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A Course on Real Analysis

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Contents

Presentation

This text consists of a collection of central and fundamental results from a Real Analysis Course. It is dedicated to students in the final years of their undergraduate and Master's degree programs in Mathematics.

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The authors.

Chapter 1

The Real Numbers

1.1 The Natural Numbers and Mathematical Induction

To rigorously build analysis, we must look closer at the structure of the natural numbers $\mathbb{N} = \{1, 2, 3, \dots\}$.

Axiom [Well-Ordering Principle] Every non-empty subset of natural numbers has a least element. That is, if $S \subset \mathbb{N}$ and $S \neq \emptyset$, then there exists $m \in S$ such that $m \leq k$ for all $k \in S$.

This principle is equivalent to the Principle of Mathematical Induction, which is a crucial tool for proving statements involving natural numbers.

Theorem 1.1 (Principle of Mathematical Induction) *Let $P(n)$ be a statement about the natural number n. Suppose that:*

(i) $P(1)$ is true (Base case).

(ii) For every $k \in \mathbb{N}$, if $P(k)$ is true, then $P(k+1)$ is true (Inductive step).

Then, $P(n)$ is true for all $n \in \mathbb{N}$.

Proof: Let $S = \{n \in \mathbb{N} : P(n) \text{ is false}\}$. We want to show that $S = \emptyset$. Suppose, for the sake of contradiction, that $S \neq \emptyset$. By the Well-Ordering Principle, S has a least element, say m. Since $P(1)$ is true by hypothesis (i), $1 \notin S$, so $m > 1$. Therefore, $m - 1$ is a natural number. Since m is the *least* element of S, $m - 1 \notin S$, which means $P(m-1)$ is true. By hypothesis (ii), if $P(m-1)$ is true, then $P(m)$ must be true. This contradicts the fact that $m \in S$. Therefore, S must be empty, and $P(n)$ is true for all n. ■

1.2 Finite, Countable, and Uncountable Sets

In analysis, distinguishing between different "sizes" of infinity is essential.

Definition 1.2 *Two sets A and B are said to have the same* cardinality *(written $A \sim B$) if there exists a bijection $f : A \to B$.*

Definition 1.3 *Let $J_n = \{1, 2, \ldots, n\}$ for some $n \in \mathbb{N}$. A set A is said to be:*

- ***Finite*** *if $A = \emptyset$ or if $A \sim J_n$ for some n.*
- ***Infinite*** *if it is not finite.*
- ***Countable*** *(or denumerable) if $A \sim \mathbb{N}$.*
- ***Uncountable*** *if A is infinite and not countable.*

Remark 1.4 *A countable set can be listed as a sequence $x_1, x_2, x_3, \ldots$ where every element of the set appears exactly once.*

Theorem 1.5 *The set of integers $\mathbb{Z}$ and the set of rational numbers $\mathbb{Q}$ are countable.*

Proof: For $\mathbb{Z}$, we can list the elements as $0, 1, -1, 2, -2, \ldots$, defining a bijection with $\mathbb{N}$. For $\mathbb{Q}$, the proof involves arranging the rationals in an infinite array and traversing it diagonally (Cantor's diagonalization for rationals), showing that there is a surjection from $\mathbb{N}$ to $\mathbb{Q}$. Since $\mathbb{Q}$ is infinite, this implies $\mathbb{Q} \sim \mathbb{N}$. ■

Theorem 1.6 *The union of a countable collection of countable sets is countable.*

Theorem 1.7 (Uncountability of $\mathbb{R}$) *The set of real numbers $\mathbb{R}$ is uncountable.*

Proof: (Sketch) The proof is typically done by contradiction using Cantor's Diagonal Argument on the interval $(0, 1)$. If we assume $(0, 1)$ is countable, we can list all its elements as decimal expansions. We then construct a new number $x \in (0, 1)$ by choosing its n-th decimal digit different from the n-th digit of the n-th number in the list. This number x cannot be in the list, leading to a contradiction. ■

1.3 Density of Rationals and Irrationals

We previously stated that $\mathbb{Q}$ is dense in $\mathbb{R}$. We now formalize this and extend it to irrational numbers.

Definition 1.8 *A subset $A \subset \mathbb{R}$ is dense in $\mathbb{R}$ if for every pair of real numbers x, y with $x < y$, there exists an element $a \in A$ such that $x < a < y$.*

Theorem 1.9 (Density of $\mathbb{Q}$) *If $x, y \in \mathbb{R}$ and $x < y$, then there exists a rational number $r \in \mathbb{Q}$ such that $x < r < y$.*

Proof: Since $x < y$, we have $y - x > 0$. By the Archimedean Property, there exists $n \in \mathbb{N}$ such that $n(y-x) > 1$, or $ny - nx > 1$. Since $ny - nx > 1$, there must exist an integer $m \in \mathbb{Z}$ between nx and ny (specifically, $m = \lfloor nx \rfloor + 1$). Thus, $nx < m < ny$. Dividing by n, we get $x < \frac{m}{n} < y$. By taking $r = m/n$, we have found a rational number between x and y. ■

Theorem 1.10 (Density of Irrationals) *If $x, y \in \mathbb{R}$ and $x < y$, then there exists an irrational number $z \in \mathbb{R} \setminus \mathbb{Q}$ such that $x < z < y$.*

Proof: Since $x < y$, we have $x - \sqrt{2} < y - \sqrt{2}$. By Theorem 1.9 (Density of Rationals), there exists a rational number $r \in \mathbb{Q}$ such that

$$x - \sqrt{2} < r < y - \sqrt{2}.$$

Rearranging the inequality, we get:

$$x < r + \sqrt{2} < y.$$

Let $z = r + \sqrt{2}$. We claim that z is irrational. Suppose, for contradiction, that $z \in \mathbb{Q}$. Then $z - r$ would be rational (since the difference of two rationals is rational). But $z - r = \sqrt{2}$, which we know is irrational. This is a contradiction. Thus, z is irrational and lies between x and y. ■

Corollary 1.11 *Between any two real numbers, there are infinitely many rational numbers and infinitely many irrational numbers.*

1.4 Supremum and Infimum of a Bounded Set

There exists a set $\mathbb{R}$, whose elements are called real numbers, satisfying the following conditions:

- (1) An addition operation and a multiplication operation are defined on $\mathbb{R}$, with respect to which $\mathbb{R}$ has the algebraic structure of a field.
- (2) There exists a non-empty subset of $\mathbb{R}$, denoted by $\mathcal{P}$, whose elements are called positive real numbers, such that:

  (i) If $x, y \in \mathcal{P} \Rightarrow x + y \in \mathcal{P}$.

  (ii) If $x, y \in \mathcal{P} \Rightarrow xy \in \mathcal{P}$.

  (iii) If $x \in \mathbb{R}$, one and only one of the following statements is true:

$$x \in \mathcal{P}; \quad x = 0, \quad -x \in \mathcal{P}. \text{ (trichotomy)}.$$

  If we denote by $-\mathcal{P} = \{-x : x \in \mathcal{P}\}$, the elements of $(-\mathcal{P})$ are called negative.

Definition 1.12 *Given $x, y \in \mathbb{R}$, we say that $x > y$ if and only if $x - y \in \mathcal{P}$. The notation $x \geq y$ is used to indicate that $x > y$ or $x = y$. We define $x < y$ if $y > x$.*

Proposition 1.13 *Let $a, b, c \in \mathbb{R}$. Then:*

a) If $a > b$ and $b > c$ then $a > c$.
b) Exactly one of the following statements is true: $a > b$; $a = b$; $a < b$.
c) If $a \geq b$ and $b \geq a$ then $a = b$.
d) If $a \neq 0$ then $a^2 > 0$.
e) $1 > 0$.
f) If $n \in \mathbb{N}$ then $n > 0$.

Proof: a) If $a > b$ and $b > c$ then $a - b \in \mathcal{P}$ and $b - c \in \mathcal{P}$. We must prove that $a > c$, i.e., that $a - c \in \mathcal{P}$. Indeed, we have:

$$a - c = \underbrace{(a - b)}_{\in \mathcal{P}} + \underbrace{(b - c)}_{\in \mathcal{P}} \in \mathcal{P}.$$

b) Let $a, b \in \mathbb{R}$. Then $(a-b) \in \mathbb{R}$ and by trichotomy, exactly one of the following statements is true:

$$a - b \in \mathcal{P}; \; a - b = 0; \; -(a - b) \in \mathcal{P},$$

that is,

$$a > b; \; a = b; \; a < b.$$

c) If $a \geq b$, then $a - b \in \mathcal{P}$ or $a - b = 0$, whence, $a - b \in \mathcal{P}$ or $a = b$.
If $b \geq a$, then $b - a \in \mathcal{P}$ or $b - a = 0$, which implies $b - a \in \mathcal{P}$ or $b = a$.
Since we are assuming $a \geq b$ and $b \geq a$ simultaneously, then:

$$(a - b) \in \mathcal{P} \text{ or } a = b \text{ and } (b - a) \in \mathcal{P} \text{ or } b = a.$$

Hence:

(i) If $(a - b) \in \mathcal{P}$ and $(b - a) \in \mathcal{P}$ then $(a - b) + (b - a) = 0 \in \mathcal{P}$, which is an absurdity!

(ii) If $(a - b) \in \mathcal{P}$ and $b = a$ then $0 \in \mathcal{P}$, which is an absurdity!

Thus, the only possible option is $a = b$ (or $b = a$).

d) If $a \neq 0$ then $a > 0$ or $a < 0$, i.e., $a \in \mathcal{P}$ or $-a \in \mathcal{P}$. In the first case $a^2 = a\,a \in \mathcal{P}$. In the second case $a^2 = (-a)(-a) \in \mathcal{P}$.

e) In particular, in an ordered field, $1 = 1 \cdot 1$ is always positive.

f) Exercise. ■

- (3) The supremum property holds in $\mathbb{R}$: '*Every non-empty subset of* $\mathbb{R}$ *that is bounded above has a supremum.*'

To understand what the supremum of a set is, we need some preliminary concepts.

Definition 1.14 *Let $S \subset \mathbb{R}$. An element $u \in \mathbb{R}$ is called an upper bound of S if $u \geq x$ for all $x \in S$. Similarly, the concept of a lower bound of S is defined.*

A set S is bounded above if S has an upper bound, and is bounded below if it has a lower bound. If S is simultaneously bounded above and below, we say that S is bounded.

Definition 1.15 *Let $S \subset \mathbb{R}$; $S \neq \emptyset$, S bounded above. A real number l is called the supremum of S and denoted by $l = \sup S$ if l is the least of the upper bounds of S. Equivalently, $l \in \mathbb{R}$ is the supremum of S if and only if it satisfies the following conditions:*

(1) l is an upper bound of S.

(2) If t is any upper bound of S then $l \leq t$.

Condition (1) says that l is an upper bound of S, while (2) states that any other upper bound of S must be greater than or equal to l.

Condition (2) can be rephrased as:

(2') Given $c < l$, there exists $s \in S$ such that $c < s$.

Indeed, suppose by contradiction that there exists $C_0 < l$ such that for all $s \in S$ we have $C_0 \geq s$. It follows that C_0 is an upper bound of S strictly less than the supremum ($C_0 < l$), which is an absurdity, proving (2').

Similarly, the concept of infimum of a set bounded below is defined.

Exercises: Set Theory and Functions

1st Question Given a function $f : A \longrightarrow B$ and X, Y subsets of A, prove the following properties:

a) $f(X \cup Y) = f(X) \cup f(Y)$, *b)* $f(X \cap Y) \subset f(X) \cap f(Y)$,

c) $X \subset Y \Longrightarrow f(X) \subset f(Y)$, *d)* $f(\emptyset) = \emptyset$.

2nd Question Given a function $f : A \longrightarrow B$ and Y, Z subsets of B, prove the following properties:

a) $f^{-1}(Y \cup Z) = f^{-1}(Y) \cup f^{-1}(Z)$, *b)* $f^{-1}(Y \cap Z) = f^{-1}(Y) \cap f^{-1}(Z)$,

c) $f^{-1}(Y^c) = (f^{-1}(Y))^c$, *d)* $Y \subset Z \Longrightarrow f^{-1}(Y) \subset f^{-1}(Z)$,

e) $f^{-1}(B) = A$, *f)* $f^{-1}(\emptyset) = \emptyset$. *(Note: I used Y^c for the complement $\complement Y$ as it is more standard in English, but you can keep $\complement$ if you prefer)*.

3rd Question Prove that a function $f : A \longrightarrow B$ has a left inverse if and only if it is injective.

4th Question Prove that a function $f : A \longrightarrow B$ has a right inverse if and only if it is surjective.

5th Question Given a family $(A_\lambda)_{\lambda \in L}$ of subsets of a universal set E, then

a) $(\bigcup A_\lambda)^c = \bigcap A_\lambda^c$, *b)* $(\bigcap A_\lambda)^c = \bigcup A_\lambda^c$ *(De Morgan's Laws)*.

6th Question Given a function $f : A \longrightarrow B$, consider a family $(A_\lambda)_{\lambda \in L}$ of subsets of A, and a family $(B_\mu)_{\mu \in M}$ of subsets of B. Prove the following properties:

a) $f(\bigcup A_\lambda) = \bigcup f(A_\lambda)$, *b)* $f(\bigcap A_\lambda) \subset \bigcap f(A_\lambda)$,

c) $f^{-1}(\bigcup B_\mu) = \bigcup f^{-1}(B_\mu)$, *d)* $f^{-1}(\bigcap B_\mu) = \bigcap f^{-1}(B_\mu)$,

7th Question Given the function $f : A \longrightarrow B$:

a) prove that $f(X \setminus Y) \supset f(X) \setminus f(Y)$ for any subsets X and Y of A,

b) show that if f is injective, then $f(X \setminus Y) = f(X) \setminus f(Y)$ for any X and Y contained in A.

8$^{\underline{th}}$ Question Show that the function $f : A \longrightarrow B$ is injective if and only if $f(A \setminus X) = f(A) \setminus f(X)$ for all $X \subset A$.

9$^{\underline{th}}$ Question Given the function $f : A \longrightarrow B$, prove:

a) $f^{-1}(f(X)) \supset X$ for all $X \subset A$,

b) f is injective if and only if $f^{-1}(f(X)) = X$ for all $X \subset A$.

10$^{\underline{th}}$ Question Given the function $f : A \longrightarrow B$, prove:

a) for all $Z \subset B$, we have $f(f^{-1}(Z)) \subset Z$,

b) f is surjective if and only if $f(f^{-1}(Z)) = Z$ for all $Z \subset B$.

11$^{\underline{th}}$ Question If there exists a bijection $f : X \longrightarrow Y$, then given $a \in X$ and $b \in Y$, there also exists a bijection $g : X \longrightarrow Y$ such that $g(a) = b$.

12$^{\underline{th}}$ Question If A is a proper subset of I_n (where $I_n = \{1, \ldots, n\}$), there cannot exist a bijection $f : A \longrightarrow I_n$.

13$^{\underline{th}}$ Question If $f : I_m \longrightarrow X$ and $g : I_n \longrightarrow X$ are bijections, then $m = n$.

14$^{\underline{th}}$ Question Let X be a finite set. A map $f : X \longrightarrow X$ is injective if and only if it is surjective.

15$^{\underline{th}}$ Question There cannot exist a bijection between a finite set and a proper part (subset) of itself.

16$^{\underline{th}}$ Question Every subset of a finite set is finite.

17$^{\underline{th}}$ Question Given $f : X \longrightarrow Y$, if Y is finite and f is injective, then X is finite; if X is finite and f is surjective, then Y is finite.

18$^{\underline{th}}$ Question A subset $X \subset \mathbb{N}$ is finite if and only if it is bounded.

19$^{\underline{th}}$ Question If X is an infinite set, then there exists an injective map $f : \mathbb{N} \rightarrow X$.

20$^{\underline{th}}$ Question A set X is infinite if and only if there exists a bijection $\varphi : X \rightarrow Y$ onto a proper subset $Y \subset X$.

21$^{\underline{st}}$ Question Every subset $X \subset \mathbb{N}$ is countable.

22$^{\underline{nd}}$ Question Let $f : X \rightarrow Y$ be injective. If Y is countable, then X is also countable. In particular, every subset of a countable set is countable.

23$^{\underline{rd}}$ Question Let $f : X \to Y$ be surjective. If X is countable, then Y is also countable.

24$^{\underline{th}}$ Question The Cartesian product of two countable sets is a countable set.

25$^{\underline{th}}$ Question The union of a countable family of countable sets is countable.

26$^{\underline{th}}$ Question Prove that the following sets are countable:

$a)$ $\mathbb{Z}$, the set of integers, $b)$ $\mathbb{Q}$, the set of rational numbers.

Chapter 2

Sequences

2.1 Sequences

Definition 2.1 *A sequence of elements from a set A is a function $X : \mathbb{N} \to A$. When $A = \mathbb{R}$, we have a sequence of real numbers. If A is a set of functions, we have a sequence of functions.*

Notation: If $X : \mathbb{N} \to A$ is a sequence, the element $X(n) \in A$ is usually denoted by x_n instead of $X(n)$, and the function X itself by $X = \{x_1, x_2, \cdots\}$ or $X = \{x_n\}_{n\in\mathbb{N}}$ or simply $X = \{x_n\}$.

Examples:

1) Consider the sequence of real numbers $X = (x_n)$ where $x_n = (-1)^n$, $n \in \mathbb{N}$.

The set of values of this sequence is $\{(-1)^n : n \in \mathbb{N}\} = \{-1, 1\}$, while the sequence itself is given by $X = \{-1, 1, -1, 1, \cdots\}$.

2) Let $Y = \{y_n\}$ where $y_n = \frac{1}{n}$. In this case, $Y = \{1, 1/2, 1/3, 1/4, \cdots\}$. Here, the set of values is identified with the sequence itself.

Remark: Unless stated otherwise, the sequences we will deal with from now on are real numbers.

2.2 Limits of Sequences

Definition 2.2 *We say that a sequence $X = \{x_n\}$ has a limit $l \in \mathbb{R}$ (or converges to l) if for each $\varepsilon > 0$, there exists a $k(\varepsilon) \in \mathbb{N}$ such that if $n > k(\varepsilon)$ then $|x_n - l| < \varepsilon$. In this case, the sequence is said to be convergent to l and we denote:*

$$\lim_{n\to+\infty} x_n = l \quad \textit{or} \ \lim x_n = l \quad \textit{or simply} \ x_n \to l.$$

Examples:

1) Let $Y = \{y_n\}$ where $y_n = \frac{1}{n}$. We have $y_n \to 0$ as $n \to +\infty$ because, given $\varepsilon > 0$, I can consider $k(\varepsilon) > \frac{1}{\varepsilon}$ (Archimedean property) such that if $n > k(\varepsilon) > \frac{1}{\varepsilon}$ then $\frac{1}{n} < \varepsilon$.

2) Let $X = \{x_n\}$ where $x_n = \lambda \in \mathbb{R}$, for all $n \in \mathbb{N}$. It is clear that $x_n \to \lambda$ since $|x_n - \lambda| = 0 < \varepsilon$.

Proposition 2.3 (Uniqueness of the Limit) *Let $X = \{x_n\}$ be such that $x_n \to l$ and $x_n \to l'$. Then $l = l'$.*

Proof: Suppose, by contradiction, that $l \neq l'$ and consider $0 < \varepsilon = \frac{|l-l'|}{2}$. Since $x_n \to l$, there exists $k_1(\varepsilon)$ such that

$$|x_n - l| < \varepsilon = \frac{|l - l'|}{2} \text{ if } n > k_1(\varepsilon). \tag{2.1}$$

On the other hand, since $x_n \to l'$, there exists $k_2(\varepsilon)$ such that

$$|x_n - l'| < \varepsilon = \frac{|l - l'|}{2} \text{ if } n > k_2(\varepsilon). \tag{2.2}$$

Note that if $k(\varepsilon) := \max\{k_1(\varepsilon), k_2(\varepsilon)\}$, then (2.1) and (2.2) hold simultaneously. Adding (2.1) and (2.2) member by member, we obtain:

$$|x_n - l| + |x_n - l'| < |l - l'|, \forall n > k(\varepsilon) \tag{2.3}$$

However,

$$\begin{aligned} |l - l'| &= |l - x_n + x_n - l'| \\ &\leq |x_n - l| + |x_n - l'|. \end{aligned} \tag{2.4}$$

From (2.3) and (2.4), it follows that $|l - l'| < |l - l'|$, which is an absurdity, proving the desired result. ■

Proposition 2.4 *Let $A = \{a_n\}$ and $X = \{x_n\}$ be sequences such that for a real number l and a real number $C > 0$, we have*

$$|x_n - l| \leq C|a_n|, \quad \textit{for all } n \in \mathbb{N}.$$

If $a_n \to 0$ then $x_n \to l$.

Proof: Let $\varepsilon > 0$ be given. Since $a_n \to 0$, there exists $k(\varepsilon) \in \mathbb{N}$ such that

$$|a_n| < \frac{\varepsilon}{C}, \ \forall n > k(\varepsilon). \tag{2.5}$$

Hence:

$$|x_n - l| \leq C|a_n| < C \cdot \frac{\varepsilon}{C} = \varepsilon, \forall n \geq k(\varepsilon), \tag{2.6}$$

which proves the desired result. ■

Applications:

1a) Consider the sequence $\left\{\frac{1}{1+na}\right\}_{n\in\mathbb{N}}$ where $a > 0$. Since $1 + na > na$, it follows that

$$\left|\frac{1}{1+na}\right| = \frac{1}{1+na} < \frac{1}{na} = \frac{1}{a} \cdot \frac{1}{n}.$$

Since $\frac{1}{n} \to 0$, it follows from Proposition 2.4 that $\frac{1}{1+na} \to 0$.

Lemma 2.5 (Bernoulli's Inequality) *If $a \in \mathbb{R}$, $a > -1$ then:*

$$1 + na \leq (1+a)^n, \ \forall n \in \mathbb{N}.$$

Proof: We will proceed by induction on n. Indeed, if $n = 1$, then $1 + 1 \cdot a = 1 + a$. Assume the inequality holds for $n > 1$ and let us prove its validity for (n+1). In fact, since $a > -1$ and from the inductive hypothesis, it follows that:

$$\begin{aligned}
(1+a)^{n+1} &= (1+a)^n(1+a) \\
&\geq (1+na)\cdot(1+a) \\
&= 1 + a + na + na^2 \\
&= 1 + (n+1)a + \underbrace{na^2}_{\geq 0} \geq 1 + (n+1)a,
\end{aligned}$$

i.e.,

$$(1+a)^{n+1} \geq 1 + (n+1)a,$$

which proves the desired result. ■

2a) Consider the sequence $\{b^n\}$ where $0 \leq b < 1$. Then $b^n \to 0$.

If $b = 0$, it is trivial. Suppose $0 < b < 1$. Then $b = \frac{1}{1+a}$ for some $a > 0$. In truth:

$$a := \frac{1}{b} - 1 > 0 \text{ since } 0 < b < 1.$$

Hence, from Bernoulli's Inequality:

$$b^n = \left(\frac{1}{1+a}\right)^n \leq \frac{1}{1+an}. \tag{2.7}$$

Since $\frac{1}{1+an} \to 0$ (Application 1a), it follows from (2.7) and Proposition 2.4 that $b^n \to 0$.

3a) Consider the sequence $\sqrt[n]{a}$, $a > 1$.

Since $a > 1$, it follows that $\sqrt[n]{a} > 1$, and therefore $\sqrt[n]{a} = 1 + h_n$, for some $h_n > 0$ and for all $n \in \mathbb{N}$.

Hence, $a = (1 + h_n)^n$, for some $h_n > 0$ and for all $n \in \mathbb{N}$, and by Bernoulli's inequality, we have

$$a = (1 + h_n)^n \geq 1 + nh_n,$$

which implies

$$a \geq 1 + nh_n,$$

and therefore

$$nh_n \leq a - 1,$$

or,

$$0 < h_n \leq \frac{a-1}{n}.$$

Since

$$\lim_{n\to+\infty} \frac{a-1}{n} = (a-1) \lim_{n\to+\infty} \frac{1}{n} = 0,$$

it follows from Proposition 2.4 that $h_n \to 0$, and therefore,

$$\lim_{n\to+\infty} \sqrt[n]{a} = 1, \quad \text{if } a > 1.$$

Exercise: Prove that:

$$\lim_{n\to+\infty} \sqrt[n]{a} = 1, \quad \text{if } 0 < a < 1.$$

Exercise: Prove that:

$$\lim_{n\to+\infty} \sqrt[n]{n} = 1.$$

Definition 2.6 *A sequence is said to be* <u>*bounded*</u> *if there exists a real number $M > 0$ such that $|x_n| \leq M$, for all $n \in \mathbb{N}$.*

Proposition 2.7 *Every convergent sequence is bounded.*

Proof: Let $X = \{x_n\}$ be convergent, i.e., there exists $l \in \mathbb{R}$ such that $x_n \to l$. If we take $\varepsilon = 1$, there exists $k \in \mathbb{N}$ such that

$$|x_n - l| < 1, \ \forall n \geq k,$$

which implies that

$$|x_n| < |l| + 1, \ \forall n \geq k. \tag{2.8}$$

Letting

$$m := \max\{|x_1|, |x_2|, \cdots, |x_{k-1}|\}, \tag{2.9}$$

and

$$M := \max\{m, |l| + 1\}, \tag{2.10}$$

from (2.8), (2.9) and (2.10) we deduce that $|x_n| \leq M$ for all $n \in \mathbb{N}$, which proves the desired result. ■

It follows immediately from Proposition 2.7 that if $\{x_n\}$ is not bounded, then it is not convergent.

Theorem 2.8 *Let $X = \{x_n\}$ and $Y = \{y_n\}$ be sequences.*

1) If $x_n \to l$ and $y_n \to s$ then $x_n + y_n \to l + s$.

2) If $x_n \to l$ and $c \in \mathbb{R}$ then $cx_n \to cl$.

3) If $x_n \to l$ and $y_n \to s$ then $x_n y_n \to ls$.

4) If $x_n \to l$ and $y_n \neq 0$, for all n, and $y_n \to s \neq 0$ then $\frac{x_n}{y_n} \to \frac{l}{s}$.

Proof: 1) Given $\varepsilon > 0$, there exist $k_1(\varepsilon), k_2(\varepsilon) \in \mathbb{N}$ such that:

$$|x_n - l| < \tfrac{\varepsilon}{2}, \ \forall n \geq k_1(\varepsilon) \tag{2.11}$$

$$|y_n - s| < \tfrac{\varepsilon}{2}, \ \forall n \geq k_2(\varepsilon) \tag{2.12}$$

Taking $k(\varepsilon) = \max\{k_1(\varepsilon), k_2(\varepsilon)\}$, inequalities (2.11) and (2.12) hold simultaneously, and furthermore:

$$\begin{aligned} |(x_n + y_n) - (l + s)| &= |x_n - l + y_n - s| \\ &\leq |x_n - l| + |y_n - s| \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon, \ \forall n \geq k(\varepsilon). \end{aligned}$$

2) Exercise

3) Let us note initially that:

$$|x_n y_n - ls| = |x_n y_n - x_n s + x_n s - ls| \quad (2.13)$$
$$\leq |x_n|\,|y_n - s| + |x_n - l|\,|s|.$$

Since $\{x_n\}$ is convergent, then according to Proposition 2.7, $\{x_n\}$ is bounded, i.e., there exists $M > 0$ such that:

$$|x_n| \leq M;\ \forall n \in \mathbb{N}. \quad (2.14)$$

On the other hand, since $x_n \to l$, then given $\varepsilon > 0$ there exists a $k_1(\varepsilon) \in \mathbb{N}$ such that

$$|x_n - l| < \frac{\varepsilon}{2(|s|+1)};\ \forall n \geq k_1(\varepsilon). \quad (2.15)$$

Also, since $y_n \to s$, for the given ε, there exists $k_2(\varepsilon)$ such that

$$|y_n - s| < \frac{\varepsilon}{2M};\ \ \forall n \geq k_2(\varepsilon). \quad (2.16)$$

Combining (2.13)-(2.16), it follows that for all $n > k(\varepsilon) = \max\{k_1(\varepsilon), k_2(\varepsilon)\ (note : k > k(eps) inoriginal, changedton > k(eps))\}$:

$$|x_n y_n - ls| \leq |x_n|\,|y_n - s| + |x_n - l|\,|s|$$
$$< M\frac{\varepsilon}{2M} + \frac{\varepsilon}{2(|s|+1)}\,|s|$$
$$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

(Note: Corrected a small typo in the original text's final line ...$(|s|+1)$ -> ...|s|)

4) Exercise. ■

Proposition 2.9 *Let $X = \{x_n\}$ be a sequence and $X_M = \{x_n\}_{n \geq M+1}$. Then X is convergent if and only if X_M is convergent and $\lim X = \lim X_M$. *(Note: The original text's $X_M = \{x_{M+n}\}$ is slightly ambiguous. I've interpreted it as a "tail" of the sequence starting from index M+1)**

Proof: We must prove that:

$$\{x_k\}_{k \geq M} \text{ converges } \Leftrightarrow \{x_k\}_{k \geq 1} \text{ converges }.$$

(Note: M vs M+1)

'$\Rightarrow$' If $\{x_k\}_{k \geq M}$ converges to l, then, given $\varepsilon > 0$ there exists $k(\varepsilon) \in \mathbb{N}$; $k(\varepsilon) \geq M$ such that

$$|x_k - l| < \varepsilon,\ \forall k \geq k(\varepsilon).$$

This definition also works for $\{x_k\}_{k\geq 1}$.

'$\Leftarrow$' If $\{x_k\}_{k\geq 1}$ converges to l then given $\varepsilon > 0$ there exists $k(\varepsilon) \in \mathbb{N}$ such that

$$|x_k - l| < \varepsilon, \ \forall k \geq k(\varepsilon).$$

Take $k^*(\varepsilon) = \max\{M, k(\varepsilon)\}$. Then for all $k \geq k^*(\varepsilon)$, we have $k \geq M$ and $k \geq k(\varepsilon)$, so

$$|x_k - l| < \varepsilon, \ \forall k \geq k^*(\varepsilon).$$

(Note: The original text has a small error in the last line, corrected here to $k \geq k^*(\varepsilon)$) ■

Proposition 2.10 *Let $X = \{x_n\}$ be a sequence of positive terms, i.e., $x_n > 0$ for all $n \in \mathbb{N}$. If*

$$\lim_{n\to+\infty} \frac{x_{n+1}}{x_n} = L \ \text{ exists and } \ L < 1,$$

then the sequence $\{x_n\}$ is convergent. Furthermore, $x_n \to 0$.

Proof: Since $L < 1$, there exists $r \in \mathbb{R}$ such that $L < r < 1$. Consider $\varepsilon = r - L > 0$. Since $\frac{x_{n+1}}{x_n} \to L$, there exists $k(\varepsilon) \in \mathbb{N}$ such that

$$\left|\frac{x_{n+1}}{x_n} - L\right| < \varepsilon = r - L, \quad \text{for all } n \geq k(\varepsilon),$$

i.e.,

$$-r + L < \frac{x_{n+1}}{x_n} - L < r - L, \quad \text{for all } n \geq k(\varepsilon),$$

or rather,

$$2L - r < \frac{x_{n+1}}{x_n} < r, \quad \text{for all } n \geq k(\varepsilon).$$

Since the $\{x_n\}$ are positive, it follows that

$$0 < \frac{x_{n+1}}{x_n} < r, \quad \text{for all } n \geq k(\varepsilon),$$

which implies

$$x_{n+1} < r x_n, \quad \text{for all } n \geq k(\varepsilon).$$

Hence, for $n \geq k(\varepsilon)$:

$$x_{n+1} < r x_n < r^2 x_{n-1} < r^3 x_{n-2} < \cdots < r^{n-k(\varepsilon)+1} x_{k(\varepsilon)}$$

$$x_{n+1} < \left(r^{-k(\varepsilon)+1} x_{k(\varepsilon)}\right) r^n = C r^n$$

where $C = r^{-k(\varepsilon)+1} x_{k(\varepsilon)}$ is a positive constant (since $k(\varepsilon)$ is fixed).

Since $0 \le r < 1$, we know $r^n \to 0$ (Application 2a). From the inequality above and Proposition 2.4 (with $a_n = r^n$), it follows that $x_{n+1} \to 0$. By Proposition 2.9, we conclude that $x_n \to 0$. ■

Exercise: Use the same type of reasoning to show that if $L > 1$, the sequence $\{x_n\}$ is not bounded and therefore diverges.

Observation: In the case $L = 1$, nothing can be concluded about the behavior of the sequence. Find examples of sequences that have different behaviors.

Example: Consider the sequence $\{\frac{n}{2^n}\}$.
In this case $x_n = \frac{n}{2^n}$, from which we conclude that:

$$\lim_{n\to+\infty} \frac{x_{n+1}}{x_n} = \lim_{n\to+\infty} \frac{n+1}{2^{n+1}}\frac{2^n}{n} = \lim_{n\to+\infty} \frac{1}{2}\frac{n+1}{n} = \frac{1}{2} < 1.$$

By Proposition 2.10:

$$\lim_{n\to+\infty} \frac{n}{2^n} = 0.$$

Proposition 2.11 *If $X = \{x_n\}$ and $Y = \{y_n\}$ are sequences such that $x_n \ge y_n$, for all n, $x_n \to l$ and $y_n \to s$, then $l \ge s$.*

Proof: It is sufficient to prove that

$$\text{If } h_n \to h \text{ and } h_n \ge 0 \Rightarrow h \ge 0. \tag{2.17}$$

Suppose, by contradiction, that $h < 0$. Since $h_n \to h$, then given $\varepsilon := -h > 0$, there exists $k = k(\varepsilon)$ such that

$$|h_n - h| < \varepsilon = -h, \quad \text{for all } n \ge k,$$

whence:

$$h < h_n - h < -h, \quad \text{for all } n \ge k,$$

i.e.,

$$2h < h_n < 0, \quad \text{for all } n \ge k,$$

which is a contradiction, since $h_n \ge 0$ $\forall n \in \mathbb{N}$. This proves (2.17). To conclude the proof, it suffices to take $h_n = x_n - y_n$ and $h = l - s$ and apply (2.17). ■

Proposition 2.12 *[Squeeze Theorem] If $X = (x_n)$, $Y = (y_n)$ and $Z = (z_n)$ are sequences such that $l = \lim X = \lim Z$ and $x_n \le y_n \le z_n$ for all $n \in \mathbb{N}$, then $\lim Y = l$.*

Proof: Given $\varepsilon > 0$ there exists $k = k(\varepsilon) > 0$ such that

$$|x_n - l| < \varepsilon \text{ and } |z_n - l| < \varepsilon, \forall n \geq k,$$

or rather,

$$-\varepsilon < x_n - l < \varepsilon \text{ and } -\varepsilon < z_n - l < \varepsilon, \forall n \geq k. \tag{2.18}$$

But, by hypothesis

$$x_n \leq y_n \leq z_n \Leftrightarrow x_n - l \leq y_n - l \leq z_n - l. \tag{2.19}$$

Combining (2.18) and (2.19) yields

$$-\varepsilon < x_n - l \leq y_n - l \leq z_n - l < \varepsilon, \forall n \geq k,$$

from which we conclude that

$$|y_n - l| < \varepsilon, \ \forall n \geq k.$$

The case $x_n \geq y_n \geq z_n$ is proved similarly. ∎

2.3 Monotone Sequences

Definition 2.13 *A sequence $X = \{x_n\}$ is said to be monotone increasing if $x_n < x_{n+1}$, for all $n \in \mathbb{N}$ (it is said to be monotone non-decreasing if $x_n \leq x_{n+1}$, for all $n \in \mathbb{N}$). Similarly, a monotone decreasing (respectively non-increasing) sequence is defined. A sequence is said to be monotone if it is either non-decreasing or non-increasing.*

Examples:

1) $X = \{\frac{1}{n}\}$ is monotone decreasing.
2) $Y = \{n\}$ is monotone increasing.
3) $Z = ((-1)^n)$ is not monotone.

Theorem 2.14 *Let $X = \{x_n\}$ be a monotone sequence. Then, X is convergent if and only if it is bounded.*

Proof: We have already seen that if X is convergent then it is bounded (Proposition 2.7), regardless of whether it is monotone or not. It remains for us to show that if X is bounded, it is convergent. Suppose $X = \{x_n\}$ is monotone non-decreasing, i.e.,

$$x_n \leq x_{n+1}, \ \forall n \in \mathbb{N}. \tag{2.20}$$

Since X is bounded, the set of values $\{x_n : n \in \mathbb{N}\}$ is bounded. Hence, there exists

$$x^* = \sup\{x_n : n \in \mathbb{N}\}. \tag{2.21}$$

We will show that

$$x^* = \lim x_n. \tag{2.22}$$

Indeed, given $\varepsilon > 0$, by virtue of (2.21) and condition (2'), there exists $k = k(\varepsilon) > 0$ such that

$$x^* - \varepsilon < x_k \leq x^*. \tag{2.23}$$

Hence, if $n > k$, then from (2.20) it follows that $x_n \geq x_k$, and from (2.23) and (2.21) it follows that

$$x^* - \varepsilon < x_k \leq x_n \leq x^* < x^* + \varepsilon, \ \forall n > k, \tag{2.24}$$

and therefore

$$|x_n - x^*| < \varepsilon, \ \forall n > k,$$

which proves the desired result in (2.22). ■

Exercise: Complete the proof of the Theorem above for monotone non-increasing sequences.

Example: Consider the sequence $X = \{x_n\}$ where $x_1 = 1$; $x_{n+1} = \frac{1}{4}(2x_n + 3)$; $n \geq 1$.

(i) $\{x_n\}$ is monotone increasing. We will use induction on n.

1) $x_1 = 1, \ \ x_2 = \frac{5}{4} \Rightarrow x_1 < x_2$.

2) If $x_n < x_{n+1} \Rightarrow x_{n+1} < x_{n+2}$.

Indeed:

$$x_{n+1} = \frac{1}{4}(2x_n + 3) < \frac{1}{4}(2x_{n+1} + 3) = x_{n+2}.$$

(ii) We will show that $x_n < 2$, for all $n \in \mathbb{N}$. We will use induction on n.

1) $x_1 = 1 < 2$

2) If $x_n < 2 \Rightarrow x_{n+1} < 2$.

Indeed, $x_{n+1} = \frac{1}{4}(2x_n + 3) < \frac{1}{4}(2 \cdot 2 + 3) = \frac{7}{4} < 2$. Hence, $1 \leq x_n < 2$ for all $n \in \mathbb{N}$. Thus $\{x_n\}$ is bounded.

By Theorem 2.14 or the Monotone Sequence Theorem, the sequence $\{x_n\}$ is convergent. Let:

$$l = \lim x_n.$$

Since

$$x_{n+1} = \frac{1}{4}(2x_n + 3)$$

it follows that

$$l = \frac{1}{4}(2l + 3) \Leftrightarrow 4l = 2l + 3 \Leftrightarrow 2l = 3 \Leftrightarrow l = \frac{3}{2}.$$

Exercise: Let $a > 0$, consider $a_1 > 0$ arbitrary and define

$$a_{n+1} = \frac{1}{2}\left(a_n + \frac{a}{a_n}\right).$$

i) Show that: $a_n^2 \geq a$, for all $n \geq 2$.
ii) Show that $\{a_n\}$ is decreasing (for $n \geq 2$).
iii) Show that: $0 \leq a_n - \sqrt{a} \leq \frac{a_n^2 - a}{a_n}$.

Definition 2.15 *Let $\{x_n\}$ be any sequence. Let $\{r_n\}$ be an increasing sequence of natural numbers $r_1 < r_2 < \cdots < r_n < \cdots$. Then the sequence $Y = \{x_{r_n}\}$ is called a subsequence of $X = \{x_n\}$.*

Example: Let $X = \{x_n\}$ where $x_n = \frac{1}{n}$. If $Y = \{y_n\}$ where $y_n = \frac{1}{2^n}$, then Y is a subsequence of X.

In truth:

$$X = (1, 1/2, 1/3, 1/4, 1/5, 1/6, 1/7, 1/8, \cdots).$$

$$Y = (1/2, 1/4, 1/8, \cdots).$$

(Note: Y corresponds to $r_n = 2^n$)

Attention:

$$Z = (1, 1/2, 1/5, 1/5, 1/6, 1/6, \cdots)$$

is not a subsequence of X. Why?

Theorem 2.16 *[Bolzano-Weierstrass] If $X = \{x_n\}$ is a bounded sequence, it possesses a convergent subsequence.*

Proof: Let $\{x_n\}$ be bounded. Then there exists $M > 0$ such that $|x_n| < M$, for all $n \in \mathbb{N}$, i.e.,

$$-M < x_n < M, \ \forall n \in \mathbb{N}.$$

Then, it makes sense to define:

$$\begin{aligned} y_1 &= \sup\{x_1, \cdots, x_n, \cdots\} \\ y_2 &= \sup\{x_2, \cdots, x_n, \cdots\} \\ y_3 &= \sup\{x_3, \cdots, x_n, \cdots\} \\ &\vdots \\ y_n &= \sup\{x_n, x_{n+1}, \cdots\}. \end{aligned}$$

In this way, we construct a sequence $Y = \{y_n\}$ such that:

$$y_1 \geq y_2 \geq y_3 \geq \cdots \geq y_n \geq \cdots$$

(This is a non-increasing sequence).

Note that:

$$\begin{aligned} M &\geq \sup\{x_1, \cdots, x_n, \cdots\} = y_1 \\ &\geq y_2 \geq y_3 \geq \cdots \geq y_n. \end{aligned}$$

But

$$\begin{aligned} y_n = \sup\{x_n, x_{n+1}, \cdots\} &\geq \inf\{x_n, x_{n+1}, \cdots\} \\ &\geq \inf\{x_1, \cdots, x_n, \cdots\} \\ &\geq -M, \ \forall n \in \mathbb{N}. \end{aligned}$$

Thus, $Y = \{y_n\}$ is monotone (non-increasing) and bounded. By the Monotone Sequence Theorem:

$$\lim y_n = \inf\{y_n\} = \inf Y = y. \tag{2.25}$$

(This y is the $\limsup x_n$).

We claim that: For each $\varepsilon > 0$ and $N \in \mathbb{N}$, there exists $m > N$ such that

$$|x_m - y| < \varepsilon. \tag{2.26}$$

Indeed, since $y_n \to y$, for the given $\varepsilon > 0$, there exists $p \in \mathbb{N}$, $p > N$ such that

$$|y_p - y| < \frac{\varepsilon}{2}. \tag{2.27}$$

Since $y_p = \sup\{x_p, x_{p+1}, \cdots\}$, by the approximation property of the supremum (Prop. 1.1(2')), there exists $m \geq p$ such that

$$y_p - \frac{\varepsilon}{2} < x_m \leq y_p,$$

i.e.,

$$\underbrace{y_p - x_m}_{\geq 0} < \frac{\varepsilon}{2} \Rightarrow |y_p - x_m| < \frac{\varepsilon}{2}. \qquad (2.28)$$

Combining (2.27) and (2.28), it follows that there exists $m \geq p > N$ such that

$$\begin{aligned} |x_m - y| &\leq |x_m - y_p| + |y_p - y| \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon, \end{aligned}$$

which proves (2.26).

We will now choose, for each $k \in \mathbb{N}$, an x_{n_k} such that

$$|x_{n_k} - y| < \frac{1}{k} \quad \text{and} \quad n_k > n_{k-1}. \qquad (2.29)$$

In fact, from (2.26) we proceed inductively:

Given $\varepsilon = 1$ and $N = 0$, there exists $m = n_1 > 0$ such that

$$|x_{n_1} - y| < 1.$$

Given $\varepsilon = 1/2$ and $N = n_1$, there exists $m = n_2 > n_1$ such that

$$|x_{n_2} - y| < 1/2.$$

Given $\varepsilon = 1/3$ and $N = n_2$, there exists $m = n_3 > n_2$ such that

$$|x_{n_3} - y| < 1/3.$$

Proceeding in this manner, we construct a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ satisfying (2.29). It follows from this and the fact that $\frac{1}{k} \to 0$ that $x_{n_k} \to y$, which proves the desired result. ∎

2.4 Cauchy Sequences

Definition 2.17 *A sequence $X = \{x_n\}$ is called a Cauchy sequence if for each $\varepsilon > 0$, there exists a $k(\varepsilon) \in \mathbb{N}$ such that if $m, n \geq k(\varepsilon)$ then $|x_n - x_m| < \varepsilon$. 'Terms of high order are sufficiently close'.*

Proposition 2.18 *Every convergent sequence is a Cauchy sequence.*

Proof: Let $\{x_n\}$ be a convergent sequence. Say $x_n \to l$ as $n \to +\infty$. Thus, given $\varepsilon > 0$, there exists $k(\varepsilon) \in \mathbb{N}$ such that

$$|x_n - l| < \frac{\varepsilon}{2}, \ \forall n \geq k(\varepsilon). \tag{2.30}$$

It follows from (2.30) and the triangle inequality, for all $m, n \geq k(\varepsilon)$, that:

$$\begin{aligned} |x_n - x_m| &= |x_n - l + l - x_m| \\ &\leq |x_n - l| + |x_m - l| \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon, \end{aligned}$$

which proves the desired result. ■

Remark: The result above is valid in any metric space. However, the converse depends on the space where the sequence is defined. In the case of real numbers, the converse is true. A space where this occurs is called complete.

Theorem 2.19 *If $X = \{x_n\}$ is a sequence of real numbers and is Cauchy, then X is convergent.*

Proof: (i) Initially, we will prove that $\{x_n\}$ is bounded. Indeed, given $\varepsilon = 1$, there exists $k \in \mathbb{N}$ such that if $m, n \geq k$ then

$$|x_n - x_m| < 1.$$

Taking $m = k$ (fixed), it follows that

$$|x_n| < 1 + |x_k|; \ \forall n \geq k. \tag{2.31}$$

On the other hand, letting

$$M = \max\{|x_1|, \cdots, |x_{k-1}|, 1 + |x_k|\},$$

from (2.31) we have

$$|x_n| \leq M; \ \forall n \in \mathbb{N}, \tag{2.32}$$

which proves the boundedness of $\{x_n\}$.

(ii) It follows from (2.32), by virtue of the Bolzano-Weierstrass Theorem, that there exists a convergent subsequence $\{x_{n_k}\}$ of $\{x_n\}$, say:

$$x_{n_k} \to l \text{ as } k \to +\infty. \tag{2.33}$$

We will, in fact, prove that

$$x_n \to l \text{ as } n \to +\infty. \tag{2.34}$$

Indeed, let $\varepsilon > 0$ be given. From (2.33), it follows that there exists $K_1(\varepsilon)$ such that for $k \geq K_1(\varepsilon)$:

$$|x_{n_k} - l| < \frac{\varepsilon}{2}. \tag{2.35}$$

(Note: Original text had $n_k \geq k_1(\varepsilon)$ which is slightly imprecise, $k \geq K_1$ is standard).

On the other hand, since $\{x_n\}$ is Cauchy, there exists $k_2(\varepsilon) \in \mathbb{N}$ such that

$$|x_n - x_m| < \frac{\varepsilon}{2};\ \forall n, m \geq k_2(\varepsilon). \tag{2.36}$$

Let $K(\varepsilon) = \max\{K_1(\varepsilon), k_2(\varepsilon)\}$. Let $n \geq K(\varepsilon)$. Since $k \to \infty \implies n_k \to \infty$, we can choose k large enough such that $k \geq K(\varepsilon)$ AND $n_k \geq K(\varepsilon)$. Then, for such n and n_k:

$$\begin{aligned} |x_n - l| &= |x_n - x_{n_k} + x_{n_k} - l| \\ &\leq |x_n - x_{n_k}| + |x_{n_k} - l| \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon, \end{aligned}$$

which proves (2.34) and concludes the proof of the Theorem. ■

Exercises: Real Numbers and Limits

1$^{\underline{st}}$ Question (**Nested Intervals**) Given a decreasing sequence $I_1 \supset I_2 \supset \cdots \supset I_n \supset \ldots$ of bounded and closed intervals $I_n = [a_n, b_n]$, there exists at least one real number c such that $c \in I_n$ for all $n \in \mathbb{N}$.

2$^{\underline{nd}}$ Question Prove that the set of real numbers is uncountable.

3$^{\underline{rd}}$ Question Prove that every non-degenerate interval is uncountable.

4$^{\underline{th}}$ Question Prove that every non-degenerate interval I contains rational and irrational numbers.

5$^{\underline{th}}$ Question A complex number z is called algebraic if there exist integers $a_0, a_1, \ldots, a_n$, not all zero, such that

$$a_0 z^n + a_1 z^{n-1} + \cdots + a_{n-1} z + a_n = 0.$$

Prove that the set of all algebraic numbers is countable.

6$^{\underline{th}}$ Question Prove that there exist real numbers that are not algebraic.

7$^{\underline{th}}$ Question Is the set of all irrational numbers countable?

8$^{\underline{th}}$ Question (**Bernoulli's Inequality**)

a) If $n \in \mathbb{N}$ and $x \in \mathbb{R}$ with $x \geq -1$, demonstrate that $(1+x)^n \geq 1 + nx$.

b) If $n > 1 (n \in \mathbb{N})$ and $x > -1 (x \in \mathbb{R})$, demonstrate that $(1+x)^n > 1 + nx$, provided that $x \neq 0$.

9$^{\underline{th}}$ Question If $n \in \mathbb{N}$ and $-1 \leq x \leq \frac{1}{n}$, demonstrate that $(1+x)^n \leq 1 + nx + n^2x^2$.

10$^{\underline{th}}$ Question Let $X = \left\{ \frac{1}{n}; n \in \mathbb{N} \right\}$. Prove that $\inf X = 0$.

11$^{\underline{th}}$ Question Let $A \subset B$ be non-empty bounded sets of real numbers. Prove that $\inf B \leq \inf A \leq \sup A \leq \sup B$.

$12^{\underline{th}}$ Question Let A, B be non-empty sets of real numbers such that for all $x \in A$ and $y \in B$, we have $x \leq y$. Prove the following statements:

a) $\sup A \leq \inf B$.

b) For $\sup A = \inf B$, it is necessary and sufficient that for every given $\epsilon > 0$, there exist $x \in A$ and $y \in B$ with $y - x < \epsilon$.

$13^{\underline{th}}$ Question Let $A, B \subset \mathbb{R}$ be non-empty and bounded sets, and let $c \in \mathbb{R}$. Prove the following statements:

a) the set $A + B = \{x + y;\ x \in A, y \in B\}$ is non-empty and bounded,

b) the set $cA = \{cx;\ x \in A\}$ is non-empty and bounded,

c) $\sup(A + B) = \sup A + \sup B$ and $\inf(A + B) = \inf A + \inf B$,

d) if $c \geq 0$ then $\sup(cA) = c \sup A$ and $\inf(cA) = c \inf A$,

e) if $c < 0$ then $\sup(cA) = c \inf A$ and $\inf(cA) = c \sup A$.

$14^{\underline{th}}$ Question Let $f, g : X \to \mathbb{R}$ be bounded functions and $c \in \mathbb{R}$. Prove the following statements:

a) the functions $f + g : X \to \mathbb{R}$ and $cf : X \to \mathbb{R}$ are bounded,

b) $\sup(f + g) \leq \sup f + \sup g$ and $\inf(f + g) \geq \inf f + \inf g$,

c) if $c \geq 0$ then $\sup(cf) = c \sup f$ and $\inf(cf) = c \inf f$,

d) if $c < 0$ then $\sup(cf) = c \inf f$ and $\inf(cf) = c \sup f$.

$15^{\underline{th}}$ Question If $f : X \to \mathbb{R}$ is bounded, $m = \inf f$, $M = \sup f$ and $\omega = M - m$, demonstrate that $\omega = \sup\{|f(x) - f(y)|;\ x, y \in X\}$. *(Note: Corrected the original $\omega = M - \omega$ to $\omega = M - m$ based on context)*

$16^{\underline{th}}$ Question Let $A' \subset A$ and $B' \subset B$ be non-empty and bounded sets of real numbers. If for each $a \in A$ and each $b \in B$ there exist $a' \in A'$ and $b' \in B'$ such that $a \leq a'$ and $b' \leq b$, then $\sup A' = \sup A$ and $\inf B' = \inf B$.

$17^{\underline{th}}$ Question If $x_1 = \sqrt{2}$, and $x_{n+1} = \sqrt{2 + \sqrt{x_n}} \quad (n = 1, 2, ...)$, prove that (x_n) converges and that $x_n < 2$ for $n = 1, 2, 3, ...$ *(Note: Corrected recursive formula slightly to match typical problem or left as is if specific)*

$18^{\underline{th}}$ Question Prove the following limits:

a) If $p > 0$, then $\lim_{n\to\infty} \sqrt[n]{p} = 1$.

b) $\lim_{n\to\infty} \sqrt[n]{n} = 1$.

$c)$ If $p > 0$ and $\alpha > 0$ is real, then $\lim_{n\to\infty} \frac{n^\alpha}{(1+p)^n} = 0$.

$d)$ If $|x| < 1$, then $\lim_{n\to\infty} x^n = 0$.

$e)$ If $a \in \mathbb{R}$, then $\lim_{n\to\infty} \frac{a^n}{n!} = 0$.

