Bol. Soc. Paran. Mat.
©SPM – E-ISSN-2175-1188
SPM: www.spm.uem.br/bspm

(3s.) **v. 2026 (44) 6** : 1–3.
ISSN-0037-8712
doi:10.5269/bspm.81141

Order Version of the Riesz Theorem

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ABSTRACT: This paper offers an order-theoretic generalization of the classical Riesz theorem within the setting of Riesz spaces, employing the notion of order compactness as a key conceptual tool (see [2]). The main result showing that a Riesz space endowed with a strictly positive linear functional φ must be finite dimensional if its unit sphere is σ-order compact. Moreover, if φ is order continuous and a box $[-a, a]$ $(0 < a$ $)$ is $\sigma-$order compact then E is finitedimensional.

Keywords: Riesz space, order compactness, strictly positive linear functional, dimension.

Contents

1. Preliminary

Recall that a partially ordered set $(X, \leqslant)$ is a set X equipped with a partial order $\leqslant$. A partially ordered set $(X, \leqslant)$ is a lattice if each pair of elements $x, y \in X$ has a supremum $(x \vee y)$ and an infimum $(x \wedge y)$. An ordered vector space E is a real vector space with an order relation $\leqslant$ that is compatible with the algebraic structure of E in the sense that it satisfies properties:
(i) $x \leqslant y$ implies $x + z \leqslant y + z$ for each $z \in E$; and
(ii) $x \leqslant y$ implies $\alpha x \leqslant \alpha y$ for each $0 \leqslant \alpha$.

An ordered vector space that is also a lattice is called a Riesz space. For a vector x in a Riesz space E, the positive part x^+, the negative part x^-, and the absolute value $|x|$ are defined by $x^+ = x \vee 0$, $x^- = x \wedge 0$ and $|x| = x \vee (-x) = x^+ + x^-$.

Recall that a vector subspace F of a Riesz space E is a Riesz subspace if it is closed under the lattice operations on E (ie. $x \in F$ implies $|x| \in F$). A subset S of a Riesz space is called solid if $|y| \leqslant |x|$ and $x \in S$ imply $y \in S$. An ideal of a Riesz space E is a solid vector subspace of E. A net $\{x_\alpha\}$ in a Riesz space is decreasing, written $x_\alpha \downarrow$, if $\alpha \leqslant \beta$ implies $x_\beta \leqslant x_\alpha$.

For the order convergence of nets, several definitions exist in the literature. We follow Aliprantis [1, p. 322].

Definition 1.1 *A net $\{x_\alpha\}$ in E is order convergent to some $x \in E$, (written $x_\alpha \xrightarrow{o} x$) if there is a net $\{y_\alpha\}$ (with the same directed set) satisfying $y_\alpha \downarrow 0$ and $|x_\alpha - x| \leqslant y_\alpha$ for each α.*

A subset S of a Riesz space is order closed if $\{x_\alpha\} \subset S$ and $x_\alpha \xrightarrow{o} x$ imply $x \in S$. A solid set S is order closed if and only if $0 \leqslant x_\alpha \uparrow x$ and $\{x_\alpha\} \subset S$ imply $x \in S$. A band is an order closed ideal.

A Riesz space E is order complete if every nonempty subset that is order bounded from above has a supremum. (Equivalently, if every nonempty subset that is bounded from below has an infimum). For a band B in an order complete Riesz space E, we denote by δ_B the generalized distance to B which is the mapping $x \rightarrow \inf\{|x - b| : b \in B\}$ from E to E^+. Obviously, δ_B is order continuous (which clearly derives from inequality $||x| - |y|| \leqslant |x - y|$ for all $x, y \in E$). Moreover, any element x of E, satisfies

$$x \in B \Leftrightarrow \delta_B(x) = 0$$

Compactness is one of the most useful tools in functional analysis. This notion is extended to Riesz spaces in [2] as follows.

2020 *Mathematics Subject Classification:* 26A51, 32F17.
Submitted December 31, 2025. Published February 17, 2026

Definition 1.2 *A subset K of a Riesz space E is:*

- *order compact if every net of E has an order convergent subnet with limit point belonging to K.*
- *$\sigma-$order compact if every sequence of E has an order convergent subsequence with limit point belonging to K.*

A linear functional $\varphi : E \to \mathbb{R}$ on a Riesz space E is strictly positive if $0 < x$ implies $0 < \varphi(x)$. For instance the Riemann integral is a strictly positive linear functional on $C[0,1]$. Amazingly, there are Riesz spaces that have no strictly positive linear functional: On $\mathbb{R}^{\mathbb{N}}$, the Riesz space of all real sequences, there are no strictly positive linear functionals. In Proposition 2.1 below, we will show that this is the case for any non-normable Riesz space.

2. Order Riesz Theorem

Riesz's theorem states that if E is a normed vector space such that the unit ball is relatively compact, iff E is finite-dimensional. We extend this result to Riesz spaces.

In the sequel E, denotes an order complete Riesz space.

Lemma 2.1 *Assume that E admits a strictly positive linear functional φ. Then E is normable and becomes a Banach lattice. Moreover, its norm is order continuous if and only if φ is also order continuous.*

Proof: Consider the mapping $|| \, ||_\varphi : E \to [0, +\infty[$ defined for all $x \in E$ by $||x||_\varphi = \varphi(|x|)$. It is easy to check that $|| \, ||_\varphi$ is a Riesz norm on E, which is also order continuous whenever φ is order continuous.

