tor_{n+1}^{R_{f_i}}(N_{f_i}, M_{f_i}) \cong \big(Tor_{n+1}^R(N, M)\big)_{f_i}$. Thus, for each i, there exists an element $s_i \in S$ such that $(s_i f_i^{n_i}) \cdot Tor_{n+1}^R(N, M) = 0$.

Since $(f_1, \ldots, f_p) \cap S \neq \varnothing$, the argument used in the proof of the local criterion for u-S-flatness applies verbatim: there exists an element $s \in S$ annihilating $Tor_{n+1}^R(N, M)$. Hence $Tor_{n+1}^R(N, M)$ is uniformly S-torsion.

As N was arbitrary, this shows that u-S-$\mathrm{fd}_R(M) \leq n$, and the proof is complete. □

The next theorem gives a precise relationship between the u-S-flat dimension of M and the ordinary flat dimensions of the localizations M_P at prime ideals disjoint from S.

Theorem 3.2 *Let R be a commutative ring, $S \subseteq R$ a multiplicative subset, and M an R-module. Then* u-S-$\mathrm{fd}_R(M) = \sup\Big\{\mathrm{fd}_{R_P}(M_P) \;\Big|\; P \in Spec(R),\ P \cap S = \varnothing\Big\}$.

Proof: Assume that u-S-$\mathrm{fd}_R(M) = n < \infty$. By definition, for every R-module N, the module $Tor_{n+1}^R(N, M)$ is u-S-torsion. Hence there exists an element $s \in S$ such that $s \cdot Tor_{n+1}^R(N, M) = 0$. Let $P \in Spec(R)$ be a prime ideal such that $P \cap S = \varnothing$. Then $s \notin P$, so s becomes invertible in the localization R_P. Localizing the above equality at P yields $\big(Tor_{n+1}^R(N, M)\big)_P = 0$. Using the compatibility of the functor Tor with localization, we obtain $Tor_{n+1}^{R_P}(N_P, M_P) = 0$ for all R_P-modules N_P. Therefore, $\mathrm{fd}_{R_P}(M_P) \leq n$. Since this holds for all prime ideals P with $P \cap S = \varnothing$, we conclude $\sup_{\substack{P\in Spec(R)\\ P\cap S=\varnothing}} \mathrm{fd}_{R_P}(M_P) \leq u$-$S$-$\mathrm{fd}_R(M)$. Conversely, assume that there exists an integer $n \geq 0$ such that $\mathrm{fd}_{R_P}(M_P) \leq n$ for all $P \in Spec(R)$ with $P \cap S = \varnothing$. We must show that u-S-$\mathrm{fd}_R(M) \leq n$.

Let N be an arbitrary R-module and set $T := Tor_{n+1}^R(N, M)$.

Suppose, for a contradiction, that $S \cap Ann_R(T) = \varnothing$, where $Ann_R(T)$ denotes the annihilator of T. By standard properties of multiplicative sets, there exists a prime ideal $P \in Spec(R)$ such that $Ann_R(T) \subseteq P$ and $P \cap S = \varnothing$.

Localizing at P, we obtain $T_P = \big(Tor_{n+1}^R(N, M)\big)_P \cong Tor_{n+1}^{R_P}(N_P, M_P)$. By the assumption $\mathrm{fd}_{R_P}(M_P) \leq n$, the right-hand side is zero. Thus $T_P = 0$.

However, since $Ann_R(T) \subseteq P$, the localization T_P cannot vanish unless $T = 0$, which contradicts the assumption $S \cap Ann_R(T) = \varnothing$. Therefore, $S \cap Ann_R(T) \neq \varnothing$. Hence there exists an element $s \in S$ such that $s \cdot T = 0$.

Since N was arbitrary, this shows that $Tor_{n+1}^R(N, M)$ is u-S-torsion for all R-modules N. By definition, this means u-S-$\mathrm{fd}_R(M) \leq n$. Combining both inequalities, we conclude that u-S-$\mathrm{fd}_R(M) = \sup_{\substack{P\in Spec(R)\\ P\cap S=\varnothing}} \mathrm{fd}_{R_P}(M_P)$. □

Theorem 3.3 *Let R be a commutative ring, $S \subseteq R$ a multiplicative subset, and $f_1, \ldots, f_p \in R$ such that $(f_1, \ldots, f_p) \cap S \neq \varnothing$. Then u-S-w.gl.dim$(R) = \sup_{1\leq i\leq p}\big(u$-$S_{f_i}$-w.gl.dim$(R_{f_i})\big)$, where S_{f_i} denotes the image of S in R_{f_i}.*

Proof: We prove the equality by comparing both sides.

First inequality. Let M be any R-module and fix $i \in \{1, \ldots, p\}$. By Proposition 3.1, localization does not increase the uniform flat dimension, hence u-S_{f_i}-$\mathrm{fd}_{R_{f_i}}(M_{f_i}) \leq u$-$S$-$\mathrm{fd}_R(M)$. Taking the supremum over all R-modules M gives u-S_{f_i}-w.gl.dim$(R_{f_i}) \leq u$-S-w.gl.dim(R). Since this holds for every i, we obtain $\sup_{1\leq i\leq p} u$-S_{f_i}-w.gl.dim$(R_{f_i}) \leq u$-S-w.gl.dim(R).

Set $n := \sup_{1\leq i\leq p} u\text{-}S_{f_i}\text{-w.gl.dim}(R_{f_i})$. Let M be any R-module. By definition of n, for each i we have $u\text{-}S_{f_i}\text{-fd}_{R_{f_i}}(M_{f_i}) \leq n$. Applying the finite local criterion for the u-S-flat dimension (Theorem 2.1), we deduce that $u\text{-}S\text{-fd}_R(M) \leq n$. Since this inequality holds for every R-module M, taking the supremum over all M yields $u\text{-}S\text{-w.gl.dim}(R) \leq n$.

Combining the two inequalities proves the desired equality. □

Theorem 3.4 *Let R be a commutative ring and $S \subseteq R$ a multiplicative subset. Then u-S-w.gl.dim$(R) = \sup_{\substack{P\in Spec(R)\\ P\cap S=\emptyset}} \operatorname{w.gl.dim}(R_P)$.*

Proof: Let M be any R-module and let P be a prime ideal such that $P\cap S = \varnothing$. , we have $\text{fd}_{R_P}(M_P) \leq u\text{-}S\text{-fd}_R(M)$. Taking the supremum over all R-modules M yields $\operatorname{w.gl.dim}(R_P) \leq u\text{-}S\text{-w.gl.dim}(R)$. Since this holds for every prime ideal P with $P \cap S = \varnothing$, we obtain $\sup_{\substack{P\in Spec(R)\\ P\cap S=\varnothing}} \operatorname{w.gl.dim}(R_P) \leq u\text{-}S\text{-w.gl.dim}(R)$.

Set $n := \sup_{\substack{P\in Spec(R)\\ P\cap S=\varnothing}} \operatorname{w.gl.dim}(R_P)$. Let M be an arbitrary R-module. By definition of n, for every prime ideal P with $P \cap S = \varnothing$ we have $\text{fd}_{R_P}(M_P) \leq n$. Applying again Theorem 3.2, we deduce that $u\text{-}S\text{-fd}_R(M) \leq n$. Since this inequality holds for all R-modules M, taking the supremum over M gives $u\text{-}S\text{-w.gl.dim}(R) \leq n$.

Combining both inequalities proves the desired equality. □

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