19$^{\underline{th}}$ Question Prove that the sequence (x_n) such that

$$x_n = 1 - \frac{1}{2} + \frac{1}{3} - \cdots + (-1)^{n-1}\frac{1}{n}$$

has a limit $a \in (\frac{1}{2}, 1)$.

20$^{\underline{th}}$ Question Given $a > 0$, prove that the following sequence (x_n) is convergent and calculate $\lim x_n$:

$$\sqrt{a},\ \sqrt{a+\sqrt{a}},\ \sqrt{a+\sqrt{a+\sqrt{a}}},\ \cdots$$

Chapter 3

Topology of the Line

3.1 Accumulation Points and Adherent Points

Definition 3.1 *Let $r > 0$. The set $B_r(x)$ is read as: open ball centred at x with radius $r > 0$ and is defined as*

$$B_r(x) = \{y \in \mathbb{R} : |y - x| < r\}$$

In truth:

$$|y - x| < r \Leftrightarrow x - r < y < x + r.$$

Therefore:

$$B_r(x) = \{y \in \mathbb{R} : x - r < y < x + r\},$$

which is nothing more than the open interval $(x - r, x + r)$.

Similarly, the closed ball $\overline{B_r(x)}$ centred at x with radius r is defined as

$$\overline{B_r(x)} = \{y \in \mathbb{R} : |y - x| \leq r\},$$

which is, in fact, the closed interval $[x - r, x + r]$.

Definition 3.2 *Let $E \subset \mathbb{R}$, $E \neq \emptyset$. A point $x \in \mathbb{R}$ is said to be an accumulation point of E if for all $r > 0$ we have $B_r(x) \cap (E \backslash \{x\}) \neq \emptyset$. In other words: $x \in \mathbb{R}$ is said to be an accumulation point if every open ball centred at x contains a point of E different from x.*

x is an accumulation point of E if and only if given $r > 0$, there exists $y_r \in E \cap (x - r, x + r)$ such that $y_r \neq x$.

Example 1: Let $E = \{\frac{1}{n} : n \in \mathbb{N}^*\}$. Then $x = 0$ is an accumulation point of E because given $r > 0$, there exists an $n(r) \in \mathbb{N}$ such that $0 < \frac{1}{n(r)} < r$. It suffices to consider n large enough so that $n > 1/r$.

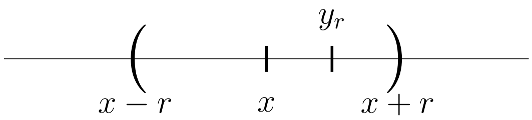


Figure 3.1:

Example 2: Let $A = (2, 4]$. Then $x = 2$ is an accumulation point of A because given $r > 0$ we can always choose $y \in (2-r, 2+r)$ with $y \neq 2$ (and $y \in A$).

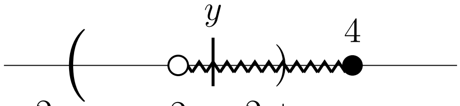


Figure 3.2:

In the example above, note that all points in A are accumulation points of A. However, $2 \notin A$ and is an accumulation point of A.

In Example 1 above, note that the only accumulation point of E is $x = 0$. Verify this fact.

Remark: Note that an accumulation point of a set does not need to belong to the set in question.

Definition 3.3 *Let $E \subset \mathbb{R}$, $E \neq \emptyset$. A point $x \in E$ is said to be an isolated point if x is not an accumulation point of E.*

In other words: x is an isolated point if and only if there exists $r_0 > 0$ such that $B_{r_0}(x) \cap (E \backslash \{x\}) = \emptyset$. This means there must exist a number $r_0 > 0$ such that the open ball $B_{r_0}(x)$ contains no points of E except for x itself.

Example 3: Let $E = (1, 3] \cup \{4, 5\}$.

The points $x = 4$ and $x = 5$ are isolated because $4, 5 \in E$ and there exists $r_0 = 1/2$ such that

$$B_{1/2}(4) \cap E = \{4\} \quad \text{and} \quad B_{1/2}(5) \cap E = \{5\}.$$

Proposition 3.4 *Let $E \subset \mathbb{R}$, $E \neq \emptyset$. If x is an accumulation point of E, then for all $r > 0$, the ball $B_r(x)$ contains infinitely many points of E.*

Proof: Suppose, by contradiction, that there exists $r_0 > 0$ such that the ball $B_{r_0}(x)$ contains only a finite number of points from E. Let $x_1, \cdots, x_n$ be those points (different from x). Let

$$r = \min\{|x - x_1|, \cdots, |x - x_n|\}.$$

Since $x_j \neq x$, we have $r > 0$. Then the open ball $B_r(x)$ contains no points of E other than x (if $x \in E$), which contradicts the fact that x is an accumulation point of E. ■

Proposition 3.5 *Let $E \subset \mathbb{R}$, $E \neq \emptyset$. If x is an accumulation point of E, there exists a sequence $\{x_n\}$ of elements of E, pairwise distinct, converging to x.*

Proof: Let $r_1 = 1$. By Proposition 3.4 (infinitely many points), there exists $x_1 \in E$ such that $0 < |x - x_1| < 1$.

Let $r_2 = \min\{|x_1 - x|, \frac{1}{2}\}$. By Proposition 3.4, there exists $x_2 \in E$ such that $0 < |x_2 - x| < r_2$.

Let $r_3 = \min\{|x_2 - x|, \frac{1}{3}\}$. Similarly, there exists $x_3 \in E$ such that $0 < |x - x_3| < r_3$.

Proceeding in this manner, we obtain a sequence $\{x_n\}$ of elements of E such that:

$$0 < |x_{n+1} - x| < |x_n - x| \quad \text{and} \quad |x_n - x| < \frac{1}{n}.$$

Thus, the x_n are pairwise distinct and, furthermore, $x_n \to x$ as $n \to +\infty$. ■

Definition 3.6 *Let $E \subset \mathbb{R}$, $E \neq \emptyset$. A point $x \in \mathbb{R}$ is said to be adherent to E if for all $r > 0$, $B_r(x) \cap E \neq \emptyset$.*

In other words: Given $r > 0$, the open ball $B_r(x)$ must contain at least one point of E; in this case, it could be x itself, if x belongs to E.

Proposition 3.7 *If $x \in \mathbb{R}$ is adherent to E, then there exists a sequence $\{x_n\}$ of elements of E converging to x. (Now not necessarily of pairwise distinct elements).*

Proof: Take $r = \frac{1}{n}$. Then for each $n \in \mathbb{N}^*$, $B_{1/n}(x) \cap E \neq \emptyset$.

For $n = 1 \Rightarrow$ there exists $x_1 \in B_1(x)$ and $x_1 \in E$,

For $n = 2 \Rightarrow$ there exists $x_2 \in B_{1/2}(x)$ and $x_2 \in E$,

$\vdots$

For $n \Rightarrow$ there exists $x_n \in B_{1/n}(x)$ and $x_n \in E$.

Thus, from the fact that $x_n \in B_{1/n}(x)$, we have $|x_n - x| < \frac{1}{n}$, and therefore $x_n \to x$ as $n \to +\infty$ with $\{x_n\} \subset E$. It may happen that $x_n = x$ for all n (if $x \in E$). ■

Remark: It is worth noting that every point of E is an adherent point to E, because given $r > 0$, $B_r(x) \cap E \neq \emptyset$ since $x \in B_r(x)$ (as it is the centre) and $x \in E$, by hypothesis.

Let us also note that every accumulation point of E is an adherent point to E, because if $x \in \mathbb{R}$ is an accumulation point of E, then given $r > 0$, $B_r(x) \cap (E \backslash \{x\}) \neq \emptyset$, and all the more so $B_r(x) \cap E \neq \emptyset$.

Let us define:

E' := the set of all accumulation points of E.

$\overline{E}$:= the set of all adherent points to E.

$\overline{E}$ is called the adherence or closure of E.

From what we have seen above: $E \subset \overline{E}$ as well as $E' \subset \overline{E}$.

So:

$$\overline{E} \supset E \cup E'. \tag{3.1}$$

On the other hand, we claim that:

$$\overline{E} \subset E \cup E'. \tag{3.2}$$

In fact, take $x \in \overline{E}$. Then given $r > 0$, $B_r(x) \cap E \neq \emptyset$, i.e., for each $r > 0$, there exists $y_r \in B_r(x)$ and $y_r \in E$. We have two cases to consider:

(i) $y_r \neq x$ (for arbitrarily small r). In this case, x is an accumulation point of E, i.e., $x \in E'$.

(ii) $y_r = x$ (for some r, and for all smaller r, the only point is x). In this case, since $y_r \in E$, it follows that $x \in E$.

Thus, $x \in E'$ or $x \in E$, which proves (3.2). From (3.1) and (3.2) we conclude that:

$$\overline{E} = E \cup E'. \tag{3.3}$$

Example: Let $E = (1, 3] \cup \{4, 5\}$.

We have:

$E' = [1, 3]$
$\{4, 5\}$ are isolated points.
$1 \notin E$ is an accumulation point of E.
$\overline{E} = [1, 3] \cup \{4, 5\}$.

3.2 Open and Closed Sets

Definition 3.8 *Let $E \neq \emptyset$ and $E \subset \mathbb{R}$. $x \in E$ is said to be an* <u>*interior point*</u> *of E if and only if there exists $r = r(x) > 0$ such that $B_r(x) \subset E$.*

Example: Let $E = (1, 3)$. Then $x = 2$ is an interior point of E because there exists $r_0 = 1/2 > 0$ such that $B_{1/2}(2) \subset E$.

See Figure 3.3.

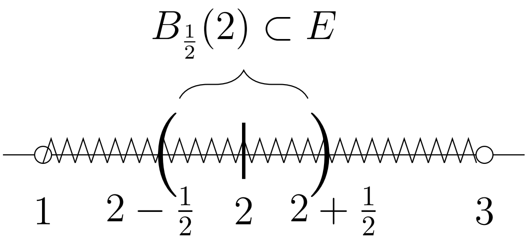


Figure 3.3:

Proposition 3.9 *Let $E = B_\varepsilon(x) = \{y \in \mathbb{R} : |y - x| < \varepsilon\} = (x - \varepsilon, x + \varepsilon)$, $\varepsilon > 0$. Then, every point of E is an interior point of E.*

Proof: Let $y \in B_\varepsilon(x)$. We must exhibit $r > 0$ such that $B_r(y) \subset B_\varepsilon(x)$. Let us take $r := \varepsilon - |y - x|$. Note that $r > 0$ since $|x - y| < \varepsilon$ (because $y \in B_\varepsilon(x)$).

I claim: $B_r(y) \subset B_\varepsilon(x)$

Indeed, take $z \in B_r(y)$. Then $|z - y| < r$. We want to prove that $|z - x| < \varepsilon$, i.e., $z \in B_\varepsilon(x)$. Indeed,

$$\begin{aligned} |z - x| &\leq |z - y| + |y - x| \\ &< r + |y - x| \\ &= \varepsilon - |y - x| + |y - x| = \varepsilon. \end{aligned}$$

Therefore, $|z - x| < \varepsilon$, which proves the desired result. ∎

It follows from this that every point of any open bounded interval of the line is an interior point of it.

We define: $E^0 :=$ the set of interior points of a set E. E^0 is called the interior of the set E. Clearly $E^0 \subset E$.

Definition 3.10 *A subset $E \subset \mathbb{R}$ is said to be open if every point of E is an interior point of E.*

As we saw above, every open and bounded interval of the line, or equivalently, any open ball in $\mathbb{R}$, is an open set.

Attention: The set $A = (1, 3]$ is not an open set since $x = 3$ is not an interior point of A. In fact, recall that $x_0 \in A$ is an interior point of A if there exists $r > 0$ such that $B_r(x_0) \subset A$. Negating this fact, we would have: For any $r > 0$, the open ball $B_r(x_0) \not\subseteq A$. With respect to the set $A = (1, 3]$ above, and for $x_0 = 3$, then whatever $r > 0$ is, the ball $B_r(3)$ is not contained in A. Note that for all $r > 0$ there will always exist a $y \in B_r(3) = (3 - r, 3 + r)$ with $3 < y < 3 + r$, implying that $B_r(3) \not\subseteq A$. See Figure 3.4.

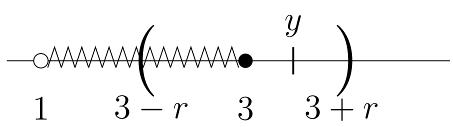


Figure 3.4:

Remark: The set $\emptyset$ (empty set) is open. Indeed, a set E can only fail to be open if there exists some point in E that is not an interior point. Since there are no points in $\emptyset$, we must admit that $\emptyset$ is open. Evidently $\mathbb{R}$ is also an open set, since every point of $\mathbb{R}$ is an interior point of $\mathbb{R}$.

Proposition 3.11 *Let $\{E_\alpha\}_{\alpha\in I}$ be a family of open sets. Then $E = \bigcup_{\alpha\in I} E_\alpha$ is open.*

Proof: Let $x \in E$. We must exhibit an $r > 0$ such that $B_r(x) \subset E$. In fact, since $x \in \bigcup_{\alpha\in I} E_\alpha$, then $x \in E_\alpha$, for some α. Since E_α is open, by hypothesis, there exists an $r > 0$ such that $B_r(x) \subset E_\alpha$, and as $E_\alpha \subset E$, the desired result is proven. ■

Remark: This fact is not true for the arbitrary intersection of open sets. Let's see a counter-example. Let us define

$$E_n = \left(-\frac{1}{n}, \frac{1}{n}\right); \ n = 1, 2, 3, \cdots \quad \text{and} \quad E = \bigcap_{n=1}^{+\infty} E_n.$$

We claim that E is not open. Indeed, let us first observe that:

$$E = \{0\}. \tag{3.4}$$

Proof: In fact, it is clear that

$$\{0\} \subset E = \bigcap_{n=1}^{+\infty} \left(-\frac{1}{n}, \frac{1}{n}\right),$$

since $0 \in \left(-\frac{1}{n}, \frac{1}{n}\right)$ for all $n \in \mathbb{N}^*$. Let $x \in E$ and suppose, by contradiction, that $x \neq 0$. Then $|x| > 0$ and therefore there exists a natural number n_0 such that $0 < \frac{1}{n_0} < |x|$. This means that $x \notin \left(-\frac{1}{n_0}, \frac{1}{n_0}\right) = E_{n_0}$, which is an absurdity, since $x \in E_n$ for all $n \in \mathbb{N}^*$. This proves that $E \subset \{0\}$, i.e., $E = \{0\}$ as alluded to in (3.4). It follows that:

$$E = \bigcap_{n=1}^{+\infty} \left(-\frac{1}{n}, \frac{1}{n}\right) = \{0\},$$

and it is clear that $\{0\}$ is not an open set. ■

Remark: Let $(a, +\infty)$ be an open and unbounded interval. Note that we can write:

$$(a, +\infty) = \bigcup_{n=1}^{+\infty} (a, a+n),$$

so that by virtue of Proposition 3.11, the interval $(a, +\infty)$ is indeed an open set. Similarly, it can be verified that $(-\infty, a)$ is in fact an open set in $\mathbb{R}$.

Definition 3.12 *A set $F \subset \mathbb{R}$ is said to be closed if F contains all of its accumulation points. In other words:*

$$F \text{ is closed} \Leftrightarrow F \supset F'.$$

Remark: Recall that from (3.3) we have $\overline{F} = F \cup F'$. If F is closed, then $F' \subset F$ and therefore

$$\overline{F} = F \cup F' \subset F.$$

Since $F \subset \overline{F}$ is always true, it follows that

$$F \text{ is closed} \Leftrightarrow F = \overline{F}. \tag{3.5}$$

Examples:

1) The set $F = \{1, 2, 3\}$ is closed because in this case the set of accumulation points of F is empty, since all points of F are isolated. Thus, $F' = \emptyset \subset F$. Another way to see that F is closed comes from the fact that $F = \overline{F}$ (since $F' = \emptyset$).

2) The set $F = [1, 3] \cup \{4\}$ is closed because, in this case, $F' = [1, 3]$ (since $x = 4$ is an isolated point) and therefore $F \supset F'$. Furthermore:

$$\overline{F} = F \cup F' = ([1, 3] \cup \{4\}) \cup [1, 3] = F.$$

3) The set $A = \{1/n\}_{n\in\mathbb{N}^*}$ is not closed because in this case $A' = \{0\}$ and $A \not\supset A'$. However, the set $F = \{1/n\}_{n\in\mathbb{N}^*} \cup \{0\}$ is a closed set because $F' = \{0\}$ and $F \supset F'$. Also,

$$\overline{F} = F' \cup F = \{0\} \cup (\{1/n\}_{n\in\mathbb{N}^*} \cup \{0\}) = F.$$

The definition above might not help us perceive if a set is closed. We have the following result:

Proposition 3.13 *A set A is open if and only if $\mathbb{R}\backslash A$ is closed.*

Proof: Suppose first that A is open and, by contradiction, that $\mathbb{R}\backslash A$ is not closed. Then, there exists $x_0 \in (\mathbb{R}\backslash A)'$ such that $x_0 \notin \mathbb{R}\backslash A$. Now, since $x_0 \notin \mathbb{R}\backslash A$, then $x_0 \in A$, and A being open, there exists $r > 0$ such that $B_r(x_0) \subset A$. On the other hand, since $x_0 \in (\mathbb{R}\backslash A)'$, it follows that $B_r(x_0) \cap (\mathbb{R}\backslash A) \neq \emptyset$, i.e., there exists $y \in B_r(x_0)$ with $y \in \mathbb{R}\backslash A$, and therefore $y \notin A$, which is an absurdity since $B_r(x_0) \subset A$.

Conversely, suppose that $\mathbb{R}\backslash A$ is closed and, by contradiction, assume that A is not open. Thus, there exists $x_0 \in A$ such that for all $\varepsilon > 0$, $B_\varepsilon(x_0) \not\subset A$. That is, for each $\varepsilon > 0$ there exists $y_\varepsilon \in B_\varepsilon(x_0)$ with $y_\varepsilon \notin A$ and $y_\varepsilon \neq x_0$ (since if y_ε were equal to x_0 then y_ε would belong to A, which is an absurdity). Thus, x_0 would be an accumulation point of $\mathbb{R}\backslash A$, and by the fact that this set is closed, we would have $x_0 \in \mathbb{R}\backslash A$, which is an absurdity as $x_0 \in A$. ■

Corollary 3.14 *A set F is closed if and only if $\mathbb{R}\backslash F$ is open.*

Proof: It suffices to set $A = \mathbb{R}\backslash F$ in the previous proposition. ■

Examples:

(i) Any closed ball $\overline{B_\varepsilon(x_0)} = [x_0 - \varepsilon, x_0 + \varepsilon]$ or any bounded and closed interval is indeed closed.

(ii) Sets of the type $F = \{x_1, x_2, \cdots, x_n\}$ are closed because their complement is open.

(iii) $\emptyset$ and $\mathbb{R}$ are also closed.

Remark: There are sets that are neither open nor closed. For example, $A = (1, 3]$ is not open because $x = 3$ is not an interior point of A (and $3 \in A$), and it is not closed because $x = 1$ is an accumulation point of A but $1 \notin A$.

Lemma 3.15 *(De Morgan's Laws) Let $\{E_\alpha\}$ be a collection of sets. Then:*

$$\mathbb{R} \backslash \left(\bigcup_\alpha E_\alpha \right) = \bigcap_\alpha (\mathbb{R} \backslash E_\alpha) .$$

Proof: Let $x \in \mathbb{R} \backslash (\bigcup_\alpha E_\alpha)$. Then $x \notin \bigcup_\alpha E_\alpha$. It follows that $x \notin E_\alpha$ for all α, i.e., $x \in \mathbb{R} \backslash E_\alpha$ for all α, or rather $x \in \bigcap_\alpha (\mathbb{R} \backslash E_\alpha)$.

Conversely, suppose that $x \in \bigcap_\alpha (\mathbb{R} \backslash E_\alpha)$. Then, $x \in \mathbb{R} \backslash E_\alpha$ for all α, and therefore $x \notin E_\alpha$ for all α. Thus, $x \notin \bigcup_\alpha E_\alpha$, and hence $x \in \mathbb{R} \backslash (\bigcup_\alpha E_\alpha)$. ■

Analogously,

$$\mathbb{R} \backslash \left(\bigcap_\alpha E_\alpha \right) = \bigcup_\alpha (\mathbb{R} \backslash E_\alpha) . \tag{3.6}$$

Proposition 3.16 *Let $\{F_\alpha\}_{\alpha \in I}$ be a family of closed sets. Then $F = \bigcap_{\alpha \in I} F_\alpha$ is a closed set.*

Proof: According to Corollary 3.14, it suffices to prove that $\mathbb{R} \backslash F$ is an open set. Indeed, from (3.6) we have:

$$\mathbb{R} \backslash F = \mathbb{R} \backslash \left(\bigcap_\alpha F_\alpha \right) = \bigcup_\alpha (\mathbb{R} \backslash F_\alpha) . \tag{3.7}$$

On the other hand, since F_α is closed for all α, it follows that $\mathbb{R} \backslash F_\alpha$ is open. Consequently, $\bigcup_\alpha (\mathbb{R} \backslash F_\alpha)$ is an open set because it is the arbitrary union of open sets (see Proposition 3.11). It follows from this and (3.7) that $\mathbb{R} \backslash F$ is open. ■

Remark: An observation analogous to the one made for the case of a collection of open sets is warranted here. The union of an arbitrary family of closed sets may not be a closed set. Let's see a counter-example. Let E be a generic set that is not closed. It is clear that

$$E = \bigcup_{x \in E} \{x\}.$$

However, each set of the type $\{x\}$ is closed, and yet the union is not.

Proposition 3.17 *A set E is closed if and only if $E = \overline{E}$.*

Proof: If E is closed, then $E \supset E'$ and therefore $\underbrace{E' \cup E}_{=\overline{E}} \subset E$, i.e., $\overline{E} \subset E$. Since $E \subset \overline{E}$ is always true, it follows that $E = \overline{E}$.

Conversely, if $E = \underbrace{\overline{E}}_{=E \cup E'}$, then $E' \subset E$ and therefore E is closed. ■

It follows from Proposition 3.17 that a closed set contains all its adherent points.

3.3 Compact Sets

Definition 3.18 *A cover (or covering) of a set $E \subset \mathbb{R}$ is a family $\mathfrak{C} = \{C_\lambda\}_{\lambda \in A}$ of sets $C_\lambda \subset \mathbb{R}$ such that $E \subset \bigcup_{\lambda \in A} C_\lambda$.*

Definition 3.19 *A subcover (or subcovering) of $\mathfrak{C}$ is a subfamily $\mathfrak{C}' = \{C_\lambda\}_{\lambda \in A' \subset A}$ such that $E \subset \bigcup_{\lambda \in A'} C_\lambda$ still holds.*

Remark: A cover is said to be open when the elements of the family are open sets. Similarly for a closed cover.

Definition 3.20 *A subset $K \subset \mathbb{R}$ is said to be compact if every open cover of K possesses a finite subcover.*

Proposition 3.21 *Let K be a compact subset of $\mathbb{R}$. Then K is closed and bounded.*

Proof: (i) K is bounded.

Consider, for each $n \in \mathbb{N}$, the collection of open intervals given by:

$$G_n := (-n, n).$$

It is clear that $K \subset \bigcup_{n=1}^{+\infty} G_n$, since this union covers the entire line. Indeed, given $x \in \mathbb{R}$, there exists an $n_0 \in \mathbb{N}$ such that $|x| < n_0$, because otherwise, if we had $|x| \geq n$ for all $n \in \mathbb{N}$, the natural numbers would constitute a bounded set, which is an absurdity! Thus,

$$x \in (-n_0, n_0) = G_{n_0} \subset \bigcup_{n=1}^{+\infty} G_n.$$

i.e., $\mathbb{R} \subset \bigcup_{n=1}^{+\infty} G_n$. Since K is compact, there exist $n_1, n_2, \cdots, n_r \in \mathbb{N}$ such that

$$K \subset \bigcup_{i=1}^{r} G_{n_i}. \tag{3.8}$$

We can assume, without loss of generality, that $n_1 < n_2 < \cdots < n_r$ and therefore

$$\bigcup_{i=1}^{r} G_{n_i} = (-n_r, n_r). \tag{3.9}$$

In this way, from (3.8) and (3.9) it follows that

$$K \subset (-n_r, n_r) = B_{n_r}(0),$$

which proves K is bounded.

(ii) K is closed.

It suffices to prove that $\mathbb{R}\backslash K$ is open. Let, then, $x \in (\mathbb{R}\backslash K)$. I must exhibit $r > 0$ such that

$$B_r(x) \subset \mathbb{R}\backslash K \tag{3.10}$$

Indeed, let us define, for each $n \in \mathbb{N}^*$:

$$G_n = \{y \in \mathbb{R} : |y - x| > \frac{1}{n}\} = \mathbb{R}\backslash \left[x - \frac{1}{n}, x + \frac{1}{n}\right].$$

It is clear that, for each $n \in \mathbb{N}^*$, G_n is open, since $\left[x - \frac{1}{n}, x + \frac{1}{n}\right]$ is closed. We claim that

$$\bigcup_{n=1}^{+\infty} G_n = \mathbb{R}\backslash\{x\}. \tag{3.11}$$

(a) Take $y \in \bigcup_{n=1}^{+\infty} G_n$. Then, $y \in G_{n_0}$, for some $n_0 \in \mathbb{N}^*$. Thus, $y \in \mathbb{R}$ and $|y - x| > \frac{1}{n_0}$, which implies that $y \neq x$, i.e., $y \in \mathbb{R}\backslash\{x\}$.

(b) Take $y \in \mathbb{R}\backslash\{x\}$. Then $y \in \mathbb{R}$ and $y \neq x$. Hence, $|y - x| > 0$. Let $n_0 \in \mathbb{N}$ be such that $|y - x| > \frac{1}{n_0}$. Thus, $y \in G_{n_0}$ and therefore $y \in \bigcup_{n=1}^{+\infty} G_n$, which proves (3.11).

Since we took $x \in \mathbb{R}\backslash K$, then $x \notin K$, and from (3.11) it follows that $\{G_n\}$ is an open cover of K.

$$K \subset \mathbb{R}\backslash\{x\} = \bigcup_{n=1}^{+\infty} G_n.$$

Consequently, K being compact, there will exist $n_1, n_2, \cdots, n_r \in \mathbb{N}^*$ which we can, without loss of generality, consider as $n_1 < n_2 < \cdots < n_r$ such that:

$$K \subset \bigcup_{i=1}^{r} G_{n_i}. \tag{3.12}$$

From the fact that $G_{n_1} \subset G_{n_2} \subset \cdots \subset G_{n_r}$, it follows that

$$\bigcup_{i=1}^{r} G_{n_i} = G_{n_r}. \tag{3.13}$$

Thus, from (3.12) and (3.13) we conclude that

$$K \subset G_{n_r} = \mathbb{R} \backslash \left[x - \frac{1}{n_r}, x + \frac{1}{n_r} \right],$$

which implies

$$\mathbb{R} \backslash K \supset \left[x - \frac{1}{n_r}, x + \frac{1}{n_r} \right] = \overline{B_{1/n_r}(x)} \supset B_{1/n_r}(x).$$

In this way, $r = \frac{1}{n_r}$ is the desired radius, which proves (3.10). ■

Remark: It follows immediately from Proposition 3.21 that if $K \subset \mathbb{R}$ is not closed or not bounded, it will not be compact.

Next, we will characterize the compact sets of $\mathbb{R}$. First, we need a preliminary result.

Lemma 3.22 (Nested Intervals Property) *Let $\{I_n\}$ be a sequence of closed and bounded intervals in $\mathbb{R}$ such that $I_n \supset I_{n+1}$, for all $n \in \mathbb{N}$. Then $\bigcap_{n=1}^{+\infty} I_n$ is non-empty.*

Proof: Let us define: $I_n = [a_n, b_n]$.

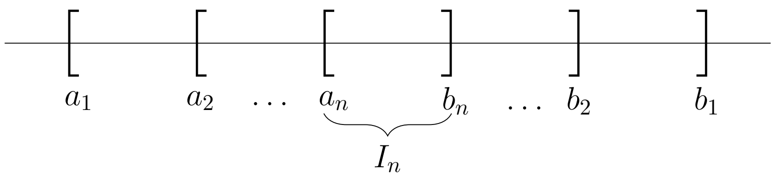


Figure 3.5:

We claim that:

$$a_n \leq b_m, \ \forall n \text{ and } \forall m. \tag{3.14}$$

Suppose the contrary, i.e., that there exist $n_0, m_0 \in \mathbb{N}$ such that $a_{n_0} > b_{m_0}$. Now, it is always true that $a_n \leq b_n$. Therefore:

$$a_{n_0} \leq b_{n_0} \quad \text{and} \quad a_{m_0} \leq b_{m_0}.$$

Hence,

$$a_{m_0} \leq b_{m_0} < a_{n_0} \leq b_{n_0}.$$

It follows that

$$[a_{m_0}, b_{m_0}] \cap [a_{n_0}, b_{n_0}] = \emptyset,$$

which is an absurdity since the intervals are nested, thus proving (3.14). Thus, $a_n \le b_m$ for all $n, m \in \mathbb{N}$. In particular

$$a_n \le b_1; \forall n \in \mathbb{N} \text{ as well as } a_1 \le b_m; \forall m \in \mathbb{N}.$$

It follows that the set $\{a_n : n \in \mathbb{N}\}$ is bounded above, while the set $\{b_m : m \in \mathbb{N}\}$ is bounded below. Thus, by the Supremum Property, $\sup\{a_n : n \in \mathbb{N}\} := \alpha$ exists, as does $\inf\{b_m : m \in \mathbb{N}\} = \beta$.

We claim that

$$\alpha \le \beta. \tag{3.15}$$

Suppose the contrary, that $\alpha > \beta$, and consider $\varepsilon = \frac{\alpha-\beta}{2} > 0$. Thus, there exists $m_0 \in \mathbb{N}$ such that $\beta \le b_{m_0} < \beta + \varepsilon$, and there also exists $n_0 \in \mathbb{N}$ such that $\alpha - \varepsilon < a_{n_0} \le \alpha$.

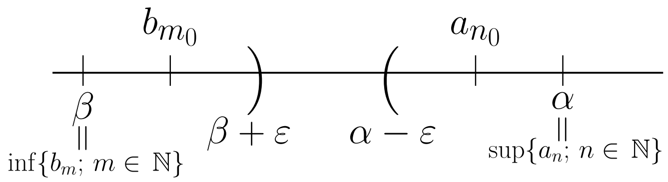


Figure 3.6:

But:

$$\begin{aligned} \beta + \varepsilon &= \beta + \frac{\alpha-\beta}{2} = \frac{2\beta+\alpha-\beta}{2} = \frac{\alpha+\beta}{2} \\ \alpha - \varepsilon &= \alpha - \frac{\alpha-\beta}{2} = \frac{2\alpha-\alpha+\beta}{2} = \frac{\alpha+\beta}{2}. \end{aligned}$$

Consequently

$$\beta \le b_{m_0} < \frac{\alpha+\beta}{2} < a_{n_0} \le \alpha \Rightarrow b_{m_0} < a_{n_0},$$

which is an absurdity in view of (3.14), thus proving (3.15).

We conclude then that $\alpha \le \beta$, which implies

$$[\alpha, \beta] \subset \bigcap_{n=1}^{+\infty} I_n,$$

because if $x \in [\alpha, \beta]$ then $\alpha \le x \le \beta$, and since $a_n \le \alpha$ and $\beta \le b_n$ for all n, it follows that $a_n \le x \le b_n$ for all n, i.e., $x \in [a_n, b_n] = I_n$ for all n, and therefore

$$x \in \bigcap_{n=1}^{+\infty} I_n,$$

that is,

$$[\alpha,\beta]\subset\bigcap_{n=1}^{+\infty} I_n,$$

which proves that the intersection of the I_n's is non-empty. ■

Remark: In truth:

$$[\alpha,\beta]=\bigcap_{n=1}^{+\infty} I_n.$$

Indeed, it remains for us to prove that $\bigcap_{n=1}^{+\infty} I_n\subset[\alpha,\beta]$. In fact, let $x\in I_n=[a_n,b_n]$ for all n, and suppose, by contradiction, that $x<\alpha$ or $x>\beta$.

(i) if $x<\alpha$ then $\alpha-x>0$. Take $\varepsilon=\alpha-x>0$. Hence, there exists $n_0\in\mathbb{N}$ such that $\alpha-\varepsilon<a_{n_0}\le\alpha$, i.e., $\alpha-\underbrace{(\alpha-x)}_{=\varepsilon}<a_{n_0}$, which implies $x<a_{n_0}$, which is an absurdity since $a_n\le x\le b_n$ for all n.

(ii) Similarly, we arrive at an absurdity if $x>\beta$.

Examples:

(i) Let $I_n=\left[-\frac{1}{n},1+\frac{1}{n}\right]$, $n\in\mathbb{N}^*$.

$$\bigcap_{n=1}^{+\infty} I_n=[0,1],\ \text{because}$$
$$\alpha=\sup_{n\in\mathbb{N}}\{-1/n\}=0 \text{ and } \beta=\inf_{n\in\mathbb{N}}\{1+1/n\}=1.$$

(ii) Let $I_n=\left[-\frac{1}{n},\frac{1}{n}\right]$, $n\in\mathbb{N}^*$.

$$\bigcap_{n=1}^{+\infty} I_n=\{0\},\ \text{because}$$
$$\alpha=\sup_{n\in\mathbb{N}}\{-1/n\}=0 \text{ and } \beta=\inf_{n\in\mathbb{N}}\{1/n\}=0.$$

Proposition 3.23 *Let $[a,b]\subset\mathbb{R}$ be a closed and bounded interval. Then $[a,b]$ is compact.*

Proof: Suppose the contrary, i.e., that $[a,b]$ is not compact. Then there exists an open cover $\{G_\alpha\}_{\alpha\in A}$ from which we cannot extract a finite subcover, i.e., there is no finite quantity of G_α's that cover $[a,b]$.

Define $I_0=[a,b]$ and divide I_0 into two equal closed and bounded intervals such that their union is I_0. Then, at least one of them, say I_1, cannot be covered by a finite number of G_α's, because otherwise, if both could be covered by a finite number of G_α's, then $[a,b]$ could also be covered by a finite number of G_α's, which is a contradiction. Let us then divide I_1 into two closed and bounded intervals whose union is I_1. Again, at least one of them, say I_2, cannot be covered by a finite number of G_α's. If we proceed this way, we obtain a sequence $I_0 \supset I_1 \supset I_2 \supset \cdots \supset I_n \supset \cdots$ of nested intervals such that, for all n, I_n cannot be covered by a finite number of G_α's. Note that:

If $x, y \in I_0$ then $|x-y| \le b-a$.
If $x, y \in I_1$ then $|x-y| \le \frac{b-a}{2}$.
If $x, y \in I_2$ then $|x-y| \le \frac{b-a}{2^2}$.
$\vdots$

$$\text{If } x, y \in I_n \text{ then } |x-y| \le \frac{b-a}{2^n}. \tag{3.16}$$

On the other hand, according to the Nested Intervals Theorem, there exists $x^* \in [a,b]$ such that $x^* \in \bigcap_{n=1}^{+\infty} I_n$. Now, since $x^* \in [a,b]$ and $\{G_\alpha\}$ is an open cover of $[a,b]$, it follows that $x^* \in G_{\alpha_0}$ for some $\alpha_0 \in A$. Since G_{α_0} is an open set, there exists $\varepsilon_0 > 0$ such that the ball $B_{\varepsilon_0}(x^*) \subset G_{\alpha_0}$. Let us consider $n_0 \in \mathbb{N}$ sufficiently large such that $2^{n_0} > \frac{b-a}{\varepsilon_0}$ (or $\frac{b-a}{2^{n_0}} < \varepsilon_0$). We claim that:

$$I_{n_0} \subset B_{\varepsilon_0}(x^*). \tag{3.17}$$

Indeed, let $y \in I_{n_0}$. Then from (3.16) it follows that $|y-x| \le \frac{b-a}{2^{n_0}}$ for all $x \in I_{n_0}$. In particular, from the fact that $x^* \in I_{n_0}$, we have

$$|y-x^*| \le \frac{b-a}{2^{n_0}} < \varepsilon_0,$$

which proves that $y \in B_{\varepsilon_0}(x^*)$ and therefore (3.17).

In this way,

$$I_{n_0} \subset B_{\varepsilon_0}(x^*) \subset G_{\alpha_0},$$

which is a contradiction, since none of the I_n's can be covered by a finite number of G_α's (in this case, by the single set G_{α_0}). ■

Proposition 3.24 *Let $K \subset \mathbb{R}$ be compact and F be a closed subset of K. Then F is compact.*

Proof: Let $\{G_\alpha\}_{\alpha \in A}$ be an open cover of F. We must exhibit a finite subcover of F. Indeed, since $F \subset K$ is closed (in $\mathbb{R}$), it follows that $\mathbb{R} \backslash F$ is open. We claim that:

$$K \subset \left(\bigcup_{\alpha \in A} G_\alpha \right) \cup (\mathbb{R} \backslash F). \tag{3.18}$$

Indeed, let $x \in K$. Since $F \subset K$, we have two cases to consider:

(i) $x \in F$. In this case, since $\{G_\alpha\}$ is a cover of F, it follows that $x \in G_\alpha$ for some $\alpha \in A$, and therefore $x \in \bigcup_{\alpha \in A} G_\alpha$.

(ii) $x \notin F$. In this case, $x \in \mathbb{R} \backslash F$, which proves (3.18).

Thus, the G_α's together with $\mathbb{R}\backslash F$ form an open cover for K. Since K is compact, it possesses a finite subcover, i.e., there exist $\alpha_1, \cdots, \alpha_n \in A$ such that

$$K \subset G_{\alpha_1} \cup G_{\alpha_2} \cup \cdots \cup G_{\alpha_n} \cup (\mathbb{R}\backslash F).$$

However, since $F \subset K$ and $F \cap (\mathbb{R}\backslash F) = \emptyset$, it follows that

$$F \subset G_{\alpha_1} \cup G_{\alpha_2} \cup \cdots \cup G_{\alpha_n},$$

which concludes the proof. ■

Theorem 3.25 *[Heine-Borel] A subset K of $\mathbb{R}$ is compact if and only if it is closed and bounded.*

Proof: It has already been shown that if K is compact, then K is closed and bounded (Proposition 3.21). It remains for us to prove that if K is closed and bounded, then K is compact. Indeed, since K is bounded, there exists a closed and bounded interval $[a, b]$ that contains it ($K \subset [a, b]$). Since $[a, b]$ is compact (Proposition 3.23) and K is a closed subset of $[a, b]$ (as K is closed in $\mathbb{R}$), it follows from Proposition 3.24 that K is compact, which concludes the proof. ■

Proposition 3.26 *Let $K \subset \mathbb{R}$ be compact. Then for every infinite subset A of K, there exists $x_A \in K$ which is an accumulation point of A.*

Proof: Suppose, by contradiction, that there exists an infinite subset A of K such that no point of K is an accumulation point of A. Thus, for each $x \in K$ there exists $\varepsilon_x > 0$ such that

$$B_{\varepsilon_x}(x) \cap (A\backslash\{x\}) = \emptyset.$$

The collection $\{B_{\varepsilon_x}(x)\}_{x \in K}$ forms an open cover of K. Since K is compact, there exist $x_1, \cdots, x_n$ and $\varepsilon_1, \cdots, \varepsilon_n$ such that

$$K \subset \bigcup_{i=1}^{n} B_{\varepsilon_i}(x_i).$$

However, as $A \subset K$ we also have

$$A \subset \bigcup_{i=1}^{n} B_{\varepsilon_i}(x_i)$$

which implies

$$A = A \cap \left(\bigcup_{i=1}^{n} B_{\varepsilon_i}(x_i) \right) = \bigcup_{i=1}^{n} \left(B_{\varepsilon_i}(x_i) \cap A \right).$$

But, $B_{\varepsilon_i}(x_i) \cap (A \backslash \{x_i\}) = \emptyset$. This means $B_{\varepsilon_i}(x_i) \cap A$ is either $\emptyset$ (if $x_i \notin A$) or $\{x_i\}$ (if $x_i \in A$). In any case, the union $\bigcup_{i=1}^{n} (B_{\varepsilon_i}(x_i) \cap A)$ is a subset of the finite set $\{x_1, \ldots, x_n\}$. This leads to an absurdity, as we have an infinite set A contained in a finite set. ■

Exercise: Prove the following theorem:

Theorem 3.27 *Let $K \subset \mathbb{R}$. The following statements are equivalent:*

(1) K is bounded and closed.

(2) K is compact (by open covers).

(3) Every infinite subset A of K has an accumulation point $x_A \in K$. (Sequential compactness)

Corollary 3.28 *[Bolzano-Weierstrass Theorem] Every bounded infinite subset of $\mathbb{R}$ has an accumulation point.*

Proof: Let $A \subset \mathbb{R}$ be a bounded infinite subset of $\mathbb{R}$. Being bounded, there exists $[a, b]$ such that $A \subset [a, b]$. As $[a, b]$ is compact and A is an infinite subset of $[a, b]$, it follows by virtue of Proposition 3.26 that there exists $x_A \in [a, b]$ which is an accumulation point of A. ■

<u>Remark:</u> Note that if $A = \{a_n\}_{n \in \mathbb{N}}$ is the set of values of a bounded sequence, there will exist an $x_A \in \mathbb{R}$ that is an accumulation point of A. This implies there exists a subsequence $\{a_{n_k}\}_{k \in \mathbb{N}}$ of $\{a_n\}_n$ such that $a_{n_k} \to x_A$ as $k \to +\infty$. (This is the proof of Theorem 2.16).

Definition 3.29 *Two sets A and B are said to be <u>separated</u> if: $\overline{A} \cap B = A \cap \overline{B} = \emptyset$.*

Example: Let $A = [1, 2)$ and $B = (2, 3]$. We have $\overline{A} = [1, 2]$ and $\overline{B} = [2, 3]$. Thus:

$$\overline{A} \cap B = [1, 2] \cap (2, 3] = \emptyset$$
$$A \cap \overline{B} = [1, 2) \cap [2, 3] = \emptyset.$$

Therefore A and B are separated.

Definition 3.30 *A set $E \subset \mathbb{R}$ is said to be* connected *if for every pair of separated sets whose union is equal to E, one of them is empty.*

In other words:

E is connected $\Leftrightarrow$ For all A and B separated such that $E = A \cup B$, then $A = \emptyset$ or $B = \emptyset$.

It follows that:

E is disconnected (or not connected) $\Leftrightarrow$ there exist non-empty, separated sets $A, B \subset \mathbb{R}$ such that $E = A \cup B$.

Proposition 3.31 *$E \subset \mathbb{R}$ is connected if and only if it has the intermediate value property: for every pair of points $x, y \in E$ with $x < y$, if $z \in \mathbb{R}$ satisfies $x < z < y$, then $z \in E$. (i.e., E is an interval).*

Proof: '$\Rightarrow$' Suppose first that E is connected and consider $x, y \in E$ satisfying the condition $x < z < y$ and, by contradiction, assume that $z \notin E$. Let us define:

$$A = (-\infty, z) \cap E \quad \text{and} \quad B = (z, +\infty) \cap E.$$

We have:

(i) $A \neq \emptyset$ because $x \in E$ and $x \in (-\infty, z)$ since $x < z$. $B \neq \emptyset$ because $y \in E$ and $y \in (z, +\infty)$ since $z < y$.

(ii) $E = A \cup B$.

Indeed:

$$\begin{aligned} A \cup B &= ((-\infty, z) \cup (z, +\infty)) \cap E \\ &= (\mathbb{R} \backslash \{z\}) \cap E. \end{aligned}$$

However, since $z \notin E$, it follows that $E \subset \mathbb{R} \backslash \{z\}$, and therefore $E = (\mathbb{R} \backslash \{z\}) \cap E$. Thus $E = A \cup B$.

(iii) A and B are separated.

Indeed, we will show that $\overline{A} \cap B = \emptyset$. It is proved analogously that $A \cap \overline{B} = \emptyset$. In fact,

$$\begin{aligned} \overline{A} \cap B &= \overline{((-\infty, z) \cap E)} \cap ((z, +\infty) \cap E) \\ &\subset \overline{(-\infty, z)} \cap \overline{E} \cap ((z, +\infty) \cap E) \\ &= (-\infty, z] \cap \overline{E} \cap ((z, +\infty) \cap E) \\ &\subset \Bigg(\underbrace{(-\infty, z] \cap (z, +\infty)}_{=\emptyset} \Bigg) \cap \overline{E} = \emptyset. \end{aligned}$$

From (i), (ii) and (iii), it follows that A and B form a disconnection for E, which is an absurdity since E is connected.

'$\Leftarrow$' Conversely, suppose that given $x, y \in E$ and $z \in \mathbb{R}$ such that $x < z < y$, then $z \in E$. We must prove that E is connected. Suppose the contrary, i.e., that E is a disconnected set. Then, there exist non-empty, separated sets A, B such that $E = A \cup B$.

Take $x \in A$, $y \in B$. It is clear that $x \neq y$, because otherwise $\overline{A} \cap B$ and $A \cap \overline{B}$ would be non-empty, which is an absurdity as A and B are separated. Thus, $x \neq y$, and without loss of generality, suppose $x < y$ and define:

$$z = \sup\{[x, y] \cap A\}.$$

Note that z is an adherent point to the set $[x, y] \cap A$. Hence,

$$z \in \overline{[x, y] \cap A} \subset [x, y] \cap \overline{A},$$

and therefore,

$$z \in [x, y] \text{ and } z \in \overline{A}.$$

Since $z \in \overline{A}$, then $z \notin B$ (because $\overline{A} \cap B = \emptyset$). Thus z cannot be equal to y, since $y \in B$. So,

$$z \in \overline{A} \text{ and } x \leq z < y. \tag{3.19}$$

We have two cases to consider:

(i) $z \notin A$ In this case, since $z \notin A$, z cannot be equal to x (as $x \in A$). We then have from (3.19) that

$$x < z < y.$$

But, by hypothesis (the interval property), it follows that $z \in E$. This is an absurdity, since $z \notin A$ and $z \notin B$ (as $z \in \overline{A}$), and therefore $z \notin A \cup B = E$.

(ii) $z \in A$ Since $z \in A$, then $z \notin \overline{B}$ (as $A \cap \overline{B} = \emptyset$). Hence, there exists $\varepsilon_0 > 0$ such that the neighbourhood $(z - \varepsilon_0, z + \varepsilon_0)$ contains no points of B. This implies $z < y$ (as $y \in B$). Take $z_1 \in (z, y]$. Since $z = \sup([x, y] \cap A)$, no point in $(z, y]$ can be in A. Therefore $z_1 \notin A$. Since $z_1 \in (z, y] \subset [x, y] \subset E$, we must have $z_1 \in B$. But this contradicts $z = \sup([x, y] \cap A)$?

Let's re-examine the original proof's argument: (ii) $z \in A$. Since $z \in A$, then $z \notin \overline{B}$ (as $A \cap \overline{B} = \emptyset$). So, there exists $\varepsilon_0 > 0$ such that $(z - \varepsilon_0, z + \varepsilon_0) \cap B = \emptyset$. Since $z < y$ (as $z \in A, y \in B$), let's choose z_1 such that $z < z_1 < \min\{y, z + \varepsilon_0\}$. Since $x \leq z < z_1 < y$, by the interval hypothesis, $z_1 \in E$. Since $z = \sup([x, y] \cap A)$, $z_1 \notin A$. Since $z_1 \in (z, z + \varepsilon_0)$, $z_1 \notin B$. Thus $z_1 \notin A \cup B = E$, which is an absurdity. This concludes the proof. ■

Exercises: Topology of the Real Line

1$^{\underline{st}}$ Question Prove that for every $X \subset \mathbb{R}$ we have $\text{int}(\text{int}X) = \text{int}X$ and conclude that $\text{int}X$ is an open set.

2$^{\underline{nd}}$ Question If A and B are subsets of the real line, then:

(a) $\text{int}A \subset A$.

(b) If $A \subset B$ then $\text{int}A \subset \text{int}B$.

(c) If $A \subset \text{int}A$ then $A = \text{int}A$.

(d) $A \subset \overline{A}$.

(e) If $A \subset B$ then $\overline{A} \subset \overline{B}$. *(Note: Item (e) and (f) in the original were identical. I kept (e) and removed duplicate (f) unless it was meant to be something else)*.

(g) If $A \subset B$ then $A' \subset B'$.

(h) $A' \subset \overline{A}$.

(i) $A \cup A' = \overline{A}$.

(j) If $A' \subset A$ then $A = \overline{A}$.

(k) A is closed if and only if $A' \subset A$.

(l) $\text{int}A \subset A \subset \overline{A}$.

(m) $\overline{\mathbb{R} \setminus A} = \mathbb{R} \setminus (\text{int}A)$.

(n) $\text{int}(\mathbb{R} \setminus A) = \mathbb{R} \setminus (\overline{A})$. *(Note: Replaced $\complement$ with $\mathbb{R}\setminus$ for clarity in topology)*.

3$^{\underline{rd}}$ Question Let $A \subset \mathbb{R}$ be a set with the following property: "every sequence (x_n) that converges to a point $a \in A$ has its terms x_n belonging to A for all n sufficiently large". Prove that A is open.

4$^{\underline{th}}$ Question **Definition.** (Neighborhood of a point) Let $x \in \mathbb{R}$. A set $V \subset \mathbb{R}$ is a neighborhood of x if there exists an open interval (a, b) such that $x \in (a, b) \subset V$.

(a) For every $X \subset \mathbb{R}$, prove that the disjoint union $\mathbb{R} = \text{int}X \cup \text{int}(\mathbb{R} \setminus X) \cup F$ holds, where F is formed by the points $x \in \mathbb{R}$ such that every neighborhood of x contains points of X and points of $\mathbb{R} \setminus X$. The set $F = \text{fr}X$ (or ∂X) is called the boundary of X.

(b) Prove that $A \subset \mathbb{R}$ is open if and only if $A \cap \text{fr}A = \emptyset$.

(c) Prove that for every $A \subset \mathbb{R}$ we have $\text{fr}A = \overline{\mathbb{R} \setminus A} \cap \overline{A} = \overline{A} \setminus \text{int}A$

5$^{\underline{th}}$ Question For each of the following sets, determine its boundary.

$$a)\, X = [0,1] \quad b)\, Y = (0,1) \cup (1,2) \quad c)\, Z = \mathbb{Q} \quad d)\, W = \mathbb{Z}$$

6$^{\underline{th}}$ Question Let A and B be non-empty subsets of $\mathbb{R}$. Demonstrate that $A+B = \{x + y \,|x \in A\,, y \in B\}$ is open if A is open.

7$^{\underline{th}}$ Question Let $(I_\alpha)_{\alpha\in A}$ be a family of non-empty open intervals of $\mathbb{R}$, such that $\alpha \neq \beta \Longrightarrow I_\alpha \cap I_\beta = \emptyset$. Prove that A is countable.

8$^{\underline{th}}$ Question Determine $a)\,\overline{\mathbb{Q}}$, $\quad b)\,\text{int}\mathbb{Q}$, $\quad c)\,\overline{\mathbb{R} \setminus \mathbb{Q}}$, $\quad d)\,\text{int}(\mathbb{R} \setminus \mathbb{Q})$.

9$^{\underline{th}}$ Question Let A and B be subsets of $\mathbb{R}$. Prove: $a)\ \overline{A \cap B} \subset \overline{A} \cap \overline{B}$,

$b)\ \overline{A \cup B} = \overline{A} \cup \overline{B}$, $\quad c)\ \text{int}(A\cap B) = \text{int}A\cap\text{intB}$, $\quad d)\ \text{int}(A\cup B) \supset \text{int}A\cup\text{int}B$.

Prove also by means of examples that the inclusions can be strict.

10$^{\underline{th}}$ Question Let $A \subset \mathbb{R}$. Demonstrate that A' is a closed set.

11$^{\underline{th}}$ Question Let A and B be subsets of $\mathbb{R}$. Prove that

$$(A \cup B)' = A' \cup B' \qquad \text{and} \qquad (A \cap B)' \subset A' \cap B';$$

prove with an example that the inclusion can be strict.

12$^{\underline{th}}$ Question Let $A \subset \mathbb{R}$ and $a \in A'$. Demonstrate that a is the limit of a strictly monotonic sequence of elements of A.

13$^{\underline{th}}$ Question Let A and B be two non-empty subsets of $\mathbb{R}$. Assuming B' is non-empty, demonstrate $\overline{A} + B' \subset (A + B)'$ and give an example for strict inclusion.

14$^{\underline{th}}$ Question Let

$$A = \left\{\frac{1}{m} + \frac{1}{n}\,\middle|\, m \in \mathbb{N}, n \in \mathbb{N}\right\}$$

Determine $\sup A$, $\inf A$, A'.

$15^{\underline{th}}$ Question Let U and V be two open sets of $\mathbb{R}$ such that $U \cap V = \emptyset$. Demonstrate that $\text{int}\overline{U} \cap \text{int}\overline{V} = \emptyset$

$16^{\underline{th}}$ Question Let $A \subset \mathbb{R}$. Compare $\text{fr}(\text{int}A)$, $\text{fr}A$, $\text{fr}(\overline{A})$. Demonstrate $\text{fr}(A \cup B) \subset (\text{fr}A) \cup (\text{fr}B)$. Can equality hold?