Conversely, assume that $|| \, ||_\varphi$ is order continuous, that is, $x_\alpha \overset{o}{\to} 0$ in E implies $\varphi(|x_\alpha|) = ||x_\alpha||_\varphi \to 0$ in $\mathbb{R}$. The conclusion follows from the fact that $|\varphi(x_\alpha)| \leqslant |\varphi|(|x_\alpha|) = \varphi(|x_\alpha|)$. □

In functional analysis, several spaces are non-normable, for which the following is given.

Proposition 2.1 *A non-normable Riesz space cannot admit a strictly positive linear form.*

Proof: A contradiction arises by applying Lemma 2.1. □

Lemma 2.2 *Let B be a band in E such that $B \neq E$. Then for every $n \in \mathbb{N}^*$, there is $a_n \in E$ such that $\varphi(|a_n|) = 1$ and $(1 - \frac{1}{n})|a_n| \leqslant \delta_B(a_n)$.*

Proof: Let $v \in E$ such that $v \notin B$. Since B is a band and E order complete, it follows that $\delta_B(v) \in E^+ - \{0\}$. By the countable infimum property satisfied by E possessing φ (see [1, Theorem 8.22, p. 326]), for every $n \in \mathbb{N}^*$ there is $x_n \in B$ such that

$$0 < \delta_B(v) \leqslant |v - x_n| < \frac{1}{1 - \frac{1}{n}} \delta_B(v)$$

Then $\varphi(|v - x_n|) \geqslant \varphi(\delta_B(v)) > 0$. Since $\ker(\varphi)$ clearly becomes an ideal, then $\varphi(v - x_n) \neq 0$. Put $a_n = \frac{1}{\varphi(|v - x_n|)}(v - x_n)$, which satisfies $\varphi(|a_n|) = 1$. Assume that $w \in B$,

$$\begin{aligned}
|a_n - w| &= \left|\frac{1}{\varphi(|v - x_n|)}(v - x_n) - w\right| \\
&= \frac{1}{\varphi(|v - x_n|)} |(v - x_n) - \varphi(|v - x_n|)\, w| \\
&\geqslant \frac{1}{\varphi(|v - x_n|)} \delta_B(v) \text{ (because } x_n + \varphi(|v - x_n|)\, w \in B) \\
&\geqslant \frac{1}{\varphi(|v - x_n|)} \left(1 - \frac{1}{n}\right) |v - x_n|
\end{aligned}$$

Since w is arbitrary in B, we deduce that

$$\delta_B(a_n) \geqslant \frac{1}{\varphi\left(|v-x_n|\right)}\left(1-\frac{1}{n}\right)|v-x_n| = (1-\frac{1}{n})|a_n|.$$

□

In what follows, we shall assume that E admits a strictly positive linear form φ. Recall that when the norm is not order continuous then an order compact subset of E is not necessarily norm-compact [2, p.297]. We denote $\mathbb{S}_\varphi$ the unit sphere of the norm $|| \, ||_\varphi$ in E.

Now we state and prove the result relating finite dimension and $\sigma-$order compactness.

Theorem 2.1 *If in a Riesz space E having a strictly positive linear form φ the unit $|| \, ||_\varphi$-sphere is $\sigma-$order compact, then E is finite-dimensional.*

Proof: Assume, to derive a contradiction, that E is infinite-dimensional, there exists a sequence $(B_n)_n$ of finite-dimensional bands such that $B_n \subsetneq B_{n+1}$. In view of Lemma 2.2, it is easy to construct a sequence $(u_n)_n$ satisfying $u_n \in B_n$, $\varphi\left(|u_n|\right) = 1$ and $|u_n - u_{n-1}| \geqslant \delta_{B_{n-1}}(u_n) \geqslant \left(1-\frac{1}{n}\right)|u_n|$. Since $||u_n||_\varphi = 1$ and the unit sphere $\sigma-$order compact, by passing to a subsequence, we may assume the existence of $b \in \mathbb{S}_\varphi$ such that $u_n \xrightarrow{o} b$. We have $\varphi(b) = 1$ and $|u_n - u_{n-1}| \geqslant \left(1-\frac{1}{n}\right)|u_n|$ implies $b = 0$, a contradiction. □

The converse of this theorem is also true and obvious.

Moreover, if the strictly positive lineare forme φ is order continuous then we have the following.

Lemma 2.3 *If there exists $0 < a \in E$ such that $[-a, a]$ is $\sigma-$order compact, then the unit sphere $\mathbb{S}_\varphi$ is $\sigma-$order compact, whenever φ is order continuous.*

Proof: Replacing φ by $\frac{1}{\varphi(a)}\varphi$, we can suppose that $\varphi(a) = 1$. Obviously $\mathbb{S}_\varphi$ is order closed subset of $[-a, a]$. It follows from [2, Proposition 2.1 (iii)] that $\mathbb{S}_\varphi$ is $\sigma-$order compact. □

Lemma 2.3 and Theorem 2.1 lead to the following.

Theorem 2.2 *Suppose that E having an order continuous strictly positive linear form φ and a box $[-a, a]$ is $\sigma-$order compact for some $0 < a \in E$, then E is finite-dimensional.*

Conflict of interest statement

On behalf of all authors, the corresponding author states that there is no conflict of interest.

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