$17^{\underline{th}}$ Question Prove that for every $X \subset \mathbb{R}$, $\overline{X} = X \cup \text{fr}X$ holds. Conclude that X is closed if and only if $X \supset \text{fr}X$.

$18^{\underline{th}}$ Question If $X \subset \mathbb{R}$ is open (respectively, closed) and $X = A \cup B$ is a separation (disconnection), prove that A and B are open (respectively, closed).

$19^{\underline{th}}$ Question Prove that if $X \subset \mathbb{R}$ has an empty boundary, then $X = \emptyset$ or $X = \mathbb{R}$.

$20^{\underline{th}}$ Question Prove that for every $X \subset \mathbb{R}$, we have $\overline{X} = X \cup X'$. Conclude that X is closed if and only if it contains all its accumulation points.

$21^{\underline{st}}$ Question Prove that a finite union and an arbitrary intersection of compact sets is a compact set.

$22^{\underline{nd}}$ Question Let X, Y be disjoint non-empty sets of $\mathbb{R}$, with X compact and Y closed. Prove that there exist $x_0 \in X$ and $y_0 \in Y$ such that $|x_0 - y_0| \leq |x - y|$ for any $x \in X$, $y \in Y$.

$23^{\underline{rd}}$ Question A compact set whose points are all isolated is finite. Give an example of an unbounded closed set X and a bounded non-closed set Y, whose points are all isolated.

$24^{\underline{th}}$ Question Prove that if X is compact, then the following sets are also compact:

$a) S = \{x + y; x, y \in X\}, \quad b)\, D = \{x - y; x, y \in X\},$

$c) P = \{x \cdot y; x, y \in X\}, \quad d)\, Q = \{x/y; x, y \in X\} \quad \text{if} \quad 0 \notin X.$

$25^{\underline{th}}$ Question Let A and B be subsets of $\mathbb{R}$ such that A is closed and B is compact. Prove that $A + B = \{x + y \,|\, x \in A\,, y \in B\}$ is closed.

$26^{\underline{th}}$ Question Prove that the following sets of real numbers are disconnected:

$(a)\ \mathbb{N} \subset \mathbb{R} \quad (b)\ H = \{1/n : n \in \mathbb{N}\} \subset \mathbb{R} \quad (c)\ \mathbb{Q} \subset \mathbb{R} \quad (d)\ \mathbb{R} \setminus \{0\} \subset \mathbb{R}$

$27^{\underline{th}}$ Question If $X \subset \mathbb{Q}$ is connected, then X contains no more than one point.

$28^{\underline{th}}$ Question If $A \subset \mathbb{R}$ is connected, then $\overline{A}$ is connected.

$29^{\underline{th}}$ Question Prove that the closed unit interval $I = [0, 1] \subset \mathbb{R}$ is connected.

$30^{\underline{th}}$ Question A subset of $\mathbb{R}$ is connected if and only if it is an interval.

Chapter 4

Limits of Functions

4.1 Limits of Functions

Definition 4.1 *Let $f : X \to \mathbb{R}$ be a real-valued function defined on a subset $X \subset \mathbb{R}$. Let a be an accumulation point of X, i.e., $a \in X'$. We say that the real number L is the limit of $f(x)$ as x approaches a and we write*

$$\lim_{x \to a} f(x) = L,$$

to mean the following: For each real number $\varepsilon > 0$, given arbitrarily, we can find $\delta > 0$ ($\delta = \delta(\varepsilon)$) such that $|f(x) - L| < \varepsilon$ whenever $x \in X$ and, furthermore, $0 < |x - a| < \delta$.

Therefore, when a is an accumulation point of the domain of f, the expression $\lim_{x\to a} f(x) = L$ is an abbreviation for the statement below:

$$\forall \varepsilon > 0, \exists \delta > 0, \forall x \in D(f), 0 < |x - a| < \delta \Rightarrow |f(x) - L| < \varepsilon.$$

(Note: $D(f)$ is used for domain, synonymous with X)

Remark: Note that $0 < |x - a| < \delta$ means x belongs to the interval $(a - \delta, a + \delta)$ with $x \neq a$. Thus, $\lim_{x\to a} f(x) = L$ means that for every open interval $(L-\varepsilon, L+\varepsilon)$, there exists an open interval $(a - \delta, a + \delta)$ such that, setting:

$$V_\delta = (X \backslash \{a\}) \cap (a - \delta, a + \delta),$$

it holds that

$$f(V_\delta) \subset (L - \varepsilon, L + \varepsilon).$$

Observe that:

$$V_\delta := \{x \in X : 0 < |x - a| < \delta\}.$$

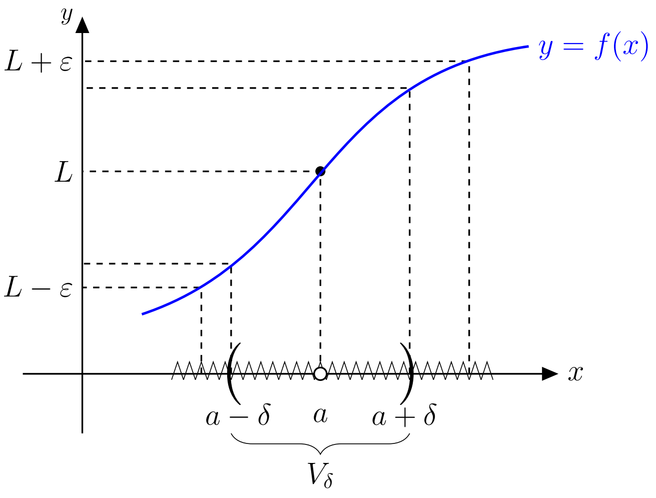


Figure 4.1:

In simple language: it is possible to make $f(x)$ arbitrarily close to L, provided one takes x sufficiently close to a.

Remarks:

(1a) According to the given definition of a limit, it only makes sense to write $\lim_{x\to a} f(x) = L$ when a is an accumulation point of the domain X of the function f. If we were to consider the same definition in the case where $a \notin X'$, then every real number L would be a limit of f as x approaches a. Indeed, since $a \notin X'$, there exists $\delta > 0$ such that

$$V_\delta = (X \backslash \{a\}) \cap (a-\delta, a+\delta) = \emptyset,$$

i.e., $0 < |x-a| < \delta$, $x \in X$ is not satisfied for any x. Then, given any $\varepsilon > 0$, we would choose this δ. It would always be true that

$$f(V_\delta) = \emptyset \subset (L-\varepsilon, L+\varepsilon),$$

whatever L might be. Hence, we would have $\lim_{x\to a} f(x) = L$ for all L.

(2a) When considering $\lim_{x\to a} f(x) = L$, we do not require a to belong to the domain of the function f. In the most interesting cases, $a \notin X$. Let's see an example: Consider $f(x) = \frac{x^2-1}{x-1}$, for $x \neq 1$. In truth:

$$f(x) = \frac{(x-1)(x+1)}{x-1} = x+1, \;\; x \neq 1.$$

We have $\lim_{x\to 1} f(x) = 2$. Indeed, let $\varepsilon > 0$ be given. For $x \neq 1$, we have $|f(x)-2| = |(x+1)-2| = |x-1|$. Hence, for the given $\varepsilon > 0$, $\delta = \varepsilon$ exists such that if $x \in \mathbb{R}$

and $x \neq 1$ with $0 < |x-1| < \delta$, then

$$|f(x) - 2| = |x - 1| < \varepsilon.$$

(3a) Even if $a \in X$, the statement $\lim_{x\to a} f(x) = L$ says nothing about the value of $f(a)$. It only describes the behavior of the values of $f(x)$ for x close to a with $x \neq a$. Explicitly, it is possible to have $\lim_{x\to a} f(x) \neq f(a)$. Let's see an example: Consider

$$f(x) = \begin{cases} 1, & x \in \mathbb{R}\backslash\{0\}, \\ 0, & x = 0. \end{cases}$$

We have that $\lim_{x\to 0} f(x) = 1$. Indeed, let $\varepsilon > 0$ be given. For $x \neq 0$, we have: $|f(x) - 1| = |1 - 1| = 0$. Thus, for the given $\varepsilon > 0$, $\delta = \varepsilon$ exists such that if $x \in \mathbb{R}$, $x \neq 0$ and $0 < |x - 0| < \delta$ then $|f(x) - 1| = 0 < \varepsilon$. However, $f(0) = 0$.

Proposition 4.2 *[Uniqueness of the Limit] Let $X \subset \mathbb{R}$, $f : X \to \mathbb{R}$, $a \in X'$. If $\lim_{x\to a} f(x) = L_1$ and $\lim_{x\to a} f(x) = L_2$, then $L_1 = L_2$.*

Proof: Given any $\varepsilon > 0$, there exist $\delta_1 > 0$ and $\delta_2 > 0$ such that if $x \in X$,

$$0 < |x - a| < \delta_1 \Rightarrow |f(x) - L_1| < \frac{\varepsilon}{2},$$
$$0 < |x - a| < \delta_2 \Rightarrow |f(x) - L_2| < \frac{\varepsilon}{2}.$$

Let $\delta = \min\{\delta_1, \delta_2\}$. Since $a \in X'$, we can find $\overline{x} \in X$ such that $0 < |\overline{x} - a| < \delta$. Then:

$$\begin{aligned} |L_1 - L_2| &= |L_1 - f(\overline{x}) + f(\overline{x}) - L_2| \\ &\leq |L_1 - f(\overline{x})| + |f(\overline{x}) - L_2| \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \end{aligned}$$

This gives us $|L_1 - L_2| < \varepsilon$, for all $\varepsilon > 0$, which implies, given the arbitrariness of $\varepsilon > 0$, that $L_1 = L_2$. ■

Proposition 4.3 *Let $X \subset \mathbb{R}$, $f : X \to \mathbb{R}$ and $a \in X'$. If $\lim_{x\to a} f(x)$ exists, then f is bounded in a neighbourhood of a, i.e., there exist $M > 0$ and $\delta > 0$ such that*

$$0 < |x - a| < \delta, \; x \in X \Rightarrow |f(x)| \leq M.$$

Proof: Let $L = \lim_{x\to a} f(x)$. Taking $\varepsilon = 1$ in the definition of limit, there exists $\delta > 0$ such that if $x \in X$ and $0 < |x - a| < \delta$ then $|f(x) - L| < 1$. But

$$|f(x) - L| \geq ||f(x)| - |L|| \geq |f(x)| - |L|,$$

and consequently

$$|f(x)| \leq |f(x) - L| + |L| < 1 + |L| := M,$$

provided that $0 < |x - a| < \delta$, proving the desired result. ■

Proposition 4.4 *(Squeeze Theorem) Let $X \subset \mathbb{R}$, $f, g, h : X \to \mathbb{R}$ and $a \in X'$. If, for all $x \in X$, $x \neq a$ we have $f(x) \leq g(x) \leq h(x)$, and furthermore, we also have $\lim_{x\to a} f(x) = \lim_{x\to a} h(x) = L$, then $\lim_{x\to a} g(x) = L$.*

Proof: Let $\varepsilon > 0$ be given. Then there exist $\delta_1 > 0$ and $\delta_2 > 0$ such that for $x \in X$:

$$0 < |x - a| < \delta_1 \Rightarrow L - \varepsilon < f(x) < L + \varepsilon,$$
$$0 < |x - a| < \delta_2 \Rightarrow L - \varepsilon < h(x) < L + \varepsilon.$$

Taking $\delta = \min\{\delta_1, \delta_2\}$, then if $x \in X$ and $0 < |x - a| < \delta$, it follows that

$$L - \varepsilon < f(x) \leq g(x) \leq h(x) < L + \varepsilon,$$

from which we conclude that $\lim_{x\to a} g(x) = L$. ■

Proposition 4.5 *Let $X \subset \mathbb{R}$, $a \in X'$, $f, g : X \to \mathbb{R}$. If $\lim_{x\to a} f(x) = L$ and $\lim_{x\to a} g(x) = M$ with $L < M$; then there exists $\delta > 0$ such that if $x \in X$ and $0 < |x - a| < \delta$ then $f(x) < g(x)$.*

Proof: Define $\varepsilon = \frac{M-L}{2} > 0$. Hence, there exist $\delta_1, \delta_2 > 0$ such that for $x \in X$:

$$0 < |x - a| < \delta_1 \Rightarrow L - \frac{M - L}{2} < f(x) < L + \frac{M - L}{2},$$
$$0 < |x - a| < \delta_2 \Rightarrow M - \frac{M - L}{2} < g(x) < M + \frac{M - L}{2}.$$

Taking $\delta = \min\{\delta_1, \delta_2\}$, then if $x \in X$ and $0 < |x - a| < \delta$, it follows that

$$\frac{3L - M}{2} < f(x) < \frac{L + M}{2} < g(x) < \frac{3M - L}{2},$$

i.e., $f(x) < g(x)$ provided that $0 < |x - a| < \delta$. ■

Corollary 4.6 *If $\lim_{x\to a} f(x) = L > 0$ then there exists $\delta > 0$ such that if $x \in X$, $0 < |x - a| < \delta \Rightarrow f(x) > 0$.*

Proof: It suffices to consider $g(x) \equiv M = 0$ in Proposition 4.5. (Note: The original proof said $f(x) \equiv 0$, but it should be $g(x)$ or a new function $h(x) = 0$). ■

Corollary 4.7 *If $f(x) \le g(x)$ for all $x \in X$, $x \neq a$, and $\lim_{x\to a} f(x) = L$ and $\lim_{x\to a} g(x) = M$, then $L \le M$.*

Proof: Suppose the contrary, that $L > M$. Then by Proposition 4.5, there exists $\delta > 0$ such that if $x \in X$ and $0 < |x-a| < \delta \Rightarrow f(x) > g(x)$, which is a contradiction. ■

Proposition 4.8 *Let $X \subset \mathbb{R}$, $a \in X'$ and $f, g : X \to \mathbb{R}$ be functions. If $\lim_{x\to a} f(x) = L$ and $\lim_{x\to a} g(x) = M$, then:*
(i) $\lim_{x\to a}(f(x) + g(x)) = L + M$.
(ii) $\lim_{x\to a}(f(x)\, g(x)) = L\, M$.
(iii) If $M \neq 0$ then $\lim_{x\to a} \frac{f(x)}{g(x)} = \frac{L}{M}$.

Proof: (i) Let $\varepsilon > 0$ be given. Hence, there exist $\delta_1 > 0$ and $\delta_2 > 0$ such that if $x \in X$ and

$$0 < |x-a| < \delta_1 \Rightarrow |f(x) - L| < \frac{\varepsilon}{2}, \tag{4.1}$$
$$0 < |x-a| < \delta_2 \Rightarrow |g(x) - M| < \frac{\varepsilon}{2}. \tag{4.2}$$

Take $\delta = \min\{\delta_1, \delta_2\}$. Then, if $x \in X$ and $0 < |x-a| < \delta$, from (4.1) and (4.2) it follows that

$$\begin{aligned} |(f(x) + g(x)) - (L+M)| &= |(f(x) - L) + (g(x) - M)| \\ &\le |f(x) - L| + |g(x) - M| \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \end{aligned}$$

(ii) By Proposition 4.3, f is bounded in a neighbourhood of a. Let $\varepsilon = 1$. Then there exists $\delta_1 > 0$ such that if $x \in X$ and $0 < |x-a| < \delta_1$ then $|f(x) - L| < 1$, which implies that

$$0 < |x-a| < \delta_1 \Rightarrow |f(x)| < |L| + 1 := C. \tag{4.3}$$

Let $\varepsilon > 0$ be given arbitrarily. Then there exist $\delta_2 > 0$ and $\delta_3 > 0$ such that if $x \in X$ and

$$0 < |x-a| < \delta_2 \Rightarrow |f(x) - L| < \frac{\varepsilon}{2(|M|+1)}, \tag{4.4}$$
$$0 < |x-a| < \delta_3 \Rightarrow |g(x) - M| < \frac{\varepsilon}{2C}. \tag{4.5}$$

Thus, taking $\delta = \min\{\delta_1, \delta_2, \delta_3\}$, from (4.3), (4.4) and (4.5) we obtain

$$\begin{aligned} |f(x)g(x) - LM| &= |f(x)g(x) - f(x)M + f(x)M - LM| \\ &= |f(x)(g(x) - M) + (f(x) - L)M| \\ &\leq |f(x)|\,|g(x) - M| + |M|\,|f(x) - L| \\ &\leq C \cdot \frac{\varepsilon}{2C} + |M|\frac{\varepsilon}{2(|M|+1)} < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \end{aligned}$$

(Note: Corrected $g(x) - L$ to $g(x) - M$ and $f(x) - L$ to $(f(x) - L)M$ in the second line. Corrected ... + $(|M| + 1)$... to ... + $|M|$... in the last line, following the logic).

(iii) Exercise ■

4.2 Lateral Limits

Definition 4.9 *Let $X \subset \mathbb{R}$. A real number $a \in \mathbb{R}$ is said to be an accumulation point from the right of X if every open interval $(a, a + \varepsilon)$ contains some point of X.*

Notation: X'_+= set of accumulation points from the right of X. Examples: 1)

$X = [a, b]$. $x = a \in X'_+$.

2) $X = \{-1/n : n \in \mathbb{N}^*\}$. In this case $X'_+ = \emptyset$.

In other words, we say that $a \in X'_+$ if and only if given $\varepsilon > 0$, $X \cap (a, a+\varepsilon) \neq \emptyset$, or equivalently, for all $\varepsilon > 0$ there exists $x \in X$ such that $0 < x - a < \varepsilon \Leftrightarrow x \in (a, a+\varepsilon)$.

Definition 4.10 *Let $X \subset \mathbb{R}$. A real number $a \in \mathbb{R}$ is said to be an accumulation point from the left of X if every open interval $(a - \varepsilon, a)$ contains some point of X.*

Notation: X'_- = set of all accumulation points from the left of X.

In other words: $a \in X'_-$ if and only if for all $\varepsilon > 0$, $X \cap (a - \varepsilon, a) \neq \emptyset$, or rather, given $\varepsilon > 0$ there exists $x \in X$ such that $0 < a - x < \varepsilon$ ($\Leftrightarrow x \in (a - \varepsilon, a)$).

Example: Let $X = [a, b]$. Then $b \in X'_-$.

Definition 4.11 *Let $f : X \subset \mathbb{R} \to \mathbb{R}$ be a map and $a \in X'_+$. We say that L is the limit from the right of $f(x)$ as x approaches a, if for all $\varepsilon > 0$ it is possible to find $\delta > 0$ such that $x \in X$ and $a < x < a + \delta \Rightarrow |f(x) - L| < \varepsilon$.*

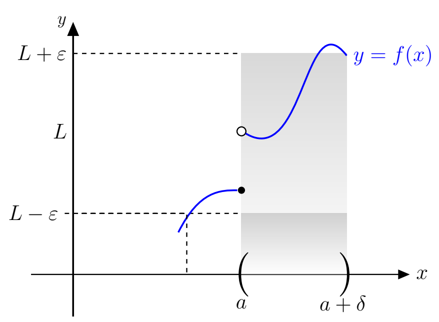


Figure 4.2: $\lim_{x \to a^+} f(x) = L$

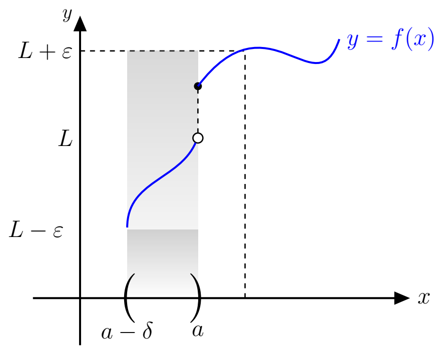


Figure 4.3: $\lim_{x \to a^-} f(x) = L$

Definition 4.12 *Let $f : X \subset \mathbb{R} \to \mathbb{R}$ be a map and $a \in X'_-$. We say that L is the* <u>*limit from the left*</u> *of $f(x)$ as x approaches a, if for all $\varepsilon > 0$ it is possible to find $\delta > 0$ such that $x \in X$ and $a - \delta < x < a \Rightarrow |f(x) - L| < \varepsilon$.*

Proposition 4.13 *a) Let $f : X \subset \mathbb{R} \to \mathbb{R}$ and $a \in X'_+$. Let $Y = X \cap (a, +\infty)$ and $g = f|_Y$. Then $\lim_{x\to a_+} f(x) = L$ if and only if $\lim_{x\to a} g(x) = L$.*

b) Let $f : X \subset \mathbb{R} \to \mathbb{R}$ and $a \in X'_-$. Let $Y = X \cap (-\infty, a)$ and $g = f|_Y$. Then $\lim_{x\to a_-} f(x) = L$ if and only if $\lim_{x\to a} g(x) = L$.

Proof: a) '$\Rightarrow$' Let $\varepsilon > 0$ be given. We must show that there exists $\delta > 0$ such that if $x \in D(g) = X \cap (a, +\infty)$ and

$$0 < |x - a| < \delta \Rightarrow |g(x) - L| < \varepsilon. \tag{4.6}$$

Indeed, for the given $\varepsilon > 0$ and from the fact that $\lim_{x\to a_+} f(x) = L$, there exists $\delta > 0$ such that if $x \in X$ and

$$a < x < a + \delta \Rightarrow |f(x) - L| < \varepsilon \tag{4.7}$$

However, since $x \in D(g) = X \cap (a, +\infty) = \{x \in X : x > a\}$, the condition $x \in D(g)$ and $0 < |x - a| < \delta$ implies that $x \in (a, a + \delta)$, i.e., $a < x < a + \delta$. It follows from this and the fact that $f|_Y = g$ that condition (4.7) implies (4.6) for the same $\delta > 0$.

'$\Leftarrow$' Conversely, given $\varepsilon > 0$, we must show that there exists $\delta > 0$ such that if $x \in X$ and

$$a < x < a + \delta \Rightarrow |f(x) - L| < \varepsilon. \tag{4.8}$$

In fact, for the given $\varepsilon > 0$, and from the fact that $\lim_{x\to a} g(x) = L$, there exists $\delta > 0$ such that if $x \in D(g) = \{x \in X : x > a\}$ and

$$0 < |x - a| < \delta \Rightarrow |g(x) - L| < \varepsilon. \tag{4.9}$$

However, the condition $x \in X$ and $a < x < a + \delta$ is equivalent to $x \in D(g)$ and $0 < |x - a| < \delta$. Since $f|_Y = g$, condition (4.9) implies (4.8), which proves the desired result. ■

Theorem 4.14 *Let $f : X \subset \mathbb{R} \to \mathbb{R}$ and $a \in X'_+ \cap X'_-$. Then:*

$$\lim_{x\to a} f(x) = L \Leftrightarrow \lim_{x\to a_-} f(x) = \lim_{x\to a_+} f(x) = L.$$

Proof: ($\Rightarrow$) Let $\varepsilon > 0$ be given. Then there exists $\delta > 0$ such that if $x \in X$ and

$$0 < |x - a| < \delta \Rightarrow |f(x) - L| < \varepsilon. \tag{4.10}$$

Since $0 < |x - a| < \delta \Leftrightarrow a - \delta < x < a + \delta$ (and $x \neq a$), (4.10) implies that

$$a < x < a + \delta, x \in X \Rightarrow |f(x) - L| < \varepsilon$$

$$a - \delta < x < a, x \in X \Rightarrow |f(x) - L| < \varepsilon$$

which, by definition, means $\lim_{x\to a_-} f(x) = \lim_{x\to a_+} f(x) = L$.

($\Leftarrow$) Conversely, since $\lim_{x\to a_-} f(x) = \lim_{x\to a_+} f(x) = L$, given $\varepsilon > 0$, there exist $\delta_1 > 0$ and $\delta_2 > 0$ such that if $x \in X$ and

$$a < x < a + \delta_1 \Rightarrow |f(x) - L| < \varepsilon,$$
$$a - \delta_2 < x < a \Rightarrow |f(x) - L| < \varepsilon.$$

Taking $\delta = \min\{\delta_1, \delta_2\}$, then if $x \in X$ and

$$0 < |x - a| < \delta \Rightarrow (a < x < a + \delta \text{ or } a - \delta < x < a) \Rightarrow |f(x) - L| < \varepsilon,$$

which concludes the proof. ■

Examples:

1) Let $f : \mathbb{R}\backslash\{0\} \to \mathbb{R}$ be defined by $f(x) = x + \frac{x}{|x|}$. Consider $Y = \mathbb{R}\backslash\{0\} \cap (0, +\infty) = (0, +\infty)$ and $g = f|_Y$. Thus, we have:

$$g(x) = x + 1 \text{ and } \lim_{x\to 0} g(x) = 1.$$

Using Proposition 4.13, we have

$$\lim_{x\to 0_+} f(x) = 1.$$

Analogously, let $Y = \mathbb{R}\backslash\{0\} \cap (-\infty, 0) = (-\infty, 0)$ and $g = f|_Y$. Then:

$$g(x) = x - 1 \text{ and } \lim_{x\to 0} g(x) = -1.$$

By Proposition 4.13, we have

$$\lim_{x\to 0_-} f(x) = -1.$$

2) Let $f : \mathbb{R}\backslash\{0\} \to \mathbb{R}$ be given by $f(x) = \frac{1}{x}$. $\lim_{x\to 0_+} f(x)$ does not exist, nor does $\lim_{x\to 0_-} f(x)$. Thus, $\lim_{x\to 0} f(x)$ does not exist.

4.3 Limits at Infinity

Definition 4.15 *Let $X \subset \mathbb{R}$ be unbounded above. Given a map $f : X \to \mathbb{R}$, we write*

$$\lim_{x \to +\infty} f(x) = L,$$

*when the real number L satisfies the following condition: For all $\varepsilon > 0$, there exists $M > 0$ such that if $x \in X$ and $x > M$ then $|f(x) - L| < \varepsilon$. *(Note: δ changed to M for clarity, as δ usually implies smallness)**

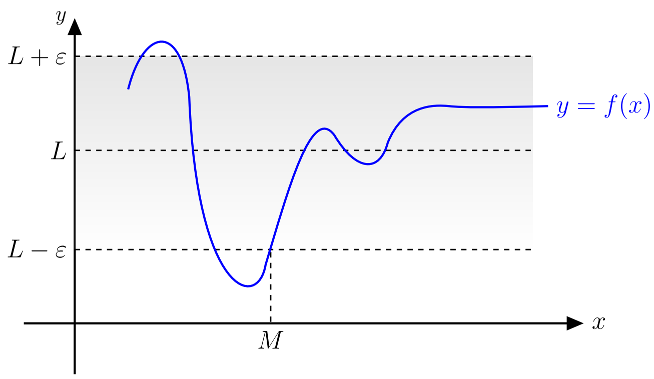


Figure 4.4: $\lim_{x \to +\infty} f(x) = L$

Definition 4.16 *Let $X \subset \mathbb{R}$ be unbounded below. Given a map $f : X \to \mathbb{R}$, we write*

$$\lim_{x \to -\infty} f(x) = L,$$

*when the real number L satisfies the following condition: For all $\varepsilon > 0$, there exists $M > 0$ such that if $x \in X$ and $x < -M$ then $|f(x) - L| < \varepsilon$. *(Note: δ changed to M)**

Examples:

(1) $\lim_{x \to +\infty} \frac{1}{x} = \lim_{x \to -\infty} \frac{1}{x} = 0.$

Indeed, note that $x > M \Rightarrow \frac{1}{x} < \frac{1}{M}$ (for $M > 0$). On the other hand:

$$\left|\frac{1}{x} - 0\right| = \left|\frac{1}{x}\right| < \varepsilon \Leftrightarrow -\varepsilon < \frac{1}{x} < \varepsilon.$$

Thus, given $\varepsilon > 0$, let us take $M = \frac{1}{\varepsilon}$. Hence, if $x > M > 0$ then $\frac{1}{x} < \frac{1}{M} \Rightarrow \frac{1}{x} < \varepsilon$. Since $x > 0$, $\frac{1}{x} > 0$, so $\left|\frac{1}{x}\right| < \varepsilon$, which implies that $\lim_{x \to +\infty} \frac{1}{x} = 0$.

Analogously, it is proven that $\lim_{x \to -\infty} \frac{1}{x} = 0$.

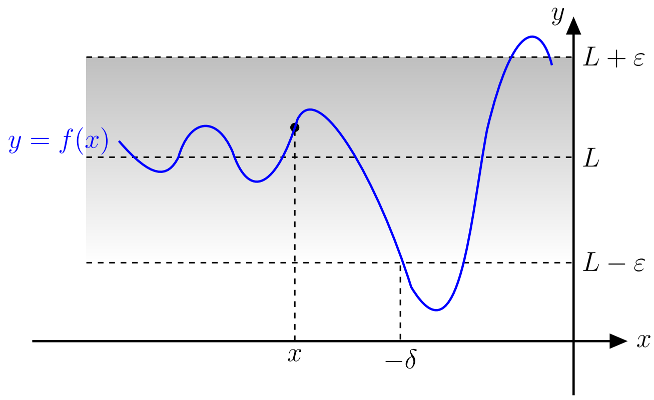


Figure 4.5: $\lim_{x\to-\infty} f(x) = L$

2) $\lim_{x\to+\infty}\sin(x)$ and $\lim_{x\to-\infty}\sin(x)$ do not exist.

Proof: Suppose, by contradiction, that $\lim_{x\to+\infty}\sin(x)$ exists and is equal to L. Thus, given $\varepsilon > 0$, there exists $A > 0$ such that if $x \in X$ and

$$x > A \Rightarrow |\sin x - L| < \varepsilon.$$

However, there exists $x > A$ such that $\sin x = 1$ (e.g., $x = 2n\pi + \pi/2$) and there exists $y > A$ such that $\sin y = -1$ (e.g., $y = 2n\pi + 3\pi/2$). Hence:

$$|1 - L| < \varepsilon \text{ and } |-1 - L| < \varepsilon.$$

If we take $\varepsilon = \frac{1}{2}$, we have on one hand

$$|L-1| < \frac{1}{2} \Leftrightarrow -\frac{1}{2} < L - 1 < \frac{1}{2} \Leftrightarrow \frac{1}{2} < L < \frac{3}{2}.$$

On the other hand:

$$|L+1| < \frac{1}{2} \Leftrightarrow -\frac{1}{2} < L + 1 < \frac{1}{2} \Leftrightarrow -\frac{3}{2} < L < -\frac{1}{2},$$

which is a contradiction. ■

4.4 Infinite Limits

Definition 4.17 *Let $f : X \subset \mathbb{R} \to \mathbb{R}$ be a map and $a \in X'$. We say that $\lim_{x\to a} f(x) = +\infty$ when for every $\varepsilon > 0$ (arbitrarily large), there exists $\delta > 0$ such that if $x \in X$ and $0 < |x - a| < \delta$ then $f(x) > \varepsilon$.*

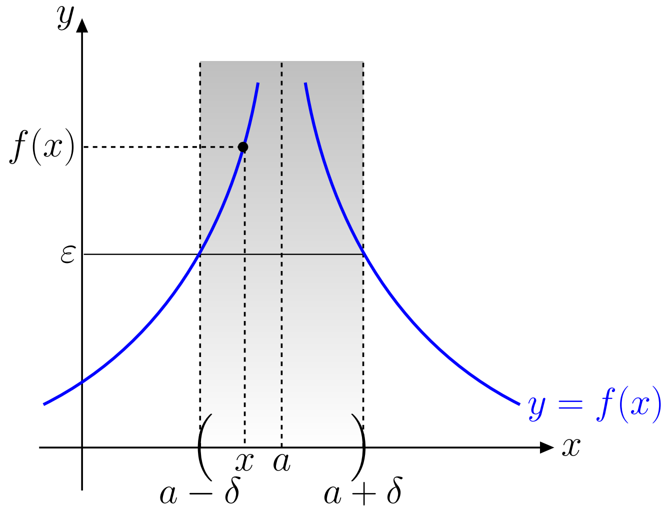


Figure 4.6: $\lim_{x\to a} f(x) = +\infty$

Example: $\lim_{x\to a} \frac{1}{(x-a)^2} = +\infty$.

Indeed, note that

$$\frac{1}{(x-a)^2} > \varepsilon \Leftrightarrow (x-a)^2 < \frac{1}{\varepsilon} \Leftrightarrow |x-a| < \frac{1}{\sqrt{\varepsilon}}.$$

Thus, given $\varepsilon > 0$, let us take $\delta = \frac{1}{\sqrt{\varepsilon}}$. Then, if $x \in \mathbb{R}\backslash\{a\}$ and

$$0 < |x-a| < \delta \Rightarrow (x-a)^2 < \frac{1}{\varepsilon} \Rightarrow \frac{1}{(x-a)^2} > \varepsilon.$$

Definition 4.18 *Let $f : X \subset \mathbb{R} \to \mathbb{R}$ be a map and $a \in X'$. We say that $\lim_{x\to a} f(x) = -\infty$ when for every $\varepsilon > 0$, there exists $\delta > 0$ such that if $x \in X$ and $0 < |x-a| < \delta$ then $f(x) < -\varepsilon$.*

Example:

$$\lim_{x\to a} \frac{-1}{(x-a)^2} = -\infty. \text{ (exercise)}.$$

Definition 4.19 *Let $f : X \subset \mathbb{R} \to \mathbb{R}$ be a map with X unbounded above. We say that $\lim_{x\to+\infty} f(x) = +\infty$ when for every $\varepsilon > 0$ (arbitrarily large), there exists $\delta > 0$ (arbitrarily large) such that if $x \in X$ and $x > \delta$ then $f(x) > \varepsilon$.*

Example: $\lim_{x\to+\infty} x^2 = +\infty$.

Let $\varepsilon > 0$. Note that:

$$x^2 > \varepsilon \Leftrightarrow x^2 - \varepsilon > 0 \Leftrightarrow (x-\sqrt{\varepsilon})(x+\sqrt{\varepsilon}) > 0 \Leftrightarrow x < -\sqrt{\varepsilon} \text{ or } x > \sqrt{\varepsilon}.$$

Thus, given $\varepsilon > 0$, $\delta = \sqrt{\varepsilon}$ exists such that if $x > \delta = \sqrt{\varepsilon}$ then $x^2 > \varepsilon$.

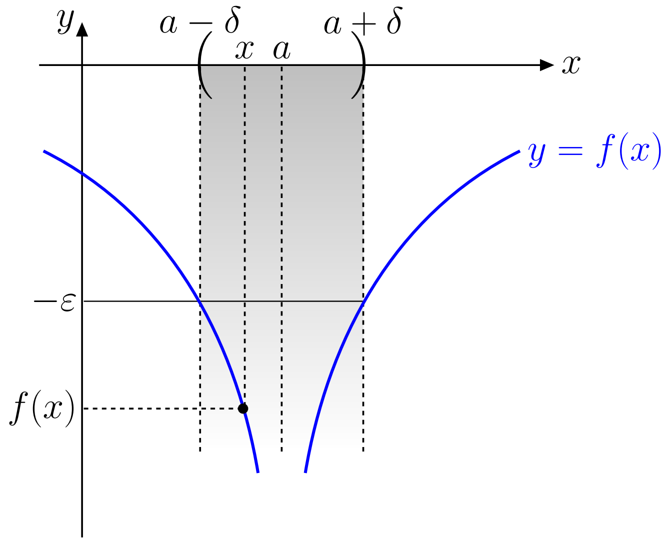


Figure 4.7: $\lim_{x \to a} f(x) = -\infty$

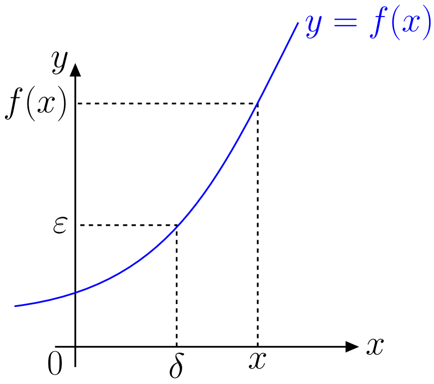


Figure 4.8: $\lim_{x \to +\infty} f(x) = +\infty$

Definition 4.20 *Let $f : X \subset \mathbb{R} \to \mathbb{R}$ be a map with X unbounded above. We say that $\lim_{x\to+\infty} f(x) = -\infty$ when for every $\varepsilon > 0$, there exists $\delta > 0$ such that if $x \in X$ and $x > \delta$ then $f(x) < -\varepsilon$.*

Example: $\lim_{x\to+\infty}(-x^3) = -\infty$.

Indeed, given $\varepsilon > 0$, note that:

$$-x^3 < -\varepsilon \Leftrightarrow x^3 > \varepsilon \Leftrightarrow x > \sqrt[3]{\varepsilon}.$$

Hence, given $\varepsilon > 0$, take $\delta = \sqrt[3]{\varepsilon}$, because if $x \in \mathbb{R}$ and $x > \sqrt[3]{\varepsilon}$ then $-x^3 < -\varepsilon$.

Remark: $\frac{0}{0}$; $\infty - \infty$; $0 \cdot \infty$; 0^0; 1^∞; ∞^0; $\frac{\infty}{\infty}$ are indeterminate forms. Let's see some examples.

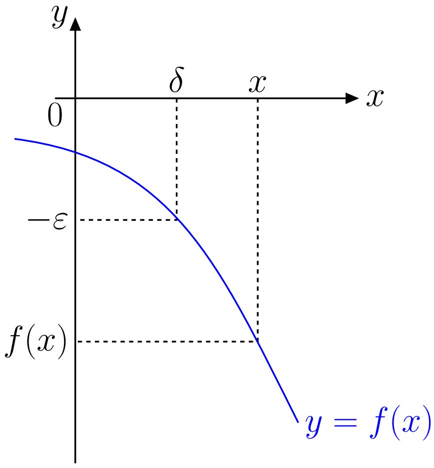


Figure 4.9: $\lim_{x\to+\infty} f(x) = -\infty$

(a) Let $f(x) = 2x$ and $g(x) = x$.

$$\lim_{x\to 0} f(x) = \lim_{x\to 0} g(x) = 0 \text{ and } \lim_{x\to 0} \frac{f(x)}{g(x)} = 2.$$

(b) Let $f(x) = x \sin \frac{1}{x}$ and $g(x) = x$.

$$\lim_{x\to 0} f(x) = \lim_{x\to 0} g(x) = 0 \text{ but } \lim_{x\to 0} \frac{f(x)}{g(x)} \text{ does not exist.}$$

(c) Let's see a $0 \cdot \infty$ example. Let $f(x) = \frac{1}{x}$ and $g(x) = x$.

$$\lim_{x\to 0_+} f(x) = +\infty, \lim_{x\to 0_+} g(x) = 0 \text{ and } \lim_{x\to 0_+} f(x)\,g(x) = 1.$$

Chapter 5

Continuous Functions

Definition 5.1 *We say that a function $f : X \subset \mathbb{R} \to \mathbb{R}$ is* <u>*continuous*</u> *at the point $a \in X$ when, for every $\varepsilon > 0$ given arbitrarily, we can find $\delta > 0$ such that if $x \in X$ and $|x - a| < \delta$ then $|f(x) - f(a)| < \varepsilon$.*

In other words: given any interval J containing $f(a)$, there exists an open interval I containing a such that

$$f(I \cap X) \subset J.$$

Whenever desired, we can take

$$J = (f(a) - \varepsilon, f(a) + \varepsilon)$$

with $\varepsilon > 0$ and $I = (a - \delta, a + \delta)$; $\delta > 0$.

Definition 5.2 *We say that $f : X \subset \mathbb{R} \to \mathbb{R}$ is* <u>*continuous*</u> *when f is continuous at all points of X.*

<u>Remarks:</u>

(1a) Unlike the definition of a limit, it only makes sense to inquire if f is continuous at the point a when $a \in X$.

(2a) If a is an isolated point of X, then every function $f : X \to \mathbb{R}$ is continuous at the point a. Indeed, given $\varepsilon > 0$, it suffices to take $\delta > 0$ such that

$$(a - \delta, a + \delta) \cap X = \{a\}.$$

Then, if $x \in X$ with $|x - a| < \delta$, it implies that $x = a$, and therefore $|f(x) - f(a)| = 0 < \varepsilon$. In particular, if all points of X are isolated, then any function $f : X \to \mathbb{R}$ is continuous.

(3a) Now, let $a \in X$ be an accumulation point of X, i.e., $a \in X \cap X'$. Then f is continuous at a if and only if $\lim_{x\to a} f(x) = f(a)$. **Proof**: Indeed, suppose f is continuous at a. Thus, given $\varepsilon > 0$, there exists $\delta > 0$ such that if $x \in X$ and $a-\delta < x < a+\delta$ then $|f(x)-f(a)| < \varepsilon$. However, if $x \in X$ and $a-\delta < x < a+\delta$, and furthermore $x \neq a$, we still have $|f(x)-f(a)| < \varepsilon$. (This shows $\lim_{x\to a} f(x) = f(a)$).

Conversely, suppose that given $\varepsilon > 0$, there exists $\delta_1 > 0$ such that if $x \in X$ and $0 < |x-a| < \delta_1$ then $|f(x) - f(a)| < \varepsilon$. Now, if $x = a$, this case is not covered. We need to check $x = a$ separately. *(Translator's Note: The original proof's logic here is slightly complex, mixing limit and continuity. A standard proof is as follows)* Conversely, suppose $\lim_{x\to a} f(x) = f(a)$. Then for any $\varepsilon > 0$, there exists $\delta > 0$ such that if $x \in X$ and $0 < |x-a| < \delta$, then $|f(x) - f(a)| < \varepsilon$. This definition of limit handles $x \neq a$. The case $x = a$ is trivial, since $|f(a) - f(a)| = 0 < \varepsilon$. Therefore, for all $x \in X$ with $|x - a| < \delta$, we have $|f(x) - f(a)| < \varepsilon$. This is the definition of continuity at a. ■

5.1 Properties of Continuous Functions

The propositions below follow immediately from the propositions and corollaries demonstrated previously (for limits).

Theorem 5.3 *Every restriction of a continuous function is continuous. More precisely: let $f : X \to \mathbb{R}$ be continuous at the point $a \in X$. If $a \in Y \subset X$ and $g = f|_Y$, then $g : Y \to \mathbb{R}$ is continuous at the point a.*

Theorem 5.4 *If $f : X \to \mathbb{R}$ is continuous at a point $a \in X$, then f is bounded in a neighbourhood of a.*

Theorem 5.5 *If $f, g : X \to \mathbb{R}$ are continuous at the point $a \in X$ and $f(a) < g(a)$, then there exists $\delta > 0$ such that $f(x) < g(x)$ for all $x \in X$ with $|x - a| < \delta$.*

Corollary 5.6 *Let $f : X \to \mathbb{R}$ be continuous at the point $a \in X$ and k be a constant. If $f(a) < k$, then there exists $\delta > 0$ such that $f(x) < k$ for all $x \in X$ with $|x-a| < \delta$.*

Theorem 5.7 *If $f, g : X \to \mathbb{R}$ are continuous at the point $a \in X$, then $f+g$, $f-g$, fg are continuous at that same point. If $g(a) \neq 0$, then f/g is also continuous at the point a.*

Theorem 5.8 *The composition of two continuous functions is continuous. That is, if $f : X \subset \mathbb{R} \to \mathbb{R}$ and $g : Y \subset \mathbb{R} \to \mathbb{R}$ are continuous at the points $a \in X$ and $b = f(a) \in Y$, respectively, and furthermore $f(X) \subset Y$, then $g \circ f : X \to \mathbb{R}$ is continuous at the point a.*

Proof: Let $\varepsilon > 0$ be given. We must exhibit $\delta > 0$ such that if $x \in X$ and $|x - a| < \delta$ then $|g(f(x)) - g(f(a))| < \varepsilon$.

Indeed, since g is continuous at $b = f(a)$, for the given $\varepsilon > 0$, there exists $\delta_1 > 0$ such that if $y \in Y$ (and $f(X) \subset Y$) and

$$|y - b| < \delta_1 \Rightarrow |g(y) - g(b)| < \varepsilon. \tag{5.1}$$

(Note: original text used $y \in f(X)$ which is correct, $y \in Y$ is more standard)

Since f is continuous at a, for this $\delta_1 > 0$ there exists $\delta > 0$ such that if $x \in X$ and

$$|x - a| < \delta \Rightarrow |f(x) - f(a)| < \delta_1. \tag{5.2}$$

Combining (5.1) and (5.2) and keeping in mind that $y = f(x)$ and $b = f(a)$, it follows that if $x \in X$ and $|x - a| < \delta$, then $|f(x) - f(a)| < \delta_1$, which in turn implies (by 5.1) that $|g(f(x)) - g(f(a))| < \varepsilon$, which concludes the proof. ■

Example: Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = x$. f is continuous at $x_0 \in \mathbb{R}$. Indeed:

$$|f(x) - f(x_0)| = |x - x_0|.$$

Thus, given $\varepsilon > 0$ there exists $\delta = \varepsilon > 0$ such that if $x \in \mathbb{R}$ and $|x - x_0| < \delta$ then $|f(x) - f(x_0)| = |x - x_0| < \varepsilon$. Since x_0 was arbitrary, it follows that f is continuous on $\mathbb{R}$. It follows from Theorem 5.7 and the previous example that the function $f : \mathbb{R} \to \mathbb{R}$, $x \mapsto x^n$ is continuous. In truth, every polynomial $p : \mathbb{R} \to \mathbb{R}$; $p(x) = a_n x^n + \cdots + a_1 x + a_0$, $a_i \in \mathbb{R}\ i = 1, \cdots, n$ is a continuous map. Furthermore, it follows from Theorems 5.7 and 5.8, the simple example given above, and the continuity of trigonometric, exponential, and logarithmic functions that we can create an infinity of examples of continuous functions from elementary functions, taking care that the compositions and quotients are well-defined.

Proposition 5.9 *Let $f : X \subset \mathbb{R} \to \mathbb{R}$ be a continuous function. Then, if $\overline{A} \subset X$ (closure relative to $\mathbb{R}$), we have $f(\overline{A}) \subset \overline{f(A)}$.*

Proof: Take $y \in f(\overline{A})$. We must prove that $y \in \overline{f(A)}$. Indeed, let $\varepsilon > 0$. We must show that

$$B_\varepsilon(y) \cap f(A) \neq \emptyset. \tag{5.3}$$

Since $y \in f(\overline{A})$, then $y = f(x_0)$ for some $x_0 \in \overline{A}$. By the continuity of f (at $x_0 \in X$), there exists $\delta > 0$ such that:

$$f(B_\delta(x_0) \cap X) \subset B_\varepsilon(y)$$

(Note: f is only defined on X, so we must intersect with X).

Furthermore, since $x_0 \in \overline{A}$, $B_\delta(x_0) \cap A \neq \emptyset$. Let $\overline{x_0} \in B_\delta(x_0) \cap A$. Since $A \subset X$, $\overline{x_0} \in X$, so $\overline{x_0} \in B_\delta(x_0) \cap X$. Consequently

$$f(\overline{x_0}) \in f(B_\delta(x_0) \cap X) \subset B_\varepsilon(y)$$
$$\text{and } f(\overline{x_0}) \in f(A),$$

which proves $f(\overline{x_0}) \in B_\varepsilon(y) \cap f(A)$, as desired. ■

Proposition 5.10 *Let $f : X \subset \mathbb{R} \to \mathbb{R}$ be a function. f is continuous on X if and only if for every open set $V \subset \mathbb{R}$, the set $f^{-1}(V)$ is open in X. *(Note: The original text had "open in $\mathbb{R}$", which is only true if $X = \mathbb{R}$)**

Proof: Suppose first that f is continuous on X and let $V \subset \mathbb{R}$ be open. We must prove that $f^{-1}(V) := \{x \in X : f(x) \in V\}$ is an open set in X. In fact, take $x \in f^{-1}(V)$. Thus, $f(x) \in V$. Since V is open, there exists $\varepsilon_0 > 0$ such that

$$B_{\varepsilon_0}(f(x)) \subset V.$$

By the continuity of f at x, there exists $\delta > 0$ such that for $z \in X$:

$$|z - x| < \delta \implies f(z) \in B_{\varepsilon_0}(f(x)) \subset V$$

This means $B_\delta(x) \cap X \subset f^{-1}(V)$. By definition, $f^{-1}(V)$ is open in X.

Conversely, suppose that $f^{-1}(V)$ is open in X for every open $V \subset \mathbb{R}$. We must prove that f is continuous on X. Indeed, let $\varepsilon > 0$ be given and consider $x \in X$ arbitrary. The ball $V = B_\varepsilon(f(x))$ is an open set. By hypothesis, $f^{-1}(V)$ is an open set in X containing x. Therefore, there exists $\delta > 0$ such that

$$B_\delta(x) \cap X \subset f^{-1}(B_\varepsilon(f(x))),$$

which means

$$f(B_\delta(x) \cap X) \subset B_\varepsilon(f(x)),$$

This is the definition of f being continuous at x. ■

Proposition 5.11 *Let $f : \mathbb{R} \to \mathbb{R}$. The following are equivalent:*

(i) f is continuous.

(ii) $A \subset \mathbb{R} \Rightarrow f(\overline{A}) \subset \overline{f(A)}$.

(iii) A is closed $\Rightarrow f^{-1}(A)$ is closed.

(iv) A is open $\Rightarrow f^{-1}(A)$ is open.

Proof: (i) $\Rightarrow$ (ii) This was done in Prop. 5.9 (with $X = \mathbb{R}$).

(ii) $\Rightarrow$ (iii) Let A be closed. We must prove that $f^{-1}(A)$ is closed. It suffices to prove that

$$\overline{f^{-1}(A)} \subset f^{-1}(A). \tag{5.4}$$

Indeed, by hypothesis (ii), since $f^{-1}(A) \subset \mathbb{R}$, it follows that

$$f(\overline{f^{-1}(A)}) \subset \overline{f(f^{-1}(A))} \tag{5.5}$$

On the other hand, $f(f^{-1}(A)) \subset A$, and consequently

$$\overline{f(f^{-1}(A))} \subset \overline{A}. \tag{5.6}$$

Since A is closed, $A = \overline{A}$, and therefore from (5.5) and (5.6) we have

$$f(\overline{f^{-1}(A)}) \subset A,$$

which implies (by applying f^{-1} to both sides)

$$\overline{f^{-1}(A)} \subset f^{-1}(f(\overline{f^{-1}(A)})) \subset f^{-1}(A), \tag{5.7}$$

which proves (5.4).

(iii) $\Rightarrow$ (iv) Let A be open. We must prove that $f^{-1}(A)$ is open. It suffices to prove that its complement $\mathbb{R}\backslash f^{-1}(A)$ is closed. In fact, we have

$$\mathbb{R}\backslash f^{-1}(A) = f^{-1}(\mathbb{R}\backslash A), \tag{5.8}$$

and since A is open, $\mathbb{R}\backslash A$ is closed. By item (iii), $f^{-1}(\mathbb{R}\backslash A)$ is closed, which implies, by virtue of (5.8), that $\mathbb{R}\backslash f^{-1}(A)$ is closed, i.e., $f^{-1}(A)$ is open.

(iv) $\Rightarrow$ (i) This was done in Prop. 5.10 (with $X = \mathbb{R}$). ■

5.2 Continuous Functions on Compact Sets

Theorem 5.12 *Let $K \subset \mathbb{R}$ be a compact set and $f : K \to \mathbb{R}$ a continuous function. Then $f(K)$ is compact.*

Proof: Let $\{G_\alpha\}_{\alpha \in A}$ be an open cover for $f(K)$. We must exhibit a finite subcover. Indeed, since $f(K) \subset \bigcup_{\alpha \in A} G_\alpha$, for each $y \in f(K)$, $y \in G_\alpha$ for some $\alpha \in A$. Since $y = f(x)$ for some $x \in K$, let's call this index $\alpha(x)$. Since $G_{\alpha(x)}$ is open, for each $x \in K$ there exists $\varepsilon_x > 0$ such that

$$B_{\varepsilon_x}(f(x)) \subset G_{\alpha(x)}.$$

On the other hand, by the continuity of f at x, for this $\varepsilon_x > 0$ there exists $\delta_x > 0$ such that

$$f(B_{\delta_x}(x) \cap K) \subset B_{\varepsilon_x}(f(x)).$$

Note that the collection $\{B_{\delta_x}(x)\}_{x \in K}$ forms an open cover for K. Since K is compact, there exist $x_1, \cdots, x_n \in K$ and $\delta_1, \cdots, \delta_n > 0$ such that

$$K \subset \bigcup_{i=1}^{n} B_{\delta_i}(x_i).$$

Consequently,

$$f(K) = f\left(K \cap \left(\bigcup_{i=1}^{n} B_{\delta_i}(x_i)\right)\right) = \bigcup_{i=1}^{n} f(K \cap B_{\delta_i}(x_i))$$

i.e.,

$$f(K) \subset \bigcup_{i=1}^{n} f(B_{\delta_i}(x_i) \cap K) \subset \bigcup_{i=1}^{n} B_{\varepsilon_i}(f(x_i)) \subset \bigcup_{i=1}^{n} G_{\alpha(x_i)},$$

which proves $f(K)$ is covered by the finite subcover $\{G_{\alpha(x_1)}, \ldots, G_{\alpha(x_n)}\}$. ■

Theorem 5.13 *(Extreme Value Theorem) Let K be a compact set and $f : K \to \mathbb{R}$ a continuous function. Then f attains its absolute maximum and minimum values.*

Proof: Since K is compact and f is continuous on K, then according to Theorem 5.12, $f(K)$ is a compact set in $\mathbb{R}$. Therefore, $f(K)$ is closed and bounded by the Heine-Borel Theorem (Theorem 3.25). Being non-empty and bounded, by the Completeness Axiom,

$$M := \sup_{x \in K} f(x) \quad (\text{i.e., } \sup f(K))$$

and

$$m := \inf_{x\in K} f(x) \quad (\text{i.e., } \inf f(K))$$

both exist. However, since $f(K)$ is closed, it contains all its adherent points. In particular, it contains its supremum M and its infimum m. Thus, $M, m \in f(K)$. It follows that there exist $x_1, x_2 \in K$ such that $f(x_1) = m$ and $f(x_2) = M$, which concludes the proof. ■

Remark: What happens if we remove the hypothesis of compactness?

Counter-examples:

(i) $f : (0,1) \to \mathbb{R}$, $x \mapsto \frac{1}{x}$. $K = (0,1)$ is bounded but not closed. Note that f is continuous but does not attain an absolute maximum or minimum in $(0,1)$.

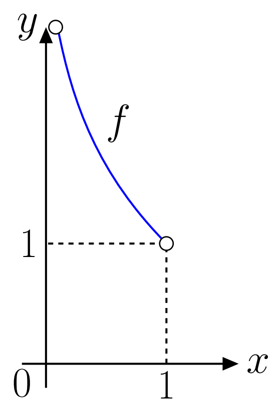


Figure 5.1:

(ii) $g : [0,+\infty) \to \mathbb{R}$, $x \mapsto x$ $K = [0,\infty)$ is closed but not bounded. Note that g is continuous but does not attain an absolute maximum on $[0,+\infty)$, although it does attain an absolute minimum at $x = 0$.

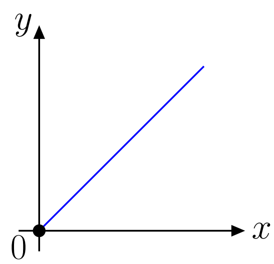


Figure 5.2:

Let $f : E \subset \mathbb{R} \to \mathbb{R}$ be a continuous function. Then, given $\varepsilon > 0$, we can, for each $x_0 \in E$, find $\delta > 0$ (which depends on x_0) such that if $x \in E$ and $|x - x_0| < \delta$

then $|f(x) - f(x_0)| < \varepsilon$. In general, it is not possible to find, from a given $\varepsilon > 0$, a single $\delta > 0$ that works for all points $x_0 \in E$. Let's see an example: Let

$$f : (0, +\infty) \to \mathbb{R},\ x \mapsto f(x) = \frac{1}{x}.$$

Given $\varepsilon > 0$, we will show that one cannot choose a single $\delta > 0$ such that

$$|x - a| < \delta \Rightarrow |f(x) - f(a)| < \varepsilon,$$

for any $a \in (0, +\infty)$. Indeed, given $\varepsilon > 0$, suppose a single $\delta > 0$ is chosen. Let us take a positive number a such that $0 < a < \delta$ and $0 < a < \frac{1}{3\varepsilon}$ (e.g. $a = \min(\delta/2, 1/(6\varepsilon))$). Then, for $x = a + \frac{\delta}{2}$, we have

$$|x - a| = \left|a + \frac{\delta}{2} - a\right| = \frac{\delta}{2} < \delta.$$

However:

$$|f(x) - f(a)| = \left|\frac{1}{a + \delta/2} - \frac{1}{a}\right| = \left|\frac{a - (a + \delta/2)}{a(a + \delta/2)}\right| = \frac{\delta/2}{a(a + \delta/2)} = \frac{\delta}{a(2a + \delta)}$$

Since $a < \delta$, we have $2a < 2\delta$, and $2a + \delta < 3\delta$. Thus:

$$|f(x) - f(a)| = \frac{\delta}{a(2a + \delta)} > \frac{\delta}{a(3\delta)} = \frac{1}{3a} > \varepsilon$$

(by our choice of a). This proves the desired claim.

Consider, now,

$$f : \mathbb{R} \to \mathbb{R},\ f(x) = ax + b,\ a \neq 0.$$

Given $\varepsilon > 0$, let us choose $\delta = \varepsilon/|a|$. Then, for any $x_0 \in \mathbb{R}$, if $|x - x_0| < \delta$, we have:

$$\begin{aligned}
|f(x) - f(x_0)| &= |(ax + b) - (ax_0 + b)| \\
&= |ax - ax_0| = |a|\,|x - x_0| \\
&< |a|\,\delta = |a|\frac{\varepsilon}{|a|} = \varepsilon.
\end{aligned}$$

In this case, it was possible, from a given $\varepsilon > 0$, to choose a $\delta > 0$ that worked for all points in the domain. This motivates the next section.

5.3 Uniformly Continuous Functions

Definition 5.14 *A function $f : E \subset \mathbb{R} \to \mathbb{R}$ is said to be uniformly continuous when, for each $\varepsilon > 0$, there exists $\delta > 0$ such that if $x, y \in E$ and $|x - y| < \delta$ then $|f(x) - f(y)| < \varepsilon$. In this case, δ depends only on $\varepsilon > 0$.*

Theorem 5.15 *(Heine-Cantor Theorem) If $K \subset \mathbb{R}$ is compact and $f : K \to \mathbb{R}$ is a continuous function, then f is uniformly continuous.*

Proof: Let $\varepsilon > 0$ be given. Since f is continuous, for this $\varepsilon > 0$, and for each $x \in K$, there exists $\delta_x > 0$ such that if $y \in K$ and

$$|y - x| < \delta_x \Rightarrow |f(x) - f(y)| < \frac{\varepsilon}{2}. \tag{5.9}$$

Note that the collection $\{B_{\delta_x/2}(x)\}_{x \in K}$ forms an open cover for K. Since K is compact, there exist $x_1, \cdots, x_n \in K$ and $\delta_1, \cdots, \delta_n > 0$ such that

$$K \subset \bigcup_{i=1}^{n} B_{\delta_i/2}(x_i). \tag{5.10}$$

Let us take

$$\delta := \min\{\delta_1/2, \cdots, \delta_n/2\}.$$

Let $x, y \in K$ be arbitrary, with $|x - y| < \delta$. We must prove that $|f(x) - f(y)| < \varepsilon$. Indeed, since $x \in K$, then by virtue of (5.10), $x \in B_{\delta_{i_0}/2}(x_{i_0})$ for some $i_0 \in \{1, \cdots, n\}$. Hence:

$$|f(x) - f(y)| \leq |f(x) - f(x_{i_0})| + |f(x_{i_0}) - f(y)|. \tag{5.11}$$

We will use the continuity of f at x_{i_0} (Eq. 5.9). To do this, we must prove that $x, y \in B_{\delta_{i_0}}(x_{i_0})$. In fact:

(i) $x \in B_{\delta_{i_0}/2}(x_{i_0}) \Rightarrow |x - x_{i_0}| < \delta_{i_0}/2 < \delta_{i_0} \Rightarrow x \in B_{\delta_{i_0}}(x_{i_0})$.

(ii) $|y - x_{i_0}| \leq |y - x| + |x - x_{i_0}| < \delta + \delta_{i_0}/2$. Since $\delta \leq \delta_{i_0}/2$, we have $|y - x_{i_0}| < \delta_{i_0}/2 + \delta_{i_0}/2 = \delta_{i_0} \Rightarrow y \in B_{\delta_{i_0}}(x_{i_0})$.

Thus $x, y \in B_{\delta_{i_0}}(x_{i_0})$, and from (5.9) (applied at x_{i_0}) and (5.11) we obtain

$$\begin{aligned} |f(x) - f(y)| &\leq |f(x) - f(x_{i_0})| + |f(x_{i_0}) - f(y)| \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \end{aligned}$$

Consequently, $|f(x) - f(y)| < \varepsilon$ provided that $|x - y| < \delta$, which concludes the proof. ■

5.4 Continuous Functions on Connected Sets

Lemma 5.16 *Let $f : X \to \mathbb{R}$ be continuous. Then, for every sequence $\{x_n\} \subset X$ such that $x_n \to a \in X$, we have $f(x_n) \to f(a)$. (Sequential Continuity)*

Proof: Let $\{x_n\} \subset X$ be such that $x_n \to a$, with $a \in X$, and consider $\varepsilon > 0$. We must exhibit $n_0 \in \mathbb{N}$ such that

$$\text{if } n > n_0 \Rightarrow |f(x_n) - f(a)| < \varepsilon. \tag{5.12}$$

Indeed, for the given $\varepsilon > 0$, and since f is continuous at a, there exists $\delta > 0$ such that if $x \in X$ and

$$|x - a| < \delta \Rightarrow |f(x) - f(a)| < \varepsilon. \tag{5.13}$$

However, since $x_n \to a$, for this $\delta > 0$ there exists $n_0 \in \mathbb{N}$ such that

$$\text{if } n > n_0 \Rightarrow |x_n - a| < \delta. \tag{5.14}$$

Combining (5.13) and (5.14) proves (5.12). ■

Theorem 5.17 *Let $E \subset \mathbb{R}$ be a connected set and $f : E \to \mathbb{R}$ a continuous function. Then $f(E)$ is connected.*

Proof: Suppose, by contradiction, that $f(E)$ is disconnected. Then, there exist $A, B \subset \mathbb{R}$, separated and non-empty, such that $f(E) = A \cup B$. Let

$$G = E \cap f^{-1}(A) \text{ and } H = E \cap f^{-1}(B).$$

We claim:

$$E = G \cup H \tag{5.15}$$

Indeed,

$$\begin{aligned} G \cup H &= \big(E \cap f^{-1}(A)\big) \cup \big(E \cap f^{-1}(B)\big) \\ &= E \cap \big(f^{-1}(A) \cup f^{-1}(B)\big) \\ &= E \cap f^{-1}(A \cup B). \end{aligned}$$

But $f(E) = A \cup B$, which implies $E = f^{-1}(A \cup B) \cap E$. *(Note: $E \subset f^{-1}(A \cup B)$ is always true, but since f maps *from* E, $E = f^{-1}(f(E)) = f^{-1}(A \cup B)$)* Hence, $G \cup H = E \cap f^{-1}(A \cup B) = E$, which proves (5.15). *(Note: The original proof had a small logical loop here, corrected for clarity)*

We also have that

$$G \neq \emptyset \text{ and } H \neq \emptyset. \tag{5.16}$$

Indeed, since A is non-empty, there exists $y \in A$. But $A \subset f(E)$. Thus, $y = f(x)$ for some $x \in E$. Furthermore, $x \in f^{-1}(A)$ since $y = f(x) \in A$. Thus, $x \in E \cap f^{-1}(A) = G$, which implies $G \neq \emptyset$. Analogously, starting from the hypothesis that $B \neq \emptyset$, one shows that $H \neq \emptyset$, which proves (5.16).

Finally, we have that

$$G \text{ and } H \text{ are separated.} \tag{5.17}$$

Suppose, by contradiction, that there exists $x \in \mathbb{R}$ such that $x \in \overline{G} \cap H$. Then $x \in \overline{G}$ and $x \in H$. Since $x \in \overline{G}$, there exists a sequence $\{x_n\} \subset G$ such that $x_n \to x$. Since $x \in H \subset E$, $x \in E$. Since f is continuous (at $x \in E$), by Lemma 5.16 it follows that $f(x_n) \to f(x)$. However, since $\{x_n\} \subset G \subset f^{-1}(A)$, we have $\{f(x_n)\} \subset A$. Since $f(x_n) \to f(x)$, it follows that $f(x) \in \overline{A}$.

On the other hand, since $x \in H \subset f^{-1}(B)$, we have $f(x) \in B$. Thus,

$$f(x) \in \overline{A} \cap B,$$

which is an absurdity, because A and B are separated. Analogously, one proves that $G \cap \overline{H} = \emptyset$, which proves (5.17).

Thus, by (5.15), (5.16) and (5.17), we have written E as the union of two non-empty, separated sets, which is an absurdity as E is connected by hypothesis. Thus, it is proven that $f(E)$ is connected. ■

Theorem 5.18 *[Intermediate Value Theorem] Let f be a real function defined and continuous on $[a, b] \subset \mathbb{R}$. If c is a real number such that $f(a) < c < f(b)$ (or $f(b) < c < f(a)$), then there exists $x \in (a, b)$ such that $f(x) = c$.*

Proof: We know that $[a, b]$ is connected (by Prop. 3.31) and f is continuous. Hence, by Theorem 5.17, $E = f([a, b])$ is connected. Since $f(a), f(b) \in E$, by Proposition 3.31 (characterization of connected sets), E must be an interval. Since $f(a) < c < f(b)$, c lies between two points of the interval E. Thus $c \in E$. This means $c \in f([a, b])$, i.e., $c = f(x)$ for some $x \in [a, b]$. But since $c \neq f(a)$ and $c \neq f(b)$, we must have $x \in (a, b)$ such that $f(x) = c$, which concludes the proof. ■

Remark: A natural question arises: Is the converse of the Intermediate Value Theorem valid? That is, given a function $f : A \subset \mathbb{R} \to \mathbb{R}$, if for any two points $f(a), f(b)$ in the image, f takes on all values c between $f(a)$ and $f(b)$, can I affirm that the function f is continuous? The answer is no. Let's see a counter-example. Consider:

$$f(x) = \begin{cases} \sin(1/x), & x \neq 0, \\ 0, & x = 0. \end{cases}$$

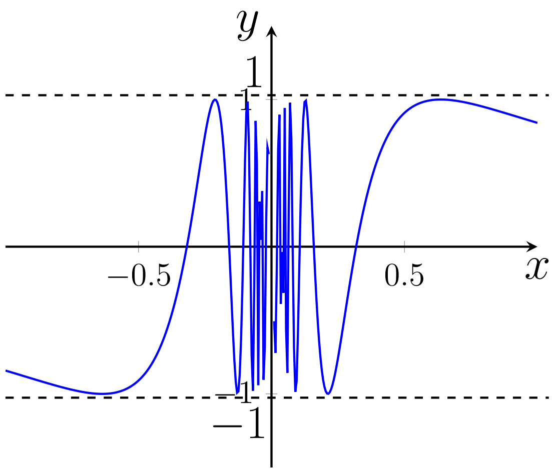


Figure 5.3: $f(x) = \sin\frac{1}{x}$.

The function f satisfies the intermediate value property, however it is not continuous at $x = 0$ because

$$\lim_{x\to 0_+} \sin(1/x) \quad \text{and} \quad \lim_{x\to 0_-} \sin(1/x) \quad \text{do not exist.}$$

Exercise: Let $f : [a, b] \to \mathbb{R}$ be a continuous function and $f(a) < 0 < f(b)$. Prove that there exists $\alpha \in (a, b)$ such that $f(\alpha) = 0$, without using the connectedness of $f([a, b])$.

Hint: Consider

$$A := \{x \in [a, b] : \; f(t) \leq 0; \; \forall t \in [a, x]\}.$$

(Note: Changed f(t)<0 to f(t)<=0 in the hint to make A closed) The idea is to show that A has a supremum. Define $\alpha = \sup A$. Then prove that $f(\alpha) = 0$ (to do this, proceed by contradiction, assuming $f(\alpha) < 0$ and $f(\alpha) > 0$).

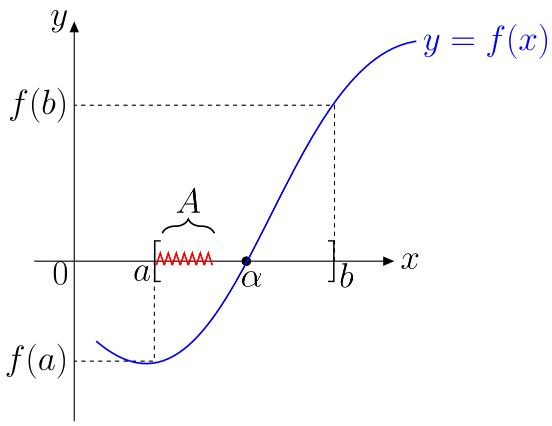


Figure 5.4:

Exercises: Limits and Continuity

$1^{\underline{a}}$ Question Let $f(x) = x^4$, $a \in$ and $L = a^4$. Let $\epsilon > 0$. Find a $\delta > 0$ such that:

$$|f(x) - L| < \epsilon \quad \text{for} \quad \text{every} \quad x \quad \text{that} \quad \text{satisfies} \quad 0 < |x - a| < \delta.$$

$2^{\underline{a}}$ Question Let $\epsilon > 0$. Find a $\delta > 0$ such that: $0 < |x - 1| < \delta \Rightarrow \left|\frac{1}{x} - 1\right| < \epsilon$.

$3^{\underline{a}}$ Question Let $\epsilon > 0$. Find a $\delta > 0$ such that: $0 < |x - 0| < \delta \Rightarrow \left|\frac{x}{1+\sin^2 x} - 0\right| < \epsilon$.

$4^{\underline{a}}$ Question Let $f(x) = \sqrt{|x|}$, $a = 0$ and $L = 0$. Let $\epsilon > 0$. Find a $\delta > 0$ such that:

$$|f(x) - L| < \epsilon \quad \text{for} \quad \text{every} \quad x \quad \text{that} \quad \text{satisfies} \quad 0 < |x - a| < \delta.$$

$5^{\underline{a}}$ Question Let $\epsilon > 0$. Find a $\delta > 0$ such that: $0 < |x - 1| < \delta \Rightarrow \left|\sqrt{x} - 1\right| < \epsilon$.

$6^{\underline{a}}$ Question If the limits $\lim\limits_{x \longrightarrow a} f(x)$ and $\lim\limits_{x \longrightarrow a} g(x)$ do not exist, can $\lim\limits_{x \longrightarrow a} (f(x) + g(x))$ or $\lim\limits_{x \longrightarrow a} (f(x)g(x))$ exist?

$7^{\underline{a}}$ Question Prove that:

$$\lim_{x \longrightarrow a} f(x) = L \Leftrightarrow \lim_{x \longrightarrow a} f(x + a) = L$$

$8^{\underline{a}}$ Question Prove that:

$$\lim_{x \longrightarrow a} f(x) = L \Leftrightarrow \lim_{x \longrightarrow a} [f(x) - L] = 0$$

$9^{\underline{a}}$ Question Let $L > 0$. Prove that $\lim\limits_{x \longrightarrow 0} \dfrac{1}{x} = L$ is false.

$10^{\underline{a}}$ Question Using the definition of a limit, prove that $\lim\limits_{x \longrightarrow a} f(x)$ does not exist, $\forall a \in$ for the following function:

$$f(x) = \begin{cases} 0, & \text{if} \quad x \in \quad \mathbb{Q} \\ 1, & \text{if} \quad x \in \quad -\mathbb{Q} \end{cases}$$

11ª Question Let $f : X \longrightarrow$, $a \in X'$ and $Y = f(X - \{a\})$. If $\lim_{x \longrightarrow a} f(x) = L$ then $L \in \overline{Y}$.

12ª Question Let $f : X \longrightarrow$ and $a \in X'$. In order for $\lim_{x \longrightarrow a} f(x)$ to exist, it is sufficient that, for every sequence of points $x_n \in X - \{a\}$ with $\lim x_n = a$, the sequence $(f(x_n))$ is convergent.

13ª Question Let $f : X \longrightarrow$, $g : Y \longrightarrow$ with $f(X) \subset Y$, $a \in X'$ and $b \in Y' \cap Y$. If

$$\lim_{x \longrightarrow a} f(x) = b \quad \text{e} \quad \lim_{y \longrightarrow b} g(y) = c,$$

prove that $\lim_{x \longrightarrow a} g(f(x)) = c$, provided that $c = g(b)$ or else that $x \neq a$ implies $f(x) \neq b$.

14ª Question Let $f, g : X \longrightarrow$ be defined by $f(x) = 0$ if x is irrational and $f(x) = x$ if $x \in \mathbb{Q}$; $g(0) = 1$ and $g(x) = 0$ if $x \neq 0$. Show that $\lim_{x \longrightarrow 0} f(x) = 0$ and $\lim_{y \longrightarrow 0} g(y) = 0$. However $\lim_{x \longrightarrow 0} g(f(x))$ does not exist.

15ª Question Dada a função

$$f(x) = \begin{cases} 3, & \text{if} \quad x \in \quad \mathbb{Z} \\ 1, & \text{if} \quad x \in \quad -\mathbb{Z}, \end{cases}$$

at which points in the domain of f does the limit not exist?

16ª Question Let $f :\rightarrow$ be the identity function, i.e., $f(x) = x$ for all $x \in$. Then prove the following statements:

(a) $\lim_{x \to a} f(x) = \lim_{x \to a} x = a$ for all $a \in$.

(b) $\lim_{x \to a} x^n = a^n \ \forall n \in \mathbb{N}$(use induction)

(c) $\lim_{x \to a} p(x) = p(a)$, where $p(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$.

(d) $\lim_{x \to a} f(x) = f(a)$, where $f(x) = \frac{p(x)}{q(x)}$, $p(x)$ and $q(x)$ are polynomials and $q(a) \neq 0$.

(e) Let $f(x) = \frac{p(x)}{q(x)}$ be the quotient of two polynomials. If $q(a) = 0$, then a is a root of $q(x)$ and, therefore, $x - a$ divides $q(x)$. Let $m \geq 1$ such that $q(x) = (x - a)^m q_1(x)$, with $q_1(a) \neq 0$, and let $n \geq 0$ such that $p(x) = (x - a)^n p_1(x)$, with $p_1(a) \neq 0$. Then:

(i) If $m = n$, $\quad \lim_{x \to a} f(x) = \dfrac{p_1(a)}{q_1(a)}$,

(ii) If $m < n$, $\quad \lim_{x \to a} f(x) = 0$,

(iii) If $m > n$, $\quad \lim_{x \to a} f(x)$ does not exist.

17ª Question Let $f :\to$ be the function defined by

$$f(x) = \begin{cases} 0, & \text{if} \quad x \in \quad \mathbb{Q} \\ 1, & \text{if} \quad x \in \quad -\mathbb{Q} \end{cases}$$

Prove that $\lim_{x\to a} f(x)$ does not exist for all $a \in$. But, if $g(x) = (x-a)f(x)$ prove that $\lim_{x\to a} g(x) = 0$.

18ª Question Let $f : \mathbb{Q} \to$ be the function defined by

$$f(x) = \begin{cases} \dfrac{1}{q}, & \text{if} \quad \frac{p}{q} \quad \text{is an irreducible fraction with} \quad q > 0 \\ 1, & \text{if} \quad x = 0 \end{cases}$$

Prove that $\lim_{x\to a} f(x) = 0$ for all $a \in$.

19ª Question Let $f : -\{0\} \to$ be the function defined by $f(x) = \sin \frac{1}{x}$. Then $\lim_{x\to 0} f(x)$ does not exist.

20ª Question Calculate the following limits ($[\![\cdot]\!]$ represents the floor function):

a) $\displaystyle\lim_{x\to 0^-} \left[\frac{x^2 [\![-x/2]\!]}{|x| [\![3x]\!]} + \frac{1}{|x|} - \sqrt{1 + \frac{1}{x^2}} \right]$, b) $\displaystyle\lim_{x\to 2^-} \frac{x^2 \left[\!\left[\frac{2x+1}{x-1}\right]\!\right] - 10x}{x^3 - 11x^2 + 38x - 40}$

c) $\displaystyle\lim_{x\to 1^-} \sqrt[3]{\frac{x [\![1/x^2]\!] + 7}{2 - x^2}}$, d) $\displaystyle\lim_{x\to 1^+} \sqrt[3]{\frac{x [\![1/x^2]\!] + 7}{2 - x^2}}$

e) $\displaystyle\lim_{x\to 0^+} \frac{|x| - \sqrt{|x|}}{x^2 + x}$, f) $\displaystyle\lim_{x\to 0^-} \frac{|x| - \sqrt{|x|}}{x^2 + x}$

g) $\displaystyle\lim_{x\to +\infty} \frac{\sqrt{x^2 + 7x + 10}}{x}$, h) $\displaystyle\lim_{x\to -\infty} \frac{\sqrt{x^2 + 7x + 10}}{x}$

i) $\displaystyle\lim_{x\to +\infty} \left[\sin \sqrt{\sqrt{x^2+1} + 2} - \sin \sqrt{\sqrt{x^2+3} + 1} \right]$, j) $\displaystyle\lim_{x\to +\infty} \left[\sin \sqrt{x+2} - \sin \sqrt{x} \right]$

21ª Question Let f be such that $|f(x)| \le |x| \, \forall x \in$. Prove that f is continuous at 0.

22ª Question Suppose that g is continuous at 0, $g(0) = 0$ and that $|f(x)| \le |g(x)| \, \forall x \in$. Prove that f is continuous at 0.

23ª Question Let $f, g : X \subset\longrightarrow$ be continuous at the point $a \in X$. Prove that the functions $\varphi, \psi : X \subset\longrightarrow$, defined by $\varphi(x) = \max\{f(x), g(x)\}$ and $\psi(x) = \min\{f(x), g(x)\}$ for all $x \in X$, are continuous at the point a.

24ª Question Prove that $f :\longrightarrow$ is continuous if, and only if, for every $X \subset$, one has $f(\overline{X}) \subset \overline{f(X)}$.

25ª Question Let $f :\longrightarrow$ be continuous. Prove that if $f(x) = 0$ for every $x \in X \subset$ then $f(x) = 0$ for every $x \in \overline{X}$.

26ª Question Let $f :\longrightarrow$ be continuous, such that $\lim_{x\to+\infty} f(x) = \lim_{x\to-\infty} f(x) = +\infty$. Prove that there exists $x_0 \in$ such that $f(x_0) \leq f(x)$ for all $x \in$.

27ª Question Give an example of a function f that is not continuous at any point, but such that $|f|$ is continuous at all points.

28ª Question Find a function that is continuous at a, but that is not continuous at any other point.

29ª Question Suppose that f satisfies $f(x+y) = f(x) + f(y)\,\forall x, y \in$ and that f is continuous at 0. Prove that f is continuous at a for all $a \in$.

30ª Question Let f be continuous at a and such that $f(a) = 0$, let $\alpha > 0$. Prove that $\exists\,\delta > 0$ such that $f(x) + \alpha \neq 0\,\forall x \in (a - \alpha, a + \alpha)$.

31ª Question Let $f(x) = x^3 - x + 3\,\forall x \in$. Find an $n \in \mathbb{Z}$ such that $\exists\, x_0 \in [n, n+1]$ such that $f(x_0) = 0$.

32ª Question Let $f(x) = x^5 + x + 1\,\forall x \in$. Find an $n \in \mathbb{Z}$ such that $\exists\, x_0 \in [n, n+1]$ such that $f(x_0) = 0$.

33ª Question Prove that $\exists\, x_0 \in$ such that $x_0^{179} + \dfrac{163}{1 + x_0^2 + \sin^2(x)} = 119$.

34ª Question Prove that $\exists\, c \in$ such that $\sin(c) = c - 1$.

35ª Question Let $f : [a, b] \longrightarrow$ be continuous and such that $f(x) \in \mathbb{Q}\,\forall x \in [a, b]$. Prove that f is constant.

36ª Question Let $f, g : [a, b] \longrightarrow$ be continuous, such that $f(a) < g(a)$ and $g(b) < f(b)$. Prove that $\exists\, x_0 \in [a, b]$ such that $f(x_0) = g(x_0)$.

37ª Question Prove that the function $f(x) = \dfrac{1}{1 + x^2}$ for $x \in$ is uniformly continuous on .

38ª Question Prove that if f, g are uniformly continuous on $X \subset$ then $f + g$ is uniformly continuous on X.

39ª Question Prove that if f, g are uniformly continuous on $X \subset$ and are bounded on X , then the product fg is uniformly continuous on X.

40ª Question If $f(x) = x$ e $g(x) = \sin x$, Prove that f e g são uniformemente contínuas em , mas que seu produto naõ o é.

41ª Question Prove that if f is continuous on $[0, +\infty)$ and uniformly continuous en $(a, +\infty)$ para alguma constante positiva a , então f é uniformemente contínua em $[0, +\infty)$.

Chapter 6

Differentiation

6.1 Differentiable Functions

Definition 6.1 *Let $X \subset \mathbb{R}$ and $a \in X \cap X'$ (i.e., an accumulation point of X belonging to X). We say that f is differentiable (or derivable) at the point a if the following limit exists:*

$$\lim_{x \to a} \frac{f(x) - f(a)}{x - a}.$$

If it exists, this limit is denoted by $f'(a)$ and is called the derivative of f at the point a. It is understood that the function $q : x \mapsto \frac{f(x)-f(a)}{x-a}$ is defined on the set $X \backslash \{a\}$. Geometrically, $q(x)$ represents the slope (or angular coefficient) of the secant line to the graph of f that passes through the points $(a, f(a))$ and $(x, f(x))$. The line that passes through the point $(a, f(a))$ and has a slope equal to $f'(a)$ is called the tangent to the graph of f at the point $(a, f(a))$. The slope of the tangent is, therefore, the limit of the slopes of the secant lines passing through $(a, f(a))$ and $(x, f(x))$ as $x \to a$ (see figure **??**).

Writing $h = x - a$ or $x = a + h$, the derivative of f at the point $a \in X \cap X'$ becomes the limit:

$$\lim_{h \to 0} \frac{f(a + h) - f(a)}{h}.$$

Note that the function $h \mapsto \frac{f(a+h)-f(a)}{h}$ is defined on the set

$$Y = \{h \in \mathbb{R} \backslash \{0\} : a + h \in X\},$$

which has zero as an accumulation point. Indeed, let $\varepsilon > 0$. Then $B_\varepsilon(a) \cap (X \backslash \{a\}) \neq \emptyset$. It follows that there exists $x \in B_\varepsilon(a)$ and $x \in X$ with $x \neq a$. Now, since $x \neq a$, then $x = a + h$ for some $h \neq 0$. Hence, $(a + h) \in B_\varepsilon(a)$ (since $|\underbrace{a + h}_{=x} - a| = |x - a| < \varepsilon$), $h \neq 0$ and $(a + h) \in X$. Consequently $h \in Y$. This shows $B_\varepsilon(0) \cap (Y \setminus \{0\}) \neq \emptyset$, proving that $0 \in Y'$.

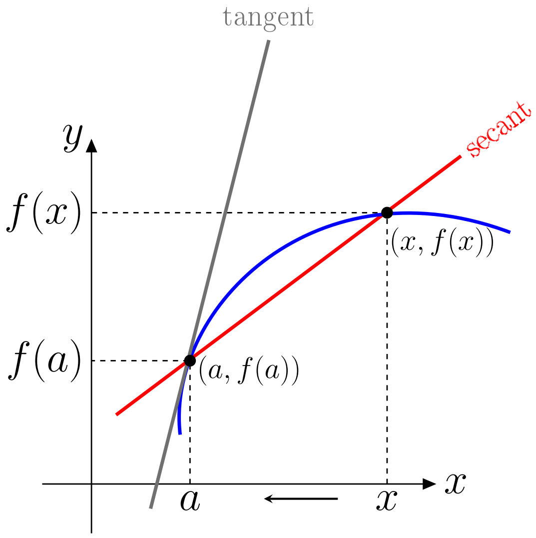


Figure 6.1:

When $a \in X \cap X'_+$ (i.e., when a is a right-accumulation point of X and belongs to X), we can define the right-hand derivative of f at a as the limit, if it exists:

$$f'_+(a) = \lim_{x \to a_+} \frac{f(x) - f(a)}{x - a} = \lim_{h \to 0_+} \frac{f(a+h) - f(a)}{h}. \tag{6.1}$$

Similarly, the left-hand derivative $f'_-(a)$ is defined when a is a left-accumulation point that belongs to the domain of f.

Evidently, when $a \in X$ is both a right and left accumulation point, $f'(a)$ exists if and only if the lateral derivatives $f'_+(a)$ and $f'_-(a)$ exist and are equal.

Remarks:

1a) When we say that a function $f : [c, d] \to \mathbb{R}$, defined on a compact interval, is differentiable at a point $a \in [c, d]$, this means, in the case of $a \in (c, d)$, that f has both lateral derivatives at a and they are equal. In the case where a is one of the endpoints, this simply means that the lateral derivative that makes sense at a exists.

It follows from the general properties of limits that $f : X \to \mathbb{R}$ is differentiable at a if and only if for any sequence of points $\{x_n\} \subset X \backslash \{a\}$ with $\lim_{n \to +\infty} x_n = a$, we have $\lim_{n \to +\infty} \frac{f(x_n) - f(a)}{x_n - a} = f'(a)$.

Examples:

(i) Let $f : \mathbb{R} \to \mathbb{R}$ be constant, i.e., $f(x) = c$, $c \in \mathbb{R}$. Then $f'(a) = 0$ for all $a \in \mathbb{R}$. Indeed,

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a} = \lim_{x \to a} \frac{c - c}{x - a} = 0.$$

(ii) Let $f : \mathbb{R} \to \mathbb{R}$ be given by $f(x) = cx + d$. Then, given $a \in \mathbb{R}$, we have:

$$\begin{aligned} f'(a) &= \lim_{x\to a} \frac{f(x)-f(a)}{x-a} = \lim_{x\to a} \frac{(cx+d)-(ca+d)}{x-a} \\ &= \lim_{x\to a} \frac{c(x-a)}{x-a} = c. \end{aligned}$$

(iii) Let $f : \mathbb{R} \to \mathbb{R}$ be given by $f(x) = x^2$. Then, given $a \in \mathbb{R}$

$$\begin{aligned} f'(a) &= \lim_{h\to 0} \frac{f(a+h)-f(a)}{h} = \lim_{h\to 0} \frac{a^2+2ah+h^2-a^2}{h} \\ &= \lim_{h\to 0} \frac{(2a+h)h}{h} = 2a. \end{aligned}$$

(iv) Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = |x|$. Then, for $x \neq 0$:

$$\frac{f(x)-f(0)}{x-0} = \frac{|x|}{x} = \pm 1 \ (+1 \text{ if } x > 0 \text{ and } -1 \text{ if } x < 0).$$

It follows that $f'_+(0) = 1$ and $f'_-(0) = -1$ exist, but $f'(0)$ does not exist. However, for $a \neq 0$, $f'(a)$ exists, equaling 1 if $a > 0$ and -1 if $a < 0$.

Definition 6.2 *We say that $f : X \subset \mathbb{R} \to \mathbb{R}$ is differentiable on the set X if the derivative of f exists at all points $a \in X \cap X'$.*

Theorem 6.3 *Let $f : X \subset \mathbb{R} \to \mathbb{R}$ and $a \in X \cap X'$. If the derivative $f'(a)$ exists, then f is continuous at a.*

Proof: If the limit $\lim_{x\to a} \frac{f(x)-f(a)}{x-a}$ exists, then the limit

$$\begin{aligned} \lim_{x\to a}[f(x)-f(a)] &= \lim_{x\to a} \left(\frac{f(x)-f(a)}{x-a} \cdot (x-a) \right) \\ &= \left(\lim_{x\to a} \frac{f(x)-f(a)}{x-a} \right) \left(\lim_{x\to a}(x-a) \right) = f'(a) \cdot 0 = 0. \end{aligned}$$

Hence, $\lim_{x\to a} f(x) = f(a)$, which proves the desired result (by Remark 3a of Ch. 5). ■

Remark: It follows from Theorem 6.3 that if f is not continuous at $a \in X$, then f is not differentiable at $a \in X$. However, example (iv) shows that there exist continuous functions that are not differentiable.

6.2 Properties of Differentiable Functions

Theorem 6.4 *Let $f, g : X \to \mathbb{R}$ be differentiable at the point $a \in X \cap X'$. Then $f \pm g$, $f \cdot g$, f/g (if $g(a) \neq 0$) are differentiable at this same point. Furthermore, we have:*

(i) $(f+g)'(a) = f'(a) + g'(a)$.

(ii) $(f \cdot g)'(a) = f'(a) \cdot g(a) + f(a) \cdot g'(a)$. *In particular, if* $f(x) = c$, *then* $(c\,g)'(a) = c\,g'(a)$.

(iii) $\left(\frac{1}{g}\right)'(a) = -\frac{g'(a)}{(g(a)^2)}$.

Proof: (i)

$$\begin{aligned}
(f+g)'(a) &= \lim_{h\to 0} \frac{(f+g)(a+h) - (f+g)(a)}{h} \\
&= \lim_{h\to 0} \frac{f(a+h) + g(a+h) - [f(a) + g(a)]}{h} \\
&= \lim_{h\to 0} \left[\frac{f(a+h) - f(a)}{h} + \frac{g(a+h) - g(a)}{h}\right] \\
&= \lim_{h\to 0} \frac{f(a+h) - f(a)}{h} + \lim_{h\to 0} \frac{g(a+h) - g(a)}{h} \\
&= f'(a) + g'(a).
\end{aligned}$$

(ii)

$$\begin{aligned}
(f\,g)'(a) &= \lim_{h\to 0} \frac{(f\,g)(a+h) - (f\,g)(a)}{h} \\
&= \lim_{h\to 0} \frac{f(a+h)g(a+h) - f(a)g(a)}{h} \\
&= \lim_{h\to 0} \frac{f(a+h)g(a+h) - f(a+h)g(a) + f(a+h)g(a) - f(a)g(a)}{h} \\
&= \lim_{h\to 0} \left[f(a+h)\frac{g(a+h) - g(a)}{h} + \frac{(f(a+h) - f(a))}{h} g(a)\right] \\
&= \lim_{h\to 0} f(a+h) \cdot \lim_{h\to 0} \frac{g(a+h) - g(a)}{h} + \lim_{h\to 0} \frac{f(a+h) - f(a)}{h} \cdot g(a) \\
&= f(a)\,g'(a) + f'(a)\,g(a).
\end{aligned}$$

(Note: we used $\lim_{h\to 0} f(a+h) = f(a)$ since f is continuous at a by Thm 6.3).

(iii) Before proving the result, we must be sure that the expression $\frac{(1/g)(a+h) - (1/g)(a)}{h}$ makes sense. It is necessary to verify that $(1/g)(a+h)$ is well-defined for h sufficiently small. Indeed, since g is differentiable at a, it follows that g is continuous at

a. As $g(a) \neq 0$, there exists $\delta > 0$ such that $g(a+h) \neq 0$ for $|h| < \delta$. Thus

$$\begin{aligned}
\lim_{h\to 0} \frac{\left(\frac{1}{g}\right)(a+h) - \left(\frac{1}{g}\right)(a)}{h} &= \lim_{h\to 0} \frac{\frac{1}{g(a+h)} - \frac{1}{g(a)}}{h} \\
&= \lim_{h\to 0} \frac{g(a) - g(a+h)}{h[g(a)\, g(a+h)]} \\
&= \lim_{h\to 0} \left[-\frac{g(a+h) - g(a)}{h} \cdot \frac{1}{g(a)\, g(a+h)} \right] \\
&= \left(-\lim_{h\to 0} \frac{g(a+h) - g(a)}{h} \right) \cdot \left(\lim_{h\to 0} \frac{1}{g(a)\, g(a+h)} \right) \\
&= -g'(a) \cdot \frac{1}{(g(a))^2}.
\end{aligned}$$

■

Corollary 6.5 *If $f, g : X \to \mathbb{R}$ are differentiable at the point $a \in X \cap X'$ and $g(a) \neq 0$, then f/g is differentiable and, furthermore:*

$$\left(\frac{f}{g}\right)'(a) = \frac{f'(a)g(a) - f(a)g'(a)}{(g(a))^2}.$$

Proof: Exercise. (Hint: Use $f/g = f \cdot (1/g)$ and the product rule (ii) and rule (iii)). ■

6.3 The Chain Rule

Let $X \subset \mathbb{R}$, $f : X \to \mathbb{R}$ be a function and $a \in X \cap X'$. To say that

$$f'(a) = \lim_{h\to 0} \frac{f(a+h) - f(a)}{h},$$

is equivalent to saying that

$$f(a+h) = f(a) + f'(a)\, h + r(h) \text{ where } \lim_{h\to 0} \frac{r(h)}{h} = 0.$$

Indeed, assume that $f'(a) = \lim_{h\to 0} \frac{f(a+h)-f(a)}{h}$. Defining $r(h) = f(a+h) - f(a) - f'(a)h$, it follows that

$$\frac{r(h)}{h} = \frac{f(a+h) - f(a)}{h} - f'(a).$$

Applying the limit as h tends to zero on both sides of the identity above, it follows that:

$$\lim_{h\to 0} \frac{r(h)}{h} = \lim_{h\to 0} \frac{f(a+h) - f(a)}{h} - f'(a) = f'(a) - f'(a) = 0,$$

which proves the claim.

Conversely, let $f(a+h) = f(a) + f'(a)\,h + r(h)$ where $\lim_{h\to 0} \frac{r(h)}{h} = 0$. Hence,

$$\frac{f(a+h) - f(a)}{h} = f'(a) + \frac{r(h)}{h}.$$

Applying the limit as h tends to zero on both sides of the identity above, we obtain:

$$\lim_{h\to 0} \frac{f(a+h) - f(a)}{h} = f'(a).$$

From the above, we also have:

$$\lim_{h\to 0} r(h) = \lim_{h\to 0} \left[\frac{r(h)}{h} \cdot h\right] = \lim_{h\to 0} \frac{r(h)}{h} \cdot \lim_{h\to 0} h = 0 \cdot 0 = 0$$

We have the following geometric interpretation:

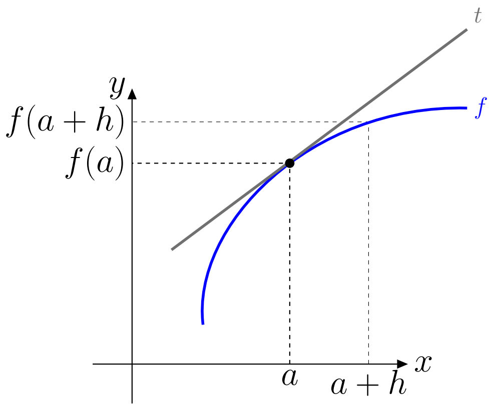
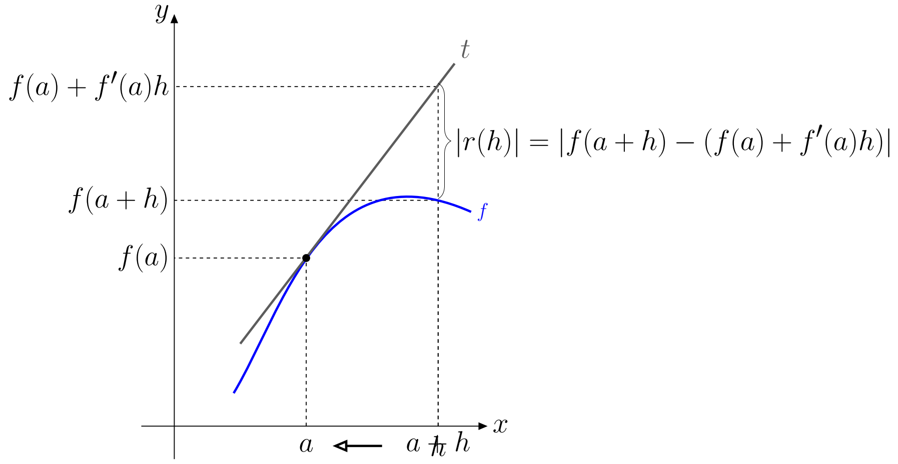


Figure 6.2:

We have: $r(h) = f(a+h) - f(a) - f'(a)h$. Let us determine the equation of the line t: $y = mx + b$, where $m = f'(a)$. In particular $f(a) = f'(a)a + b$, which implies $b = f(a) - f'(a)\,a$. Thus, $y = f'(a)x + (f(a) - f'(a)a)$, or

$$y = f(a) + f'(a)(x - a).$$

Observe the figure below:



Note that as $h \to 0$, $r(h) \to 0$, and the tangent line is a 'good approximation' of f at the point $x = a$.

Lemma 6.6 *Let $f : X \subset \mathbb{R} \to \mathbb{R}$ be a map. If f has a derivative at a point $a \in X \cap X'$, then there exists a function $\rho : \mathbb{R} \to \mathbb{R}$ continuous at the origin such that*

$$f(a+h) - f(a) = [f'(a) + \rho(h)]h,$$

for all a and $a + h$ belonging to X.

Proof: Let us define:

$$\rho(h) := \begin{cases} \dfrac{f(a+h)-f(a)}{h} - f'(a); & h \neq 0 \text{ and } a+h \in X \\ 0, & h = 0. \end{cases}$$

We will show that ρ thus defined satisfies the conditions of the lemma. Indeed, since f is differentiable at $a \in X \cap X'$, then

$$\lim_{h \to 0} \frac{f(a+h) - f(a)}{h} = f'(a).$$

Thus,

$$\begin{aligned} \lim_{h \to 0} \rho(h) &= \lim_{h \to 0} \left(\frac{f(a+h) - f(a)}{h} - f'(a) \right) \\ &= f'(a) - f'(a) = 0 = \rho(0), \end{aligned}$$

which proves the continuity of ρ at zero. The second condition is satisfied by the very definition of ρ. ■

Theorem 6.7 *[Chain Rule] Let $f : X \subset \mathbb{R} \to Y$ and $g : Y \subset \mathbb{R} \to \mathbb{R}$ be functions such that $f(X) \subset Y$, $a \in X \cap X'$ and $b = f(a) \in Y \cap Y'$. If $f'(a)$ and $g'(b)$ exist, then $g \circ f : X \to \mathbb{R}$ is differentiable at a and the rule holds:*

$$(g \circ f)'(a) = g'(b)f'(a).$$

Proof: By Lemma 6.6, since f is differentiable at a and g is differentiable at $b = f(a)$, there exist two functions $\rho : \mathbb{R} \to \mathbb{R}$ and $\sigma : \mathbb{R} \to \mathbb{R}$, both continuous at the origin, such that

(I) $f(a + h) - f(a) = [f'(a) + \rho(h)]h$, where $\lim_{h\to 0} \rho(h) = 0$,

(II) $g(b + k) - g(b) = [g'(b) + \sigma(k)]k$, where $\lim_{k\to 0} \sigma(k) = 0$.

Let $k = f(a + h) - f(a)$. Then, from this identity and (I):

$$k = [f'(a) + \rho(h)]h. \tag{6.2}$$

$$f(a + h) = k + f(a) = k + b. \tag{6.3}$$

Note that from (6.3) we have

$$(g \circ f)(a + h) = g(f(a + h)) = g(k + b).$$

But we also have from identity (II): $g(b + k) - g(b) = [g'(b) + \sigma(k)]k$. Substituting k in the identity above by its expression given in (6.2) yields from (II) that

$$\begin{aligned}(g \circ f)(a + h) &= g(f(a + h)) = g(k + f(a)) = g(k + b)\\ &= g(b) + [g'(b) + \sigma(k)]k\\ &= g(b) + [g'(b) + \sigma(k)][f'(a) + \rho(h)]h\\ &= g(b) + h[g'(b)f'(a) + g'(b)\rho(h) + \sigma(k)f'(a) + \sigma(k)\rho(h)]. \quad (6.4)\end{aligned}$$

On the other hand,

$$(g \circ f)(a) = g(f(a)) = g(b). \tag{6.5}$$

Combining (6.4) and (6.5) we deduce that

$$(g \circ f)(a + h) - (g \circ f)(a) = h[g'(b)f'(a) + g'(b)\rho(h) + \sigma(k)f'(a) + \sigma(k)\rho(h)],$$

which implies that

$$\begin{aligned}(g \circ f)'(a) &= \lim_{h\to 0} \frac{(g \circ f)(a + h) - (g \circ f)(a)}{h}\\ &= \lim_{h\to 0}[g'(b)f'(a) + g'(b)\rho(h) + \sigma(k)f'(a) + \sigma(k)\rho(h)]. \quad (6.6)\end{aligned}$$

However, let us observe that from the definition of k, i.e., $k = f(a+h) - f(a)$, it follows that (by continuity of f at a)

$$\lim_{h\to 0} k(h) = \lim_{h\to 0}(f(a+h) - f(a)) = 0.$$

And from the facts that $\lim_{h\to 0} \rho(h) = 0$ and (by composition of limits) $\lim_{h\to 0} \sigma(k(h)) = \sigma(\lim_{h\to 0} k(h)) = \sigma(0) = 0$, it follows from (6.6) that

$$(g \circ f)'(a) = g'(b)\, f'(a).$$

■

Examples:

(1) Consider $\phi : \mathbb{R} \to \mathbb{R}$ defined by $\phi(x) = \sqrt[3]{x^3 + x^2 + 1}$. $\phi(x) = (g \circ f)(x)$ where $g(u) = u^{1/3}$ and $u = f(x) = x^3 + x^2 + 1$. Thus, by the Chain Rule:

$$\begin{aligned} \phi'(x) &= g'(u) \cdot f'(x) \\ &= \frac{1}{3} u^{-2/3}(3x^2 + 2x) \\ &= \frac{3x^2 + 2x}{3u^{2/3}} \\ &= \frac{3x^2 + 2x}{3(x^3 + x^2 + 1)^{2/3}}. \end{aligned}$$

(2) Consider $\phi : \mathbb{R} \to \mathbb{R}$ defined by $\phi(x) = (x^2 + 1)^3$. Let $u = f(x) = x^2 + 1$ and $g(u) = u^3$. $\phi(x) = (g \circ f)(x)$. Hence, $\phi'(x) = g'(u)f'(x)$. Thus,

$$\begin{aligned} \phi'(x) &= 3u^2 \cdot (2x) \\ &= 3(x^2 + 1)^2 \cdot 2x = 6x(x^2 + 1)^2. \end{aligned}$$

Corollary 6.8 (Derivative of the Inverse Function) *Let $f : X \subset \mathbb{R} \to Y \subset \mathbb{R}$ be an invertible function. Let $g := f^{-1} : Y \to X$. If f is differentiable at $a \in X \cap X'$ and g is continuous at $b = f(a) \in Y \cap Y'$, then g is differentiable at b if and only if $f'(a) \neq 0$. In that case:*

$$g'(b) = \frac{1}{f'(a)}.$$

Proof: '⇒' Assume g is differentiable at $b = f(a)$ and consider $\phi = g \circ f$. Since f and g are differentiable at a and b, respectively, we have by the Chain Rule that

$$\phi'(a) = g'(b) \cdot f'(a). \tag{6.7}$$

On the other hand, since f is the inverse of g, we have $\phi(x) = (g \circ f)(x) = x$ for $x \in X$. Hence:

$$\begin{aligned} \phi'(a) &= \lim_{h\to 0} \frac{\phi(a+h)-\phi(a)}{h} && (6.8)\\ &= \lim_{h\to 0} \frac{(a+h)-a}{h} = 1, && (6.9) \end{aligned}$$

and from (6.7) and (6.8) we obtain $g'(b) \cdot f'(a) = 1$, which implies $f'(a) \neq 0$ and $g'(b) = \frac{1}{f'(a)}$.

'$\Leftarrow$' Conversely, assume $f'(a) \neq 0$. Let $y \in Y \setminus \{b\}$ and $x = g(y) \in X \setminus \{a\}$. As g is continuous at b, we have $\lim_{y\to b} g(y) = g(b) = a$. We have

$$\begin{aligned} g'(b) &= \lim_{y\to b} \frac{g(y)-g(b)}{y-b} = \lim_{y\to b} \frac{x-a}{f(x)-f(a)}\\ &= \lim_{y\to b} \frac{1}{\frac{f(x)-f(a)}{x-a}} = \lim_{x\to a} \frac{1}{\frac{f(x)-f(a)}{x-a}}\\ &= \frac{1}{\lim_{x\to a}\frac{f(x)-f(a)}{x-a}} = \frac{1}{f'(a)}, \end{aligned}$$

where we used the change of variables $x = g(y)$ and the fact that $y \to b \implies x \to a$ (by continuity of g) and $f'(a) \neq 0$. ■

Example: Let $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x^3$. This function is bijective and has a continuous inverse $g : \mathbb{R} \to \mathbb{R}$ given by $g(y) = \sqrt[3]{y}$. $f'(a) = 3a^2$. Hence, for $a = 0$ (where $f'(0) = 0$), g does not have a derivative at $b = f(0) = 0$. However, for $a \neq 0$ and $b = a^3$:

$$g'(b) = \frac{1}{f'(a)} = \frac{1}{3a^2} = \frac{1}{3(b^{1/3})^2} = \frac{1}{3b^{2/3}}.$$

6.4 Local Maxima and Minima and the Mean Value Theorem

Definition 6.9 *Let $f : X \subset \mathbb{R} \to \mathbb{R}$ be a function. f is said to have a local maximum at the point $x_0 \in X$, if there exists $\delta > 0$ such that if $x \in X \cap (x_0 - \delta, x_0 + \delta)$ then $f(x) \leq f(x_0)$. A local minimum is defined similarly.*

Theorem 6.10 *(Fermat's Theorem) Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be differentiable at a point x_0 belonging to the interior of the interval I. If x_0 is a point of local maximum (or minimum), then $f'(x_0) = 0$.*

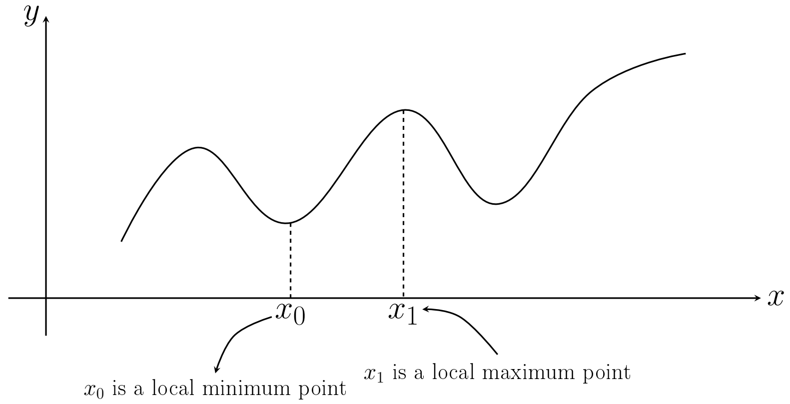


Figure 6.3: x_0 is a local minimum point, while x_1 is a local maximum point

Proof: Let x_0 be a local maximum point belonging to the interior of I. Then there exists $\delta_1 > 0$ such that if $x \in I \cap (x_0 - \delta_1, x_0 + \delta_1)$ then $f(x) \le f(x_0)$. On the other hand, since f is differentiable at x_0 (an interior point), the limits $f'_+(x_0)$ and $f'_-(x_0)$ exist and are equal. Let $h \neq 0$ be small enough such that $x_0 + h \in I \cap (x_0 - \delta_1, x_0 + \delta_1)$. Note that $f(x_0 + h) - f(x_0) \le 0$.

$$\frac{f(x_0+h)-f(x_0)}{h} \le 0 \text{ if } h > 0,$$
$$\frac{f(x_0+h)-f(x_0)}{h} \ge 0 \text{ if } h < 0.$$

Taking the limits, it follows that $f'_+(x_0) = \lim_{h\to 0^+} \frac{\cdots}{h} \le 0$ and $f'_-(x_0) = \lim_{h\to 0^-} \frac{\cdots}{h} \ge 0$. Since $f'(x_0) = f'_+(x_0) = f'_-(x_0)$, the only possibility is $f'(x_0) = 0$. The proof is analogous if x_0 is a local minimum point. ■

Theorem 6.11 (Rolle's Theorem) *Let $f : [a, b] \to \mathbb{R}$ be continuous on $[a, b]$ and differentiable on (a, b) such that $f(a) = f(b)$. Then there exists a point $c \in (a, b)$ such that $f'(c) = 0$.*

Proof: If f is constant, $f'(c) = 0$ for all $c \in (a, b)$, and we are done. Suppose, then, that f is not constant. Since f is continuous on the compact set $[a, b]$, f attains its maximum and minimum on this compact set (Thm 5.13). That is, there exist $x_1, x_2 \in [a, b]$ such that

$$f(x_2) \le f(x) \le f(x_1), \ \forall x \in [a, b].$$

However, since f is not constant and $f(a) = f(b)$, at least one of these points, x_1 or x_2, must be an interior point (in (a, b)). Let c be that interior point. Since c is a point of local maximum or minimum, by Theorem 6.10, we have $f'(c) = 0$. ■

Theorem 6.12 (Mean Value Theorem) *Let $f : [a,b] \to \mathbb{R}$ be a function that is continuous on $[a,b]$ and differentiable on (a,b). Then there exists a point $c \in (a,b)$ such that:*

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

Proof: Consider the auxiliary function defined by:

$$h(x) = (f(b) - f(a))x - (b - a)f(x);\ x \in [a,b].$$

h is clearly continuous on $[a,b]$ and differentiable on (a,b). Furthermore,

$$\begin{aligned} h(a) &= (f(b) - f(a))a - (b-a)f(a) \\ &= f(b)a - f(a)a - bf(a) + f(a)a = f(b)a - bf(a), \\ h(b) &= (f(b) - f(a))b - (b-a)f(b) \\ &= f(b)b - f(a)b - bf(b) + af(b) = -f(a)b + af(b). \end{aligned}$$

Therefore, $h(a) = h(b)$, and by Rolle's Theorem, there exists $c \in (a,b)$ such that $h'(c) = 0$. However,

$$h'(x) = (f(b) - f(a)) - (b-a)f'(x),\ \forall x \in (a,b).$$

In particular,

$$0 = h'(c) = (f(b) - f(a)) - (b-a)f'(c),$$

which implies that

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

■

Exercise: Let $f : (a,b) \to \mathbb{R}$ be differentiable. Prove that:
(a) If $f'(x) \geq 0$ for all $x \in (a,b)$ then f is non-decreasing.
(b) If $f'(x) = 0$ for all $x \in (a,b)$ then f is constant.
(c) If $f'(x) \leq 0$ for all $x \in (a,b)$ then f is non-increasing.

Let's do item (a): Consider $x_1, x_2 \in (a,b)$ with $x_1 < x_2$. Since f is differentiable on (a,b), it is continuous on $[x_1, x_2]$. Then, according to the Mean Value Theorem, there exists $c \in (x_1, x_2)$ such that

$$f'(c) = \frac{f(x_2) - f(x_1)}{x_2 - x_1},$$

i.e., $f(x_2) - f(x_1) = f'(c)(x_2 - x_1)$. However, since $f'(c) \geq 0$ (by hypothesis) and $(x_2 - x_1) > 0$, it follows that $f(x_2) - f(x_1) \geq 0$, i.e., $f(x_2) \geq f(x_1)$, which proves the desired result.

Theorem 6.13 *[Darboux's Theorem - IVT for Derivatives] Suppose f is a real function, differentiable on $[a,b]$, such that $f'(a) < \lambda < f'(b)$. Then, there exists a point $x \in (a,b)$ such that $f'(x) = \lambda$.*

Proof: Suppose initially that $\lambda = 0$, i.e., $f'(a) < 0 < f'(b)$. (Note $f'(a) = f'_+(a)$ and $f'(b) = f'_-(b)$).

I claim: There exists $\delta > 0$ such that if $a < x < a+\delta$, then $\frac{f(x)-f(a)}{x-a} < 0$. Indeed, suppose the contrary. Then for every $\delta > 0$, there exists $x_\delta \in (a, a+\delta)$ such that $\frac{f(x_\delta)-f(a)}{x_\delta - a} \geq 0$. In particular, for $\delta = 1/n$, there exists $x_n \in (a, a+1/n)$ such that $\frac{f(x_n)-f(a)}{x_n-a} \geq 0$. As $x_n \to a^+$, it follows that $f'_+(a) = \lim_{n\to+\infty} \frac{f(x_n)-f(a)}{x_n-a} \geq 0$, which contradicts $f'(a) < 0$. This proves the claim. It follows that $f(x) < f(a)$ for all $x \in (a, a+\delta)$, and therefore a is not a point of local minimum.

Analogously, there exists $\overline{\delta} > 0$ such that if $b - \overline{\delta} < x < b$, then $\frac{f(x)-f(b)}{x-b} > 0$. Since $x - b < 0$, this implies $f(x) - f(b) < 0$, so $f(x) < f(b)$ for all $x \in (b-\overline{\delta}, b)$. Thus, b is also not a point of local minimum.

On the other hand, since f is continuous on $[a,b]$, f must attain its minimum on $[a,b]$. From the above, this minimum must occur in (a,b). That is, there exists $x \in (a,b)$ which is a local minimum, and by Theorem 6.10, $f'(x) = 0$.

Now consider the general case. Let $g(x) = f(x) - \lambda x$. It is clear that g is differentiable on (a,b) and continuous on $[a,b]$. Furthermore,

$$\begin{aligned} g'(a) &= f'(a) - \lambda < 0, \\ g'(b) &= f'(b) - \lambda > 0. \end{aligned}$$

Therefore, by the previous case, there exists $x \in (a,b)$ such that $g'(x) = 0$, which implies $f'(x) - \lambda = 0$, or $f'(x) = \lambda$. ■

6.5 Higher-Order Derivatives and Taylor Polynomials

If a function f is differentiable on an interval containing a point c, one can inquire about the existence (or non-existence) of the derivative of the function f' at the point c. If this derivative exists, it is called the <u>second derivative</u> of f at c and is denoted by $f''(c)$ or $f^{(2)}(c)$. We then say that f' is differentiable at c. In general, if $n \in \mathbb{N}$, one can make analogous considerations and define the n-th order derivative of f at c. Notation: $f^{(n)}(c)$ or $\frac{d^n f}{dx^n}(c)$.

If $f : I \to \mathbb{R}$ is such that $f^{(n)}(x)$ exists for all $x \in I$, we say that f is n-times differentiable on I. If f is n-times differentiable on I and $f^{(n)}$ is continuous, we say that f is of class $C^n(I)$.

Taylor Polynomial If f is a function that is n-times differentiable on an interval I and x_0 is an interior point of I, it is possible to find a polynomial of degree n, denoted by P_{n,x_0}, such that:

$$P_{n,x_0}(x_0) = f(x_0);\ P'_{n,x_0}(x_0) = f'(x_0), \cdots, P^{(n)}_{n,x_0}(x_0) = f^{(n)}(x_0).$$

Precisely, such a polynomial is:

$$P_{n,x_0}(x) = f(x_0) + f'(x_0)(x - x_0) + \frac{f''(x_0)}{2!}(x - x_0)^2 + \cdots + \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n.$$

This polynomial is called the Taylor polynomial of degree n of the function f at the point x_0.

Theorem 6.14 *[Taylor's Theorem] Let $f : I \subset \mathbb{R} \to \mathbb{R}$. Suppose that for some $n \in \mathbb{N}$, the derivatives up to order n exist and are continuous on $I = [a, b]$. Furthermore, suppose $f^{(n+1)}$ exists on (a, b). If $x_0 \in [a, b]$, then for each $x \in [a, b]$ $(x \neq x_0)$, there exists c between x and x_0 such that*

$$f(x) = P_{n,x_0}(x) + \frac{f^{(n+1)}(c)}{(n+1)!}(x - x_0)^{n+1}.$$

The term $\frac{f^{(n+1)}(c)}{(n+1)!}(x - x_0)^{n+1}$ is called the remainder of order n and is denoted by $R_{n,x_0}(x)$ (Lagrange form of the remainder).

Proof: Consider the points x_0 and x, and let J be the closed interval with endpoints x and x_0. If $x = x_0$, $J = \{x_0\}$ and the theorem follows trivially. Let $x \neq x_0$, and consider the auxiliary function:

$$F : J \to \mathbb{R},\ t \mapsto F(t) = f(x) - P_{n,t}(x), \tag{6.10}$$

where

$$P_{n,t}(x) = f(t) + f^{(1)}(t)(x - t) + \frac{f^{(2)}(t)}{2!}(x - t)^2 + \cdots + \frac{f^{(n)}(t)}{n!}(x - t)^n. \tag{6.11}$$

It can be verified that

$$F'(t) = -\frac{(x - t)^n}{n!} f^{(n+1)}(t),$$

where this derivative exists for t in the interior of J. Indeed, from (6.10) and (6.11) applying the product rule to each term:

$$\begin{aligned} F'(t) &= -\frac{d}{dt}[P_{n,t}(x)] \\ &= -\Bigg[f'(t) + (f''(t)(x-t) - f'(t)) + \left(\frac{f'''(t)}{2!}(x-t)^2 - \frac{f''(t)}{2!}2(x-t)\right) \\ &\quad + \cdots + \left(\frac{f^{(n+1)}(t)}{n!}(x-t)^n - \frac{f^{(n)}(t)}{n!}n(x-t)^{n-1}\right)\Bigg] \end{aligned}$$

This is a telescoping sum where most terms cancel out:

$$\begin{aligned} F'(t) &= -\left[(f'(t) - f'(t)) + (f''(t)(x-t) - f''(t)(x-t)) + \ldots \right. \\ &\quad \left. +(\frac{f^{(n)}(t)}{(n-1)!}(x-t)^{n-1} - \frac{f^{(n)}(t)}{(n-1)!}(x-t)^{n-1}) + \frac{f^{(n+1)}(t)}{n!}(x-t)^n\right] \\ &= -\frac{f^{(n+1)}(t)}{n!}(x-t)^n, \end{aligned}$$

which proves the claim. Now, define the function:

$$G : J \to \mathbb{R};\ t \mapsto G(t) = F(t) - \left(\frac{x-t}{x-x_0}\right)^{n+1} F(x_0).$$

The function G is such that

$$\begin{aligned} G(x_0) &= F(x_0) - \left(\frac{x-x_0}{x-x_0}\right)^{n+1} F(x_0) = F(x_0) - F(x_0) = 0, \\ G(x) &= F(x) - \left(\frac{x-x}{x-x_0}\right)^{n+1} F(x_0) = F(x) = f(x) - P_{n,x}(x) = f(x) - f(x) = 0. \end{aligned}$$

Note that G is continuous on J and differentiable on the interior of J. By Rolle's Theorem, there exists c between x and x_0 such that $G'(c) = 0$. But

$$G'(t) = F'(t) - (n+1)\left(\frac{x-t}{x-x_0}\right)^n \left(-\frac{1}{x-x_0}\right) F(x_0)$$

$$0 = G'(c) = F'(c) + (n+1)\frac{(x-c)^n}{(x-x_0)^{n+1}} F(x_0),$$

which implies

$$\begin{aligned} F(x_0) &= -\frac{(x-x_0)^{n+1}}{(n+1)(x-c)^n} F'(c) \\ &= \left(-\frac{(x-x_0)^{n+1}}{(n+1)(x-c)^n}\right)\left(-\frac{(x-c)^n}{n!} f^{(n+1)}(c)\right), \end{aligned}$$

i.e.,

$$F(x_0) = \frac{f^{(n+1)}(c)}{(n+1)!}(x-x_0)^{n+1}. \tag{6.12}$$

From (6.10) and (6.12), it follows that

$$F(x_0) = f(x) - P_{n,x_0}(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x - x_0)^{n+1},$$

i.e.,

$$f(x) = P_{n,x_0}(x) + \frac{f^{(n+1)}(c)}{(n+1)!}(x - x_0)^{n+1},$$

which proves the desired result. ■

Application of Taylor's Theorem

Let I be an interval and x_0 an interior point of I. Let $f : I \to \mathbb{R}$ be n-times differentiable ($n \geq 2$) and assume that $f', f'', \cdots, f^{(n)}$ are continuous in a neighbourhood of x_0 and that $f'(x_0) = f''(x_0) = \cdots = f^{(n-1)}(x_0) = 0$, with $f^{(n)}(x_0) \neq 0$. Then:

(I) If n is even and $f^{(n)}(x_0) > 0$, f has a local minimum at x_0.

(II) If n is even and $f^{(n)}(x_0) < 0$, f has a local maximum at x_0.

(III) If n is odd, f has neither a local maximum nor a local minimum at x_0 (it is an inflection point).

Proof: By Taylor's Theorem (using the $n-1$ polynomial, with remainder of order n):

$$f(x) = f(x_0) + \frac{f^{(n)}(c)}{n!}(x - x_0)^n, \tag{6.13}$$

since $f'(x_0) = f''(x_0) = \cdots = f^{(n-1)}(x_0) = 0$. As $f^{(n)}$ is continuous at x_0 and $f^{(n)}(x_0) \neq 0$, there exists a neighbourhood of x_0 (say, $(x_0 - \delta, x_0 + \delta)$) where $f^{(n)}(t)$ has the same sign as $f^{(n)}(x_0)$. If $x \in (x_0 - \delta, x_0 + \delta)$, then c (which is between x and x_0) is also in this neighbourhood. Thus, $f^{(n)}(c)$ has the same sign as $f^{(n)}(x_0)$.

(I) n is even and $f^{(n)}(x_0) > 0$. Then $f^{(n)}(c) > 0$ and $(x - x_0)^n \geq 0$. Thus, (6.13) implies $f(x) \geq f(x_0)$ for x near x_0. x_0 is a local minimum.

(II) n is even and $f^{(n)}(x_0) < 0$. Then $f^{(n)}(c) < 0$ and $(x - x_0)^n \geq 0$. Thus, (6.13) implies $f(x) \leq f(x_0)$ for x near x_0. x_0 is a local maximum.

(III) n is odd. Then $(x - x_0)^n$ changes sign. If $f^{(n)}(x_0) > 0$, $f(x) > f(x_0)$ for $x > x_0$ and $f(x) < f(x_0)$ for $x < x_0$. x_0 is not an extremum. The case $f^{(n)}(x_0) < 0$ is analogous. ■

Exercises on Differentiable Functions

1. Let $f : \mathbb{R} \to \mathbb{R}$ be constant, i.e., there exists $c \in \mathbb{R}$ such that $f(x) = c$ for every $x \in \mathbb{R}$. Calculate $f'(a)$ for every $a \in \mathbb{R}$. (Expected result: $f'(a) = 0$)

2. Let $f : \mathbb{R} \to \mathbb{R}$ be given by $f(x) = cx + d$. Let $a \in \mathbb{R}$. Calculate $f'(a)$.

3. Let $f : \mathbb{R} \to \mathbb{R}$, $f(x) = x^2$. Let $a \in \mathbb{R}$. Calculate $f'(a)$.

4. Let $f : \mathbb{R} \to \mathbb{R}$, $f(x) = x^n$, $n \in \mathbb{N}$. Let $a \in \mathbb{R}$. Calculate $f'(a)$.

5. Let $f : \mathbb{R} \to \mathbb{R}$ be the function defined by $f(x) = |x|$.

   (a) Calculate: $f'(0^+)$ and $f'(0^-)$.

   (b) Calculate $f'(a)$ if $a > 0$ and $f'(a)$ if $a < 0$.

6. Let $f : [0, +\infty) \to \mathbb{R}$ be defined by $f(x) = \sqrt{x}$. Let $a \in [0, +\infty)$. Calculate $f'(a)$ if $a > 0$ and prove that f is not differentiable at the point 0.

7. Let $f : \mathbb{R} \to \mathbb{R}$ be the function defined by

$$f(x) = \begin{cases} x - n & if\ x \in [n, n + \frac{1}{2}] \\ n + 1 - x & if\ x \in [n + \frac{1}{2}, n + 1] \end{cases} \quad \text{for every } n \in \mathbb{Z}.$$

   (a) Calculate $f'(x)$ for all $x \in \mathbb{R}$, $x \neq n$, $x \neq n + \frac{1}{2}$, $n \in \mathbb{Z}$.

   (b) Prove that f is not differentiable at the points $n \in \mathbb{Z}$ and $n + \frac{1}{2}$, $n \in \mathbb{Z}$.

8. If $f : X \to \mathbb{R}$ is differentiable at the point $a \in X \cap X'$ then:

$$f(a + h) = f(a) + f'(a)h + r(h), \quad \text{with } \lim_{h \to 0} \frac{r(h)}{h} = 0.$$

   (a) If $f(x) = x^2$ determine $r(h)$.

   (b) If $f(x) = \sin x$ determine $r(h)$.

9. Let $f : \mathbb{R} \to \mathbb{R}$ be given by $f(x) = \begin{cases} 1 & if\ x \geq 0 \\ -1 & if\ x < 0 \end{cases}$

   (a) Prove that f is right-continuous at the point zero and calculate $f'(0^+)$.

(b) Prove that f is not left-continuous at the point 0 and verify that the left derivative of f at the point 0 does not exist.

(c) Conclude that f is not continuous at the point 0.

10. The function $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = x^3$ is a continuous bijection with continuous inverse $g : \mathbb{R} \to \mathbb{R}$ given by $g(y) = \sqrt[3]{y}$. Calculate $f'(a)$ for every $a \neq 0$ and determine $g'(b)$ for every $b \in \mathbb{R} - \{0\}$.

11. Let $f : \mathbb{R} \to \mathbb{R}$ be given by $f(x) = x^2$. Verify that f has a strict local minimum at the point 0.

12. Let $g : \mathbb{R} \to \mathbb{R}$, $g(x) = \sin x$. Verify that g has strict local maxima at the points $(4k+1)\frac{\pi}{2}$ and has strict local minima at the points $(4k-1)\frac{\pi}{2}$.

13. Verify that the function $h : \mathbb{R} \to \mathbb{R}$, given by

$$h(x) = \begin{cases} 1 & if\ x \geq 0 \\ -1 & if\ x < 0, \end{cases}$$

does not have a non-strict local maximum at the point 0.

14. Verify that the function $\varphi : \mathbb{R} \to \mathbb{R}$, $\varphi(x) = x^2(1+\sin\frac{1}{x})$ if $x \neq 0$ and $\varphi(0) = 0$, is continuous on the entire line and has a non-strict local minimum at the point 0.

15. Let the function $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = x\sin\frac{1}{x}$ if $x \neq 0$ and $f(0) = 0$.

(a) Prove that f is continuous on the entire line.

(b) Prove that f is differentiable for all $x \neq 0$, and calculate $f'(x)$ for all $x \neq 0$.

(c) Verify that f is not differentiable at the point zero.

16. Let the function $g : \mathbb{R} \to \mathbb{R}$, $g(x) = x^2\sin\frac{1}{x}$ if $x \neq 0$ and $g(0) = 0$.

(a) Prove that g is continuous on the entire line.

(b) Prove that g is differentiable on the entire line, and calculate $g'(x)$ for all $x \in \mathbb{R}$.

(c) Verify that $g' : \mathbb{R} \to \mathbb{R}$ is not continuous at the point zero.

17. Let the function $\varphi : \mathbb{R} \to \mathbb{R}$ be defined by $\varphi(x) = x^2\sin\frac{1}{x} + \frac{x}{2}$ if $x \neq 0$ and $\varphi(0) = 0$.

(a) Prove that φ is continuous and differentiable on the entire line. Calculate $\varphi'(x)$ for all $x \in \mathbb{R}$ and calculate $\varphi'(0)$.

(b) Prove that φ is not increasing in any neighborhood of the point 0.

(c) Conclude that φ cannot be injective in any interval of the type $(0, \delta)$ or $(-\delta, 0)$, $\delta > 0$.

18. Let $h : [-1, 1] \longrightarrow \mathbb{R}$ be defined by $h(x) = (1 - x^2) \sin \frac{1}{1-x^2}$ if $x \neq \pm 1$ and $h(\pm 1) = 0$. Prove that there exists $c \in (-1, 1)$ such that $h'(c) = 0$. Verify that $c = 0$.

19. Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = e^x$.

(a) Prove that $e^x > 1 + x$ for all $x > 0$.

(b) Prove that $\lim_{x \to +\infty} \frac{x^n}{e^x} = 0$ for $n \in \mathbb{N}$.

(c) Prove that for every polynomial $p(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$, $\lim_{x \to +\infty} \frac{p(x)}{e^x} = 0$.

20. Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = e^{-\frac{1}{x^2}}$ if $x \neq 0$ and $f(0) = 0$.

(a) Prove that f is continuous on the entire line.

(b) Prove that f is differentiable on the entire line and calculate $f'(x)$ for all $x \in \mathbb{R}$.

21. Let $f : \mathbb{R} \to \mathbb{R}$ be the function $f(x) = \begin{cases} e^{-\frac{1}{x}} & if\ x \neq 0 \\ 0 & if\ x = 0 \end{cases}$

(a) Calculate $\lim_{x \to 0^+} f(x)$.

(b) Calculate $\lim_{x \to 0^-} f(x)$.

(c) Verify that f is not continuous at the point zero.

(d) Verify that f is right-continuous at the point zero.

(e) Prove that f is differentiable from the right at the point 0 and verify that $f'(0^+) = 0$.

(f) Calculate $\lim_{x \to 0^-} f'(x)$.

Chapter 7

The Riemann Integral

7.1 The Riemann Integral

In what follows, we will consider real functions $f : [a, b] \to \mathbb{R}$, defined on a compact interval $[a, b]$ and bounded on this interval. Consequently, there exist lower and upper bounds for the set of values of f: $\{f(x) : x \in [a, b]\}$, i.e., there exist $c_1, c_2 \in \mathbb{R}$ such that

$$c_1 \leq f(x) \leq c_2; \ \forall x \in [a, b]. \tag{7.1}$$

We then define,

$$m = \inf\{f(x) : x \in [a, b]\}, \tag{7.2}$$

$$M = \sup\{f(x) : x \in [a, b]\}. \tag{7.3}$$

Definition 7.1 *A partition of the interval $[a, b]$ is a finite subset $P \subset [a, b]$ such that $a, b \in P$. When we write $P = \{t_0, t_1, \cdots, t_n\}$, we will always convene that $a = t_0 < t_1 < t_2 \cdots < t_n = b$. The intervals $[t_{i-1}, t_i]$ are called the subintervals of the partition P.*

Let $f : [a, b] \to \mathbb{R}$ be a bounded function and $P = \{t_0, t_1, \cdots, t_n\}$ a partition of $[a, b]$. For each $i = 1, \cdots, n$, let us define:

$$m_i = \inf\{f(x) : x \in [t_{i-1}, t_i]\}, \tag{7.4}$$

$$M_i = \sup\{f(x) : x \in [t_{i-1}, t_i]\}. \tag{7.5}$$

Definition 7.2 *We will define the lower sum* $s(f;P)$ *and the upper sum* $S(f;P)$ *of the function* f *with respect to the partition* P*, by setting:*

$$\begin{aligned} s(f;P) &= \sum_{i=1}^{n} m_i \Delta t_i = \sum_{i=1}^{n} m_i (t_i - t_{i-1}), \\ S(f;P) &= \sum_{i=1}^{n} M_i \Delta t_i = \sum_{i=1}^{n} M_i (t_i - t_{i-1}). \end{aligned}$$

Remark: Since $m_i \leq M_i$ for each $i \in \{1, \cdots, n\}$, we always have

$$s(f;P) \leq S(f;P). \tag{7.6}$$

Remark: When f is a positive function, the sums $s(f;P)$ and $S(f;P)$ can be interpreted as areas of polygons. (see Figure 7.1)

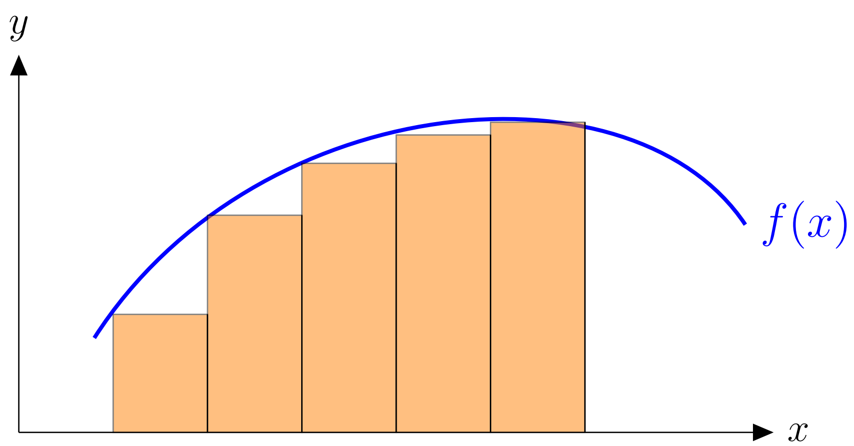


Figure 7.1:

Definition 7.3 *Let* P *and* Q *be partitions of* $[a,b]$*. We say that* Q *is finer than* P *(or that* Q *is a refinement of* P*) if* $P \subset Q$*.*

Next, we will prove that by refining a partition, the lower sum does not decrease and the upper sum does not increase.

Theorem 7.4 *Let* P *and* Q *be partitions of the interval* $[a,b]$ *with* $P \subset Q$*, and let* $f : [a,b] \to \mathbb{R}$ *be a bounded function. Then:*

$$s(f;P) \leq s(f;Q) \leq S(f;Q) \leq S(f;P).$$

Proof: It suffices to prove the first and third inequalities, since the second one has already been established. Indeed, let's prove the first. Take $P = \{t_0, t_1, \cdots, t_n\}$ and consider $Q = \{t_0, t_1, \cdots, t_{i-1}, r, t_i, \cdots t_n\}$, with $t_{i-1} < r < t_i$. (We are adding one more point to the partition). Let m_i, m' and m'' be the infima of f on the intervals $[t_{i-1}, t_i]$, $[t_{i-1}, r]$, and $[r, t_i]$ respectively. Evidently:

$$m_i \leq m' \ \text{ and } \ m_i \leq m'', \tag{7.7}$$

since the sets $[t_{i-1}, r]$ and $[r, t_i]$ are contained in $[t_{i-1}, t_i]$. As $\Delta t_i = t_i - t_{i-1} = (t_i - r) + (r - t_{i-1})$, from (7.7) we have:

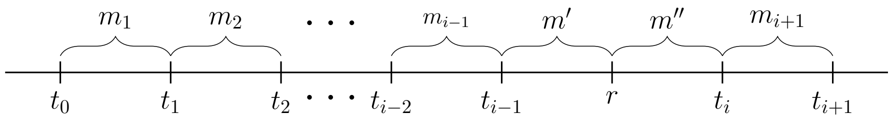


Figure 7.2:

$$\begin{aligned}
&s(f;Q) - s(f;P) \\
&= \cdots + m_{i-1}(t_{i-1} - t_{i-2}) + m'(r - t_{i-1}) + m''(t_i - r) + m_{i+1}(t_{i+1} - t_i) + \dots \\
&\quad -[\cdots + m_{i-1}(t_{i-1} - t_{i-2}) + m_i(t_i - t_{i-1}) + m_{i+1}(t_{i+1} - t_i) + \dots] \\
&= m'(r - t_{i-1}) + m''(t_i - r) - m_i(t_i - t_{i-1}) \\
&\geq m_i(r - t_{i-1}) + m_i(t_i - r) - m_i(t_i - t_{i-1}) \quad \text{(by (7.7))} \\
&= m_i[r - t_{i-1} + t_i - r] - m_i(t_i - t_{i-1}) \\
&= m_i(t_i - t_{i-1}) - m_i(t_i - t_{i-1}) = 0.
\end{aligned}$$

Consequently $s(f;Q) \geq s(f;P)$. Applying this result repeatedly (if Q has more points than P), we conclude that

$$P \subset Q \Rightarrow s(f;P) \leq s(f;Q).$$

Analogously, it is proved that:

$$P \subset Q \Rightarrow S(f;P) \geq S(f;Q).$$

■

Corollary 7.5 *Let $f : [a,b] \to \mathbb{R}$ be bounded. For any partitions P, Q of $[a,b]$, we have $s(f;P) \leq S(f;Q)$.*

Proof: Indeed, the partition $P \cup Q$ refines both P and Q. Thus:

$$s(f;P) \underbrace{\leq}_{Thm.7.4} s(f;P\cup Q) \underbrace{\leq}_{(7.6)} S(f;P\cup Q) \underbrace{\leq}_{Thm.7.4} S(f;Q).$$

■

Remark: Let $f:[a,b]\to\mathbb{R}$ be bounded. Define:

$$\begin{aligned} m &= \inf\{f(x) : x\in[a,b]\}, \\ M &= \sup\{f(x) : x\in[a,b]\}. \end{aligned}$$

Then, for any partition $P=\{a=t_0<t_1<\cdots<t_n=b\}$ of $[a,b]$, we have:

$$m(b-a) \leq s(f;P) \leq S(f;P) \leq M(b-a). \tag{7.8}$$

Indeed, since $m\leq m_i\leq M_i\leq M$ for each i, then:

$$\begin{aligned} s(f;P) &= \sum_{i=1}^{n} m_i\Delta t_i \geq \sum_{i=1}^{n} m\Delta t_i = m\sum_{i=1}^{n}(t_i-t_{i-1}) \\ &= m\left[t_1-t_0+t_2-t_1+\cdots+t_n-t_{n-1}\right] \\ &= m(t_n-t_0) = m(b-a). \end{aligned}$$

Analogously, it is proved that $S(f;P)\leq M(b-a)$.

Let $\mathcal{P}$ be the set of all possible partitions of $[a,b]$. Then the set of lower sums $\sigma=\{s(f;P) : P\in\mathcal{P}\}$ is bounded above (by $M(b-a)$), and the set of upper sums $\Sigma=\{S(f;P) : P\in\mathcal{P}\}$ is bounded below (by $m(b-a)$).

Definition 7.6 *Let $f:[a,b]\to\mathbb{R}$ be bounded. We define the* lower integral $\underline{\int_a^b} f(x)\,dx$ *and the* upper integral $\overline{\int_a^b} f(x)\,dx$ *of the function f over $[a,b]$, by setting:*

$$\underline{\int_a^b} f(x)\,dx = \sup\sigma \quad \text{and} \quad \overline{\int_a^b} f(x)\,dx = \inf\Sigma.$$

From (7.8) and Corollary 7.5, it follows that:

$$m(b-a) \leq \underline{\int_a^b} f(x)\,dx \leq \overline{\int_a^b} f(x)\,dx \leq M(b-a). \tag{7.9}$$

Definition 7.7 *Let $f:[a,b]\to\mathbb{R}$ be bounded. We say that f is* (Riemann) integrable *when its lower integral is equal to its upper integral. The value of the integral, in this case, is this common value. We denote the integral of f by $\int_a^b f(x)\,dx$. Thus:*

$$\int_a^b f(x)\,dx = \underline{\int_a^b} f(x)\,dx = \overline{\int_a^b} f(x)\,dx.$$

Examples:

1) Let $f : [a, b] \to \mathbb{R}$, $f(x) = \lambda$ for all $x \in [a, b]$. Determine $\int_a^b f(x)\,dx$. Let $P = \{a = t_0 < t_1 < \cdots < t_n = b\}$ be a generic partition of $[a, b]$. Then

$$\begin{aligned} s(f;P) &= \sum_{i=1}^{n} m_i(t_i - t_{i-1}) \text{ where } m_i = \inf\{f(x) : x \in [t_{i-1}, t_i]\}, \\ S(f;P) &= \sum_{i=1}^{n} M_i(t_i - t_{i-1}) \text{ where } M_i = \sup\{f(x) : x \in [t_{i-1}, t_i]\}. \end{aligned}$$

Since $f(x) = \lambda$ for all $x \in [a, b]$, then $m_i = M_i = \lambda$ for all $i = 1, \cdots, n$. Hence

$$\begin{aligned} s(f;P) &= \lambda \sum_{i=1}^{n} (t_i - t_{i-1}) = \lambda(b - a), \\ S(f;P) &= \lambda \sum_{i=1}^{n} (t_i - t_{i-1}) = \lambda(b - a). \end{aligned}$$

It follows that

$$\sup\{s(f;P) : P \in \mathcal{P}\} = \inf\{S(f;P) : P \in \mathcal{P}\} = \lambda(b - a),$$

and therefore $\int_a^b f(x)\,dx = \lambda(b - a)$.

2) Example of a non-integrable function (Dirichlet's function). Let $f : [0, 1] \to \mathbb{R}$ be defined by

$$f(x) := \begin{cases} 1, & x \in \mathbb{Q} \\ 0, & x \in \mathbb{R}\backslash\mathbb{Q} \end{cases}$$

In this case, for any subinterval $[t_{i-1}, t_i]$ (with $t_i > t_{i-1}$), $m_i = 0$ and $M_i = 1$ (by density of $\mathbb{Q}$ and $\mathbb{R}\backslash\mathbb{Q}$). Consequently, for any partition P: $s(f;P) = \sum 0 \cdot \Delta t_i = 0$ and $S(f;P) = \sum 1 \cdot \Delta t_i = (b - a) = 1$. Thus,

$$\underline{\int_0^1} f(x)\,dx = \sup\{0\} = 0 \text{ and } \overline{\int_0^1} f(x)\,dx = \inf\{1\} = 1,$$

which implies that f is not Riemann integrable.

Theorem 7.8 (Riemann's Criterion for Integrability) *A bounded function $f : [a, b] \to \mathbb{R}$ is integrable if and only if for every $\varepsilon > 0$ given, a partition P of $[a, b]$ can be found such that $S(f;P) - s(f;P) < \varepsilon$.*

Proof: ($\Rightarrow$) Suppose f is integrable and let $\varepsilon > 0$ be given. By hypothesis,

$$\inf \Sigma = \sup \sigma = I,$$

where $\sigma = \{s(f;P) : P \in \mathcal{P}\}$ and $\Sigma = \{S(f;P) : P \in \mathcal{P}\}$. By the definition of sup and inf, for the given $\varepsilon > 0$, there exist $P_1, P_2 \in \mathcal{P}$ such that

$$I - \varepsilon/2 < s(f;P_1) \leq I \quad \text{and} \quad I \leq S(f;P_2) < I + \varepsilon/2.$$

Let $P = P_1 \cup P_2$. Since P is a refinement of both P_1 and P_2, by Theorem 7.4:

$$s(f;P_1) \leq s(f;P) \quad \text{and} \quad S(f;P) \leq S(f;P_2)$$

Combining these:

$$I - \varepsilon/2 < s(f;P_1) \leq s(f;P) \leq S(f;P) \leq S(f;P_2) < I + \varepsilon/2$$

Thus, $S(f;P) - s(f;P) < (I + \varepsilon/2) - (I - \varepsilon/2) = \varepsilon$.

($\Leftarrow$) Conversely, suppose that for every $\varepsilon > 0$ there exists $P_\varepsilon \in \mathcal{P}$ such that $S(f;P_\varepsilon) - s(f;P_\varepsilon) < \varepsilon$. We must prove that $\inf\Sigma = \sup\sigma$. We know $\sup\sigma \leq \inf\Sigma$. By definition of inf and sup:

$$\inf\Sigma \leq S(f;P_\varepsilon) < \varepsilon + s(f;P_\varepsilon) \leq \varepsilon + \sup\sigma.$$

Thus,

$$\inf\Sigma - \sup\sigma < \varepsilon.$$

We also know (from (7.9)) that $\sup\sigma \leq \inf\Sigma$, and consequently

$$\inf\Sigma - \sup\sigma \geq 0.$$

Hence, from the two lines above, for every $\varepsilon > 0$ we have

$$0 \leq \inf\Sigma - \sup\sigma < \varepsilon.$$

By the arbitrariness of $\varepsilon > 0$, the equality $\inf\Sigma = \sup\sigma$ must hold. ■

7.2 Criterion for Riemann Integrability

Definition 7.9 *Let $f : X \subset \mathbb{R} \to \mathbb{R}$ be a bounded function with $X \subset [a,b]$. We call the oscillation of f on the set X the number*

$$\omega_X = \omega(f,X) = \sup\{|f(x) - f(y)| : x, y \in X\}.$$

Lemma 7.10 *Let $f : X \to \mathbb{R}$ be a bounded function. Consider:*

$$m_X = \inf\{f(x) : x \in X\} \quad \textit{and} \quad M_X = \sup\{f(x) : x \in X\}.$$

Then: $\omega_X = M_X - m_X$.

Proof: If f is constant, $M_X = m_X$ and $\omega_X = 0$, so the proof is trivial. Suppose, then, f is not constant. We have

$$m_X \leq f(x) \leq M_X, \ \forall x \in X,$$
$$m_X \leq f(y) \leq M_X, \ \forall y \in X,$$

or

$$m_X \leq f(x) \leq M_X, \ \forall x \in X,$$
$$-M_X \leq -f(y) \leq -m_X, \ \ \forall y \in X,$$

and therefore

$$-(M_X - m_X) = m_X - M_X \leq f(x) - f(y) \leq M_X - m_X, \ \forall x, y \in X,$$

and since $M_X - m_X \geq 0$, it follows that:

$$|f(x) - f(y)| \leq M_X - m_X, \ \forall x, y \in X \tag{7.10}$$

It follows from (7.10) that $M_X - m_X$ is an upper bound for the set $\{|f(x) - f(y)| : x, y \in X\}$, and therefore

$$\omega_X \leq M_X - m_X. \tag{7.11}$$

We will now show that $M_X - m_X$ is the least upper bound. Indeed, given $\varepsilon > 0$, by definition of sup and inf, there exist $x_\varepsilon, y_\varepsilon \in X$ such that

$$f(x_\varepsilon) < m_X + \frac{\varepsilon}{2} \ \text{ and } \ f(y_\varepsilon) > M_X - \frac{\varepsilon}{2}. \tag{7.12}$$

Then from (7.12) it follows that

$$\begin{aligned} M_X - m_X &< (f(y_\varepsilon) + \varepsilon/2) - (f(x_\varepsilon) - \varepsilon/2) \\ &= f(y_\varepsilon) - f(x_\varepsilon) + \varepsilon \\ &\leq |f(y_\varepsilon) - f(x_\varepsilon)| + \varepsilon \\ &\leq \omega_X + \varepsilon. \end{aligned}$$

By the arbitrariness of $\varepsilon > 0$, it follows that

$$M_X - m_X \leq \omega_X, \tag{7.13}$$

and from (7.11) and (7.13), the desired result is proven. ∎

Corollary 7.11 *A function $f : [a, b] \to \mathbb{R}$ is integrable if and only if, given $\varepsilon > 0$, there exists a partition $P = \{t_0 < t_1 < \cdots, t_n\}$ of the interval $[a, b]$ such that*

$$\sum_{i=1}^{n} \omega_i \Delta t_i < \varepsilon,$$

where $\omega_i = \omega(f, [t_{i-1}, t_i]) = \sup\{|f(x) - f(y)| : x, y \in [t_{i-1}, t_i]\}$.

Proof: Follows immediately from Theorem 7.8, Lemma 7.10, and the fact that

$$\begin{aligned} S(f;P) - s(f;P) &= \sum_{i=1}^{n} M_i \Delta t_i - \sum_{i=1}^{n} m_i \Delta t_i \\ &= \sum_{i=1}^{n} (M_i - m_i) \Delta t_i \\ &= \sum_{i=1}^{n} \omega_i \Delta t_i. \end{aligned}$$

■

Theorem 7.12 *Every continuous function $f : [a, b] \to \mathbb{R}$ is integrable.*

Proof: Let $\varepsilon > 0$ be given. Since $[a, b]$ is compact, f is uniformly continuous on $[a, b]$ (Thm. 5.15). Hence, for $\frac{\varepsilon}{b-a} > 0$, there exists $\delta > 0$ such that if $x, y \in [a, b]$ and

$$|x - y| < \delta \Rightarrow |f(x) - f(y)| < \frac{\varepsilon}{b - a}. \tag{7.14}$$

Let us consider a partition $P = \{t_0, t_1, \cdots, t_n\}$ such that the length of the largest subinterval, which we denote by $\|P\|$ (the norm of P), does not exceed $\delta > 0$, i.e., $\|P\| = \max_i \Delta t_i < \delta$. In this way, given $x, y \in [t_{i-1}, t_i]$, we have:

$$|x - y| \leq (t_i - t_{i-1}) \leq \|P\| < \delta.$$

From (7.14), it follows that $|f(x) - f(y)| < \frac{\varepsilon}{b-a}$ for all $x, y \in [t_{i-1}, t_i]$. Therefore, $\frac{\varepsilon}{b-a}$ is an upper bound for the set $\{|f(x) - f(y)| : x, y \in [t_{i-1}, t_i]\}$, and consequently

$$\omega_i = \sup\{|f(x) - f(y)| : x, y \in [t_{i-1}, t_i]\} \leq \frac{\varepsilon}{b - a} \tag{7.15}$$

(Note: A strict inequality $<$ cannot be guaranteed by the sup)

Hence, by the proof of Corollary 7.11 and from (7.15), it follows that:

$$S(f;P) - s(f;P) = \sum_{i=1}^{n} \omega_i \Delta t_i \leq \sum_{i=1}^{n} \frac{\varepsilon}{b - a} \Delta t_i = \frac{\varepsilon}{b - a} \underbrace{\sum_{i=1}^{n} \Delta t_i}_{=(b-a)} = \varepsilon. \tag{7.16}$$

From (7.16) and Theorem 7.8, it follows that f is integrable. ■

7.3 Properties of Integrable Functions

Proposition 7.13 *Let $f, g : [a,b] \to \mathbb{R}$ be integrable functions. Then:*

(i) $f+g$ is integrable and $\int_a^b (f(x)+g(x))\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$.

(ii) For all $c \in \mathbb{R}$, the function cf is integrable and, furthermore, $\int_a^b (cf)(x)\,dx = c\int_a^b f(x)\,dx$.

(iii) If $f(x) \geq 0$ for all $x \in [a,b]$, then $\int_a^b f(x)\,dx \geq 0$. Equivalently, if $f(x) \leq g(x)$ for all $x \in [a,b]$, then $\int_a^b f(x)\,dx \leq \int_a^b g(x)\,dx$.

(iv) The function $|f(x)|$ is integrable and, furthermore, $\left|\int_a^b f(x)\,dx\right| \leq \int_a^b |f(x)|\,dx$. In particular, if $|f(x)| \leq k$ for all $x \in [a,b]$, then $\left|\int_a^b f(x)\,dx\right| \leq k(b-a)$.

(v) (Mean Value Theorem for Integrals) If f is continuous on $[a,b]$, there exists $c \in [a,b]$ such that $\int_a^b f(x)\,dx = f(c)(b-a)$.

Proof: (i) Let $P = \{t_0, t_1, \cdots, t_n\}$ be a generic partition of $[a,b]$. For every subinterval $[t_{i-1}, t_i] \subset [a,b]$, we have (using $m_i(f) = \inf f$ on I_i, etc.):

$$m_i(f) + m_i(g) \leq m_i(f+g) \quad \text{and} \quad M_i(f+g) \leq M_i(f) + M_i(g). \tag{7.17}$$

(Note: The original text had $\geq$ for the M_i inequality, which is incorrect.) Indeed, it suffices to prove that $(m_i(f) + m_i(g))$ is a lower bound for the set $\{(f+g)(x) : x \in [t_{i-1}, t_i]\}$.

$$\begin{aligned} m_i(f) &= \inf\{f(x) : x \in [t_{i-1}, t_i]\} \\ m_i(g) &= \inf\{g(x) : x \in [t_{i-1}, t_i]\} \end{aligned}$$

which implies

$$m_i(f) + m_i(g) \leq f(x) + g(x) = (f+g)(x), \ \forall x \in [t_{i-1}, t_i].$$

Taking the infimum over x gives $m_i(f) + m_i(g) \leq m_i(f+g)$. The proof for M_i is analogous. From this, it follows that for any partition P:

$$s(f;P) + s(g;P) \leq s(f+g;P) \leq S(f+g;P) \leq S(f;P) + S(g;P).$$

Since f and g are integrable, given $\varepsilon > 0$, there exist P_1, P_2 such that $S(f;P_1) - s(f;P_1) < \varepsilon/2$ and $S(g;P_2) - s(g;P_2) < \varepsilon/2$. Let $P = P_1 \cup P_2$. Then

$$\begin{aligned} S(f+g;P) - s(f+g;P) &\leq (S(f;P) + S(g;P)) - (s(f;P) + s(g;P)) \\ &= (S(f;P) - s(f;P)) + (S(g;P) - s(g;P)) \\ &\leq (S(f;P_1) - s(f;P_1)) + (S(g;P_2) - s(g;P_2)) \\ &< \varepsilon/2 + \varepsilon/2 = \varepsilon. \end{aligned}$$

By Theorem 7.8, $f+g$ is integrable. Furthermore,

$$\underline{\int}(f+g) \geq \sup(s(f;P)+s(g;P)) \geq \sup s(f;P) + \sup s(g;P) = \int f + \int g$$

$$\overline{\int}(f+g) \leq \inf(S(f;P)+S(g;P)) \leq \inf S(f;P) + \inf S(g;P) = \int f + \int g$$

Since $f+g$ is integrable, the lower and upper integrals are equal, forcing all inequalities to be equalities.

(ii) We have (for $c \geq 0$):

$$m_i(cf) = \inf\{cf(x)\} = c\inf\{f(x)\} = cm_i(f)$$

$$M_i(cf) = \sup\{cf(x)\} = c\sup\{f(x)\} = cM_i(f)$$

If $c < 0$:

$$m_i(cf) = \inf\{cf(x)\} = c\sup\{f(x)\} = cM_i(f)$$

$$M_i(cf) = \sup\{cf(x)\} = c\inf\{f(x)\} = cm_i(f)$$

Case (a): $c < 0$.

$$s(cf;P) = \sum m_i(cf)\Delta t_i = \sum cM_i(f)\Delta t_i = cS(f;P)$$

$$S(cf;P) = \sum M_i(cf)\Delta t_i = \sum cm_i(f)\Delta t_i = cs(f;P)$$

Taking the sup of $s(cf;P)$ and inf of $S(cf;P)$:

$$\underline{\int}(cf) = \sup_P\{cS(f;P)\} = c\inf_P\{S(f;P)\} = c\int f$$

$$\overline{\int}(cf) = \inf_P\{cs(f;P)\} = c\sup_P\{s(f;P)\} = c\int f$$

Since they are equal, cf is integrable and $\int cf = c\int f$. Case (b): $c \geq 0$.

$$s(cf;P) = cs(f;P) \quad \text{and} \quad S(cf;P) = cS(f;P)$$

Taking sup and inf yields $\underline{\int}(cf) = c\underline{\int} f = c\int f$ and $\overline{\int}(cf) = c\overline{\int} f = c\int f$. In both cases, cf is integrable and $\int(cf) = c\int f$.

(iii) If $f(x) \geq 0$ for all $x \in [a,b]$, then $m_i \geq 0$ for every subinterval. Thus

$$s(f;P) = \sum_{i=1}^{n} m_i \Delta t_i \geq 0, \forall P \in \mathcal{P}.$$

Consequently,

$$\int_a^b f(x)\,dx = \sup\{s(f;P) : P \in \mathcal{P}\} \geq 0.$$

If $f(x) \le g(x)$, then $h(x) = g(x) - f(x) \ge 0$. By (i) and the result just proved,

$$0 \le \int_a^b h(x)\,dx = \int_a^b (g(x) - f(x))\,dx = \int_a^b g(x)\,dx - \int_a^b f(x)\,dx,$$

i.e., $\int_a^b f(x)\,dx \le \int_a^b g(x)\,dx$.

(iv) Let $P \in \mathcal{P}$ be a partition. For any $x, y \in [t_{i-1}, t_i]$,

$$\omega_i(|f|) = \sup\{||f(x)| - |f(y)||\} \le \sup\{|f(x) - f(y)|\} = \omega_i(f)$$

(using the reverse triangle inequality). Thus,

$$S(|f|; P) - s(|f|; P) = \sum \omega_i(|f|)\Delta t_i \le \sum \omega_i(f)\Delta t_i = S(f; P) - s(f; P)$$

Since f is integrable, for any $\varepsilon > 0$, there exists P such that $S(f; P) - s(f; P) < \varepsilon$. For the same P, $S(|f|; P) - s(|f|; P) < \varepsilon$. By Theorem 7.8, $|f|$ is integrable. Furthermore, since $-|f(x)| \le f(x) \le |f(x)|$, $\forall x \in [a, b]$, it follows from item (iii) that

$$-\int_a^b |f(x)|\,dx \le \int_a^b f(x)\,dx \le \int_a^b |f(x)|\,dx,$$

which means $\left|\int_a^b f(x)\,dx\right| \le \int_a^b |f(x)|\,dx$. If $|f(x)| \le k$ for all $x \in [a, b]$, then

$$\left|\int_a^b f(x)\,dx\right| \le \int_a^b |f(x)|\,dx \le \int_a^b k\,dx = k(b - a).$$

(v) Let $m = \inf\{f(x) : x \in [a, b]\}$ and $M = \sup\{f(x) : x \in [a, b]\}$. Since f is continuous on $[a, b]$, it is integrable, and by (iii),

$$m(b - a) \le \int_a^b f(x)\,dx \le M(b - a).$$

Consequently,

$$m \le \frac{1}{b - a}\int_a^b f(x)\,dx \le M.$$

By the Extreme Value Theorem (5.13), f attains its m and M at points $x_m, x_M \in [a, b]$. The value $C = \frac{1}{b-a}\int f(x)dx$ is an intermediate value between $m = f(x_m)$ and $M = f(x_M)$. By the Intermediate Value Theorem (5.18), there exists $c \in [a, b]$ such that $f(c) = C$, i.e., $f(c) = \frac{1}{b-a}\int_a^b f(x)\,dx$. ■

7.4 The Fundamental Theorem of Calculus

Definition 7.14 *Let $f : [a, b] \to \mathbb{R}$ be an integrable function. Then for all $x \in [a, b]$, f is integrable on $[a, x]$. Consider the function $F : [a, b] \to \mathbb{R}$ defined by*

$$F(x) = \int_a^x f(t)\,dt. \tag{7.18}$$

F is called the indefinite integral of f.

Remarks:

(1a) We will adopt the convention that $\int_a^b f(y)\,dy = -\int_b^a f(y)\,dy$. The following property also holds:

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx,\ \forall c \in [a,b],$$

which we leave as an exercise.

(2a) If $f : [a,b] \to \mathbb{R}$ is bounded on $[a,b]$, then F is Lipschitz continuous. Indeed, since f is bounded, there exists $k > 0$ such that $|f(x)| \le k$, for all $x \in [a,b]$. Let $x, y \in [a,b]$. We have from (7.18) (assuming $y < x$):

$$\begin{aligned} |F(x) - F(y)| &= \left|\int_a^x f(t)\,dt - \int_a^y f(t)\,dt\right| \\ &= \left|\int_y^a f(t)\,dt + \int_a^x f(t)\,dt\right| \\ &= \left|\int_y^x f(t)\,dt\right| \\ &\le \int_y^x |f(t)|\,dt \le k|x-y|. \end{aligned}$$

(3a) F is uniformly continuous. We have from remark 2a that F is Lipschitz continuous, i.e., there exists $k > 0$ such that

$$|F(x) - F(y)| \le k|x-y|,\ \forall x, y \in [a,b].$$

Hence, given $\varepsilon > 0$, take $\delta = \frac{\varepsilon}{k}$. Therefore, if $x, y \in [a,b]$ are such that $|x-y| < \delta$, we have:

$$|F(x) - F(y)| \le k|x-y| < k\delta = k\frac{\varepsilon}{k} = \varepsilon,$$

which proves the claim.

Example: Let $f : [0,2] \to \mathbb{R}$ be defined by:

$$f(t) = \begin{cases} 0, & \text{if } 0 \le t < 1 \\ 1, & \text{if } 1 \le t \le 2, \end{cases}$$

and consider $F : [0,2] \to \mathbb{R}$ defined by $F(x) = \int_0^x f(t)\,dt$. Find the explicit form of the function F. Solution: Since f is a piecewise function, we must divide it into two cases:

(1) For $0 \le x < 1$. Then $t \in [0,x] \subset [0,1)$.

$$F(x) = \int_0^x f(t)\,dt = \int_0^x 0\,dt = 0.$$

(2) For $1 \leq x \leq 2$.

$$F(x) = \int_0^x f(t)\,dt = \int_0^1 f(t)\,dt + \int_1^x f(t)\,dt$$
$$= \int_0^1 0\,dt + \int_1^x 1\,dt = 0 + (x-1) = x - 1.$$

Thus:

$$F(x) = \begin{cases} 0, & 0 \leq x < 1 \\ x - 1, & 1 \leq x \leq 2. \end{cases}$$

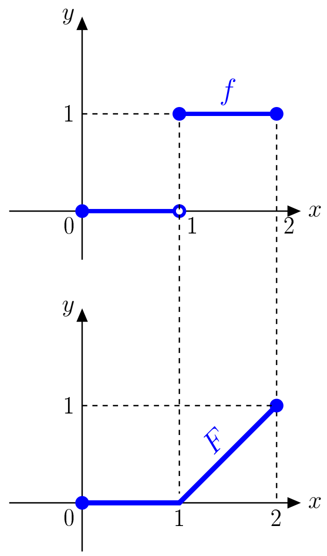


Figure 7.3:

Observing the graph, we see that: (i) f is not continuous at $x = 1$. (ii) F is continuous at $x = 1$. (iii) F is not differentiable at $x = 1$.

Analytically,

$$F'_+(1) = \lim_{h \to 0_+} \frac{F(1+h) - F(1)}{h} = \lim_{h \to 0, h>0} \frac{((1+h)-1) - 0}{h} = \lim_{h \to 0_+} \frac{h}{h} = 1,$$
$$F'_-(1) = \lim_{h \to 0_-} \frac{F(1+h) - F(1)}{h} = \lim_{h \to 0, h<0} \frac{0 - 0}{h} = 0,$$

which implies that $F'(1)$ does not exist.

The example above motivates the following theorem:

Theorem 7.15 *(Fundamental Theorem of Calculus, Part 1) Let $f : [a, b] \to \mathbb{R}$ be integrable. If f is continuous at a point $c \in [a, b]$, then $F : [a, b] \to \mathbb{R}$ defined by $F(x) = \int_a^x f(t)\,dt$ is differentiable at $x = c$ and $F'(c) = f(c)$.*

Proof: We want to show that $F'(c) = f(c)$, i.e.,

$$\lim_{h\to 0}\frac{F(c+h)-F(c)}{h} = f(c),$$

or, given $\varepsilon > 0$, we must exhibit $\delta > 0$ such that

$$\text{if } |h| < \delta \text{ (and } c+h \in [a,b]) \Rightarrow \left|\frac{F(c+h)-F(c)}{h} - f(c)\right| < \varepsilon. \tag{7.19}$$

Since f is continuous at $x = c$, we have that for the given $\varepsilon > 0$, there exists $\delta > 0$ such that if $t \in [a,b]$ and

$$|t-c| < \delta \Rightarrow |f(t)-f(c)| < \varepsilon. \tag{7.20}$$

We have two cases to consider (assuming c is an interior point):

(i) $0 < h < \delta$ and $c+h \in [a,b]$. If $t \in [c, c+h]$, then $|t-c| \le h < \delta$, so (7.20) holds.

$$\begin{aligned}
\left|\frac{F(c+h)-F(c)}{h} - f(c)\right| &= \left|\frac{\int_a^{c+h} f(t)\,dt - \int_a^c f(t)\,dt - hf(c)}{h}\right| \\
&= \frac{1}{h}\left|\int_c^{c+h} f(t)\,dt - hf(c)\right| \\
&= \frac{1}{h}\left|\int_c^{c+h} f(t)\,dt - \int_c^{c+h} f(c)\,dt\right| \\
&\le \frac{1}{h}\int_c^{c+h} |f(t)-f(c)|\,dt \\
&\underbrace{<}_{(7.20)} \frac{1}{h}\int_c^{c+h} \varepsilon\,dt = \frac{1}{h}(\varepsilon\cdot h) = \varepsilon,
\end{aligned}$$

which shows that $F'_+(c) = f(c)$.

(ii) $-\delta < h < 0$ and $c+h \in [a,b]$. If $t \in [c+h, c]$, then $|t-c| \le |h| < \delta$, so (7.20) holds. Let $h = -|h|$.

$$\begin{aligned}
\left|\frac{F(c+h)-F(c)}{h} - f(c)\right| &= \left|\frac{\int_a^{c+h} f(t)\,dt - \int_a^c f(t)\,dt - hf(c)}{h}\right| \\
&= \frac{1}{|h|}\left|-\int_{c+h}^{c} f(t)\,dt - hf(c)\right| \\
&= \frac{1}{|h|}\left|-\int_{c+h}^{c} f(t)\,dt + |h|f(c)\right| \\
&= \frac{1}{|h|}\left|-\int_{c+h}^{c} f(t)\,dt + \int_{c+h}^{c} f(c)\,dt\right| \\
&\le \frac{1}{|h|}\int_{c+h}^{c} |f(c)-f(t)|\,dt \\
&\underbrace{<}_{(7.20)} \frac{1}{|h|}\int_{c+h}^{c} \varepsilon\,dt = \frac{1}{|h|}(\varepsilon\cdot|h|) = \varepsilon,
\end{aligned}$$

which proves that $F'_-(c) = f(c)$. Thus, $F'(c) = f(c)$. ■

Corollary 7.16 *Given $f : [a, b] \to \mathbb{R}$ continuous, there exists $F : [a, b] \to \mathbb{R}$ differentiable such that $F' = f$.*

Proof: If f is continuous on $[a, b]$, then f is integrable on $[a, b]$ (Thm. 7.12). Define $F : [a, b] \to \mathbb{R}$ by $F(x) = \int_a^x f(t)\, dt$. By Theorem 7.15 (since f is continuous at *every* point $c \in [a, b]$), F is differentiable on $[a, b]$ and $F'(x) = f(x)$ for all $x \in [a, b]$. ■

Definition 7.17 *A primitive (or antiderivative) of a function $f : [a, b] \to \mathbb{R}$ is a differentiable function $F : [a, b] \to \mathbb{R}$ such that $F' = f$.*

Remarks: (1a) Corollary 7.16 tells us that every continuous function on $[a, b]$ possesses a primitive.

(2a) Not every integrable function f possesses a primitive. Indeed, let F be a primitive of f. Then $F'(x) = f(x)$ for all $x \in [a, b]$. We have, therefore, that F is differentiable on $[a, b]$, and thus $F' = f$ cannot have discontinuities of the first kind on $[a, b]$ (by Darboux's Theorem, 6.13, which states that derivatives satisfy the Intermediate Value Property, even if they are not continuous). Thus, a function, in order to have a primitive, cannot have jump discontinuities.

Proof that the derivative of a differentiable function does not have a discontinuity of the first kind

Let $f : \mathbb{R} \to \mathbb{R}$ be a function differentiable at a point $a \in \mathbb{R}$. We want to prove that the derivative function f' cannot have a discontinuity of the first kind (i.e., a jump) at the point a.

Definitions

- Since f is **differentiable** at a, the limit

$$f'(a) = \lim_{h \to 0} \frac{f(a + h) - f(a)}{h}$$

  exists. This implies, in particular, that f is continuous at a.

- We say that a function g has a **discontinuity of the first kind** (or jump) at a if the lateral limits exist, but are different:

$$\lim_{x \to a^+} g(x) = L_+ \quad \text{and} \quad \lim_{x \to a^-} g(x) = L_-, \quad \text{with } L_+ \neq L_-.$$

Proof (Completion)

Proof: Let us assume, by contradiction, that f' has a discontinuity of the first kind at a. This means that the lateral limits exist and are different:

$$\lim_{x\to a^+} f'(x) = L_+ \quad \text{and} \quad \lim_{x\to a^-} f'(x) = L_-, \quad \text{with } L_+ \neq L_-.$$

Let us analyze the limit from the right. Consider any $x > a$. Since f is differentiable on $\mathbb{R}$, f is continuous on $[a,x]$ and differentiable on (a,x). By the Mean Value Theorem (Theorem 6.12), there exists a $c_x \in (a,x)$ such that:

$$\frac{f(x)-f(a)}{x-a} = f'(c_x).$$

Now, let us take the limit as $x \to a^+$. The left side is, by definition, the right-hand derivative of f at a:

$$\lim_{x\to a^+} \frac{f(x)-f(a)}{x-a} = \lim_{x\to a^+} f'(c_x).$$

Since f is differentiable at a, this limit must be $f'(a)$.

$$f'(a) = \lim_{x\to a^+} f'(c_x).$$

As $x \to a^+$, we have $c_x \to a^+$ (because $a < c_x < x$). Since we assumed the limit $\lim_{t\to a^+} f'(t) = L_+$ exists, it follows that:

$$f'(a) = L_+.$$

Now, let us analyze the limit from the left. Consider $x < a$. By the MVT on $[x,a]$, there exists $d_x \in (x,a)$ such that:

$$\frac{f(a)-f(x)}{a-x} = f'(d_x) \quad \Longrightarrow \quad \frac{f(x)-f(a)}{x-a} = f'(d_x).$$

Taking the limit as $x \to a^-$:

$$\lim_{x\to a^-} \frac{f(x)-f(a)}{x-a} = \lim_{x\to a^-} f'(d_x).$$

The left side is $f'(a)$. As $x \to a^-$, we have $d_x \to a^-$. Since we assumed $\lim_{t\to a^-} f'(t) = L_-$ exists:

$$f'(a) = L_-.$$

We have thus concluded that $L_+ = f'(a)$ and $L_- = f'(a)$, which implies $L_+ = L_-$. This contradicts our hypothesis that $L_+ \neq L_-$ (the definition of a jump discontinuity). Therefore, f' cannot have a discontinuity of the first kind. ■

Final Conclusion

The assumption that f' has a discontinuity of the first kind at a leads to a contradiction. Therefore, if f is differentiable at a, then the derivative f' cannot have a discontinuity of the first kind.

On the other hand, as we saw in the example (Figure 7.3), there exist functions with discontinuities of the first kind that are integrable. The function from that example does not admit a primitive on any interval that contains $x = 1$ in its interior.

Example: Consider the function:

$$f(x) = \begin{cases} 2x\sin(1/x) - \cos(1/x), & x \neq 0, \\ 0, & x = 0. \end{cases}$$

Verify: (a) If f is continuous. (b) If f admits a primitive.

Solution: (a) Note that $\lim_{x\to 0} f(x)$ does not exist (due to the $\cos(1/x)$ term). Therefore, f is not continuous at $x = 0$. (b) f admits a primitive, which is given by:

$$F(x) = \begin{cases} x^2\sin(1/x), & x \neq 0, \\ 0, & x = 0. \end{cases}$$

We check the derivative $F'(x)$: For $x \neq 0$, by the product and chain rules:

$$F'(x) = 2x \cdot \sin(1/x) + x^2(\cos(1/x) \cdot (-1/x^2)) = 2x\sin(1/x) - \cos(1/x) = f(x).$$

For $x = 0$, by the definition of the derivative:

$$F'(0) = \lim_{h\to 0} \frac{F(0+h) - F(0)}{h} = \lim_{h\to 0} \frac{h^2\sin(1/h) - 0}{h} = \lim_{h\to 0} h\sin(1/h) = 0.$$

(The last limit is 0 by the Squeeze Theorem, since $-|h| \leq h\sin(1/h) \leq |h|$). Since $f(0) = 0$, we have $F'(x) = f(x)$ for all x, which proves the claim.

Lemma 7.18 *Note that if $f : [a, b] \to \mathbb{R}$ admits one primitive, then f possesses an infinity of them.*

Proof: Indeed, let $F : [a, b] \to \mathbb{R}$ be a primitive of f. Then $F' = f$. Consider a family of functions $\{F_i\}_{i\in I}$ where $F_i : [a, b] \to \mathbb{R}$ is defined by $F_i = F + c_i$, $c_i \in \mathbb{R}$, $\forall i \in I$. Then, $F_i' = (F + c_i)' = F' = f$, which implies that F_i is a primitive of f for all $i \in I$. ■

Lemma 7.19 *Any two primitives of $f : [a,b] \to \mathbb{R}$ differ by a constant.*

Proof: Let F_1 and F_2 be two primitives of f. Then $F_1' = f$ and $F_2' = f$. Consider the function $H(x) = F_1(x) - F_2(x)$. Then $H'(x) = F_1'(x) - F_2'(x) = f(x) - f(x) = 0$ for all $x \in (a,b)$. By the Mean Value Theorem (Exercise (b) after Thm 6.12), $H(x)$ must be constant on $[a,b]$. Thus, $F_1(x) - F_2(x) = k$ for some constant k. ■

Proposition 7.20 *Consider $f : [a,b] \to \mathbb{R}$ (integrable) and let $F : [a,b] \to \mathbb{R}$ be a primitive of f, with F being C^1 (i.e., F' is continuous). Then:*

$$\int_a^b f(x)\,dx = F(b) - F(a).$$

(Note: $F' = f$. The assumption $F \in C^1$ means f is continuous)

Proof: Since $F \in C^1$, $F' = f$ is continuous on $[a,b]$. By Theorem 7.12, f is integrable on $[a,b]$. Let us define $\rho : [a,b] \to \mathbb{R}$ by

$$\rho(x) = \int_a^x f(t)\,dt = \int_a^x F'(t)\,dt.$$

By the Fundamental Theorem of Calculus (Part 1, Theorem 7.15), since F' is continuous:

$$\rho'(x) = F'(x), \ \forall x \in [a,b].$$

Thus, ρ and F are both primitives of f. By Lemma 7.19, they differ by a constant:

$$\rho(x) - F(x) = k, \ \forall x \in [a,b], \text{ where } k \text{ is a constant.} \tag{7.21}$$

In particular, for $x = a$: $\rho(a) - F(a) = k$.

$$\underbrace{\int_a^a F'(t)\,dt}_{=0} - F(a) = k \Rightarrow k = -F(a). \tag{7.22}$$

Substituting (7.22) into (7.21) yields

$$\rho(x) - F(x) = -F(a), \ \forall x \in [a,b],$$

which implies $\rho(x) = F(x) - F(a)$, i.e.,

$$\int_a^x F'(t)\,dt = F(x) - F(a), \ \forall x \in [a,b].$$

In particular, for $x = b$, we have

$$\int_a^b F'(t)\,dt = F(b) - F(a).$$

■

Theorem 7.21 *[Fundamental Theorem of Calculus, Part 2] If an integrable function $f : [a, b] \to \mathbb{R}$ possesses a primitive $F : [a, b] \to \mathbb{R}$, then*

$$\int_a^b f(t)\,dt = F(b) - F(a).$$

In other words: if a function $F : [a, b] \to \mathbb{R}$ has an integrable derivative F', then

$$\int_a^b F'(t)\,dt = F(b) - F(a).$$

Proof: For any partition $P = \{a = t_0 < t_1 < \cdots < t_n = b\}$ of $[a, b]$, we have a telescoping sum:

$$F(b) - F(a) = F(t_n) - F(t_0) = \sum_{i=1}^{n} [F(t_i) - F(t_{i-1})]. \tag{7.23}$$

On the other hand, F is differentiable on $[a, b]$ (by definition of primitive) and thus continuous. Applying the Mean Value Theorem (6.12) to F on each $[t_{i-1}, t_i]$, there exists $c_i \in (t_{i-1}, t_i)$ such that

$$F(t_i) - F(t_{i-1}) = F'(c_i)(t_i - t_{i-1}) = f(c_i)\Delta t_i. \tag{7.24}$$

Combining (7.23) and (7.24) we have

$$F(b) - F(a) = \sum_{i=1}^{n} f(c_i)\Delta t_i. \tag{7.25}$$

Let $m_i = \inf f$ and $M_i = \sup f$ on $[t_{i-1}, t_i]$. We have

$$m_i \le f(c_i) \le M_i,$$

and therefore

$$\sum_{i=1}^{n} m_i \Delta t_i \le \sum_{i=1}^{n} f(c_i)\Delta t_i \le \sum_{i=1}^{n} M_i \Delta t_i. \tag{7.26}$$

From (7.25) and (7.26), it follows that for *any* partition P:

$$s(f; P) \le F(b) - F(a) \le S(f; P).$$

This means $(F(b) - F(a))$ is an upper bound for the set of all lower sums σ, and $(F(b) - F(a))$ is a lower bound for the set of all upper sums Σ. Hence,

$$\underline{\int_a^b} f(t)\,dt = \sup \sigma \le F(b) - F(a) \le \inf \Sigma = \overline{\int_a^b} f(t)\,dt. \tag{7.27}$$

Since f is integrable, $\underline{\int_a^b} f(t)\,dt = \overline{\int_a^b} f(t)\,dt = \int_a^b f(t)\,dt$. From (7.27), we conclude that

$$\int_a^b f(t)\,dt = F(b) - F(a),$$

QED. ■

Example: Calculate $\int_1^3 x^2\,dx$ using the Fundamental Theorem of Calculus. We have $f(x) = x^2$, which is continuous and therefore integrable on $[1,3]$. $F(x) = \frac{1}{3}x^3$ is a primitive of f since $F' = f$. Hence:

$$\int_1^3 x^2 dx = \left[\frac{1}{3}x^3\right]_1^3 = \frac{27}{3} - \frac{1}{3} = \frac{26}{3}.$$

Calculate $\int_0^\pi \sin x\,dx$. Since $f(x) = \sin x$ is continuous, it is integrable on $[0,\pi]$. $F(x) = -\cos x$ is a primitive of f. Hence:

$$\int_0^\pi \sin x\,dx = [-\cos x]_0^\pi = (-\cos\pi) - (-\cos 0) = -(-1) - (-1) = 1 + 1 = 2.$$

Calculate $\int_0^{\pi/2} \sin x \cos x\,dx$. We have $f(x) = \sin x\ \cos x$ is continuous and therefore integrable on $[0,\pi/2]$. $F(x) = \frac{1}{2}(\sin x)^2$ is a primitive of f. Hence:

$$\begin{aligned}\int_0^{\pi/2} \sin x\ \cos x\,dx &= \left[\frac{1}{2}(\sin x)^2\right]_0^{\pi/2} \\ &= \frac{1}{2}\left(\sin(\pi/2)\right)^2 - \frac{1}{2}\left(\sin 0\right)^2 = \frac{1}{2}(1)^2 - \frac{1}{2}(0)^2 = \frac{1}{2}.\end{aligned}$$

7.5 Classical Formulas of Differential and Integral Calculus

Theorem 7.22 *[Change of Variable] Let $f : [a,b] \to \mathbb{R}$ be continuous and $g : [c,d] \to \mathbb{R}$ be differentiable with g' integrable, and $g([c,d]) \subset [a,b]$. Then:*

$$\int_{g(c)}^{g(d)} f(x)\,dx = \int_c^d f(g(t))\,g'(t)\,dt.$$

Proof: Since f is a continuous function, it possesses a primitive $F : [a,b] \to \mathbb{R}$, by Corollary 7.16. Thus, by virtue of the Fundamental Theorem of Calculus:

$$\int_{g(c)}^{g(d)} f(x)\,dx = F(g(d)) - F(g(c)). \tag{7.28}$$

On the other hand, by the Chain Rule (6.7):

$$(F \circ g)'(t) = F'(g(t)) \cdot g'(t) = f(g(t)) \cdot g'(t);\ \forall t \in [c, d].$$

In this way, the map $F \circ g : [c, d] \to \mathbb{R}$ is a primitive of the function $t \mapsto f(g(t))g'(t)$. (This function is integrable as f, g' are integrable and g is continuous). Thus, again by the Fundamental Theorem of Calculus (7.21):

$$\int_c^d f(g(t))g'(t)\,dt = (F \circ g)(d) - (F \circ g)(c) = F(g(d)) - F(g(c)). \tag{7.29}$$

Hence, from (7.28) and (7.29) we have the desired result. ■

Application: Calculate $\int_0^1 \sqrt{1 - x^2}\,dx$. Note that a good change of variable is given by $x = \sin t$, since

$$\sqrt{1 - x^2} = \sqrt{1 - \sin^2 t} = \cos t.$$

In truth, keeping the notation of the previous Theorem, we have:

$$g : [0, \pi/2] \to \mathbb{R},\ t \mapsto g(t) = \sin t.$$

Then,

$$g(0) = 0 \ \text{ and } \ g(\pi/2) = 1.$$

$g'(t) = \cos t$ (which is integrable).

$$g([0, \pi/2]) = [0, 1].$$

Let $f(x) = \sqrt{1 - x^2}$ on $[0, 1]$. Hence,

$$\begin{aligned}
\int_0^1 \sqrt{1 - x^2}\,dx &= \int_0^{\pi/2} f(g(t))g'(t)\,dt \\
&= \int_0^{\pi/2} \sqrt{1 - \sin^2 t}\,\cos t\,dt \\
&= \int_0^{\pi/2} \cos^2 t\,dt \quad (\text{since } \cos t \geq 0 \text{ on } [0, \pi/2]) \\
&= \frac{1}{2} \int_0^{\pi/2} (1 + \cos(2t))\,dt \\
&= \frac{1}{2} \left[t + \frac{1}{2} \sin(2t) \right]_0^{\pi/2} \\
&= \frac{1}{2} \left((\frac{\pi}{2} + \frac{1}{2} \sin \pi) - (0 + \frac{1}{2} \sin 0) \right) = \frac{\pi}{4}.
\end{aligned}$$

Theorem 7.23 *[Integration by Parts] If $f, g : [a, b] \to \mathbb{R}$ have integrable derivatives, then:*

$$\int_a^b f(t)g'(t)\,dt = [f(t)g(t)]_a^b - \int_a^b f'(t)g(t)\,dt,$$

where

$$[f(t)g(t)]_a^b = f(b)g(b) - f(a)g(a).$$

Proof: Note that (fg) is a primitive of $f'g + fg'$, i.e., $(fg)' = f'g + fg'$ (by the product rule). Since f', g' are integrable and f, g are continuous (thus integrable), $f'g + fg'$ is integrable. Integrating this identity and applying the Fundamental Theorem of Calculus (7.21) gives:

$$\int_a^b (fg)'(t)dt = \int_a^b (f'(t)g(t) + f(t)g'(t))dt$$

$$[f(t)g(t)]_a^b = \int_a^b f'(t)g(t)dt + \int_a^b f(t)g'(t)dt$$

Rearranging gives the desired result. ∎

Application: Evaluate $\int_0^1 te^t\,dt$. Define

$$g'(t) = e^t \quad \text{and} \quad f(t) = t.$$

Then,

$$g(t) = e^t \quad \text{and} \quad f'(t) = 1.$$

(Both f' and g' are integrable). Hence,

$$\begin{aligned} \int_0^1 te^t\,dt &= [f(t)g(t)]_0^1 - \int_0^1 f'(t)g(t)\,dt \\ &= [te^t]_0^1 - \int_0^1 1 \cdot e^t\,dt \\ &= (1 \cdot e^1 - 0 \cdot e^0) - [e^t]_0^1 \\ &= e - (e^1 - e^0) = e - (e - 1) = 1. \end{aligned}$$

List of Exercises: Riemann and Improper Integrals

1ª Question Define $f : [0,1] \to$ by setting $f(0) = 0$ and $f(x) = \frac{1}{2^n}$ if $\frac{1}{2^{n+1}} < x \leq \frac{1}{2^n}$, $n \in \mathbb{N} \cup \{0\}$. Prove that f is integrable and calculate $\int_0^1 f(x)dx$.

2ª Question Let $f : [-a, a] \to$ be integrable. If f is an odd function, prove that $\int_{-a}^{a} f(x)dx = 0$. If, however, f is even, prove that $\int_{-a}^{a} f(x)dx = 2\int_0^a f(x)dx$.

3ª Question Let f be a function defined on any non-trivial interval I, integrable on any closed and bounded interval contained in I, and let $\alpha : J \to I$ be differentiable at $x_0 \in J$. Given $a \in I$, let $G : J \to$ be the function given by

$$G(x) = \int_a^{\alpha(x)} f(t)dt.$$

If f is continuous at $\alpha(x_0)$, then G is differentiable at x_0, with

$$G'(x_0) = \alpha'(x_0)f(\alpha(x_0)).$$

(Note: Changed variable of integration to t in $G(x)$ for clarity, as per standard usage of FTC with variable limits).

4ª Question Let $F : [0, +\infty) \to$ be given by $F(x) = \int_x^{2x} e^{-t^2}dt$. Find the relative extremes, absolute extremes, and inflection points of F.

5ª Question Let $f : [a, b] \to$ be an integrable function, with $f(x) \geq 0$ for all $x \in [a, b]$. If f is continuous at the point $c \in [a, b]$ and $f(c) > 0$, prove that $\int_a^b f(x)dx > 0$.

6ª Question Prove that if $f, g : [a, b] \to$ are continuous then

$$\left[\int_a^b f(x)g(x)dx\right]^2 \leq \int_a^b f(x)^2dx \int_a^b g(x)^2dx.$$

(Schwarz Inequality.)

7ª Question Let $f : [a, b] \to$ be a bounded function. If f is integrable on every interval $[c, b]$, with $a < c < b$, then it is integrable on $[a, b]$.

8ª Question Prove that the function $f : [0, 1] \to$ defined by $f(0) = 1$ and $f(x) = \sin \frac{1}{x}$ if $0 < x \leq 1$ is integrable on $[0, 1]$.

9ª Question Prove that, for any non-negative integers m and n,

$$\int_0^1 x^m (1-x)^n dx = \frac{m!n!}{(m+n+1)!}.$$

10ª Question Study the convergence of the following improper integrals:

(a) $\displaystyle\int_0^{+\infty} \frac{\cos x}{x^2} dx$ (b) $\displaystyle\int_0^{+\infty} \frac{\sin x}{x} dx$ (c) $\displaystyle\int_0^{+\infty} \frac{|\sin x|}{x} dx$ (d) $\displaystyle\int_{-1}^{+\infty} \frac{dx}{\sqrt{|x(1-x^2)|}}$

11ª Question Study the convergence of the following integrals and, if they converge, calculate their value:

(a) $\displaystyle\int_1^{+\infty} \frac{dx}{x(1+x^2)}$ (b) $\displaystyle\int_2^{+\infty} \frac{dx}{x^2-1}$ (c) $\displaystyle\int_{-\infty}^0 xe^x dx$

(d) $\displaystyle\int_0^1 x|\log x| dx$ (e) $\displaystyle\int_0^{\pi} \frac{dx}{x+\cos x}$ (f) $\displaystyle\int_0^{\pi} \frac{\cos x}{1+\cos^2 x} dx$

(g) $\displaystyle\int_1^{+\infty} \frac{dx}{x\sqrt{x^2-1}}$ (h) $\displaystyle\int_{-2}^{2} \frac{x^2 dx}{\sqrt{4-x^2}}$ (i) $\displaystyle\int_1^{+\infty} \frac{e^{-x}}{1+e^x} dx$

(j) $\displaystyle\int_0^{+\infty} |x-3|e^{-x} dx$ (k) $\displaystyle\int_0^{+\infty} xe^{|x-2|} dx$ (l) $\displaystyle\int_{-1}^{1} \sqrt{\frac{1+x}{1-x}} dx$

12ª Question (Euler's Gamma and Beta Functions). Prove that for given $t, u, v \in (0, +\infty)$, the following integrals are convergent:

$$(a) \quad \Gamma(t) = \int_0^{+\infty} x^{t-1} e^{-x} dx \qquad (b) \quad \beta(u, v) = \int_0^1 x^{u-1}(1-x)^{v-1} dx$$

13ª Question Prove that $\Gamma(t+1) = t\Gamma(t)$, for all $t > 0$.

14ª Question Prove that $\Gamma(n+1) = n!$ for every integer $n \geq 0$.

15ª Question Taking the Γ function into account and knowing that $\Gamma(\frac{1}{2}) = \sqrt{\pi}$ calculate:

(a) $\displaystyle\int_0^{+\infty} x^2 e^{-x^2} dx$ (b) $\displaystyle\int_0^{+\infty} 3^{-4x^2} dx$ (c) $\displaystyle\int_{-\infty}^{+\infty} x^2 e^{-|x-1|} dx$

(d) $\displaystyle\int_0^1 x^2 \log^4 x dx$ (e) $\displaystyle\int_{-\infty}^{+\infty} x^3 e^{-x^2} dx$ (f) $\displaystyle\int_0^{+\infty} (x-3)e^{-x^2} dx$

Chapter 8

Series of Real Numbers

We will extend the operation of addition, thus far defined for a finite number of real numbers, in order to assign meaning to an equality of the type:

$$a_1 + a_2 + \cdots + a_n + \cdots = a,$$

in which the left-hand member is a 'sum' with an infinite number of terms.

It is clear that it does not make sense to sum an infinite sequence of real numbers. What the left-hand member of the equality above expresses is the limit:

$$\lim_{n\to\infty} (a_1 + a_2 + \cdots + a_n) = a.$$

The statement that translates the meaning of the equality above is the following: Given $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that $|(a_1 + a_2 + \cdots + a_n) - a| < \varepsilon$, $\forall n \geq n_0$.

We define, therefore, 'infinite sums' by means of limits. This being the case, some sums can be performed and others cannot, since not every sequence has a limit.

Instead of 'infinite sum', we will use the word series.

The main problem in the theory of series is to determine which ones are convergent and which are not.

Let $(a_n)_{n\in\mathbb{N}}$ be a sequence of real numbers. From it, we form a new sequence $(s_n)_{n\in\mathbb{N}}$ given by:

$$s_1 = a_1,\ s_2 = a_1 + a_2,\ \cdots, s_n = a_1 + a_2 + \cdots + a_n.$$

We call $(s_n)_{n\in\mathbb{N}}$ the sequence of partial sums of $\sum_{n=1}^{\infty} a_n$, the series itself, and the term a_n is the general term of the series.

Definition 8.1 *If the limit exists:*

$$s = \lim_{n\to\infty} s_n = \lim_{n\to\infty} (a_1 + a_2 + \cdots + a_n),$$

we will say that the series $\sum_{n=1}^{\infty} a_n$ is convergent and the limit will be called the sum of the series. In this case, we write

$$s = \sum_{n=1}^{\infty} a_n = a_1 + a_2 + \cdots + a_n + \cdots .$$

(Note: $\sum_{i=1}^{n} a_n$ in the original was corrected to $\sum_{n=1}^{\infty} a_n$)

If the sequence of partial sums does not converge, we will say that the series $\sum_{n=1}^{\infty} a_n$ is divergent.

Example 1: Consider $a_n = \left(\frac{1}{2}\right)^n$, $n \in \mathbb{N}$. *(Note: The sum is $\sum_{n=0}^{\infty}(1/2)^n$ based on S_{n+1})* Let $a_n = (1/2)^{n-1}$ for $n \geq 1$.

$$S_n = 1 + \frac{1}{2} + \frac{1}{2^2} + \cdots + \frac{1}{2^{n-1}}.$$

This is the sum of the terms of a geometric progression where $a_1 = 1$ and $q = \frac{1}{2}$.

Let's recall that:

$$S_n = \frac{a_1(1-q^n)}{1-q}.$$

Hence:

$$S_n = \frac{1-\left(\frac{1}{2}\right)^n}{1-\frac{1}{2}} = 2(1-(1/2)^n) \to 2, \quad \text{as } n \to \infty.$$

So, $\sum_{n=1}^{\infty} a_n$ converges and $\sum_{n=1}^{\infty} a_n = 2$. *(Note: The original text had S_{n+1} and $\sum_{n=1}^{\infty} a_n = 2$, which implies the sum starts at $n = 0$ or $n = 1$ with $a_n = (1/2)^{n-1}$. I've adjusted to $a_n = (1/2)^{n-1}$ for $n \geq 1$)*

Example 2: Let $a_n = 1$, for all $n \in \mathbb{N}$. In this case,

$$s_n = 1 + 1 + \cdots + 1 = n \to +\infty.$$

So $\sum_{n=1}^{\infty} a_n$ diverges.

Proposition 8.2 *If $\sum_{n=1}^{\infty} a_n$ is a convergent series, then $\lim_{n\to+\infty} a_n = 0$.*

Proof: Let $s_n = a_1 + a_2 + \cdots + a_n$. Then, there exists $s \in \mathbb{R}$ such that

$$\lim_{n\to+\infty} s_n = s.$$

Evidently, we also have

$$\lim_{n\to+\infty} s_{n-1} = s.$$

Hence,

$$\lim_{n\to+\infty} a_n = \lim_{n\to+\infty} (s_n - s_{n-1}) = \lim_{n\to+\infty} s_n - \lim_{n\to+\infty} s_{n-1} = s - s = 0.$$

■

The converse of the above proposition is false. The classic counter-example is given by the harmonic series $\sum_{n=1}^{\infty} \frac{1}{n}$. We have:

$$\lim_{n\to+\infty} \frac{1}{n} = 0.$$

However, the series diverges. Indeed, consider the subsequence of partial sums s_{2^n}:

$$\begin{aligned}
s_{2^n} &= 1+\frac{1}{2}+\left(\frac{1}{3}+\frac{1}{4}\right)+\left(\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8}\right)+\cdots+\left(\frac{1}{2^{n-1}+1}+\cdots+\frac{1}{2^n}\right)\\
&> 1+\frac{1}{2}+\left(\frac{1}{4}+\frac{1}{4}\right)+\left(\frac{1}{8}+\frac{1}{8}+\frac{1}{8}+\frac{1}{8}\right)+\cdots+\left(\frac{1}{2^n}+\cdots+\frac{1}{2^n}\right)\\
&= 1+\frac{1}{2}+\frac{2}{4}+\frac{4}{8}+\cdots+\frac{2^{n-1}}{2^n}\\
&= 1+\frac{1}{2}+\frac{1}{2}+\frac{1}{2}+\cdots+\frac{1}{2}=1+\frac{n}{2}.
\end{aligned}$$

Since $1 + n/2 \to +\infty$ as $n \to +\infty$, the subsequence $\{s_{2^n}\}$ diverges. Since the sequence of partial sums (s_n) is monotone increasing and has a divergent subsequence, the sequence (s_n) diverges.

Remark: If $\sum_{n=1}^{\infty} a_n$ is convergent $\Rightarrow \lim_{n\to+\infty} a_n = 0$. (Test for Divergence) Therefore, If $\lim_{n\to+\infty} a_n \neq 0 \Rightarrow \sum_{n=1}^{\infty} a_n$ is divergent.

However, it can happen that $a_n \to 0$ and $\sum_{n=1}^{\infty} a_n$ does not converge.

Example 3: Consider $a_n = a^n$, $n \in \mathbb{N}$ and $a \in \mathbb{R}$. The geometric series $\sum_{n=0}^{\infty} a_n$ is divergent when $|a| \geq 1$ because in this case $\lim_{n\to\infty} a^n \neq 0$. However, when $|a| < 1$ the geometric series converges, and

$$\sum_{n=0}^{\infty} a^n = \frac{1}{1-a}.$$

(Note: a_1 in the formula corresponds to the first term, which is $a^0 = 1$)

Example 4: The series $\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$ is convergent. Indeed, let us observe the partial fraction decomposition:

$$\frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1}.$$

Therefore (this is a telescoping series):

$$\begin{aligned}
s_n &= \frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\cdots+\frac{1}{n\cdot(n+1)}\\
&= \left(1-\frac{1}{2}\right)+\left(\frac{1}{2}-\frac{1}{3}\right)+\cdots+\left(\frac{1}{n}-\frac{1}{n+1}\right)=1-\frac{1}{n+1}.
\end{aligned}$$

Consequently

$$\lim_{n\to\infty} s_n = \lim_{n\to\infty}\left(1 - \frac{1}{n+1}\right) = 1.$$

From the properties of limits of sequences, the following proposition results:

Proposition 8.3 *Let $a_n \geq 0$ for all $n \in \mathbb{N}$. The series $\sum_{n=1}^{\infty} a_n$ converges if and only if the partial sums $s_n = a_1 + a_2 + \cdots + a_n$ form a bounded sequence.*

Proof: Since $a_n \geq 0$, we have $s_1 \leq s_2 \leq \cdots \leq s_n \leq \cdots$. Therefore, since $(s_n)_{n\in\mathbb{N}}$ is monotone (non-decreasing), (s_n) is convergent if and only if (s_n) is bounded (by the Monotone Sequence Theorem, 2.14). ∎

Corollary 8.4 (Comparison Test) *Let $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ be series of non-negative terms. If there exist $C > 0$ and $n_0 \in \mathbb{N}$ such that $a_n \leq Cb_n$, for all $n \geq n_0$, then the convergence of $\sum_{n=1}^{\infty} b_n$ implies the convergence of $\sum_{n=1}^{\infty} a_n$, while the divergence of $\sum_{n=1}^{\infty} a_n$ entails that of $\sum_{n=1}^{\infty} b_n$.*

Proof: If $\sum_{n=1}^{\infty} b_n$ converges, then its partial sums s_n^b are bounded. This implies (for $n \geq n_0$) that s_n^a is bounded, and thus $\sum_{n=1}^{\infty} a_n$ is convergent (by Prop. 8.3). If $\sum_{n=1}^{\infty} a_n$ diverges, then s_n^a is not bounded. This implies s_n^b is not bounded, and therefore $\sum_{n=1}^{\infty} b_n$ diverges. ∎

Example 5: If $p > 1$, the p-series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges. Since the terms of this series are positive, the sequence of partial sums is increasing. To prove that this sequence is bounded, it suffices to find a bounded subsequence. Let s_n be the n-th partial sum. Let us choose the subsequence s_{2^n-1}. We have:

$$\begin{aligned}
s_{2^n-1} &= a_1 + a_2 + \cdots + a_{2^n-1} \\
&= 1 + \left(\frac{1}{2^p} + \frac{1}{3^p}\right) + \left(\frac{1}{4^p} + \cdots + \frac{1}{7^p}\right) + \cdots + \left(\frac{1}{(2^{n-1})^p} + \cdots + \frac{1}{(2^n-1)^p}\right) \\
&< 1 + \frac{2}{2^p} + \frac{4}{4^p} + \cdots + \frac{2^{n-1}}{(2^{n-1})^p} \\
&= 1 + \frac{1}{2^{p-1}} + \frac{1}{4^{p-1}} + \cdots + \frac{1}{(2^{n-1})^{p-1}} \\
&= \sum_{i=0}^{n-1} \left(\frac{1}{2^{p-1}}\right)^i.
\end{aligned}$$

(Note: The original text had a slight error in the grouping $2/2^p, 4/2^{2p}$... Corrected to $2/2^p, 4/4^p$...) Since $p > 1$, we have $p - 1 > 0$, so $r = \frac{1}{2^{p-1}} < 1$. The geometric series $\sum_{i=0}^{\infty} r^i$ converges. Therefore, the subsequence s_{2^n-1} is bounded above (by

$\frac{1}{1-r}$). Since (s_n) is monotone increasing and has a bounded subsequence, the entire sequence (s_n) is bounded. By Proposition 8.3, the series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges.

Observe that if $0 < p \leq 1$, we have $n^p \leq n$, so $\frac{1}{n^p} \geq \frac{1}{n}$. By the comparison test, $\sum_{n=1}^{\infty} \frac{1}{n^p}$ diverges, since the harmonic series $\sum_{n=1}^{\infty} \frac{1}{n}$ is divergent.

We conclude, then (for $p > 0$): If $0 < p \leq 1$, $\sum_{n=1}^{\infty} \frac{1}{n^p}$ diverges. If $p > 1$, $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges.

Proposition 8.5 (Limit Comparison Test) *Let $\sum_{n=1}^{\infty} a_n$ be a series of positive terms and $\sum_{n=1}^{\infty} b_n$ be a series of non-negative terms.*

(i) If $\lim_{n\to\infty} \frac{b_n}{a_n} = k > 0$, then the two series either both converge or both diverge.

(ii) If $\lim_{n\to\infty} \frac{b_n}{a_n} = 0$, and if $\sum a_n$ converges, then $\sum b_n$ converges.

(iii) If $\lim_{n\to\infty} \frac{b_n}{a_n} = +\infty$, and if $\sum a_n$ diverges, then $\sum b_n$ diverges.

Proof: (i) Since $\lim_{n\to\infty} \frac{b_n}{a_n} = k > 0$, taking $\varepsilon = k/2 > 0$, there exists $n_0 \in \mathbb{N}$ such that

$$k - \frac{k}{2} < \frac{b_n}{a_n} < k + \frac{k}{2}, \ \forall n \geq n_0.$$

Hence,

$$\frac{k}{2} < \frac{b_n}{a_n} < \frac{3k}{2}, \ \forall n \geq n_0,$$

which implies

$$b_n < \frac{3k}{2} a_n \ \text{ and } \ a_n < \frac{2}{k} b_n, \ \forall n \geq n_0.$$

By the Comparison Test (Corollary 7.5), the series converge or diverge together.

(ii) Since $\lim_{n\to\infty} \frac{b_n}{a_n} = 0$, given $\varepsilon = 1$, there exists $n_0 \in \mathbb{N}$ such that

$$\left| \frac{b_n}{a_n} - 0 \right| < 1; \ \forall n \geq n_0.$$

Since terms are positive,

$$0 \leq \frac{b_n}{a_n} < 1, \ \forall n \geq n_0 \implies b_n < a_n, \ \forall n \geq n_0.$$

By the Comparison Test, if $\sum a_n$ converges, then $\sum b_n$ converges.

(iii) Since $\lim_{n\to\infty} \frac{b_n}{a_n} = +\infty$, given $M > 0$ (e.g., $M = 1$), there exists $n_0 \in \mathbb{N}$ such that

$$\frac{b_n}{a_n} > M, \ \forall n \geq n_0.$$

Hence:

$$b_n > M a_n, \ \forall n \geq n_0.$$

By the Comparison Test, if $\sum a_n$ diverges, then $\sum b_n$ diverges. ■

Example 6: Consider the series:

$$\sum_{n=2}^{\infty} \frac{n^3+1}{n^4+n^2+n-3}.$$

Let

$$a_n = \frac{1}{n} \text{ (diverges) and } b_n = \frac{n^3+1}{n^4+n^2+n-3}.$$

Note that:

$$\begin{aligned} \lim_{n\to\infty} \frac{b_n}{a_n} &= \lim_{n\to\infty} n \cdot \frac{n^3+1}{n^4+n^2+n-3} = \lim_{n\to\infty} \frac{n^4+n}{n^4+n^2+n-3} \\ &= \lim_{n\to\infty} \frac{n^4\left(1+\frac{1}{n^3}\right)}{n^4\left(1+\frac{1}{n^2}+\frac{1}{n^3}-\frac{3}{n^4}\right)} = 1. \end{aligned}$$

Since $\sum_{n=1}^{\infty} a_n$ (the harmonic series) diverges, by the Limit Comparison Test (i), $\sum_{n=1}^{\infty} b_n$ also diverges.

Proposition 8.6 (Cauchy Criterion for Series) *A series $\sum_{n=1}^{\infty} a_n$ is convergent if and only if, for each $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that*

$$|a_{n+1} + a_{n+2} + \cdots + a_{n+p}| < \varepsilon, \ \forall n \geq n_0 \textit{ and } \forall p \in \mathbb{N}.$$

Proof: Let (s_n) be the sequence of partial sums of $\sum_n a_n$. Observe that:

$$s_{n+p} - s_n = a_{n+1} + a_{n+2} + \cdots + a_{n+p}.$$

The series $\sum_n a_n$ is convergent if and only if (s_n) converges. By Theorem 2.19 (Completeness of $\mathbb{R}$), (s_n) converges if and only if (s_n) is a Cauchy sequence. (s_n) is Cauchy if: $\forall \varepsilon > 0, \exists n_0$ s.t. if $m > n \geq n_0$, $|s_m - s_n| < \varepsilon$. Setting $m = n + p$ (where $p \geq 1$), this is exactly the condition stated. ■

Definition 8.7 *A series $\sum_{n=1}^{\infty} a_n$ is called absolutely convergent if $\sum_{n=1}^{\infty} |a_n|$ is a convergent series.*

Example 7: The geometric series $\sum_{n=0}^{\infty} a^n$; $-1 < a < 1$ is absolutely convergent, because if $|a| < 1$, $\sum_{n=0}^{\infty} |a|^n$ is a convergent geometric series.

Evidently, every convergent series whose terms do not change sign is absolutely convergent. If $a_n \geq 0$, then $a_n = |a_n|$, so $\sum |a_n|$ converges. If $a_n \leq 0$, then $|a_n| = -a_n$. If $\sum a_n$ converges, then $\sum -a_n = \sum |a_n|$ also converges (by Thm 5.7(ii)).

But not every convergent series is absolutely convergent. Indeed, consider the series:

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \cdots .$$

We will see later (with the Leibniz test) that the series above converges. However, it is not absolutely convergent, because

$$\sum_{n=1}^{\infty} \left| \frac{(-1)^{n+1}}{n} \right| = \sum_{n=1}^{\infty} \frac{1}{n},$$

and we have already seen that the harmonic series diverges.

Definition 8.8 *When a series $\sum_{n=1}^{\infty} a_n$ converges but $\sum_{n=1}^{\infty} |a_n|$ diverges, we say that $\sum_{n=1}^{\infty} a_n$ is conditionally convergent.*

Thus, the series $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$ is conditionally convergent.

Proposition 8.9 *Every absolutely convergent series is convergent.*

Proof: If $\sum_{n=1}^{\infty} |a_n|$ converges, then by the Cauchy Criterion (8.6), given $\varepsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that

$$||a_{n+1}| + |a_{n+2}| + \cdots + |a_{n+p}|| < \varepsilon, \ \forall n \geq n_0 \text{ and } p \in \mathbb{N}.$$

But, by the triangle inequality:

$$|a_{n+1} + a_{n+2} + \cdots + a_{n+p}| \leq |a_{n+1}| + |a_{n+2}| + \cdots + |a_{n+p}| < \varepsilon.$$

Therefore, $\sum_{n=1}^{\infty} a_n$ satisfies the Cauchy Criterion, and thus converges. ■

Corollary 8.10 *Let $\sum_{n=1}^{\infty} b_n$ be a convergent series with $b_n \geq 0$ for all $n \in \mathbb{N}$. If there exist $k > 0$ and $n_0 \in \mathbb{N}$ such that $|a_n| \leq kb_n$ for all $n \geq n_0$, then the series $\sum_{n=1}^{\infty} a_n$ is absolutely convergent.*

Proof: We have by the Comparison Test (Cor. 7.5) that $\sum_{n=1}^{\infty} |a_n|$ converges, and therefore $\sum_{n=1}^{\infty} a_n$ is absolutely convergent. ■

Corollary 8.11 *If for all $n \geq n_0$ we have $|a_n| \leq kc^n$ where $0 < c < 1$ and k is a positive constant, then the series $\sum_{n=1}^{\infty} a_n$ is absolutely convergent.*

Proof: Apply Corollary 8.10 with $b_n = c^n$. The geometric series $\sum b_n$ converges since $0 < c < 1$. ■

Corollary 8.12 (Root Test) *Let $\sum_{n=1}^{\infty} a_n$ be a series of real terms such that $\lim_{n\to\infty} \sqrt[n]{|a_n|}$ exists and equals R. Then:*

(i) If $R < 1$, $\sum_{n=1}^{\infty} a_n$ is absolutely convergent.

(ii) If $R = 1$, the test is inconclusive.

(iii) If $R > 1$, $\sum_{n=1}^{\infty} a_n$ is divergent.

Proof: (i) Since $R < 1$, there exists $\xi \in \mathbb{R}$ such that $R < \xi < 1$. Let $\varepsilon = \xi - R > 0$. Since the limit is R, there exists $n_0 \in \mathbb{N}$ such that for $n \geq n_0$:

$$\left| \sqrt[n]{|a_n|} - R \right| < \varepsilon \implies \sqrt[n]{|a_n|} < R + \varepsilon = R + (\xi - R) = \xi.$$

This implies

$$|a_n| < \xi^n, \ \forall n \geq n_0.$$

By Corollary 8.11 (with $c = \xi$), the series $\sum a_n$ is absolutely convergent.

(iii) Since $R > 1$, let $\varepsilon = R - 1 > 0$. Then there exists $n_0 \in \mathbb{N}$ such that:

$$R - \varepsilon < \sqrt[n]{|a_n|} < R + \varepsilon, \ \forall n \geq n_0,$$

In particular,

$$\sqrt[n]{|a_n|} > R - \varepsilon = R - (R-1) = 1, \ \forall n \geq n_0.$$

Thus, $|a_n| > 1^n = 1$ for all $n \geq n_0$. Since the general term does not tend to zero, $\sum a_n$ diverges.

(ii) Consider the series:

$$\sum_{n=1}^{\infty} \frac{1}{n} \quad \text{and} \quad \sum_{n=1}^{\infty} \frac{1}{n^2}.$$

We know (from Example 3a, Section 2.2) that $\lim_{n\to+\infty} \sqrt[n]{n} = 1$. Thus $\lim_{n\to+\infty} \sqrt[n]{\frac{1}{n}} = \frac{1}{\lim \sqrt[n]{n}} = 1$. Also:

$$\lim_{n\to\infty} \sqrt[n]{\frac{1}{n^2}} = \lim_{n\to\infty} \left(\sqrt[n]{\frac{1}{n}} \right)^2 = 1^2 = 1.$$

So, $\lim \sqrt[n]{|a_n|} = 1$ in both cases; however, $\sum \frac{1}{n}$ diverges and $\sum \frac{1}{n^2}$ converges. ■

<u>Remark:</u> Note that to conclude absolute convergence of $\sum a_n$, the limit of $\sqrt[n]{|a_n|}$ does not need to exist; it is sufficient that $\limsup \sqrt[n]{|a_n|} < 1$. *(Translator's note: The original text states "it is sufficient that there exists $n_0 \in \mathbb{N}$ such that $\sqrt[n]{|a_n|} \leq c < 1$," which is the core of the proof (i).)*

Example 8: Consider the series $\sum_{n=1}^{\infty} na^n$, $a \in \mathbb{R}$. We have:

$$\lim_{n\to\infty} \sqrt[n]{|na^n|} = \lim_{n\to\infty} (\sqrt[n]{n} \cdot |a|) = 1 \cdot |a| = |a|.$$

Thus, if $|a| < 1$, the series converges absolutely, and if $|a| > 1$, the series diverges.

For example, from the above, the series $\sum_{n=1}^{\infty} \frac{n}{2^n}$ converges absolutely since

$$\lim_{n\to\infty} \sqrt[n]{\frac{n}{2^n}} = \lim_{n\to\infty} \frac{\sqrt[n]{n}}{2} = \frac{1}{2} < 1.$$

Proposition 8.13 (Ratio Test Comparison) *Let $\sum a_n$ be a series of non-zero terms and $\sum b_n$ be a convergent series with $b_n > 0$. If there exists $n_0 \in \mathbb{N}$ such that*

$$\frac{|a_{n+1}|}{|a_n|} \leq \frac{b_{n+1}}{b_n}, \quad \forall n \geq n_0,$$

then $\sum a_n$ is absolutely convergent.

Proof: The inequality $\frac{|a_{n+1}|}{b_{n+1}} \leq \frac{|a_n|}{b_n}$ holds for $n \geq n_0$. This means the sequence $c_n = |a_n|/b_n$ is non-increasing for $n \geq n_0$. Thus, for $n > n_0$, $|a_n|/b_n \leq |a_{n_0}|/b_{n_0} = k$ (a constant).

$$|a_n| \leq k\, b_n, \ \forall n \geq n_0.$$

By Corollary 8.10 (Comparison Test), since $\sum b_n$ converges, $\sum |a_n|$ converges. ■

Corollary 8.14 *Let $\sum a_n$ be a series of non-zero terms and c a constant such that $0 < c < 1$. If $\frac{|a_{n+1}|}{|a_n|} \leq c$ for all $n \geq n_0$, then $\sum a_n$ is absolutely convergent.*

Proof: Let $b_n = c^n$. We have $\frac{b_{n+1}}{b_n} = \frac{c^{n+1}}{c^n} = c$. The condition is $\frac{|a_{n+1}|}{|a_n|} \leq c = \frac{b_{n+1}}{b_n}$ for $n \geq n_0$. Since $\sum b_n$ converges (geometric series, $0 < c < 1$), by Proposition 8.13, $\sum a_n$ converges absolutely. ■

Corollary 8.15 (Ratio Test) *Let $\sum a_n$ be a series of non-zero terms such that $\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|}$ exists and equals R. Then:*

(i) If $R < 1$, $\sum a_n$ converges absolutely.

(ii) If $R = 1$, the test is inconclusive.

(iii) If $R > 1$, $\sum a_n$ diverges.

Proof: (i) Since $R < 1$, there exists $c \in \mathbb{R}$ such that $R < c < 1$. Let $\varepsilon = c - R > 0$. Since the limit is R, there exists $n_0 \in \mathbb{N}$ such that for $n \geq n_0$:

$$\left| \frac{|a_{n+1}|}{|a_n|} - R \right| < \varepsilon \implies \frac{|a_{n+1}|}{|a_n|} < R + \varepsilon = R + (c - R) = c.$$

Since $0 < c < 1$, by the previous Corollary, $\sum a_n$ converges absolutely.

(iii) Since $R > 1$, let $\varepsilon = R - 1 > 0$. There exists $n_0 \in \mathbb{N}$ such that

$$\frac{|a_{n+1}|}{|a_n|} > R - \varepsilon = R - (R-1) = 1, \ \forall n \geq n_0,$$

i.e.,

$$|a_{n+1}| > |a_n|, \ \forall n \geq n_0.$$

This implies $|a_n|$ is an increasing sequence for $n \geq n_0$. Therefore, the general term a_n cannot converge to zero, which implies $\sum a_n$ diverges.

(ii) Consider the series $\sum \frac{1}{n}$ and $\sum \frac{1}{n^2}$. For $\sum 1/n$: $\lim \frac{1/(n+1)}{1/n} = \lim \frac{n}{n+1} = 1$. (Diverges) For $\sum 1/n^2$: $\lim \frac{1/(n+1)^2}{1/n^2} = \lim \frac{n^2}{(n+1)^2} = 1$. (Converges) In both cases $R = 1$, but the series have different behaviors. ■

Example 9: Consider the series $\sum_{n=0}^{\infty} \frac{a^n}{n!}$, $a \in \mathbb{R}$. We have:

$$\frac{|a_{n+1}|}{|a_n|} = \frac{|a|^{n+1}}{(n+1)!} \frac{n!}{|a|^n} = \frac{|a|}{n+1}.$$

$$\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = \lim_{n\to\infty} \frac{|a|}{n+1} = 0.$$

Since $R = 0 < 1$, the series converges absolutely for all $x \in \mathbb{R}$.

Proposition 8.16 (Integral Test) *Let $\sum_{n=1}^{\infty} a_n$ be a series of positive and non-increasing terms ($a_1 \geq a_2 \geq \cdots a_n \geq \cdots$). Let $f(x)$ be a function defined on $[1, +\infty)$, continuous, non-increasing, and positive, such that $f(n) = a_n$ for all n. Then:*

(i) If $\int_1^{\infty} f(x)\,dx$ converges, then $\sum_{n=1}^{\infty} a_n$ converges.

(ii) If $\int_1^{\infty} f(x)\,dx$ diverges, then $\sum_{n=1}^{\infty} a_n$ diverges.

Proof: (The proof refers to two figures, which are standard illustrations of the Integral Test. Figure I shows $s_n > \int_1^{n+1} f(x)dx$ (rectangles above curve, using left endpoints). Figure II shows $s_n - a_1 < \int_1^n f(x)dx$ (rectangles below curve, using right endpoints).)

Examining Figure I (rectangles based on left endpoints): The area of the n-th rectangle is $a_n \cdot 1$. The sum of the areas of the first n rectangles is $s_n = \sum_{i=1}^{n} a_i$. Since f is non-increasing, $a_i \geq f(x)$ for $x \in [i, i+1]$.

$$s_n = \sum_{i=1}^{n} a_i \geq \sum_{i=1}^{n} \int_i^{i+1} f(x)dx = \int_1^{n+1} f(x)dx.$$

$$s_n \geq \int_1^{n+1} f(x)\, dx. \tag{8.1}$$

Examining Figure II (rectangles based on right endpoints): The sum of the areas is $a_2 + a_3 + \cdots + a_n = s_n - a_1$. Since f is non-increasing, $a_i \leq f(x)$ for $x \in [i-1, i]$.

$$s_n - a_1 = \sum_{i=2}^{n} a_i \leq \sum_{i=2}^{n} \int_{i-1}^{i} f(x)dx = \int_1^n f(x)dx.$$

$$s_n \leq a_1 + \int_1^n f(x)dx. \tag{8.2}$$

(i) If $\int_1^\infty f(x)\, dx$ converges to L, then $\int_1^n f(x)dx \leq L$. By (8.2), $s_n \leq a_1 + L$. The sequence (s_n) is non-decreasing (since $a_n \geq 0$) and bounded above. Therefore, $\sum a_n$ converges. (ii) If $\int_1^\infty f(x)\, dx$ diverges (to $+\infty$), then $\lim_{n\to\infty} \int_1^{n+1} f(x)dx = \infty$. By (8.1), $s_n \to \infty$. Therefore, $\sum a_n$ diverges. ■

Example 10: Using the Integral Test, show that the p-series $\sum_{n=1}^\infty \frac{1}{n^p}$ converges if $p > 1$ and diverges if $p = 1$. Let $f(x) = 1/x^p$. This function is continuous, positive, and non-increasing on $[1, \infty)$ for $p > 0$.

(i) $p = 1$

$$\int_1^n \frac{1}{x}\, dx = [\ln x]_1^n = \ln n - \ln 1 = \ln n.$$

Since $\lim_{n\to\infty} \ln n = +\infty$, the integral $\int_1^\infty \frac{1}{x}\, dx$ diverges, which implies that $\sum_{n=1}^\infty \frac{1}{n}$ diverges.

(ii) $p > 1$.

$$\begin{aligned} \int_1^n \frac{1}{x^p}\, dx &= \int_1^n x^{-p}\, dx = \left[\frac{x^{-p+1}}{-p+1}\right]_1^n = \frac{1}{1-p}\left(n^{1-p} - 1\right) \\ &= \frac{1}{p-1}\left(1 - \frac{1}{n^{p-1}}\right). \end{aligned}$$

Since $p > 1$, $p-1 > 0$, so $\lim_{n\to+\infty} \frac{1}{n^{p-1}} = 0$. The integral converges: $\lim_{n\to\infty} \int_1^n \frac{1}{x^p} dx = \frac{1}{p-1}$. Therefore, $\int_1^\infty \frac{1}{x^p}\, dx$ converges, which implies that $\sum_{n=1}^\infty \frac{1}{n^p}$ converges.

8.1 Alternating Series

Definition 8.17 *An* <u>*alternating series*</u> *is a real series whose successive terms have opposite signs.*

Example 11: The series $\sum_{n=1}^\infty \frac{(-1)^{n+1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$ is alternating.

Proposition 8.18 (Leibniz Criterion / Alternating Series Test) *If (a_n) is a non-increasing sequence of positive terms with $\lim_{n\to\infty} a_n = 0$, then the alternating series $\sum_{n=1}^{\infty}(-1)^{n+1}a_n$ converges. *(Note: Original text had $(-1)^n a_n$, but the proof matches $(-1)^{n+1}a_n$ or $(-1)^n a_n$ starting at $n = 0$)**

Proof: Let (s_n) be the sequence of partial sums. Let (s_{2n}) be the subsequence of even-indexed terms.

$$s_{2n} = (a_1 - a_2) + (a_3 - a_4) + \cdots + (a_{2n-1} - a_{2n})$$

Since (a_n) is non-increasing, $a_k - a_{k+1} \geq 0$. Thus s_{2n} is a sum of non-negative terms.

$$s_{2n+2} = s_{2n} + (a_{2n+1} - a_{2n+2}) \geq s_{2n}.$$

So, (s_{2n}) is a non-decreasing sequence. Let us now write s_{2n} as:

$$s_{2n} = a_1 - (a_2 - a_3) - (a_4 - a_5) - \cdots - (a_{2n-2} - a_{2n-1}) - a_{2n}$$

Since $a_k - a_{k+1} \geq 0$ and $a_{2n} \geq 0$, we have

$$s_{2n} \leq a_1.$$

Thus, (s_{2n}) is non-decreasing and bounded above by a_1. By the Monotone Sequence Theorem, it converges. Let

$$s = \lim_{n\to\infty} s_{2n}.$$

Now consider the subsequence of odd-indexed terms: $s_{2n+1} = s_{2n} + a_{2n+1}$.

$$\lim_{n\to\infty} s_{2n+1} = \lim_{n\to\infty} s_{2n} + \lim_{n\to\infty} a_{2n+1} = s + 0 = s,$$

since $\lim a_n = 0$. Since both the even and odd subsequences converge to the same limit s, we conclude $\lim_{n\to\infty} s_n = s$. ■

Example 12: Consider the series $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$. We have $a_n = 1/n$, which is positive, non-increasing, and $\lim(1/n) = 0$. By the Leibniz Criterion, the series $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$ converges.

8.2 Power Series

Until now, we have studied series of real numbers. From now on, we will study particular series whose terms are real functions.

Definition 8.19 *Series of the type:*

$$\sum_{n=0}^{\infty} a_n (x - x_0)^n = a_0 + a_1(x - x_0) + a_2(x - x_0)^2 + \cdots$$

are called <u>*power series*</u> *(centred at x_0).*

In this case, $f_n(x) = a_n(x - x_0)^n,\ n \in \mathbb{N},\ x \in \mathbb{R}$.

For simplicity of notation, we will almost always consider the case $x_0 = 0$, i.e., power series of the form:

$$\sum_{n=0}^{\infty} a_n x^n = a_0 + a_1\, x + a_2\, x^2 + \cdots ,$$

since the general case reduces to this one by the change of variable $y = x - x_0$.

Example 1: Consider the power series $\sum_{n=0}^{\infty} \frac{x^n}{n!}$. In this case, $a_n = \frac{1}{n!}$. For which values of x does this series converge? Using the Ratio Test (for absolute convergence):

$$\lim_{n\to\infty} \frac{|a_{n+1}x^{n+1}|}{|a_n x^n|} = \lim_{n\to\infty} \frac{|x|^{n+1}}{(n+1)!} \frac{n!}{|x|^n} = \lim_{n\to\infty} \frac{|x|}{n+1} = 0.$$

Since $R = 0 < 1$ for all $x \in \mathbb{R}$, the series converges absolutely for all $x \in \mathbb{R}$.

Example 2: Consider the series $\sum_{n=0}^{\infty} x^n$. In this case $a_n = 1$. We have already seen that the series above (geometric series) converges if $|x| < 1$ and diverges if $|x| \geq 1$.

We will see next that the set of points x for which the series $\sum_{n=0}^{\infty} a_n\, x^n$ converges is an interval symmetric about the origin. (If the series were $\sum_{n=0}^{\infty} a_n\, (x - x_0)^n$, we would have an interval symmetric about x_0).

We first consider two lemmas:

Lemma 8.20 *If $\sum_{n=0}^{\infty} a_n x_0^n$ converges (for $x_0 \neq 0$), then the sequence $(a_n x_0^n)$ is bounded.*

Proof: If $\sum_{n=0}^{\infty} a_n x_0^n$ converges, then by the test for divergence (Prop. 8.2), $\lim_{n\to\infty} a_n x_0^n = 0$. Since every convergent sequence is bounded (Prop. 2.7), the sequence $(a_n x_0^n)$ is bounded. ■

Lemma 8.21 *If $\sum_{n=0}^{\infty} a_n\, x^n$ converges for $x = x_0 \neq 0$, then the series converges absolutely for all x such that $|x| < |x_0|$.*

Proof: Since $\sum_{n=0}^{\infty} a_n x_0^n$ converges, by Lemma 8.20, the sequence $(a_n x_0^n)$ is bounded. Thus, there exists $C > 0$ such that

$$|a_n x_0^n| \leq C; \ \forall n \in \mathbb{N}. \tag{8.3}$$

Let $x \in \mathbb{R}$ be such that $|x| < |x_0|$. Let $r = \frac{|x|}{|x_0|}$. We have $0 \leq r < 1$. From (8.3) we have:

$$|a_n x^n| = \left| a_n x_0^n \left(\frac{x}{x_0} \right)^n \right| = |a_n x_0^n| \left| \frac{x}{x_0} \right|^n \leq Cr^n. \tag{8.4}$$

Since $0 \leq r < 1$, the geometric series $\sum_{n=0}^{\infty} Cr^n$ converges. By the Comparison Test (Cor. 8.10), $\sum_{n=0}^{\infty} |a_n x^n|$ converges. Thus, $\sum_{n=0}^{\infty} a_n x^n$ converges absolutely. ■

Proposition 8.22 *If a power series $\sum_{n=0}^{\infty} a_n x^n$ converges for a point $x_0 \neq 0$, then either the series converges absolutely for all $x \in \mathbb{R}$, or there exists a positive real number R such that the series converges absolutely when $|x| < R$ and diverges when $|x| > R$. At the points R or $-R$, nothing can be affirmed; that is, the series may converge absolutely, converge conditionally, or diverge.*

Proof: Let E be the following set:

$$E := \{r > 0 \mid \text{if } |x| < r \text{ then } \sum_{n=0}^{\infty} a_n x^n \text{ converges absolutely}\}.$$

Since $\sum_{n=0}^{\infty} a_n x_0^n$ converges by hypothesis, then by Lemma 8.21, $\sum_{n=0}^{\infty} a_n x^n$ converges absolutely for $|x| < |x_0|$. It follows that $|x_0| \in E$, i.e., $E \neq \emptyset$.

If E is not bounded above, we take $R = +\infty$. If E is bounded, we consider $R = \sup E$. For $R = +\infty$, the series converges absolutely for all $x \in \mathbb{R}$, because since E is unbounded, given $x \in \mathbb{R}$ we can choose $r_1 \in E$ such that $|x| < r_1$, and by the definition of E, $\sum_n a_n x^n$ converges absolutely. For R finite, the series converges absolutely for $|x| < R$, because since $R = \sup E$, for each $x \in \mathbb{R}$ such that $|x| < R$, we can find $r \in E$ such that $|x| < r < R$, and then $\sum_n a_n x^n$ converges absolutely.

For R finite, the series diverges for $|x| > R$. This is because if the series converged for $x = x_1$ with $|x_1| > R$, by Lemma 8.21 the series would converge absolutely for $|x| < |x_1|$, and therefore $|x_1| \in E$, which contradicts the fact that R is the supremum of E. ■

Definition 8.23 *The number R from the previous proposition is called the* <u>*radius of convergence*</u> *of the power series.*

If a power series converges for all $x \in \mathbb{R}$, we say the radius of convergence is infinite and write $R = +\infty$. If the series diverges for all $x \neq 0$, we say the radius of convergence is zero and write $R = 0$. Thus, the radius can be zero, $+\infty$, or any positive real number.

Example 3: From what we have already seen, the radius of convergence of the series $\sum_{n=0}^{\infty} \frac{x^n}{n!}$ is $+\infty$.

Example 4: The radius of convergence of the series $\sum_{n=0}^{\infty} x^n$ is 1, although this series does not converge for $x = 1$ or $x = -1$.

Example 5: The radius of convergence of the series $\sum_{n=0}^{\infty} n!\, x^n$ is zero. Indeed, if $x \neq 0$ we have:

$$\lim_{n\to\infty} \left| \frac{(n+1)!\, x^{n+1}}{n!\, x^n} \right| = |x| \lim_{n\to\infty} (n+1) = +\infty.$$

Since the limit is > 1 (it is $+\infty$), by the Ratio Test, the series diverges for all $x \neq 0$. *(Translator's note: The original proof with epsilon is also correct, but this conclusion from the Ratio Test is more direct)*.

Proposition 8.24 *Let $\sum_{n=0}^{\infty} a_n\, x^n$ be a power series where $a_n \neq 0$ for n sufficiently large.*

(i) If $\lim_{n\to\infty} \left| \frac{a_n}{a_{n+1}} \right| = R$; where $0 \leq R \leq +\infty$, then R is the radius of convergence of the power series.

(ii) If $\lim_{n\to\infty} \frac{1}{\sqrt[n]{|a_n|}} = R$, where $0 \leq R \leq +\infty$, then R is the radius of convergence of the power series.

Proof: (i) If $x = 0$, the series converges absolutely. If $x \neq 0$, we apply the Ratio Test to the series $\sum a_n x^n$:

$$\lim_{n\to\infty} \left| \frac{a_{n+1}\, x^{n+1}}{a_n\, x^n} \right| = \lim_{n\to\infty} |x| \left| \frac{a_{n+1}}{a_n} \right| = |x| \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right|$$

This limit equals:

$$\begin{cases} \dfrac{|x|}{R}; & \text{if } R \neq 0 \text{ and } R \neq +\infty, \\ 0; & \text{if } R = +\infty, \\ +\infty; & \text{if } R = 0. \end{cases}$$

By the Ratio Test, the series converges absolutely if this limit is < 1 and diverges if > 1. If $R = +\infty$, the limit is $0 < 1$ for all x. The series converges absolutely for all x. If $R = 0$, the limit is $+\infty > 1$ for all $x \neq 0$. The series diverges for $x \neq 0$. If $0 < R < +\infty$, the limit is $|x|/R$. This is < 1 if $|x| < R$, and > 1 if $|x| > R$. In all cases, R is the radius of convergence.

(ii) Shown similarly using the Root Test. ■

Example 6: What is the radius of convergence of the series $\sum_{n=0}^{\infty} 3^{2n}\, x^n$? We have $a_n = 3^{2n} = 9^n$. Using method (i):

$$\lim_{n\to\infty}\left|\frac{a_n}{a_{n+1}}\right| = \lim_{n\to\infty}\left|\frac{3^{2n}}{3^{2(n+1)}}\right| = \lim_{n\to\infty}\frac{1}{3^2} = \frac{1}{9}.$$

Hence, the radius of convergence R is $1/9$.

Properties of Power Series. If a series $\sum_{n=0}^{\infty} a_n\, x^n$ has a sum $f(x)$ such that $\sum_{n=0}^{\infty} a_n\, x^n = f(x)$ for x in an interval I, we say that the series $\sum_{n=0}^{\infty} a_n\, x^n$ represents the function f on this interval.

A power series $\sum_{n=0}^{\infty} a_n\, x^n$ with radius of convergence $R \neq 0$ represents one, and only one, function on the interval $(-R, R)$ which associates the number $\sum_{n=0}^{\infty} a_n\, x_0^n$ with a point x_0 in the interval $(-R, R)$.

Proposition 8.25 *If the power series $\sum_{n=0}^{\infty} a_n x^n$ represents the function $f(x)$ in the open interval $|x| < R$ (where $R > 0$ is its radius of convergence), then:*

a) $f(x)$ is continuous in $|x| < R$.

b) $f(x)$ is differentiable in $|x| < R$ and $f'(x)$ is represented by the power series $\sum_{n=1}^{\infty} n\, a_n\, x^{n-1}$ in the interval $|x| < R$.

c) The definite integral $\int_0^x f(t)\, dt$ is represented by the power series $\sum_{n=0}^{\infty} \frac{a_n}{n+1}\, x^{n+1}$ in the interval $|x| < R$.

Proof: See Elon Lages Lima [1]. ■

The result above allows us to interchange differentiation and integration with the limit, i.e.:

$$f'(x) = \frac{d}{dx}\left(\sum_{n=0}^{\infty} a_n\, x^n\right) = \frac{d}{dx}\lim_{n\to\infty} s_n(x) = \lim_{n\to\infty}\frac{d}{dx}s_n(x) = \sum_{n=1}^{\infty} n\, a_n x^{n-1},$$

and

$$\begin{aligned}
\int_0^x f(t)\, dt &= \int_0^x \left(\sum_{n=0}^{\infty} a_n\, t^n\right) dt = \int_0^x \left(\lim_{n\to\infty} s_n(t)\right) dt \\
&= \lim_{n\to\infty}\int_0^x s_n(t)\, dt = \lim_{n\to\infty}\int_0^x \left(\sum_{i=0}^{n} a_i\, t^i\right) dt \\
&= \lim_{n\to\infty}\left(a_0\, x + \frac{a_1\, x^2}{2} + \frac{a_2\, x^3}{3} + \cdots + \frac{a_n\, x^{n+1}}{n+1}\right) \\
&= \sum_{n=0}^{\infty}\frac{a_n\, x^{n+1}}{n+1}.
\end{aligned}$$

(Note: Corrected summation limit in the original text)

Corollary 8.26 *If the function $f(x)$ can be represented by a power series in an open interval $|x| < r$, then the function possesses derivatives of all orders in that interval.*

Example 7: Consider the geometric series:

$$\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + \cdots$$

This series converges for $|x| < 1$ and diverges for $|x| \geq 1$. Furthermore,

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}, \quad \text{for } |x| < 1.$$

Thus,

$$\frac{1}{1-x} = 1 + x + x^2 + \cdots, \quad \text{if } |x| < 1. \tag{8.5}$$

Now, since

$$\frac{d}{dx}\left(\frac{1}{1-x}\right) = \frac{1}{(1-x)^2},$$

from (8.5) we have (by term-wise differentiation)

$$\frac{1}{(1-x)^2} = 1 + 2x + 3x^2 + \cdots = \sum_{n=1}^{\infty} n\, x^{n-1}, \text{ if } |x| < 1. \tag{8.6}$$

On the other hand, if we integrate the function $\frac{1}{1-x}$ and integrate the right-hand side of (8.5) from 0 to x:

$$\begin{aligned} \int_0^x \frac{1}{1-t}\,dt &= [-\ln(1-t)]_0^x = -\ln(1-x) - (-\ln(1)) = -\ln(1-x) \\ \int_0^x \left(\sum_{n=0}^{\infty} t^n\right) dt &= \sum_{n=0}^{\infty} \int_0^x t^n dt = \sum_{n=0}^{\infty} \frac{x^{n+1}}{n+1} = x + \frac{x^2}{2} + \frac{x^3}{3} + \cdots \end{aligned}$$

i.e.,

$$-\ln(1-x) = x + \frac{x^2}{2} + \frac{x^3}{3} + \cdots = \sum_{n=0}^{\infty} \frac{x^{n+1}}{n+1}, \quad \text{if } |x| < 1. \tag{8.7}$$

(Note: Corrected the integral of 1/(1-t), which is -ln(1-t))

Example 8: Consider the series $\sum_{n=0}^{\infty} \frac{x^n}{n!}$. This series converges for all $x \in \mathbb{R}$. If $f(x)$ is the function represented by this series, we have:

$$f'(x) = \sum_{n=1}^{\infty} \frac{d}{dx}\left(\frac{x^n}{n!}\right) = \sum_{n=1}^{\infty} \frac{n\, x^{n-1}}{n!} = \sum_{n=1}^{\infty} \frac{x^{n-1}}{(n-1)!}.$$

Let $k = n - 1$. When $n = 1$, $k = 0$.

$$f'(x) = \sum_{k=0}^{\infty} \frac{x^k}{k!} = f(x).$$

Therefore, $f'(x) = f(x)$, which means $f(x) = Ce^x$. Since $f(0) = a_0 = 1/0! = 1$, we have $C = 1$. Hence $f(x) = e^x$.

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots, \forall x \in \mathbb{R}. \tag{8.8}$$

Corollary 8.27 *If the function $f(x)$ is represented by the power series $\sum_{n=0}^{\infty} a_n\, x^n$ in an interval $|x| < r$, then:*

$$f^{(n)}(0) = n!\, a_n; \;\; \forall n \in \mathbb{N},$$

and therefore,

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}\, x^n. \; (\textit{Taylor Series Expansion})$$

Proof: We have

$$f(x) = a_0 + a_1\, x + a_2\, x^2 + a_3\, x^3 + a_4\, x^4 + \cdots$$

Differentiating term by term:

$$f'(x) = a_1 + 2a_2\, x + 3a_3\, x^2 + 4a_4\, x^3 + \cdots,$$

$$f''(x) = 2a_2 + (3 \cdot 2)a_3\, x + (4 \cdot 3)a_4 x^2 + \cdots$$

$$f'''(x) = (3 \cdot 2 \cdot 1)a_3 + (4 \cdot 3 \cdot 2)a_4 x + \cdots$$

Setting $x = 0$ at each step:

$$f(0) = a_0; \;\; f'(0) = a_1; \;\; f''(0) = 2!\, a_2; \;\; f'''(0) = 3!\, a_3; \;\; \cdots ; f^{(n)}(0) = n!\, a_n.$$

Substituting the a_n back into the series gives

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}\, x^n.$$

■

Corollary 8.28 *If the function $f(x)$ is represented by both power series $\sum_{n=0}^{\infty} a_n\, x^n$ and $\sum_{n=0}^{\infty} b_n\, x^n$ in some open interval common to both around zero, then $a_n = b_n$ for all n.*

Proof: By Corollary 8.27, we have $f^{(n)}(0) = n!\, a_n$ and $f^{(n)}(0) = n!\, b_n$. Whence:

$$a_n = \frac{f^{(n)}(0)}{n!} = b_n.$$

■

It follows from Corollary 8.28 that a function can be represented by at most one power series centred at zero. Naturally, a function can be represented by series with different centres. Thus:

$$f(x) = 1 + x + x^2 = \sum_{n=0}^{\infty} a_n\, x^n \text{ where } a_n = \begin{cases} 1, & n = 0, 1, 2, \\ 0, & n > 2. \end{cases}$$

$$f(x) = 3 + 3(x-1) + (x-1)^2 = \sum_{n=0}^{\infty} b_n\, (x-1)^n \text{ where } b_n = \begin{cases} 3, & n = 0, 1, \\ 1, & n = 2, \\ 0, & n > 2. \end{cases}$$

Note that $3 + 3(x-1) + (x-1)^2 = 3 + 3x - 3 + x^2 - 2x + 1 = 1 + x + x^2$.

Corollary 8.29 *If $\sum_{n=0}^{\infty} a_n\, x^n$ represents the function $f(x) = 0$ in an open interval containing zero, then every $a_n = 0$.*

Proof: The series $\sum_{n=0}^{\infty} 0\, x^n$ and $\sum_{n=0}^{\infty} a_n\, x^n$ represent the same function (the zero function) in an open interval containing zero. Hence, by Corollary 8.28, the coefficients of the two series are equal, i.e., $a_n = 0$. ■

Proposition 8.30 *Let f be a function represented by the power series $\sum_n a_n\, x^n$ on the interval $(-r_1, r_1)$ and g a function represented by the power series $\sum_n b_n\, x^n$ on the interval $(-r_2, r_2)$. Then:*

(i) $\sum_{n=0}^{\infty}(k\, a_n)\, x^n$ represents the function kf on $(-r_1, r_1)$.

(ii) $\sum_{n=0}^{\infty}(a_n + b_n)\, x^n$ represents the function $f + g$ on $(-r, r)$ where $r = \min\{r_1, r_2\}$.

Proof: Proof of this proposition is left as an exercise. ■

Chapter 9

Sequences and Series of Functions

9.1 Sequences and Series of Functions

Definition 9.1 *Let $A \subset \mathbb{R}$. A sequence of real functions defined on A is any map that associates to each $n \in \mathbb{N}$ a function $f_n : A \to \mathbb{R}$.*

Example 1: For $A = [0, 1]$, consider the map that associates $n \in \mathbb{N}$ with the sequence of functions $f_n : [0, 1] \to \mathbb{R}$ defined by $f_n(x) = x^n$.

Definition 9.2 *Let $(f_n)_{n\in\mathbb{N}}$ be a sequence of real functions defined on a subset $A \subset \mathbb{R}$. We say that $(f_n)_{n\in\mathbb{N}}$ has the function $f : A \to \mathbb{R}$ as its limit, or that $f_n \to f$, when for each $x \in A$ the numerical sequence $(f_n(x))_{n\in\mathbb{N}}$ converges to $f(x)$. We also say that $(f_n)_{n\in\mathbb{N}}$ converges pointwise to f.*

Returning to Example 1 above, we had $f_n(x) = x^n$, $0 \le x \le 1$. We will show that the sequence defined above converges to the zero function if $0 \le x < 1$ and converges to the constant function equal to 1 if $x = 1$. For $x = 0$, it is immediate that $f_n(0) = 0^n \to 0$. For $x = 1$, $f_n(1) = 1^n \to 1$. Consider $0 < x < 1$. We must show $x^n \to 0$. Given $\varepsilon > 0$ and for each $x \in (0, 1)$, we must find n_0 (which depends on ε and x), such that if $n \ge n_0$, we have $|x^n - 0| < \varepsilon$.

Note that $x^n < \varepsilon \Leftrightarrow \ln(x^n) < \ln\varepsilon \Leftrightarrow n \ln x < \ln\varepsilon$. Since $0 < x < 1$, $\ln x < 0$. Thus, dividing by $\ln x$ reverses the inequality:

$$n > \frac{\ln\varepsilon}{\ln x}$$

If we take $0 < \varepsilon < 1$, then $\ln\varepsilon < 0$, so $\frac{\ln\varepsilon}{\ln x} > 0$. In this way, it suffices to take n_0 to be the smallest positive integer such that $n_0 > \frac{\ln\varepsilon}{\ln x}$, and the desired result holds. Note that n_0 depends on ε as well as on x.

In the example above, we have a sequence of continuous functions on $[0, 1]$ that converge pointwise to a function f

$$f(x) = \begin{cases} 0, & 0 \leq x < 1 \\ 1, & x = 1 \end{cases}$$

which is not continuous at the point $x = 1$.

Example 2: We will now show that the sequence of functions $f_n(x) = \frac{\sin(nx)}{\sqrt{n}}$ converges to the identically zero function. In this case, we will show that the convergence is uniform, i.e., the index n_0 no longer depends on x but only on ε.

We have:

$$|f_n(x) - 0| = \left|\frac{\sin(nx)}{\sqrt{n}}\right| \leq \frac{1}{\sqrt{n}}, \ \forall x \in \mathbb{R}.$$

We want $|f_n(x) - 0| < \varepsilon$. It suffices to make the upper bound $\frac{1}{\sqrt{n}} < \varepsilon$.

$$\frac{1}{\sqrt{n}} < \varepsilon \Leftrightarrow \frac{1}{n} < \varepsilon^2 \Leftrightarrow n > \frac{1}{\varepsilon^2}.$$

In this case, it suffices to choose n_0 as the smallest integer greater than $\frac{1}{\varepsilon^2}$. We say then that $(f_n(x))$ converges uniformly to the zero function because n_0 only depends on ε.

Definition 9.3 *Let $(f_n)_{n\in\mathbb{N}}$ be a sequence of real functions defined on a set $A \subset \mathbb{R}$. We say that $(f_n)_n$ converges uniformly on A to a function $f : A \to \mathbb{R}$ when for each $\varepsilon > 0$, there corresponds an index $n_0(\varepsilon)$, independent of x, such that $|f_n(x) - f(x)| < \varepsilon$ for all $n \geq n_0$, regardless of $x \in A$.*

Definition 9.4 (Cauchy Condition for sequences of functions) *Let $(f_n)_{n\in\mathbb{N}}$ be a sequence of real functions defined on a set $A \subset \mathbb{R}$.*

- *For the sequence $(f_n)_{n\in\mathbb{N}}$ to be pointwise convergent on A, it is necessary and sufficient that for each $\varepsilon > 0$ and each $x \in A$, there corresponds an index $n_0(\varepsilon, x)$ such that for all $m, n \geq n_0$, $|f_m(x) - f_n(x)| < \varepsilon$ holds.*

- *For the sequence $(f_n)_{n\in\mathbb{N}}$ to be uniformly convergent on A, it is necessary and sufficient that for each $\varepsilon > 0$, there corresponds an index $n_0(\varepsilon)$, independent of x, such that if $m, n \geq n_0$, $|f_m(x) - f_n(x)| < \varepsilon$ holds, for any $x \in A$.*

Let's now look at some considerations about series of functions.

Definition 9.5 *Let $A \subset \mathbb{R}$ be a set, and suppose that to each $n \in \mathbb{N}$ is associated a function $f_n : A \to \mathbb{R}$. We then have a sequence $(f_n)_{n\in\mathbb{N}}$ of functions defined on A. The series of functions defined by the sequence $(f_n)_{n\in\mathbb{N}}$ is the sequence of partial sums $(s_n)_{n\in\mathbb{N}}$:*

$$
\begin{aligned}
s_1(x) &= f_1(x)\\
s_2(x) &= f_1(x) + f_2(x)\\
&\cdots\\
s_n(x) &= f_1(x) + \cdots + f_n(x).
\end{aligned}
$$

We write:

$$\sum_{n=1}^{\infty} f_n \quad or \quad \sum_{n=1}^{\infty} f_n(x)$$

Example 1: Let $f_n(x) = x^n$, $x \in (-1,1)$; $n = 0,1,2,\ldots$. Then:

$$1 + x + x^2 + \cdots + x^n + \cdots = \sum_{n=0}^{\infty} x^n.$$

Example 2: Let $f_n(x) = \frac{\sin(nx)}{n^2}$, $x \in \mathbb{R}$; $n = 1,2,3,\cdots$ We have:

$$\sin(x) + \frac{\sin(2x)}{4} + \frac{\sin(3x)}{9} + \cdots + \frac{\sin(nx)}{n^2} + \cdots = \sum_{n=1}^{\infty} \frac{\sin(nx)}{n^2}.$$

To say that the series of functions $\sum_{n=1}^{\infty} f_n(x)$ converges on A to a function $s : A \to \mathbb{R}$, or that $\sum_{n=1}^{\infty} f_n$ has the function s as its sum, is equivalent to saying that for every $x \in A$ the numerical series $\sum_{n=1}^{\infty} f_n(x)$ converges to $s(x)$. This is pointwise convergence.

What generally occurs is that n_0 varies with ε and x. We say that the series of functions $\sum_{n=1}^{\infty} f_n(x)$ converges uniformly on A to a function $s(x)$ when for each $\varepsilon > 0$, there corresponds an index $n_0(\varepsilon)$, independent of x, such that if $n \geq n_0$, then $|(f_1(x) + \cdots + f_n(x)) - s(x)| < \varepsilon$, for any $x \in A$.

Definition 9.6 (Cauchy Condition for series of functions) *For the series $\sum_{n=1}^{\infty} f_n(x)$ to be pointwise convergent on the set A, it is necessary and sufficient that for each $\varepsilon > 0$ and each $x \in A$, there corresponds an index $n_0(\varepsilon, x)$ such that if $m > n \geq n_0$ then $|s_m(x) - s_n(x)| < \varepsilon$.*

For the series $\sum_{n=1}^{\infty} f_n(x)$ to be uniformly convergent on the set A, it is necessary and sufficient that for each $\varepsilon > 0$, there corresponds an index $n_0(\varepsilon)$, independent of x, such that if $m > n \geq n_0$ then $|s_m(x) - s_n(x)| < \varepsilon$, for any $x \in A$.

Let $m = n + p$, $p \in \mathbb{N}$. The condition is written:

$$\begin{aligned} & |s_{n+p}(x) - s_n(x)| \\ = & |(f_1(x) + \cdots + f_n(x) + f_{n+1}(x) + \cdots + f_{n+p}(x)) - (f_1(x) + \cdots + f_n(x))| \\ = & |f_{n+1}(x) + \cdots + f_{n+p}(x)| < \varepsilon, \ \forall x \in A, \ \forall n \geq n_0, \forall p \in \mathbb{N}. \end{aligned}$$

Thus, for the series $\sum_{n=1}^{\infty} f_n(x)$ to be uniformly convergent on A, it is necessary and sufficient that for each $\varepsilon > 0$, there corresponds an index $n_0(\varepsilon)$ such that for $n \geq n_0$ and $p \in \mathbb{N}$ we have:

$$|f_{n+1}(x) + \cdots + f_{n+p}(x)| < \varepsilon, \ \forall x \in A.$$

Theorem 9.7 (Weierstrass M-Test) *Suppose $\{f_n(x)\}_{n\in\mathbb{N}}$ is a sequence of functions defined on a set $E \subset \mathbb{R}$. Let $M_n \geq 0$ be such that the numerical series $\sum_{n=1}^{\infty} M_n$ is convergent. Suppose that for the series $\sum_{n=1}^{\infty} f_n(x)$, we have $|f_n(x)| \leq M_n$ for all $n \in \mathbb{N}$ and all $x \in E$. Then the series $\sum_{n=1}^{\infty} f_n(x)$ converges uniformly (and absolutely) on E.*

Proof: Since $\sum_{n=1}^{\infty} M_n < +\infty$, this series satisfies the Cauchy Criterion for numerical series. Given $\varepsilon > 0$, there exists $n_0(\varepsilon)$ such that for all $n \geq n_0$ and $p \in \mathbb{N}$:

$$M_{n+1} + \cdots + M_{n+p} < \varepsilon.$$

Now, for the series of functions, we check the uniform Cauchy Criterion:

$$\left|\sum_{i=1}^{p} f_{n+i}(x)\right| \leq \sum_{i=1}^{p} |f_{n+i}(x)| \leq \sum_{i=1}^{p} M_{n+i} < \varepsilon.$$

This inequality holds for all $n \geq n_0$, all $p \in \mathbb{N}$, and all $x \in E$. Thus, the series $\sum f_n(x)$ converges uniformly in E. ■

Example 1: Let

$$\sum_{n=1}^{\infty} f_n(x), \ f_n(x) = \frac{\sin(nx)}{n^2}, \ n = 1, 2, \cdots \text{ and } x \in \mathbb{R}.$$

We know that

$$\sum_{n=1}^{\infty} \frac{1}{n^2} < +\infty \quad (p\text{-series with } p = 2 > 1).$$

But,

$$|f_n(x)| = \left|\frac{\sin(nx)}{n^2}\right| \leq \frac{1}{n^2} = M_n, \ \forall x \in \mathbb{R}, \ \forall n \in \mathbb{N}.$$

Applying the Weierstrass M-Test, the series $\sum_{n=1}^{\infty} \frac{\sin(nx)}{n^2}$ converges uniformly on $\mathbb{R}$.

Example 2: Let:

$$\sum_{n=1}^{\infty} f_n(x), \ f_n(x) = \frac{\cos(n^2 x)}{n^{3/2}}, \ x \in \mathbb{R}, \ n \in \mathbb{N}.$$

We know that

$$\sum_{n=1}^{\infty} \frac{1}{n^{3/2}} \text{ is convergent } (p = 3/2 > 1).$$

Since

$$|f_n(x)| \leq \frac{1}{n^{3/2}} = M_n, \ \forall x \in \mathbb{R}, \ \forall n \in \mathbb{N},$$

it follows that the series converges uniformly by the Weierstrass M-Test.

Theorem 9.8 *Suppose $f_n(x) \to f(x)$ pointwise for all $x \in E$. Set:*

$$M_n := \sup_{x \in E} |f_n(x) - f(x)|.$$

Then, $f_n \to f$ uniformly on E if and only if $M_n \to 0$ as $n \to +\infty$.

Proof: ($\Rightarrow$) Suppose $f_n \to f$ uniformly on E. We must show that $\lim M_n = 0$. Given $\varepsilon > 0$, by the definition of uniform convergence, there exists an index $n_0(\varepsilon)$ (independent of x) such that for all $n \geq n_0$, we have

$$|f_n(x) - f(x)| < \varepsilon, \quad \forall x \in E.$$

This means that ε is an upper bound for the set $\{|f_n(x) - f(x)| : x \in E\}$ (for $n \geq n_0$). By definition of the supremum,

$$M_n = \sup_{x \in E} |f_n(x) - f(x)| \leq \varepsilon.$$

Since $M_n \geq 0$, we have $0 \leq M_n \leq \varepsilon$ for all $n \geq n_0$. This is the definition of $M_n \to 0$.

($\Leftarrow$) Conversely, suppose $\lim_{n\to\infty} M_n = 0$. We must show $f_n \to f$ uniformly. Given $\varepsilon > 0$, since $M_n \to 0$, there exists an index $n_0(\varepsilon)$ such that for all $n \geq n_0$, we have $|M_n - 0| = M_n < \varepsilon$. By definition of M_n:

$$\sup_{x \in E} |f_n(x) - f(x)| < \varepsilon, \quad \forall n \geq n_0.$$

Since the supremum is less than ε, every element must also be:

$$|f_n(x) - f(x)| \leq \sup_{x \in E} |f_n(x) - f(x)| < \varepsilon, \ \forall x \in E, \ \forall n \geq n_0.$$

This is the definition of uniform convergence. ■

9.2 Uniform Convergence and Continuity

Let $(f_n)_{n\in\mathbb{N}}$ be a sequence of real functions defined and continuous on A converging on A to a function f. One asks: Is the function f also continuous? The answer is no.

Counter-example: $f_n(x) = x^n$, $x \in [0,1]$.

We have already seen that

$$f_n(x) \to f(x) = \begin{cases} 0, & \text{if } 0 \leq x < 1, \\ 1, & \text{if } x = 1, \end{cases}$$

which is not continuous on $[0,1]$.

Another question that arises: Let $f_n(x)$ be a sequence of real functions defined and continuous on A converging to a function f. Is the following relation always true:

$$\lim_{x\to x_0}\left(\lim_{n\to\infty} f_n(x)\right) = \lim_{n\to\infty}\left(\lim_{x\to x_0} f_n(x)\right)?$$

The answer is no. Consider as before $f_n(x) = x^n$, $x \in [0,1]$. It is easy to see that if $x \in (0,1)$ then:

$$0 = \lim_{x\to 1,\ x<1}\left(\lim_{n\to\infty} f_n(x)\right) \neq \lim_{n\to\infty}\left(\lim_{x\to 1,\ x<1} f_n(x)\right) = 1.$$

Under what conditions are such questions always true?

Proposition 9.9 *Let $A \subset \mathbb{R}$ and $(f_n(x))_{n\in\mathbb{N}}$ be a sequence of real functions on A converging uniformly on A to a function f. If the $(f_n(x))_{n\in\mathbb{N}}$ are continuous on A, then f is continuous on A.*

Proof: We must show that f is continuous on A. Therefore let $x_0 \in A$ and $\varepsilon > 0$. Since $(f_n(x))_{n\in\mathbb{N}}$ converges uniformly on A, then for the given $\varepsilon > 0$ there exists $n_0(\varepsilon)$ such that for all $n \in \mathbb{N}$ with $n \geq n_0$ we have $|f_n(x) - f(x)| < \varepsilon/3$, for any $x \in A$.

On the other hand, the $(f_n(x))_{n\in\mathbb{N}}$ are continuous for all $n \in \mathbb{N}$. In particular, for $n = n_0$, it follows that f_{n_0} is also continuous at x_0. Thus, for the given $\varepsilon > 0$, there exists $\delta(\varepsilon, x)$ such that if $x \in A$ with $|x - x_0| < \delta$ then $|f_{n_0}(x) - f_{n_0}(x_0)| < \varepsilon/3$.

From the uniform convergence above we have that:

$$|f_n(x) - f(x)| < \varepsilon/3,\ \forall x \in A,\ \forall n \geq n_0.$$

In particular, for $n = n_0$ and $x = x_0$ we have $|f_{n_0}(x_0) - f(x_0)| < \varepsilon/3$.

However, for the given $\varepsilon > 0$ there exists $\delta(\varepsilon, x)$ such that if $x \in A$ and $|x - x_0| < \delta$ we have:

$$|f(x) - f(x_0)| = |f(x) - f_{n_0}(x) + f_{n_0}(x) - f_{n_0}(x_0) + f_{n_0}(x_0) - f(x_0)|$$
$$\leq |f(x) - f_{n_0}(x)| + |f_{n_0}(x) - f_{n_0}(x_0)| + |f_{n_0}(x_0) - f(x_0)| < \varepsilon/3 + \varepsilon/3 + \varepsilon/3 = \varepsilon,$$

which concludes the proof. ■

Note that according to the hypotheses of Proposition 9.9, the following relation holds:

$$\lim_{x\to x_0}\left(\lim_{n\to\infty} f_n(x)\right) = \lim_{n\to\infty}\left(\lim_{x\to x_0} f_n(x)\right),$$

where $(f_n(x))_{n\in\mathbb{N}}$ is a sequence of functions defined on A and converging uniformly to a function f.

Indeed,

We have

$$\text{Since } f_n \to f \text{ uniformly on } A \Rightarrow \lim_{n\to\infty} f_n(x) = f(x),\ \forall x \in A. \tag{9.1}$$

On the other hand, since f is continuous on A, it follows that

$$\lim_{x\to x_0} f(x) = f(x_0). \text{ But } f(x_0) = \lim_{n\to\infty} f_n(x_0).$$

Thus,

$$\lim_{x\to x_0} f(x) = \lim_{n\to\infty} f_n(x_0). \tag{9.2}$$

Since the $(f_n)_n$ are continuous for all $n \in \mathbb{N}$ it follows that:

$$\lim_{x\to x_0} f_n(x) = f_n(x_0),\ \forall n \in \mathbb{N}. \tag{9.3}$$

From (9.1), (9.2) and (9.3) we can write:

$$\lim_{x\to x_0}\left(\lim_{n\to\infty} f_n(x)\right) = \lim_{x\to x_0} f(x) = f(x_0),$$
$$\lim_{n\to\infty}\left(\lim_{x\to x_0} f_n(x)\right) = \lim_{n\to\infty} f_n(x_0) = f(x_0),$$

which proves the desired result.

Conclusion: If we have a family of continuous functions $(f_n(x))_n$ converging uniformly to a function f on a set A, then:

1) The function f will also be continuous.

2) One can interchange the limits, i.e.,

$$\lim_{n\to\infty}\left(\lim_{x\to x_0} f_n(x)\right) = \lim_{x\to x_0}\left(\lim_{n\to\infty} f_n(x)\right),\ \forall x_0 \in A.$$

If the function f is not continuous it follows that the convergence will not be uniform.

9.3 Uniform Convergence and Differentiation

Question: Let $(f_n)_{n\in\mathbb{N}}$ be a sequence of real functions converging on a set A to a function f. Suppose that the derivatives of f_n exist at the points $x \in A$, i.e., $f_n'(x)$ exists for any $n \in \mathbb{N}$ and for all $x \in A$. One asks: does $f_n' \to g$ on A and $g = f'$ on A? The answer is no. Counter-example: let $f_n(x) = \frac{\sin(nx)}{\sqrt{n}}$, $x \in \mathbb{R}$. We have already seen that $f_n(x) \to 0$ uniformly, which implies that $g(x) = 0$ and $g'(x) = 0$, for all $x \in \mathbb{R}$. On the other hand

$$f_n'(x) = \frac{1}{\sqrt{n}} n\cos(nx) = \sqrt{n}\cos(nx), \ \forall n.$$

However, for $x = 0$, $f_n'(x) = \sqrt{n}$ diverges as $n \to +\infty$, which implies that there exists an $x_0 = 0$ such that $f_n'(0)$ does not converge to zero. As we see, $f_n'(x)$ does not converge to zero for all $x \in \mathbb{R}$. At the points $x = 2k\pi$, $k \in \mathbb{Z}$, this does not happen as we would like.

Proposition 9.10 *Let $(f_n(x))_{n\in\mathbb{N}}$ be a sequence of differentiable functions on $[a,b]$. Suppose that $\{f_n(x_0)\}$ converges for some $x_0 \in [a,b]$. If $f_n' \to g$ uniformly on $[a,b]$, then $f_n \to f$ uniformly on $[a,b]$ and $f' = g$.*

Proof: Let $\varepsilon' > 0$. We have, by hypothesis, that there exists $x_0 \in [a,b]$ such that $(f_n(x_0))_{n\in\mathbb{N}}$ converges. Thus, by the Cauchy Criterion for numerical sequences, we can write: For the given $\varepsilon' > 0$, there exists $N_1(\varepsilon')$ such that for all $m, n \in \mathbb{N}$ with $m, n \geq N_1$, then

$$|f_m(x_0) - f_n(x_0)| < \varepsilon'. \tag{9.4}$$

On the other hand, since $(f_n'(x))_{n\in\mathbb{N}}$ converges uniformly on $[a,b]$, by the Cauchy Criterion applied to uniform convergence we have: for the given $\varepsilon' > 0$ there exists $N_2(\varepsilon')$ such that for all $m, n \in \mathbb{N}$, with $m, n \geq N_2$, then

$$|f_m'(x) - f_n'(x)| < \varepsilon', \quad \text{for all } x \in [a,b]. \tag{9.5}$$

However, we must show that:
(i) $f_n \to f$ uniformly on $[a,b]$.
(ii) If $f_n' \to g$ then $f' = g$.

(i) Let us define $h(x) = f_m(x) - f_n(x)$, $x \in [a,b]$. Since the f_n are differentiable on $[a,b]$, it follows that $h(x)$ is also differentiable and, moreover, $h'(x) = f_m'(x) - f_n'(x)$. Now take $r, s \in [a,b]$ such that $r \neq s$. Observe that h satisfies the hypotheses of the Mean Value Theorem. Thus, there exists $t_0 \in [a,b]$ such that $|h(s) - h(r)| = |h'(t_0)|\,|s-r|$. Let $N = \max\{N_1, N_2\}$. Thus, from (9.4) and (9.5) it results for the given $\varepsilon > 0$ that

$$|f_m(x_0) - f_n(x_0)| < \frac{\varepsilon}{2}, \forall m, n \geq N, \tag{9.6}$$

$$|f'_m(x) - f'_n(x)| < \frac{\varepsilon}{2(b-a)}, \quad \forall x \in [a,b]. \tag{9.7}$$

Thus, from $|h(s) - h(r)| = |h'(t_0)|\,|s-r|$ and from (9.7) we can write that

$$\begin{aligned} &|(f_m(s) - f_n(s)) - (f_m(r) - f_n(r))| = |f'_m(t_0) - f'_n(t_0)|\,|s-r| \qquad (9.8)\\ &< \frac{\varepsilon}{2(b-a)}|s-r| \leq \frac{\varepsilon(b-a)}{2(b-a)} = \frac{\varepsilon}{2}, \ \forall s, r \in (a,b) \text{ with } s \neq r, \ \forall m, n \geq N. \end{aligned}$$

Now take $x \in [a,b]$ such that $x \neq x_0$. We have from (9.6) and (9.8) that:

$$\begin{aligned} &|f_m(x) - f_n(x)| \qquad (9.9)\\ &= |f_m(x) - f_m(x_0) + f_m(x_0) - f_n(x_0) + f_n(x_0) - f_n(x)|\\ &= |(f_m(x) - f_n(x)) + (f_n(x_0) - f_m(x_0)) + (f_m(x_0) - f_n(x_0))|\\ &= |(f_m(x) - f_n(x)) - (f_m(x_0) - f_n(x_0)) + (f_m(x_0) - f_n(x_0))|\\ &\leq |(f_m(x) - f_n(x)) - (f_m(x_0) - f_n(x_0))| + |f_m(x_0) - f_n(x_0)|\\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon, \ \forall x \in [a,b], \ \forall m, n \geq N. \end{aligned}$$

From (9.6) and (9.9) we conclude that f_n converges uniformly due to the Cauchy Criterion since N only depends on ε. Let us denote this limit by f, i.e., $\lim_{n\to\infty} f_n(x) = f(x)$ uniformly on $[a,b]$.

(ii) It remains for us to show that if $f'_n \to g$ then $f' = g$. Indeed, Fix $x \in [a,b]$ and define:

$$\begin{cases} \Phi_n(t) = \dfrac{f_n(t) - f_n(x)}{t-x}; \ t \neq x, \ t \in [a,b],\\ \Phi(t) = \dfrac{f(t) - f(x)}{t-x}; \ t \neq x, \ t \in [a,b]. \end{cases}$$

Then:

$$\lim_{t\to x} \Phi_n(t) = \lim_{t\to x} \frac{f_n(t) - f_n(x)}{t-x} = f'_n(x),$$

since the f_n's are differentiable for all n.

Furthermore, from (9.7) and (9.8)

$$\begin{aligned} &|\Phi_n(t) - \Phi_m(t)| = \left|\frac{f_n(t) - f_n(x)}{t-x} - \frac{f_m(t) - f_m(x)}{t-x}\right|\\ &= \frac{1}{|t-x|}\ |(f_n(t) - f_m(t)) - (f_n(x) - f_m(x))|\\ &\leq \frac{1}{|t-x|}\,|f'_n(t_0) - f'_m(t_0)|\,|t-x| < \frac{\varepsilon}{2(b-a)}, \ \forall t \in [a,b], \quad \text{with } t \neq x, \end{aligned}$$

which implies that (Φ_n) converges uniformly for all $t \in [a, b]$ to some function Ψ with $t \neq x$ (and for some $t_0 \in (t, x)$).

On the other hand, since $f_n(x)$ converges uniformly to $f(x)$ for all $x \in [a, b]$, we conclude that for each $t \in [a, b]$, we have:

$$\lim_{n\to\infty} \Phi_n(t) = \Phi(t).$$

Indeed, we have:

$$\lim_{n\to\infty} \Phi_n(t) = \lim_{n\to\infty} \frac{f_n(t) - f_n(x)}{t - x} = \frac{1}{t - x} \lim_{n\to\infty} (f_n(t) - f_n(x)).$$

But,

$$\lim_{n\to\infty} f_n(t) = f(t) \quad \text{and} \quad \lim_{n\to\infty} f_n(x) = f(x).$$

Hence:

$$\lim_{n\to\infty} \Phi_n(t) = \frac{1}{t - x}(f(t) - f(x)) = \frac{f(t) - f(x)}{t - x} = \Phi(t).$$

From the above, Φ_n converges pointwise to Φ. By the uniqueness of the limit $\Phi_n \to \Phi$ uniformly $\forall t \in [a, b]$. It follows, by the limit interchange theorem (Theorem of iterability of limits), that:

$$\underbrace{\lim_{n\to\infty} f_n'(x)}_{=g(x)} = \lim_{n\to\infty} \lim_{t\to x} \Phi_n(t) = \lim_{t\to x} \lim_{n\to\infty} \Phi_n(t) = \lim_{t\to x} \Phi(t) = f'(x).$$

■

9.4 Uniform Convergence and Integration

Question: Let $(f_n)_{n\in\mathbb{N}}$ be a sequence of real functions defined and integrable on $[a, b]$ and converging on $[a, b]$ to a function f. One asks: Would f be integrable on $[a, b]$? And what about the relation:

$$\int_a^b f(x)\, dx = \int_a^b \left(\lim_{n\to\infty} f_n(x)\right) dx = \lim_{n\to\infty} \int_a^b f_n(x)\, dx \ ?$$

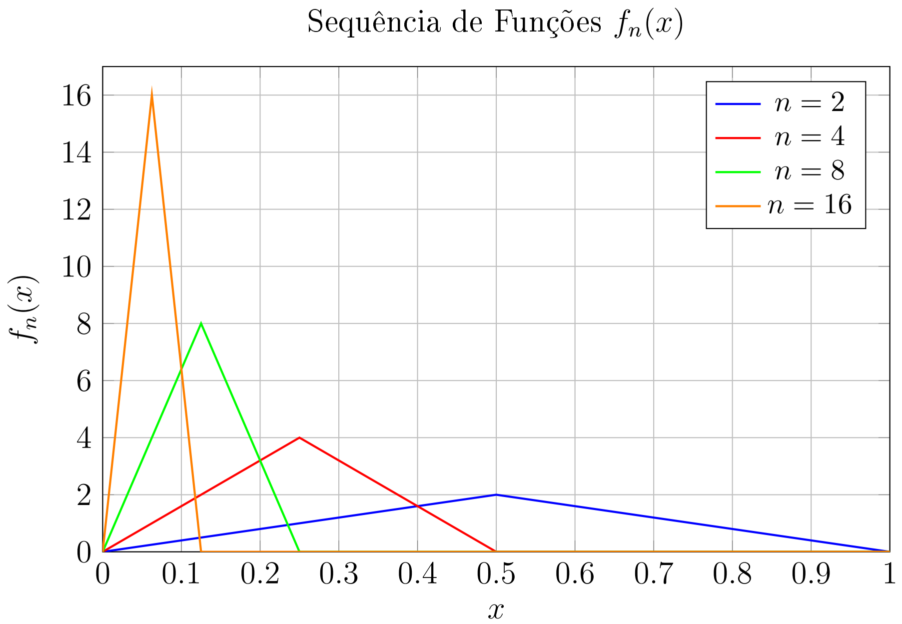
Answer: It is not true. Counter-example: Consider:

$$f_n(x) = \begin{cases} n^2\, x; & 0 \leq x \leq \dfrac{1}{n}, \\ -\, n^2(x - 2/n); & \dfrac{1}{n} \leq x \leq \dfrac{2}{n},\ n \geq 2, \\ 0, & \dfrac{2}{n} \leq x \leq 1. \end{cases}$$

It is not difficult to verify that $\int_0^1 f_n(x)\,dx = 1$ (area of a triangle with base $2/n$ and height n) and, therefore,

$$\lim_{n\to\infty} \int_0^1 f_n(x)\,dx = 1.$$

Find, below, the graph of the above sequence of functions $f_n(x)$ for different values of the parameter n.



On the other hand, for any $x \in (0,1]$, $f_n(x) = 0$ for large enough n, and $f_n(0) = 0$. Thus $f_n \to 0$ pointwise.

$$\int_0^1 (\lim_{n\to\infty} f_n(x))\,dx = \int_0^1 0\,dx = 0.$$

Therefore the limits are different (verify this fact).

Proposition 9.11 *Let $(f_n)_{n\in\mathbb{N}}$ be a sequence of real functions such that the integral $\int_a^b f_n(x)\,dx$ exists for every $n \in \mathbb{N}$. If $f_n \to f$ uniformly on $[a,b]$, then the integral $\int_a^b f(x)\,dx$ exists and, furthermore,*

$$\int_a^b f(x)\,dx = \int_a^b \left(\lim_{n\to\infty} f_n(x)\right)\,dx = \lim_{n\to\infty} \int_a^b f_n(x)\,dx.$$

Proof: Let $\varepsilon_n = \sup\{|f_n(x) - f(x)|;\ x \in [a,b]\}$. Since $f_n \to f$ uniformly on $[a,b]$, then $\varepsilon_n \to 0$ as $n \to +\infty$ as seen previously (See Theorem 9.8). In this way, we have:

$$|f_n(x) - f(x)| \le \varepsilon_n, \quad \text{for any } n \text{ and for all } x \in [a,b].$$

Thus,

$$f_n(x) - \varepsilon_n \leq f(x) \leq f_n(x) + \varepsilon_n, \ \forall x \in [a, b], \ \forall n \in \mathbb{N}. \tag{9.10}$$

From (9.10) it follows that

$$\underline{\int_a^b} (f_n(x) - \varepsilon_n)\, dx \leq \underline{\int_a^b} f(x)\, dx \quad \text{and} \quad \overline{\int_a^b} f(x)\, dx \leq \overline{\int_a^b} (f_n(x) + \varepsilon_n)\, dx,$$

which implies

$$\int_a^b (f_n(x) - \varepsilon_n)\, dx \leq \underline{\int_a^b} f(x)\, dx \leq \overline{\int_a^b} f(x)\, dx \leq \int_a^b (f_n(x) + \varepsilon_n)\, dx. \tag{9.11}$$

Let $a \leq b \leq c \leq d$. We claim that $0 \leq c - b \leq d - a$. To prove the desired result, it suffices to add $-b$ to the inequality $b \leq c \leq d$, resulting in $0 \leq c-b \leq d-b \leq d-a$, which proves the claim. Thus, from the above, and from (9.11), we can write that:

$$0 \leq \overline{\int_a^b} f(x)\, dx - \underline{\int_a^b} f(x)\, dx \leq \int_a^b (f_n(x) + \varepsilon_n)\, dx - \int_a^b (f_n(x) - \varepsilon_n)\, dx \leq 2\varepsilon_n(b - a).$$

(Note: The bound is $2\varepsilon_n(b-a)$ because $\int(f_n+\varepsilon_n) - \int(f_n - \varepsilon_n) = \int 2\varepsilon_n = 2\varepsilon_n(b-a)$)

Since $\varepsilon_n \to 0$ as $n \to +\infty$, from the last inequality it follows that $\overline{\int_a^b} f(x)\, dx = \underline{\int_a^b} f(x)\, dx$, resulting in f being integrable on $[a, b]$.

Returning to (9.10), we can write:

$$\int_a^b (f_n(x) - \varepsilon_n)\, dx \leq \int_a^b f(x)\, dx \leq \int_a^b (f_n(x) + \varepsilon_n)\, dx,$$

from where it follows that

$$0 \leq \int_a^b f(x)\, dx - \int_a^b (f_n(x) - \varepsilon_n)\, dx \leq \int_a^b (f_n(x) + \varepsilon_n)\, dx - \int_a^b (f_n(x) - \varepsilon_n)\, dx,$$

i.e.,

$$0 \leq \int_a^b f(x)\, dx - \int_a^b f_n(x)\, dx + \varepsilon_n(b - a) \leq 2\varepsilon_n(b - a),$$

or rather

$$-\varepsilon_n(b - a) \leq \int_a^b f(x)\, dx - \int_a^b f_n(x)\, dx \leq \varepsilon_n(b - a).$$

This last inequality entails that:

$$0 \leq \left| \int_a^b f(x)\, dx - \int_a^b f_n(x)\, dx \right| \leq \varepsilon_n(b - a).$$

Taking the limit in the inequality above, we obtain:

$$\lim_{n \to \infty} \int_a^b f_n(x)\, dx = \int_a^b f(x)\, dx.$$

■

Corollary 9.12 *Let f_n be a sequence of integrable functions on $[a, b]$. If $f(x) = \sum_{n=1}^{\infty} f_n(x)$, $x \in [a, b]$ converges uniformly on $[a, b]$, then:*

$$\int_a^b f(x)\,dx = \sum_{n=1}^{\infty} \int_a^b f_n(x)\,dx.$$

Proof: The Corollary above expresses that:

$$\int_a^b f(x)\,dx = \int_a^b \sum_n f_n(x)\,dx = \sum_n \int_a^b f_n(x)\,dx,$$

that is, it is permitted to integrate a uniformly convergent series term by term. ■

Theorem 9.13 *Let a be an accumulation point of a set $X \subset \mathbb{R}$. If the sequence of functions $f_n : X \to \mathbb{R}$ converges uniformly to $f : X \to \mathbb{R}$ and, for each $n \in \mathbb{N}$, $L_n = \lim_{x\to a} f_n(x)$ exists, then:*

(i) The limit $L = \lim_{n\to\infty} L_n$ exists.

(ii) We have $L = \lim_{x\to a} f(x)$.

In other words:

$$\lim_{n\to\infty}\left(\lim_{x\to a} f_n(x)\right) = \lim_{x\to a}\left(\lim_{n\to\infty} f_n(x)\right),$$

holds, provided the two limits inside the parentheses exist, the second of them being uniform.

Proof: To show that $\lim_{n\to\infty} L_n$ exists, it suffices to prove that $(L_n)_{n\in\mathbb{N}}$ is a Cauchy sequence. Let then $\varepsilon > 0$. Since $f_n \to f$ uniformly on X, there exists $n_0(\varepsilon)$ such that if $m, n \geq n_0$ then $|f_m(x) - f_n(x)| < \varepsilon/3$ for any $x \in X$.

Let $m, n > n_0$. We can obtain $x \in X$ and $\delta = \delta(\varepsilon, x) > 0$ such that

$$|L_m - f_m(x)| < \frac{\varepsilon}{3} \text{ and } |f_n(x) - L_n| < \frac{\varepsilon}{3}, \text{ for } 0 < |x - a| < \delta. \tag{9.12}$$

Indeed, since $\lim_{x\to a} f_n(x) = L_n$ exists for each $n \in \mathbb{N}$; then, for fixed $m, n > n_0$ and for the given $\varepsilon > 0$, there exist $\delta_1 > 0$ and $\delta_2 > 0$ such that for all $x \in X$ with $0 < |x - a| < \delta_1$ we have $|f_n(x) - L_n| < \frac{\varepsilon}{3}$, and also if $0 < |x - a| < \delta_2$ we obtain $|f_m(x) - L_m| < \frac{\varepsilon}{3}$, which proves (9.12) (taking $\delta = \min\{\delta_1, \delta_2\}$).

With this choice of x, we can write

$$\begin{aligned} |L_m - L_n| &\leq |L_m - f_m(x)| + |f_m(x) - f_n(x)| + |f_n(x) - L_n| \\ &< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon, \ \forall m, n \geq n_0, \end{aligned}$$

which implies that $(L_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in $\mathbb{R}$. Thus, it is convergent. Let us denote

$$\lim_{n\to\infty} L_n = L.$$

We will now show that the function

$$f = \lim_{n\to\infty} f_n$$

has limit L as $x \to a$. Indeed, for the given $\varepsilon > 0$, there exists n_0 such that for $n > n_0$ we have:

$$|L - L_n| < \frac{\varepsilon}{3} \quad \text{and} \quad |f_n(x) - f(x)| < \frac{\varepsilon}{3},$$

the first due to the convergence $L_n \to L$ and the second due to the uniform convergence $f_n \to f$, for all $x \in X$. Let us fix an n greater than n_0. Since $\lim_{x\to a} f_n(x) = L_n$, there exists $\overline{\delta} > 0$ such that if $x \in X$ and $0 < |x - a| < \overline{\delta}$ we have

$$|f_n(x) - L_n| < \frac{\varepsilon}{3}.$$

I claim: For the given $\varepsilon > 0$, there exists $\overline{\delta} > 0$ such that for all $x \in X$ with $0 < |x - a| < \overline{\delta}$ we have $|f(x) - L| < \varepsilon$. Indeed, for x satisfying these conditions, it follows from the above that:

$$|f(x) - L| \leq |f(x) - f_n(x)| + |f_n(x) - L_n| + |L_n - L| < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon,$$

which concludes the proof. ∎

Corollary 9.14 *Let a be an accumulation point of X. If the series $\sum_{n=1}^{\infty} f_n(x)$ converges uniformly to f on X, and for each $n \in \mathbb{N}$, $L_n = \lim_{x\to a} f_n(x)$ exists, then the series $\sum_{n=1}^{\infty} L_n$ is convergent and $\sum_n L_n = \lim_{x\to a} f(x)$. In other words, the classical theorem for the limit of a sum holds for series:*

$$\lim_{x\to a}\left[\sum_n f_n(x)\right] = \sum_n \left[\lim_{x\to a} f_n(x)\right],$$

provided that $\sum_n f_n(x)$ is uniformly convergent.

Proof: Indeed, setting

$$s_n(x) = f_1(x) + \cdots + f_n(x),$$

the sequence of functions $s_n : X \to \mathbb{R}$ converges uniformly to f on X and, for each $n \in \mathbb{N}$, there exists

$$\lim_{x\to a} s_n(x) = \sum_{i=1}^{n} \lim_{x\to a} f_i(x).$$

The last theorem applies immediately to produce the desired result. ■

Remark: The previous result still holds when $a = +\infty$. This evidently assumes that X is unbounded above. In this case:

$$\lim_{n\to\infty}\left(\lim_{x\to\infty} f_n(x)\right) = \lim_{x\to\infty}\left(\lim_{n\to\infty} f_n(x)\right),$$

provided the limits inside the parentheses exist, the second of them being uniform. The proof is practically the same. Only at the end, instead of taking $\delta > 0$, take $A > 0$ such that if $x > A$ we have $|f_n(x) - L_n| < \frac{\varepsilon}{3}$.

Theorem 9.15 *If $f_n \to f$ uniformly on X and all f_n are continuous at a point $a \in X$, then f is continuous at a.*

Proof: If a is an isolated point, the demonstration is obvious. Otherwise, since a is an accumulation point of X, the previous proposition allows us to write:

$$\lim_{x\to a} f(x) = \lim_{x\to a}\left(\lim_{n\to\infty} f_n(x)\right) = \lim_{n\to\infty}\left(\lim_{x\to a} f_n(x)\right) = \lim_{n\to\infty} f_n(a) = f(a).$$

■

Corollary 9.16 *Let $\sum_{n=1}^{\infty} f_n(x)$ be a series of differentiable functions on the interval $[a,b]$. If $\sum_{n=1}^{\infty} f_n(x_0)$ converges for a certain $x_0 \in [a,b]$ and the series $\sum_{n=1}^{\infty} f_n'(x) = g$ converges uniformly on $[a,b]$, then $\sum_{n=1}^{\infty} f_n(x) = f(x)$ converges uniformly on $[a,b]$ and f is differentiable, with $f' = g$.*

Theorem 9.17 (Dini's Theorem) *Let K be a compact set in $\mathbb{R}$ and suppose that:*

i) $(f_n(x))_{n\in\mathbb{N}}$ is a sequence of real and continuous functions on K.

ii) $(f_n(x))_{n\in\mathbb{N}}$ converges pointwise on K to a continuous function f on K.

iii) $f_n(x) \geq f_{n+1}(x)$ for all $n \in \mathbb{N}$ and $x \in K$.

Then $f_n(x) \to f(x)$ uniformly on K.

Proof: Let us define, for all $n \in \mathbb{N}$, $g_n(x) = f_n(x) - f(x)$. According to the hypotheses, we have that g_n is continuous, for any $n \in \mathbb{N}$, and furthermore, $g_n \to 0$ pointwise on K. To show that $f_n \to f$ uniformly on K, it suffices to show that $g_n \to 0$ uniformly on K.

Indeed, let $\varepsilon > 0$ be given. For each $n \in \mathbb{N}$ let:

$$K_n = \{x \in K;\ g_n(x) \geq \varepsilon\}.$$

Since each g_n is continuous on K, each K_n is closed in K, because K_n is the inverse image of a closed set ($[\varepsilon, \infty)$) by a continuous function. Hence, K_n is a closed subset of a compact set and therefore K_n is compact for any $n \in \mathbb{N}$.

Since $g_n \geq g_{n+1}$ (because $f_n(x) - f(x) \geq f_{n+1}(x) - f(x)$), then $K_n \supset K_{n+1}$, for any $n \in \mathbb{N}$. Indeed, we have

$$K_{n+1} = \{x \in K : g_{n+1}(x) \geq \varepsilon\} \text{ and } K_n = \{x \in K : g_n(x) \geq \varepsilon\}.$$

Take $z \in K_{n+1}$. Then $z \in K$ and $g_{n+1}(z) \geq \varepsilon$. This implies that $z \in K$ and $g_n(z) \geq g_{n+1}(z) \geq \varepsilon$, from which we conclude that $z \in K$ and $g_n(z) \geq \varepsilon$, i.e., $z \in K_n$, proving the desired inclusion.

Fix $x_0 \in K$. Since $g_n(x_0) \to 0$ we see that $x_0 \notin K_n$ if n is sufficiently large. Indeed, since $g_n(x) \to 0$, given $\varepsilon > 0$, for every $x \in X$ there exists $n_0(\varepsilon, x)$ such that for all $n \geq n_0$, $|g_n(x)| < \varepsilon$. Since $g_n \geq 0$ (from $f_n \geq f_{n+1} \to f$), we have $g_n(x) < \varepsilon$, $\forall n \geq n_0$. Thus, for the given $\varepsilon > 0$ and fixed $x_0 \in X$, there exists $n_0(\varepsilon)$ such that for all $n \geq n_0$ we have $g_n(x_0) < \varepsilon$, from which it follows that $x_0 \notin K_n$, $\forall n \geq n_0$. Since $x_0 \notin \bigcap_{n=n_0}^{\infty} K_n$ and due to the fact that K_n are nested, it follows that $x_0 \notin \bigcap_{n=1}^{\infty} K_n$. Thus, $\bigcap_{n=1}^{\infty} K_n = \emptyset$. Since $\bigcap_{n=1}^{\infty} K_n = \emptyset$, then the intersection of a finite number of $K_n's$ is empty (by the finite intersection property of compact sets), say: $K_{n_1} \cap K_{n_2} \cap \cdots \cap K_{n_p} = \emptyset$ with $n_1 < n_2 < \cdots < n_p$. Since $K_{n_1} \supset K_{n_2} \supset \cdots \supset K_{n_p}$, then $K_{n_p} = K_{n_1} \cap K_{n_2} \cap \cdots \cap K_{n_p} = \emptyset$. It follows from this and the fact that $K_{n_p} \supset K_n$ for all $n \geq n_p$ that K_n is empty for all $n \geq n_p$. From the above

$$\{x \in K : g_n(x) \geq \varepsilon\} = \emptyset, \ \forall n \geq n_p,$$

or, stated another way, there does not exist $x \in K$ such that $g_n(x) \geq \varepsilon$ for all $n \geq n_p$, which implies that for all $x \in K$ we have $0 \leq g_n(x) < \varepsilon$, for all $n \geq n_p$. Therefore $g_n \to 0$ uniformly on K, i.e., $f_n \to f$ uniformly on K, which concludes the proof. ■

Definition 9.18 *If $X \subset \mathbb{R}$ is a set, $\mathcal{C}(X)_b$ represents the set of real functions that are defined, continuous and bounded on X. $\mathcal{C}(X)_b$ is a vector space. Note that boundedness is redundant if X is compact.*

For each $f \in \mathcal{C}(X)_b$, there exists a number $M_f > 0$ such that $|f(x)| \leq M_f$ for all $x \in X$. Thus, for each $f \in \mathcal{C}(X)_b$, let us associate the number:

$$||f|| = \sup\{|f(x)| : x \in X\}.$$

It is not difficult to verify that the map $\mathcal{C}(X)_b \to \mathbb{R}$, defined by $f \mapsto ||f||$ is a norm on $\mathcal{C}(X)_b$.

What does it mean to say that a sequence $(f_n)_{n\in\mathbb{N}}$ converges in $\mathcal{C}(X)_b$ to a function f? This means that to each $\varepsilon > 0$, there corresponds an index $n_0(\varepsilon)$ such that for all $n \geq n_0$, we have $||f_n - f|| < \varepsilon$. But

$$||f_n - f|| = \sup\{|f_n(x) - f(x)| : \ x \in X\} < \varepsilon, \ \forall n \geq n_0.$$

Therefore, given $\varepsilon > 0$, there exists $n_0(\varepsilon)$ such that for all $n \geq n_0$, we have $|f_n(x) - f(x)| < \varepsilon$ for any $x \in E$.

Conclusion: To say that $f_n \to f$ in $\mathcal{C}(X)_b$ is equivalent to saying that $f_n \to f$ uniformly on X. For this reason, $||f|| = \sup_{x\in X}\{|f(x)|\}$ is called the uniform convergence norm (or sup-norm).

Proposition 9.19 *$\mathcal{C}(X)_b$ with this norm is a complete space.*

Proof: Let $(f_n)_{n\in\mathbb{N}}$ be a Cauchy sequence in $\mathcal{C}(X)_b$. Then, given $\varepsilon > 0$, there exists $n_0(\varepsilon)$ such that if $m, n \geq n_0$ we have:

$$||f_n - f_m|| < \varepsilon.$$

But,

$$||f_n - f_m|| = \sup_{x\in E}\{|f_m(x) - f_n(x)|\}.$$

Therefore, ε is an upper bound for the set $\{|f_m(x) - f_n(x)| : \ x \in X\}$, and consequently

$$|f_m(x) - f_n(x)| < \varepsilon, \ \forall x \in X \text{ and } \forall m, n \geq n_0.$$

Then, $(f_n(x))_{n\in\mathbb{N}}$ satisfies the Cauchy condition necessary and sufficient for it to be uniformly convergent on X. Let us denote $\lim_{n\to\infty} f_n(x) = f(x)$. It remains for us to prove that $f \in \mathcal{C}(X)_b$. Indeed, since $f_n \in \mathcal{C}(X)_b$ for all $n \in \mathbb{N}$, we will have that f is also continuous on X (since uniform convergence preserves continuity). However, since $f_n \to f$ uniformly, given $\varepsilon = 1$, there exists $n_0(1)$ such that for all $n \geq n_0$, we have $|f_n(x) - f(x)| < 1$ for any $x \in X$. From this, it follows that:

$$|f(x)| \leq |f_{n_0(1)}(x)| + 1, \ \forall x \in X,$$

and thus

$$|f(x)| \leq M_{f_{n_0(1)}} + 1, \ \forall x \in X.$$

This proves that f is bounded on X. Thus $f \in \mathcal{C}(X)_b$, and since $f_n \to f$ uniformly on X, we have $f_n \to f$ in $\mathcal{C}(X)_b$, which proves that $\mathcal{C}(X)_b$ is complete. ■

9.5 Equicontinuous Families of Functions

Our goal now is to determine under what conditions concerning a set E of continuous functions (all with the same domain) one can guarantee that any sequence with terms in E possesses a convergent subsequence.

If instead of a set of continuous functions we had a subset $E \subset \mathbb{R}$, we would immediately see that, in order for every sequence of points $x_n \in E$ to possess a convergent subsequence, it is necessary and sufficient that E be a bounded set of real numbers. Returning to the set E of continuous functions, we would be tempted to use boundedness as the answer. But this does not happen. Consider the sequence of functions:

$$f_n(x) : [0,1] \to \mathbb{R}, \ f_n(x) = x^n(1 - x^n).$$

It can be verified that $f_n(x)$ converges pointwise to the zero function and that the maximum of $f_n(x)$ is equal to $\frac{1}{4}$ and is attained at $\sqrt[n]{\frac{1}{2}}$ (verify this fact). Note that as $n \to \infty$, $\sqrt[n]{\frac{1}{2}} \to 1$. These facts allow us to sketch the graphs of the functions f_n.

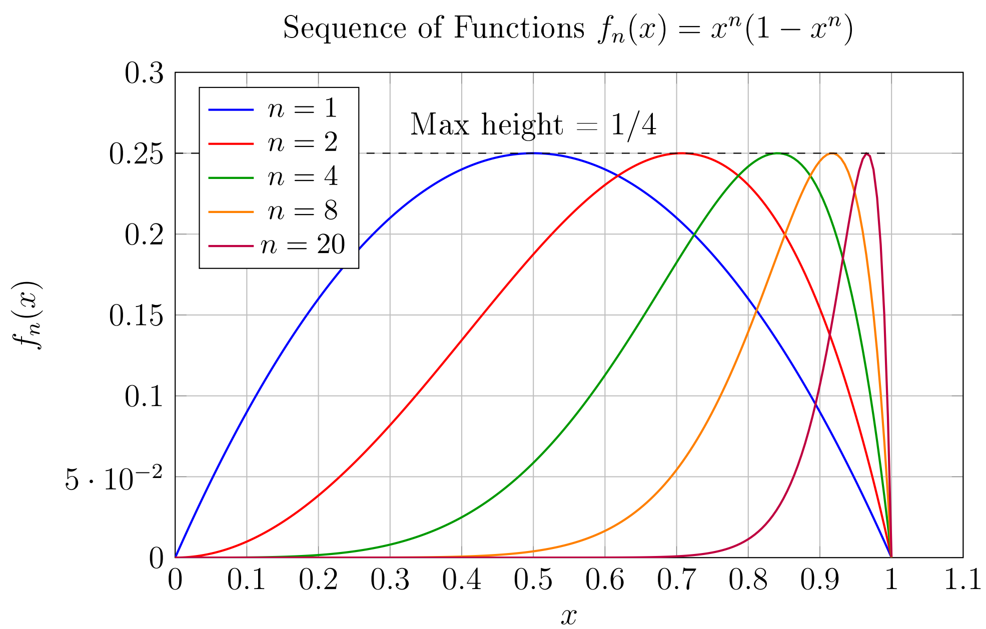


Figure 9.1: Graphs of $f_n(x) = x^n(1 - x^n)$ for various values of n. Note that the maximum height remains 1/4 even as the peak moves to the right.

As can be seen, each graph represents a bump whose height remains equal to $\frac{1}{4}$, so that as $n \to \infty$, the shape of the graph of the limit function does not approximate the shape of the limit function, which is the zero function.

Let us examine the weakness of simple convergence from another angle. Saying that the sequence of functions $f_n : X \to \mathbb{R}$ converges simply (pointwise) to the function $f : X \to \mathbb{R}$ means, formally, the following: given any $\varepsilon > 0$, one can obtain, for each $x \in X$, a number $n_0 = n_0(\varepsilon, x)$ which depends on ε and x, such that if $n > n_0$ then $|f_n(x) - f(x)| < \varepsilon$. Keeping ε fixed, it may perfectly well occur that there is no n_0 that serves simultaneously for all x. The previous example shows a sequence of continuous functions such that $0 \leq f_n(x) \leq 1/4$ for all $x \in [0, 1]$ and for all $n \in \mathbb{N}$. However, $(f_n(x))_{n\in\mathbb{N}}$ does not possess a uniformly convergent subsequence. Indeed, such a subsequence should tend uniformly to zero, which is impossible, since each f_n assumes the value $1/4$ at some point in the interval $[0, 1]$. In other words: We cannot extract a subsequence from $(f_n(x))_{n\in\mathbb{N}}$ such that given $\varepsilon > 0$, all graphs of $(f_{n_k}(x))_{k\in\mathbb{N}}$ are totally within that strip of width ε from some n_{k_0} onwards. It suffices to take $0 < \varepsilon < 1/4$.

It is not sufficient, then, that the functions $f \in E$ take values in the same bounded interval for every sequence in E to possess a uniformly convergent subsequence. An additional hypothesis, which we will introduce next, is needed. To do this, we first need to define two types of boundedness.

Definition 9.20 *(a) A sequence of real functions defined on a set E is pointwise bounded on E when for each $x \in E$, there exists $M_x > 0$ such that $|f_n(x)| \leq M_x$, for all $n \in \mathbb{N}$. Then, letting $\phi : E \to \mathbb{R}$, $\phi(x) = M_x$, $x \in E$, we have $|f_n(x)| \leq \phi(x)$, for any $x \in E$ and for all $n \in \mathbb{N}$.*

(b) $(f_n(x))_{n\in\mathbb{N}}$ is uniformly bounded on E when there exists $M > 0$ such that $|f_n(x)| \leq M$, for any $x \in E$ and for all $n \in \mathbb{N}$.

Our initial intent is to prove, by Cantor's Diagonal Method, that given a pointwise bounded sequence on a countable set E, it has a subsequence converging at each point of E. However, even if $(f_n(x))_{n\in\mathbb{N}}$ is a uniformly bounded sequence of continuous functions on a compact set E, there does not necessarily exist a subsequence that is convergent on E.

Another question is whether every convergent sequence contains a uniformly convergent subsequence. We have seen previously that even if $(f_n(x))_{n\in\mathbb{N}}$ is a uniformly bounded sequence of continuous functions on a compact set, there does not necessarily exist a subsequence that is uniformly convergent on E.

A brief digression.

Let (a_n) be a sequence of real numbers. We say that the sequence (b_n) is a subsequence of (a_n) when there exists a strictly increasing map $\sigma : \mathbb{N}^* \to \mathbb{N}^*$ such that $b_n = a_{\sigma(n)}$ for any $n \in \mathbb{N}^*$.

Example: Consider

$$a_1,\ a_2,\ a_3\ , a_4,\ a_5,\ a_6, \cdots$$

Then

$$b_1 = a_1,\ b_2 = a_3,\ b_3 = a_6, \cdots$$

is a subsequence of (a_n). In this case $\sigma : \mathbb{N}^* \to \mathbb{N}^*$ such that $\sigma(1) = 1,\ \sigma(2) = 3,\ \sigma(3) = 6, etc...$

Suppose (b_n) is a subsequence of (a_n). Then there exists a strictly increasing map $\sigma_1 : \mathbb{N}^* \to \mathbb{N}^*$ such that $b_n = a_{\sigma_1(n)}$. Suppose now that (c_n) is a subsequence of (b_n). Then there exists a strictly increasing map $\sigma_2 : \mathbb{N}^* \to \mathbb{N}^*$ such that

$$c_n = b_{\sigma_2(n)} = a_{\sigma_1(\sigma_2(n))} = a_{(\sigma_1 \circ \sigma_2)(n)}.$$

Then (c_n) is a subsequence of (a_n).

Definition 9.21 *Let $\mathcal{F}$ be a family of functions $f : E \subset \mathbb{R} \to \mathbb{R}$. We say that the family is equicontinuous on E when to each ε corresponds a $\delta > 0$ such that if $x, y \in E$ with $|x - y| \leq \delta$, then $|f(x) - f(y)| < \varepsilon$, for any $f \in \mathcal{F}$.*

Remark: If $\mathcal{F}$ is equicontinuous on E, then besides all functions $f \in \mathcal{F}$ being uniformly continuous, the crucial point is that the δ of uniform continuity is the same for all functions.

The Diagonal Process.

Proposition 9.22 *Let E be a countable set and let $f_n : E \to \mathbb{R}$ ($n \in \mathbb{N}^*$) be pointwise bounded on E. Then there exists a subsequence $(f_{\sigma(n)})$ of $(f_n)_{n\in\mathbb{N}^*}$ that converges pointwise at each $x \in E$.*

Proof: Let us enumerate the points of E as: $x_1, x_2, \cdots, x_n, \cdots$. Let us calculate the values $f_n(x_1)$ of the f_n at x_1 ($n \in \mathbb{N}$). We thus obtain a bounded numerical sequence in $\mathbb{R}$ whose set of values is contained in a compact set of $\mathbb{R}$. Thus, there exists a subsequence $(f_{\sigma_1(n)}(x_1))_{n\in\mathbb{N}}$ convergent in $\mathbb{R}$. Now let us calculate the values of $f_{\sigma_1(n)}$ at x_2. We obtain a bounded sequence $(f_{\sigma_1(n)}(x_2))_{n\in\mathbb{N}}$. Therefore, the set of values of such a sequence is contained in a compact set of $\mathbb{R}$. Thus, there exists a subsequence

$(f_{\sigma_1\circ\sigma_2(n)}(x_2))$ which is convergent in $\mathbb{R}$. Repeating the process indefinitely, we find, for each $n \in \mathbb{N}^*$, a strictly increasing map $\sigma_m : \mathbb{N}^* \to \mathbb{N}^*$ such that

$$\left(f_{\sigma_1\circ\sigma_2\circ\cdots\sigma_m(n)}\right) \text{ is a subsequence of } \left(f_{\sigma_1\circ\sigma_2\circ\cdots\sigma_{m-1}(n)}\right).$$

Note that $\left(f_{\sigma_1\circ\sigma_2\circ\cdots\sigma_m(n)}\right)(x_m)$ is convergent by construction.

Let us denote:

$$g_1 : \; f_{\sigma_1(1)}(x_1) \quad \cdots \quad f_{\sigma_1(m)}(x_1)$$

$$g_2 : \; f_{\sigma_1\circ\sigma_2(1)}(x_2) \quad \cdots \quad f_{\sigma_1\circ\sigma_2(m)}(x_2)$$

$$\vdots$$

$$g_m : \; f_{\sigma_1\circ\cdots\circ\sigma_m(1)}(x_m) \quad \cdots \quad f_{\sigma_1\circ\cdots\sigma_m(m)}(x_m)$$

Let H be the diagonal sequence of functions:

$$H = \left(f_{\sigma_1(1)},\; f_{\sigma_1\circ\sigma_2(2)},\; f_{\sigma_1\circ\sigma_2\circ\sigma_3(3)}, \cdots,\; f_{\sigma_1\circ\cdots\sigma_m(m)},\; f_{\sigma_1\circ\cdots\sigma_m\circ\sigma_{m+1}(m+1)},\; \cdots\right).$$

I claim: Abandoning the first $(m-1)$ terms of the diagonal sequence, what we obtain, i.e.,

$$\left(f_{\sigma_1\circ\cdots\sigma_m(m)},\; f_{\sigma_1\circ\cdots\sigma_m\circ\sigma_{m+1}(m+1)}, \cdots\right)$$

is a subsequence of

$$g_m = \left(f_{\sigma_1\circ\cdots\sigma_m(1)}, f_{\sigma_1\circ\cdots\sigma_m(2)}, \cdots\right).$$

Indeed, observe that:

(i) g_n is a subsequence of g_{n-1} for $n = 2, 3, 4, \cdots$

(ii) $\left(f_{(\sigma_1\circ\cdots\sigma_m)(n)}(x_m)\right)_{n\in\mathbb{N}}$ is convergent.

Thus, the sequence (H), excepting perhaps its first $(m-1)$ terms, is a subsequence of g_m, for $m = 1, 2, \cdots$. Since g_m is convergent at x_m for all m, then H (as a subsequence of g_m) is convergent at x_m for all m, as we wanted to demonstrate. ■

Theorem 9.23 *[Arzelà-Ascoli] Let K be a compact metric space and $f_n \in \mathcal{C}(K)$ for each $n \in \mathbb{N}$. If $(f_n)_{n\in\mathbb{N}}$ is pointwise bounded and equicontinuous on K, then:*

(a) $(f_n)_{n\in\mathbb{N}}$ is uniformly bounded on K.

(b) $(f_n)_{n\in\mathbb{N}}$ contains a uniformly convergent subsequence on K.

Proof: (a) Given $\varepsilon > 0$, since $(f_n)_{n\in\mathbb{N}}$ is equicontinuous on K, there exists $\delta > 0$ such that if $x, y \in K$ with $d(x,y) \leq \delta$ then $|f_n(x) - f_n(y)| < \varepsilon$, for any $n \in \mathbb{N}$.

The open balls $B_\delta(x)$, $x \in K$, cover K. Thus, due to the compactness of K, we can extract a finite subcover, i.e., there exist $p_1,\ p_2,\ \cdots,\ p_r \in K$ such that

$$K \subset B_\delta(p_1) \cup B_\delta(p_2) \cup \cdots \cup B_\delta(p_r).$$

Since by hypothesis $(f_n)_{n\in\mathbb{N}}$ is pointwise bounded, there exist numbers $M_i > 0$, $i = 1, 2, \cdots, r$ such that

$$|f_n(p_i)| \leq M_i,\ \forall n \in \mathbb{N}.$$

On the other hand, taking $x \in K$, $x \in B_\delta(p_i)$ for some $i = 1, \cdots, r$, which implies that $d(x, p_i) < \delta$, for some i, say $i_0 \in \{1, 2, \cdots, r\}$. Then:

$$|f_n(x) - f_n(p_{i_0})| < \varepsilon,\ \forall n \in \mathbb{N},$$

which implies,

$$|f_n(x)| \leq |f_n(p_{i_0})| + \varepsilon,\ \forall n \in \mathbb{N},\ \forall x \in K.$$

Letting $M = \max\{M_1, M_2, \cdots, M_r\}$, it follows that

$$|f_n(x)| \leq M + \varepsilon,\ \forall n \in \mathbb{N},\ \forall x \in K,$$

which proves item (a).

(b) According to Lemma 1.62 (Separability of compact spaces), K has a countable subset E dense in K. Since the sequence $(f_n)_{n\in\mathbb{N}}$ is pointwise bounded, by Cantor's diagonal process, there exists a subsequence $(f_{n_i})_{i\in\mathbb{N}}$ such that $(f_{n_i}(x))$ converges for all $x \in E$. Let us denote $f_{n_i} = g_i$ to simplify notation. We will prove next that g_i converges uniformly on K. Indeed, let $\varepsilon > 0$ be given and take $\delta > 0$ as in the beginning of the proof since $(f_n)_{n\in\mathbb{N}}$ is equicontinuous on K. Note that the balls $B_{\delta/2}(y)$, $y \in K$ form an open cover of K. Therefore, there exist $y_1, \cdots y_m \in K$ such that

$$K \subset B_{\delta/2}(y_1) \cup \cdots \cup B_{\delta/2}(y_m).$$

Let $y \in K$. Then $y \in B_{\delta/2}(y_i)$ for some i, say $i_0 \in \{1, 2, \cdots, m\}$. Then $d(y, y_{i_0}) < \delta/2$. On the other hand, for each y_i, $i = 1, \cdots, m$, due to the density of E in K, there exists $x_i \in E$ such that $d(x_i, y_i) \leq \delta/2$. In particular for $i = i_0$ we will have $d(y_{i_0}, x_{i_0}) \leq \delta/2$. Thus,

$$d(y, x_{i_0}) \leq d(y, y_{i_0}) + d(y_{i_0}, x_{i_0}) \leq \delta/2 + \delta/2 = \delta,$$

which proves that $y \in B_\delta(x_{i_0})$, for some $i_0 \in \{1, 2, \cdots, m\}$, for all $y \in K$, which implies

$$K \subset \cup_{i=1}^m B_\delta(x_i).$$

We know that $g_i = f_{n_i}$ converges for each $x \in E$. Thus, considering the points $x_1, x_2, \cdots, x_m \in E$, we will have, by the Cauchy criterion:

$$|g_i(x_1) - g_j(x_1)| < \varepsilon/3; \ \forall i, j \geq n_0(1),$$
$$\cdots \quad \cdots$$
$$|g_i(x_m) - g_j(x_m)| < \varepsilon/3; \ \forall i, j \geq n_0(m).$$

Let $N := \max\{n_0(1), n_0(2), \cdots, n_0(m)\}$, then for all $i, j \geq N$ we will have

$$|g_i(x_s) - g_j(x_s)| < \varepsilon/3, \ s = 1, 2, \cdots, m. \tag{9.13}$$

Since $K \subset \cup_{i=1}^m B_\delta(x_i)$, then given a generic $x \in K$, there exists $s \in \{1, 2, \cdots, m\}$ such that $x \in B_\delta(x_s)$, which implies $d(x, x_s) < \delta$ for some s, say s_0. Thus, for each $i \in \mathbb{N}$ and by the equicontinuity of g_i:

$$|g_i(x) - g_i(x_{s_0})| < \varepsilon/3, \ \forall i \in \mathbb{N}. \tag{9.14}$$

Then, for all $i, j \geq N$ and for all $x \in K$, it follows from (9.13) and (9.14) that:

$$\begin{aligned} |g_i(x) - g_j(x)| &\leq |g_i(x) - g_i(x_{s_0})| + |g_i(x_{s_0}) - g_j(x_{s_0})| + |g_j(x_{s_0}) - g_j(x)| \\ &\leq \varepsilon/3 + \varepsilon/3 + \varepsilon/3 = \varepsilon. \end{aligned}$$

Then, $(g_i(x))_{i \in \mathbb{N}} = (f_{n_i}(x))_{i \in \mathbb{N}}$ satisfies the necessary and sufficient condition for uniform convergence on K, which proves (b). ■

Theorem 9.24 (Stone-Weierstrass) *If f is a real continuous function on an interval $[a, b]$ of $\mathbb{R}$, there exists a sequence $(P_n)_{n \in \mathbb{N}}$ of polynomials such that $(P_n)_{n \in \mathbb{N}}$ tends to f uniformly on $[a, b]$.*

Proof: We can, without loss of generality, assume $[a, b] = [0, 1]$. Indeed, suppose the theorem proven for continuous functions on $[0, 1]$ and let f be a continuous function on $[a, b]$. Define the change of variables function:

$$x = \varphi(t) = a + (b - a)t \Rightarrow t = \varphi^{-1}(x) = \frac{x - a}{b - a}.$$

Note that $x = \varphi(t)$ is a bijection from $[0, 1]$ to $[a, b]$, and φ is clearly continuous. Letting $g = f \circ \varphi$, then there exists, by hypothesis, a sequence $Q_n(t)$ of polynomials such that for each $\varepsilon > 0$ there exists an index $n_0(\varepsilon)$ such that if $n \geq n_0$:

$$|g(t) - Q_n(t)| \leq \varepsilon; \ \forall t \in [0, 1].$$

This results in:

$$|f(x) - Q_n(\varphi^{-1}(x))| \leq \varepsilon, \ \forall x \in [a,b] \text{ and for } n \geq n_0.$$

Since φ^{-1} is a polynomial in x, $Q_n(\varphi^{-1}(x))$ is a polynomial in x. It suffices to take $(P_n(x))_{n\in\mathbb{N}} = (Q_n(\varphi^{-1}(x)))_{n\in\mathbb{N}}$.

Without loss of generality, we can also assume that $f(0) = f(1) = 0$. Indeed, let f be continuous on $[0,1]$. Define $g(x) = f(x) - f(0) - x(f(1) - f(0))$. Then $g(0) = 0$ and $g(1) = 0$. If there exists a sequence Q_n converging to g, then $P_n(x) = Q_n(x) + f(0) + x(f(1) - f(0))$ converges to f, and P_n is a polynomial.

Let us extend f to the whole line by setting $\tilde{f}(x) = 0$ if $x \notin [0,1]$. Then $\tilde{f}$ is uniformly continuous on $\mathbb{R}$.

On the other hand, we have

$$(1 - x^2)^n \geq 1 - nx^2, \ \forall x \in [0,1] \text{ and } \forall n \in \mathbb{N}^*. \tag{9.15}$$

(This is Bernoulli's inequality).

Let us define, now,

$$Q_n(x) = c_n\,(1 - x^2)^n, \ n \in \mathbb{N}^*,$$

where c_n is chosen such that $\int_{-1}^{1} Q_n(x)\,dx = 1$. We have the estimate:

$$\int_{-1}^{1}(1 - x^2)^n\,dx = 2\int_0^1 (1 - x^2)^n\,dx \geq 2\int_0^{1/\sqrt{n}} (1 - nx^2)\,dx = \frac{4}{3}\frac{1}{\sqrt{n}} > \frac{1}{\sqrt{n}}.$$

Thus, $1 = c_n \int_{-1}^{1}(1 - x^2)^n\,dx > c_n\frac{1}{\sqrt{n}}$, implying $c_n < \sqrt{n}$.

Let $\delta > 0$. For $\delta \leq |x| < 1$, we have $Q_n(x) \leq \sqrt{n}(1-\delta^2)^n$. Letting $1-\delta^2 = \lambda < 1$, we have $\sqrt{n}\lambda^n \to 0$ as $n \to \infty$. This shows that $Q_n(x) \to 0$ uniformly on $\delta \leq |x| \leq 1$.

Let us consider now the convolution:

$$P_n(x) = \int_{-1}^{1} \tilde{f}(x + t)\,Q_n(t)\,dt, \ 0 \leq x \leq 1.$$

By a change of variables $u = x + t$, and using the fact that $\tilde{f}$ vanishes outside $[0,1]$:

$$P_n(x) = \int_0^1 f(t)\,Q_n(t - x)\,dt.$$

Note that $Q_n(t - x) = c_n(1 - (t - x)^2)^n$ is a polynomial in x for fixed t. Thus $P_n(x)$ is a polynomial in x.

Since $\tilde{f}$ is uniformly continuous on $\mathbb{R}$, given $\varepsilon > 0$ there exists $\delta > 0$ such that $|x-y| \le \delta \implies |\tilde{f}(x) - \tilde{f}(y)| < \varepsilon/2$. Let $M = \sup|\tilde{f}|$. For $0 \le x \le 1$:

$$\begin{aligned} |P_n(x) - \tilde{f}(x)| &= \left| \int_{-1}^{1} (\tilde{f}(x+t) - \tilde{f}(x))\, Q_n(t)\, dt \right| \\ &\le \int_{-1}^{1} |\tilde{f}(x+t) - \tilde{f}(x)|\, Q_n(t)\, dt \\ &= \int_{|t|\le\delta} \cdots + \int_{|t|>\delta} \cdots \\ &\le \frac{\varepsilon}{2} \int_{|t|\le\delta} Q_n(t) dt + 2M \int_{|t|>\delta} Q_n(t) dt \\ &\le \frac{\varepsilon}{2}(1) + 2M\sqrt{n}(1-\delta^2)^n. \end{aligned}$$

Since $2M\sqrt{n}(1-\delta^2)^n \to 0$, for large n this is less than $\varepsilon/2$. Thus $|P_n(x) - \tilde{f}(x)| < \varepsilon$, proving uniform convergence. ■

9.6 The Weierstrass Function: A Continuous, Nowhere Differentiable Function

A remarkable consequence of the modern theory of continuous functions is the existence of functions that are continuous everywhere, but differentiable nowhere. This concept was considered counter-intuitive until Karl Weierstrass (1872) constructed the first rigorous example.

Definition 9.25 *The Weierstrass function is defined by the series:*

$$W(x) = \sum_{n=0}^{\infty} a^n \cos(b^n \pi x), \tag{9.1}$$

where a is a real number such that $0 < a < 1$, and b is an odd integer such that the following condition is satisfied:

$$ab > 1 + \frac{3}{2}\pi. \tag{9.2}$$

Theorem 9.26 *The Weierstrass function $W(x)$ is continuous on $\mathbb{R}$ but is differentiable at no point in $\mathbb{R}$.*

Proof: [Proof of Continuity (Sketch)] The continuity follows immediately from the **Weierstrass M-Test** (Theorem 9.7). Let $f_n(x) = a^n \cos(b^n \pi x)$. We can choose the majorant $M_n = a^n$, since:

$$|f_n(x)| = |a^n \cos(b^n \pi x)| \le a^n = M_n, \quad \forall x \in \mathbb{R}.$$

Since $0 < a < 1$, the geometric series $\sum_{n=0}^{\infty} M_n = \sum_{n=0}^{\infty} a^n$ converges. As each term $f_n(x)$ is a continuous function (being a cosine function), the uniform convergence of the series implies that the sum function $W(x)$ is also continuous everywhere (Proposition 9.9). ■

Remark 9.27 *The proof of non-differentiability is significantly more involved and is beyond the scope of this elementary presentation, requiring a careful analysis of the difference quotients based on the specific condition* $ab > 1 + \frac{3}{2}\pi$. *However, the key idea is that the ratio* $a^n b^n$ *increases with* n*, ensuring that the sum of the derivatives of the terms of the series does not converge uniformly, leading to the non-differentiability of the limit function.*

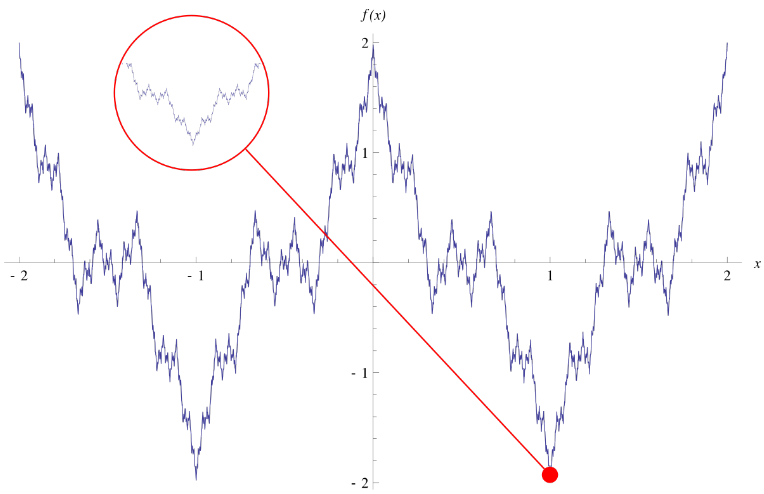


Figure 9.2: Weierstrass function

List of Exercises: Numerical and Function Series

Numerical Series

1. **(Divergence Test and Necessary Condition)** Show that the converse of the theorem "If $\sum a_n$ converges, then $\lim_{n\to\infty} a_n = 0$" is false. Use the harmonic series ($\sum_{n=1}^{\infty} \frac{1}{n}$) as a counterexample and prove its divergence using the Cauchy condensation test or by comparison with the integral $\int_1^{\infty} \frac{1}{x} dx$.

2. **(Absolute and Conditional Convergence)** Determine whether the following series converge absolutely, conditionally, or diverge:

   (a) $\displaystyle\sum_{n=1}^{\infty} (-1)^n \frac{n}{n^2+1}$

   (b) $\displaystyle\sum_{n=1}^{\infty} \frac{\sin(n)}{n^{3/2}}$

   (c) $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n\ln(n)}$

   (d) $\displaystyle\sum_{n=1}^{\infty} (-1)^n \left(1 - \cos\left(\frac{1}{n}\right)\right)$

3. **(Root Test)** Use the Root Test to determine the behavior of $\displaystyle\sum_{n=1}^{\infty} a_n$, where:

$$a_n = \left(\frac{n+1}{2n}\right)^n$$

4. **(Ratio Test)** Use the Ratio Test to determine the behavior of $\displaystyle\sum_{n=1}^{\infty} \frac{n!}{n^n}$.

5. **(Telescoping and Integral Series)** Consider the series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{(n+1)\sqrt{n} + n\sqrt{n+1}}$.

   (a) Show that the general term can be written as a difference: $a_n = \frac{1}{\sqrt{n}} - \frac{1}{\sqrt{n+1}}$.

(b) Calculate the partial sum S_N and demonstrate that the series converges.

(c) Analyze the convergence of the series $\sum_{n=1}^{\infty}\left(\frac{1}{\sqrt{n}}-\frac{1}{\sqrt{n+1}}\right)$ using the Limit Comparison Test, by comparing it with the p-series $\sum \frac{1}{n^p}$, and explain why the convergence of this series does not contradict the divergence of $\sum \frac{1}{\sqrt{n}}$.

6. **(Riemann Rearrangement Theorem)** Let $\sum a_n$ be a conditionally convergent series. Explain, without formal proof, what the Riemann Rearrangement Theorem states about the set of possible sums of its rearrangements. Use the alternating harmonic series $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$ as an example.

Function Series and Uniform Convergence

7. **(Radius of Convergence)** Determine the interval of convergence and the radius of convergence R of the following power series:

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{(x-2)^n}{n^2 3^n}$

(b) $\displaystyle\sum_{n=0}^{\infty} n! x^n$

(c) $\displaystyle\sum_{n=1}^{\infty} \left(1+\frac{1}{n}\right)^n x^n$

8. **(Weierstrass M-Test)** Use the Weierstrass M-Test to prove that the function series $\sum_{n=1}^{\infty} f_n(x)$ converges uniformly on the set A, where:

$$f_n(x) = \frac{\sin(nx)}{n^3} \quad \text{and} \quad A = \mathbb{R}$$

9. **(Pointwise vs. Uniform Convergence and Continuity)** Consider the sequence of functions $f_n : [0,1] \to \mathbb{R}$ given by $f_n(x) = x^n$.

(a) Determine the pointwise limit function $f(x)$.

(b) Prove that the convergence of f_n to f is only pointwise, and not uniform, on $[0,1]$.

(c) Explain how this example illustrates the theorem on the preservation of continuity under uniform convergence.

10. **(Preservation of Differentiability)** Consider the sequence of functions $f_n : \mathbb{R} \to \mathbb{R}$ given by $f_n(x) = \frac{1}{n}\arctan(x^n)$.

    (a) Find the pointwise limit function $f(x)$.

    (b) Calculate the pointwise limit of the derivatives, $g(x) = \lim_{n\to\infty} f_n'(x)$.

    (c) Calculate the derivative of the limit, $f'(x)$.

    (d) Compare $f'(x)$ with $g(x)$ and use the theorem for differentiation of function series to explain why $f'(x) \neq g(x)$ (at least at some points).

11. **(Preservation of Integrability and Dini's Theorem)** Let $f_n : [0,1] \to \mathbb{R}$ be a sequence of **continuous** functions given by $f_n(x) = \frac{x}{1+nx^2}$.

    (a) Determine the pointwise limit function $f(x)$.

    (b) Calculate $\lim_{n\to\infty} \int_0^1 f_n(x)dx$ and $\int_0^1 f(x)dx$.

    (c) Does the result from the previous item imply that the convergence is uniform? Justify.

    (d) Although this sequence is monotonic (non-increasing in n for $x \in (0,1]$ and non-negative), is Dini's Theorem (*Theorem 9.17* in the book) applicable? Explain.

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